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PROBABILITIES OF EXCESSIVE DEVIATIONS OF
SIMPLE LINEAR RANK STATISTICS

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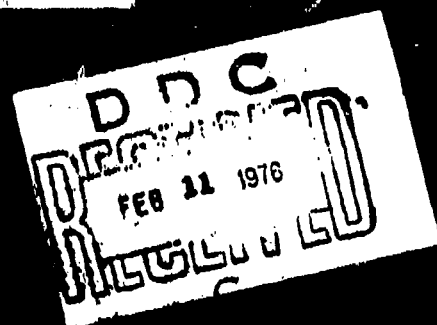
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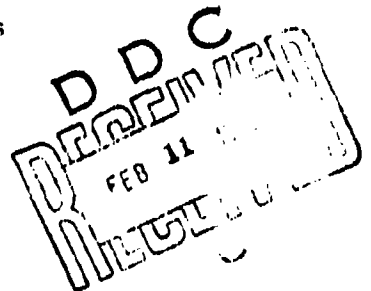
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SUMMARY

Probabilities of Excessive Deviations of Simple Linear Rank Statistics

Let $\{S_N\}$ be a sequence of r.v.'s whose asymptotic distribution is $N(0, \sigma_N^2)$ and let $\{x_N\}$ be a sequence of constants with $x_N \rightarrow \infty$. A (righthand) excessive deviation is an event of the form $\{S_N > x_N \sigma_N\}$. The asymptotic normality of S_N tells us $P\{S_N > x_N \sigma_N\} \rightarrow 0$, $N \rightarrow \infty$, but not the rate of this convergence. These rates are needed for the evaluation of Bahadur ($x_N = x\sqrt{N}$) and Bayes Risk ($x_N = x\sqrt{\log N}$) efficiencies. When S_N is a k -sample linear rank statistic (see Hájek and Šidák (1967)), and $x_N^2 = o(N)$, we show $\log P\{S_N > x_N \sigma_N\} \sim -x_N^2/2$, assuming the null hypothesis. In the two-sample case, we establish also that $P\{S_N > x_N \sigma_N\} = o\{\exp[-x_N^2/2 + J(x_N, \lambda_N)]\}$, when $N\lambda_N$ ($0 < \lambda_N < 1$) is the size of the first sample and J is a function whose behavior is analyzed for various x_N and λ_N . For example, if $x_N^4 = o(N)$, then $J(x_N, \lambda_N) = o(1)$ as $N \rightarrow \infty$.

1. Introduction.

Simple linear rank statistics arise in a variety of situations, particularly in the problem of testing the equality of two or more distributions by non-parametric tests. While the asymptotic distributions of simple linear rank statistics have been studied extensively (see, e.g., Hájek and Šidák (1967)), investigations concerning their large deviation properties have been made only recently (Stone (1967, 1968, 1969) and Woodworth (1970)). In this paper we study the rates of convergence to zero of null probabilities of excessive deviations of k -sample ($k \geq 2$) simple linear rank statistics. We begin in this section with discussions of the notions of an excessive deviation and of a simple linear rank statistic. The results are in Section 2 and the proofs are in Section 3.

The concept of an excessive deviation of a random variable will be discussed here in a general setting. Let $\{S_N, N > 1\}$ be a sequence of random variables with positive finite variances and

let $\{\sigma_N, N \geq 1\}$ be a sequence of positive constants such that

$$(1.1) \quad \sup_x |P(S_N \leq x\sigma_N) - \Phi(x)| \rightarrow 0, N \rightarrow \infty,$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Let $\{x_N, N \geq 1\}$ be a sequence of nonnegative real numbers.

We call $P_N(x_N) = P(S_N > x_N\sigma_N)$ the probability of a (right-hand) deviation of size x_N of S_N . A trivial consequence of (1.1) is

$$(1.2) \quad P_N(x_N) \sim \Phi(-x_N), N \rightarrow \infty,$$

provided $x_N = O(1)$. However, when $x_N \rightarrow \infty$, (1.2) is clearly no longer a direct consequence of (1.1), which leads one to ask: given $\{S_N\}$ and $\{x_N\}$, where $x_N \rightarrow \infty$ at some specified rate, does (1.2) hold? If not, what exactly is the asymptotic behavior of $P_N(x_N)$? To answer these questions, tools more refined than the central limit theorem are needed.

We will now introduce some terminology, due to Rubin and Sethuraman (1965a). If $x_N = O(1)$, the event $\{S_N > x_N\sigma_N\}$ is called an ordinary deviation of S_N , while, if $x_N \rightarrow \infty$, it is an excessive deviation of S_N . Two special cases of excessive deviations have

separate names, because of the applications they have found in statistics. The case where $x_N^2/\log N \rightarrow c^2$, $0 < c < \infty$ is a moderate deviation of S_N , which arises in the study of Bayes risk efficiency (see Rubin and Sethuraman (1965b) and Clickner (1972)). The most extensively studied excessive deviation is the large deviation, in which $x_N^2/N \rightarrow c^2$. This attention arises from the fact that probabilities of large deviations must be evaluated in order to compute Bahadur efficiencies (see Bahadur (1960)).

Much of the previous work on excessive deviations has been for sums of independent random variables. When S_N is the mean of i.i.d. random variables with zero mean, unit variance and finite moment generating function, and $\sigma_N = N^{-1/2}$, Gramér (1938) has shown

$$(1.3) \quad P_N(x_N) \sim \phi(-x_N) \exp[x_N^3 N^{-1/2} \lambda(x_N N^{-1/2})],$$

where $x_N \rightarrow \infty$, $x_N^2 = O(N)$, and where $\lambda(z)$ is a function which admits a convergent power series expansion for small $|z|$. A corollary of (1.3) is that (1.2) holds if $x_N^6 = o(N)$.

Feller (1943) generalized Cramér's (1938) result to non-identically distributed random variables. Rubin and Sethuraman (1965a) considered only moderate deviations but were able to relax the requirement of a moment generating function. Other work on the problem has been done by Chernoff (1958), Linnik (1960, 1961, 1962), and others.

Only right-hand deviations $\{S_N > x_N \sigma_N\}$ have been discussed here. This practice will be continued throughout this paper. All results for right-hand excessive deviations have immediate extensions to left-hand deviations $\{S_N < -x_N \sigma_N\}$ and two-sided deviations $\{|S_N| > x_N \sigma_N\}$.

We will now define a simple linear rank statistic. Let $X_{11}, \dots, X_{1n_1}, \dots, X_{k1}, \dots, X_{kn_k}$ be a sequence of independent and continuous random variables, where $k \geq 2$, $n_i \geq 1$, $i = 1, \dots, k$ and $n_1 + \dots + n_k = N$. Let $R_{11}, \dots, R_{kn_k}$ be the ranks for the combined sample $X_{11}, \dots, X_{kn_k}$. Consider the problem of testing the null hypothesis H : $X_{11}, \dots, X_{kn_k}$ are identically distributed versus the k -sample alternative with density

$$(1.4) \quad q(x_{11}, \dots, x_{kn_k}) = \prod_{i=1}^k \prod_{j=1}^{n_i} f(x_{ij} - \Delta_i),$$

where f is a known density and $\Delta_1, \dots, \Delta_k$ are known constants.

A locally most powerful test of H versus q is based on the k-sample simple linear rank statistic (see Hájek and Šidák (1967) p. 69).

$$(1.5) \quad S_N = \sum_{i=1}^k \Delta_i \sum_{j=1}^{n_i} a_N(R_{ij})$$

where $a_N(1), \dots, a_N(N)$ are a sequence of constants, called scores.

These simple linear rank statistics include many of the more common and important rank statistics. For example, if $a_N(i) = i/(N+1)$, $i=1, \dots, N$, we obtain a k-sample extension of the classical Wilcoxon rank-sum statistic. When $a_N(i) = 1$ for $i > (N+1)/2$ and -1 for $i \leq (N+1)/2$, S_N becomes a median test. With $a_N(i) = E[\Phi^{-1}(U_N^{(i)})]$, $i=1, \dots, N$, where $U_N^{(1)} < \dots < U_N^{(N)}$ are the order statistics for a sample of size N from the uniform $(0,1)$ distribution, S_N is the Fisher-Yates-Terry-Hoeffding normal scores statistic.

To the authors' knowledge, all previous work on excessive deviations of linear rank statistics has been for the case of large deviations only and under the null hypothesis H (Stone (1967, 1968, 1969) and Woodworth (1970)). In this paper we will be considering only excessive deviations that are not large (i.e. $x_N \rightarrow \infty$ but $x_N^2/N \rightarrow 0$) and only the null hypothesis H . Thus there is no overlap of the present paper with Stone's or Woodworth's work. In fact, our results fill in a gap between their work and the work of these many authors who have studied

the asymptotic normality of simple linear rank statistics (see Hájek and Šidák (1967) for references.

2. Main Results.

We need some notation. Let $\lambda_{Ni} = n_i/N$, $i=1, \dots, k$. Let $\mu_{\Delta N} = \lambda_{N1}\Delta_1 + \dots + \lambda_{Nk}\Delta_k$ and

$$(2.1) \quad \sigma_{\Delta N}^2 = \sum_{i=1}^k \lambda_{Ni}(\Delta_i - \mu_{\Delta N})^2.$$

We shall assume

$$(2.2) \quad 0 < \liminf_{N \rightarrow \infty} \lambda_{Ni} \leq \limsup_{N \rightarrow \infty} \lambda_{Ni} < 1, \quad i = 1, \dots, k,$$

and $\sigma_{\Delta N}^2 > 0$ for all $N \geq k$. Further, let $\sum_{j=1}^N a_N(j) = 0$, and $\sum_{j=1}^N a_N^2(j) = N$.

Let S_N be as defined in (1.5). Throughout the rest of this paper we will assume that the null hypothesis H holds. Then $ES_N = 0$ and $\text{var } S_N = N^2 \sigma_{\Delta N}^2 / (N-1)$. We define $P_N(x_N)$ to be

so we define $P_N(x_N)$ to be

$$(2.3) \quad P_N(x_N) = P\{S_N \geq x_N \sigma_{\Delta N}^{1/2}\}.$$

We can now state our main results.

Theorem 2.1. Let S_N be of the form (1.5), assume the null hypothesis H obtains and define $P_N(x_N)$ by (2.3), where $x_N \rightarrow \infty$ and $x_N^2/N \rightarrow 0$. Assume

$$(2.4) \quad x_N^2 \max_{1 \leq j \leq N} a_N^2(j) = o(N)$$

and

$$(2.5) \quad \gamma_N^3 = \sum_{j=1}^N |a_N(j)|^3/N = o(N^{1/2}/x_N).$$

Then, as $N \rightarrow \infty$,

$$(2.6) \quad \log P_N(x_N) \sim -x_N^2/2.$$

This crude estimate of $P_N(x_N)$ can be improved somewhat in the two-sample case ($k = 2$), provided (2.4) and (2.5) are strengthened a little. When $k = 2$, without loss of generality, we may set $\Delta_1 = 1$, $\Delta_2 = 0$, which gives (with $n=n_1$),

$$(2.7) \quad S_N = \sum_{j=1}^n a_N(R_{1j}).$$

Letting $\lambda_N = \lambda_{N1}$ and $\bar{\lambda}_N = \lambda_{N2}$, we get $\mu_{\Delta N} = \lambda_N$ and $\sigma_{\Delta N}^2 = \lambda_N \bar{\lambda}_N$, so (2.3) becomes

$$(2.8) \quad P_N(x_N) = P\{S_N \geq x_N (N \lambda_N \bar{\lambda}_N)^{1/2}\}.$$

We now state our additional results for the two-sample case.

Observe that (2.9) includes both (2.4) and (2.5).

Theorem 2.2. Let the null hypothesis H obtain, let S_N be of the form (2.7), and define $P_N(x_N)$ by (2.8), where $x_N \rightarrow \infty$, $x_N^2/N \rightarrow 0$.

If

$$(2.9) \quad \gamma_N^3 = o(N^{1/2}/x_N^3),$$

then, as $N \rightarrow \infty$,

$$(2.10) \quad P_N(x_N) = o\{\exp[-x_N^2/2 + J(x_N, \lambda_N)]\},$$

where

$$(2.11) \quad J(x_N, \lambda_N) = (x_N^2/2)(1 - \lambda_N \bar{\lambda}_N / p_N q_N) + N I(\lambda_N, p_N),$$

where

$$(2.12) \quad I(\lambda, p) = \lambda \log(\lambda/p) + \bar{\lambda} \log(\bar{\lambda}/q),$$

where $q = 1-p$, $q_N = 1-p_N$, and p_N is the unique solution of the equation

$$(2.13) \quad n = \sum_{j=1}^N \{1 + (q_N/p_N) \exp[-x_N^2(j) (\lambda_N \bar{\lambda}_N / N)^{1/2} / p_N q_N]\}^{-1}.$$

Theorems 2.1 and 2.2 are the best results obtainable with pre-

scently available methods. However, it would be desirable to improve on these results to obtain a better estimate of $P_N(x_N)$, one comparable to Cramér's (1938) result (1.3). To do this in the two-sample case a certain rate of convergence to normality of the normed distribution $\Phi_N(\cdot)$ of the mean of certain dependent random variables arising in the proof of Theorem 2.2 is required. This convergence rate has not been established. Let

$$(2.14) \quad \Delta_N = \sup_x |\Phi_N(x) - \Phi(x)|,$$

where $\Phi_N(\cdot)$ is defined in (3.39). It follows from Hájek (1964) that $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$. His result is reproduced here as Lemma 3.7. The required convergence rate is $\Delta_N x_N = o(1)$ where $x_N \rightarrow \infty$ (See (3.46)). If it can be verified that this rate holds then the following estimate is obtainable:

$$(2.15) \quad P_N(x_N) \sim \Phi(-x_N) \exp[J(x_N, \lambda_N)], \quad N \rightarrow \infty.$$

To extend Theorem 2.2 to the case $k > 2$, a multivariate generalization of Hájek's (1964) asymptotic normality result is required. Unfortunately, it too, is not yet available.

The conditions of Theorems 2.1 and 2.2 are stronger than those required to establish the asymptotic normality of simple linear rank statistics S_N (See, e.g., Hájek and Šidák (1967) pp. 193-195). This is to be expected since these theorems give

more detailed and precise information about the asymptotic behavior of S_N than does a statement of asymptotic normality. Even so, it is natural to ask whether these stronger conditions are very restrictive, in terms of potential applications. Many simple linear rank statistics S_N have scores of either of the forms

$$(2.16) \quad a_N(i) = E \phi(U_N^{(i)})$$

or

$$(2.17) \quad \hat{a}_N(i) = \phi(i/N+1),$$

where ϕ is a non-constant function on $(0,1)$ and, further, S_N is asymptotically normal if $\int_0^1 \phi^2(u) du < \infty$ (See, e.g., Hájek and Sidák (1967) Chapter V). Clearly, if

$$(2.18) \quad \int_0^1 |\phi(u)|^3 du < \infty,$$

then (2.9) holds for the scores a_N and $\hat{a}_N$ and we may apply Theorems 2.1 and 2.2 to S_N . Some simple linear rank statistics that have scores of the form (2.16) or (2.17) where ϕ satisfies (2.18) are: median, Wilcoxon, Van der Waerden and Fisher-Yates-Terry-Hoeffding, all tests for location shift, and Capen, Klotz, Ansari-Bradley, quartile, and Savage, all tests for a shift in scale. See Clickner (1972) for details.

It is not immediately obvious from Theorem 2.2 as presented, exactly how fast $P_N(x_N)$ tends to zero when x_N grows at a specified rate, say $x_N^2 = \log N$, or perhaps $x_N^6 = N^3$. It is necessary to analyze $J(x_N, \lambda_N)$ for various x_N and λ_N to see more clearly the behavior of $P_N(x_N)$. This is done in Corollaries 2.5 and 2.6 following the preliminary Lemmas 2.3 and 2.4.

Lemma 2.3. Let p_N be the solution of equation (2.13). Let (2.9) hold. Then

$$(2.19) \quad p_N = \lambda_N + \sum_{k=1}^{\infty} c_k(\lambda_N) (x_N^2/N)^k,$$

where $\{c_k(\lambda), k \geq 1\}$ is a sequence of functions of λ whose first two elements are

$$(2.20) \quad c_1(\lambda) = \frac{1}{2} - \lambda,$$

$$(2.21) \quad c_2(\lambda) = (\lambda - \frac{1}{2})(1 - 1/(2\lambda \bar{\lambda})).$$

Further, if $\lambda_N = \frac{1}{2} + O(N^{-1})$, then $p_N = \frac{1}{2} + O(N^{-1})$.

Lemma 2.4. Define $I(\lambda, p)$ by (2.12) for $0 < \lambda < 1$ and $0 < p < 1$.

If $|\lambda - p| < \min(p, q)$, then

$$(2.22) \quad I(\lambda, p) = \frac{1}{2} \left(\frac{\lambda - p}{pq} \right)^2 (2pq - \bar{\lambda}p^2 + \lambda q^2) + \sum_{i=3}^{\infty} \frac{(\lambda - p)^i}{i} \left[\frac{\bar{\lambda}}{q^i} + \frac{\lambda}{(-p)^i} \right].$$

Corollary 2.5. In Theorem 2.2 assume $\lambda_N = \lambda_0 + O(1/N)$. Then

$$J(x_N, \lambda_N) = o(1).$$

Corollary 2.6. Under the conditions of Theorem 2.2,

$$(2.23) \quad J(x_N, \lambda_N) = \frac{x_N^4}{N} \sum_{i=0}^{\infty} d_i(\lambda_N) \left(\frac{x_N^2}{N}\right)^i,$$

where $(d_i(\lambda), i \geq 0)$ is a sequence of functions of λ whose first element is

$$(2.24) \quad d_0(\lambda) = -(1-2\lambda)^2/8\lambda \bar{\lambda}.$$

Further, if, for some integer $k = 1, 2, \dots$,

$$(2.25) \quad \liminf(x_N^{2(k+1)})^{-k} > 0 \text{ and } x_N^{2(k+2)} = o(x_N^{k+1}),$$

then

$$(2.26) \quad J(x_N, \lambda_N) = o(1), \text{ if } k = 0$$

$$= \frac{x_N^4}{N} \sum_{i=0}^{k-1} d_i(\lambda_N) \left(\frac{x_N^2}{N}\right)^i + o(1), \text{ if } k > 0.$$

3. Proofs.

We begin the proofs with some preliminaries and four lemmas which will be used in proving both Theorems 2.1 and 2.2.

To simplify notation, we will often write a_i for $a_N(i)$, λ_i for λ_{Ni} , σ_Δ for $\sigma_{\Delta N}$, etc., suppressing the dependence on N . Let $W_{(1)} < \dots < W_{(N)}$ be the order statistics for the combined sample $X_{11}, \dots, X_{kn_k}$, and let

$$(3.1) \quad C_{ij} = \begin{cases} 1 & \text{if } W_{(j)} \text{ is from the } i\text{-th sample,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, k$, $j = 1, \dots, N$. Clearly, $C_{i1} + \dots + C_{iN} = n_i$, $i = 1, \dots, k$. Then, by rearranging the sum,

$$(3.2) \quad S_N = \sum_{j=1}^N a_j \sum_{i=1}^k \Delta_i C_{ij}.$$

Let $\Psi = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ be a set consisting of k points with $z = (z_1, \dots, z_k)$ a typical element and let $Z^{(1)}, \dots, Z^{(N)}$ be a sequence of independent and identically distributed random vectors taking values in Ψ with probabilities $p_1, \dots, p_k$, $p_i > 0$, $i = 1, \dots, k$, $p_1 + \dots + p_k = 1$, that is,

$$(3.3) \quad P\{Z^{(1)} = z\} = \prod_{i=1}^k p_i^{z_i}.$$

Let $Z(N) = Z^{(1)} + \dots + Z^{(N)} = [Z_1(N), \dots, Z_k(N)]$. Define a statistic T_N by

$$(3.4) \quad T_N = \sum_{j=1}^N u_j \sum_{i=1}^k \Delta_i Z_i^{(j)},$$

and let t_N be a realization of T_N . The key to these proofs is the observation that

$$(3.5) \quad P\{S_N = t_N\} = P\{T_N = t_N | Z(N) = \underline{n}\},$$

where $\underline{n} = (n_1, \dots, n_k)$, and, further, (3.5) is an identity in the probabilities $p_1, \dots, p_k$.

For each $h > 0$, define a new joint distribution for the random vectors $Z^{(1)}, \dots, Z^{(N)}$ by

$$(3.6) \quad Q\{Z^{(j)} = z^{(j)}, j=1, \dots, N\} = \prod_{j=1}^N \prod_{i=1}^k q_{ij}^{z_i^{(j)}}$$

where

$$(3.7) \quad q_{ij} = p_i e^{h a_j \Delta_i} \left[\sum_{i=1}^k p_i e^{h a_j \Delta_i} \right]^{-1}.$$

Observe that, under Q , $Z^{(1)}, \dots, Z^{(N)}$ are independent random vectors taking values in Ψ , but are not identically distributed.

Lemma 3.1. Let S_N be given by (3.2), T_N by (3.4), and $P_N(x_N)$ by (2.3). For any $h > 0$ and any $p_1, \dots, p_k$ with $p_i > 0$, $i = 1, \dots, k$, $p_1 + \dots + p_k = 1$, we have

$$(3.8) \quad P_N(x_N) = \frac{Q\{Z(N) = n\}}{P\{Z(N) = n\}} \prod_{j=1}^N \left[\sum_{i=1}^k p_i e^{h a_{ij} \Delta_i} \right] \cdot A_N,$$

where

$$(3.9) \quad A_N = \sum_{x_N} e^{-h t_N} Q\{T_N = t_N | Z(N) = n\},$$

and $\sum_{x_N}$ denotes summation over those t_N satisfying $t_N \geq x_N \sigma \Delta^{1/2}$.

Proof. From (3.6) and (3.7),

$$(3.10) \quad P_N(x_N) = [P\{Z(N) = n\}]^{-1} \sum_{x_N} P\{T_N = t_N, Z(N) = n\} \\ = \frac{\prod_{j=1}^N \left[\sum_{i=1}^k p_i e^{h a_{ij} \Delta_i} \right]}{P\{Z(N) = n\}} \sum_{x_N} e^{-h t_N} Q\{T_N = t_N, Z(N) = n\}.$$

which is equal to the right hand side of (3.8). Lemma 3.1 is now proved.

Observe that (3.8) is an identity in the $k+1$ arbitrary variables $p_1, \dots, p_k$ and h . We will later exploit this fact by making convenient choices for these quantities. But first we will obtain asymptotic approximations to all the factors in the right-hand side of (3.8) except A_N . We begin with $P\{Z(N) = n\}$.

Lemma 3.2. For any $p_1, \dots, p_k$ such that $p_i > 0, i = 1, \dots, k$ and $p_1 + \dots + p_k = 1$, we have, under (2.2), as $N \rightarrow \infty$,

$$(3.11) \quad P(Z(N)=\underline{n}) \sim (2\pi N)^{(1-k)/2} \prod_{i=1}^k \lambda_i^{-1/2} \exp[-N \sum_{i=1}^k \lambda_i \log(\lambda_i/p_i)].$$

Proof. Apply Stirling's formula.

Lemma 3.3. Let $q_{i\ell} = (q_{i1} + \dots + q_{iN})/N$ and $\underline{\Sigma}_j = (\sigma_{i\ell}^{(j)}, i, \ell = 1, \dots, k-1)$, where

$$\begin{aligned} \sigma_{i\ell}^{(j)} &= q_{ij}(1-q_{ij}) & \text{if } i = \ell \\ &= -q_{ij}q_{\ell j} & \text{if } i \neq \ell. \end{aligned}$$

Assume $\lim_{N \rightarrow \infty} [\sum_{j=1}^N (N^{-1} \underline{\Sigma}_j)] = \underline{\Sigma}$ (say) is a positive definite matrix. Then we have, uniformly in $\underline{n}$,

$$(3.12) \quad (2\pi N)^{(k-1)/2} (\det \underline{\Sigma})^{1/2} Q(Z(N)=\underline{n}) - \exp\{-\frac{1}{2} \underline{\xi}'_N \underline{\Sigma}^{-1} \underline{\xi}_N\} \rightarrow 0, N \rightarrow \infty,$$

where $\underline{\xi}'_N = \{N^{-1/2}(n_1 - Nq_{1\cdot}), \dots, N^{-1/2}(n_{k-1} - Nq_{k-1\cdot})\}$.

Proof. Let $\underline{\Xi}_N = \{N^{-1/2}[Z_1(N) - Nq_{1\cdot}], \dots, N^{-1/2}[Z_{k-1}(N) - Nq_{k-1\cdot}]\}$.

Clearly $\underline{\Xi}_N$ is asymptotically normal with zero mean vector and covariance matrix $\underline{\Sigma}$. Rvaceva (1954) has proven a local limit theorem for sums of i.i.d. lattice vectors. To prove (3.12) we follow her argument in outline, varying the details to handle our non-identically distributed vectors and taking advantage of the special structure of $Z(N)$. See Clickner (1972) for the details.

We now need some notation. Let E_Q , var_Q , etc., denote expectation, variance, etc. under Q . Then

$$(3.13) \quad E_Q T_N = \sum_{j=1}^N a_j \sum_{i=1}^k \Delta_i q_{ij}$$

and

$$(3.14) \quad v_T^2 = \text{var}_Q T_N = \sum_{j=1}^N a_j^2 \sum_{i=1}^k \Delta_i^2 q_{ij} (1 - q_{ij}).$$

Also, define $v_\Delta = p_1 \Delta_1 + \dots + p_k \Delta_k$ and

$$(3.15) \quad \tau_\Delta^2 = \sum_{i=1}^k p_i (\Delta_i - v_\Delta)^2.$$

The motivation for the bound on h in Lemma 3.4 will be made clear in (3.24) and (3.25).

Lemma 3.4. Let $h = h_N$, where $h_N \leq Kx_N/N^{1/2}$, for some K , $0 < K < \infty$.

Assume $x_N \rightarrow \infty$, $x_N^2/N \rightarrow 0$, and (2.4) obtains. Then, as $N \rightarrow \infty$,

$$(3.16) \quad \sum_{j=1}^N \log \left(\sum_{i=1}^k p_i e^{h_N a_j \Delta_i} \right) = N h_N^2 \tau_\Delta^2 / 2 + O(N h_N^3 \gamma_N^3),$$

$$(3.17) \quad q_{ij} = p_i + \frac{1}{2} p_i h_N^2 [(\Delta_i - v_\Delta)^2 - \tau_\Delta^2] + O(h_N^3 \gamma_N^3), \quad i=1, \dots, k,$$

$$(3.18) \quad E_Q T_N = N h_N \tau_\Delta^2 + O(N h_N^2 \gamma_N^3),$$

$$(3.19) \quad v_T^2 = N \sum_{i=1}^k \Delta_i^2 p_i (1 - p_i) + O(N h_N \gamma_N^3),$$

$$(3.20) \quad \sum_{j=1}^N q_{ij}(1-q_{ij})/N = p_i(1-p_i) + O(h_N^2), \quad i=1, \dots, k,$$

$$(3.21) \quad \sum_{j=1}^N q_{ij}q_{lj}/N = p_i p_l + O(h_N^2), \quad i \neq l.$$

Proof. Consider the left-hand side of (3.16):

$$\begin{aligned} \sum_{i=1}^k p_i e^{h_N a_j \Delta_i} &= 1 + h_N a_j v_\Delta + \frac{1}{2} h_N^2 a_j^2 \sum_{i=1}^k p_i \Delta_i^2 \\ &\quad + \frac{\theta}{6} h_N^3 |a_j|^3 \sum_{i=1}^k |\Delta_i|^3 e^{h_N |a_j \Delta_i|}, \end{aligned}$$

where $|\theta| \leq 1$. Then

$$\begin{aligned} \sum_{j=1}^N \log \left(\sum_{i=1}^k p_i e^{h_N a_j \Delta_i} \right) &= \sum_{j=1}^N \{ h_N a_j v_\Delta + \frac{1}{2} h_N^2 a_j^2 \Delta^2 + O(h_N^3 |a_j|^3) \} \\ &= \frac{1}{2} h_N^2 N T_\Delta^2 + O(h_N^3 \sum_{j=1}^N |a_j|^3) \\ &= N h_N^2 T_\Delta^2 / 2 + O(N h_N^3 \gamma_N^3). \end{aligned}$$

Expressions (3.17) - (3.21) follow similarly. The proof of Lemma 3.4 is complete.

Proof of Theorem 2.1

In addition to Lemmas 3.1-3.4, we need two lemmas specifically for Theorem 2.1.

Lemma 3.5. Define A_N by (3.9). Then for each N and any $h > 0$,

$$(3.22) \quad 0 \geq \log A_N + hx_N \sigma_{\Delta} N^{\frac{1}{2}} \\ \geq -4V_T h + \log\{1 - E_Q[(T_N - 2V_T - x_N \sigma_{\Delta} N^{\frac{1}{2}})^2 | Z(N) = \underline{n}] / 4V_T^2\}.$$

Proof. Clearly $A_N \leq \exp[-hx_N \sigma_{\Delta} N^{\frac{1}{2}}]$. On the other hand,

$$A_N \geq \exp[-hx_N \sigma_{\Delta} N^{\frac{1}{2}} - 4V_T h] Q(x_N \sigma_{\Delta} N^{\frac{1}{2}} \leq T_N \leq x_N \sigma_{\Delta} N^{\frac{1}{2}} + 4V_T | Z(N) = \underline{n}) \\ \geq \exp[-hx_N \sigma_{\Delta} N^{\frac{1}{2}} - 4V_T h] \{1 - E_Q[(T_N - x_N \sigma_{\Delta} N^{\frac{1}{2}} - 2V_T)^2 | Z(N) = \underline{n}] / 4V_T^2\},$$

by Chebyshev's inequality. Lemma 3.5 follows.

We will now choose values for $p_1, \dots, p_k$ and h for the k -sample case. Let

$$(3.23) \quad p_i = \lambda_i, \quad i = 1, \dots, k$$

and let $h = h_N$ be the unique solution of

$$(3.24) \quad E_Q T_H = x_N \sigma_{\Delta} N^{\frac{1}{2}} + 2V_T.$$

This choice of h_N maximizes a term in the lower bound of A_N in Lemma 3.5. A simple argument shows that h_N is well defined by (3.24), and, further

$$(3.25) \quad h_N \sim x_N / \sigma_\Delta N^{1/2}.$$

Lemma 3.6. Let (3.23) hold and let h_N be the solution of (3.24), satisfying (3.25). Then, as $N \rightarrow \infty$,

$$(3.26) \quad E_Q(T_N | Z(N) = \underline{n}) \sim E_Q T_N$$

and

$$(3.27) \quad \text{var}_Q(T_N | Z(N) = \underline{n}) \sim v_T^2.$$

Proof. Consider, for $i = 1, \dots, k$, $j = 1, \dots, N$,

$$(3.28) \quad Q(Z_i^{(j)} = 1 | Z(N) = \underline{n}) = q_{1j} \frac{Q(Z(N) = \underline{n}_1)}{Q(Z(N) = \underline{n})},$$

where $\underline{n}_1 = (n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_k)$, $i = 1, \dots, k$.

From (3.17), (3.20), (3.21) and (3.23), $\underline{\Sigma} = \lim N^{-1} \sum_{j=1}^N \underline{\Sigma}_j$ has elements of the form $\lambda_i(1-\lambda_i)$ on the main diagonal and $-\lambda_i\lambda_j$ off the main diagonal. Since $\sum_{i=1}^k \lambda_i = 1$, $\det \underline{\Sigma} = \prod_{i=1}^k \lambda_i$. Hence, by Lemma 3.3,

$$(3.29) \quad \frac{Q(Z(N) = \underline{n}_1)}{Q(Z(N) = \underline{n})} \sim \exp\left\{\frac{N}{2\lambda_i} [(\lambda_i - q_{1j})^2 - (\lambda_i - 1/N \cdot q_{1j})^2]\right\}$$

$$= \exp\{(\lambda_i - q_{1j})/\lambda_i - 1/2N\lambda_i\}$$

$$= \exp(O(x_N^2/N)),$$

uniformly in $i = 1, \dots, k, j = 1, \dots, N$. Hence,

$$(3.30) \quad E_Q(T_N | Z(N) = \underline{n}) = \sum_{j=1}^N a_j \sum_{i=1}^k \Delta_i Q(Z_i^{(j)} = 1 | Z(N) = \underline{n}) \sim E_Q T_N.$$

Similarly, for $i, i' = 1, \dots, k, j, j' = 1, \dots, N, j \neq j'$,

$$(3.31) \quad Q(Z_i^{(j)} = 1, Z_{i'}^{(j')} = 1 | Z(N) = \underline{n}) \sim a_{ij} a_{i'j'},$$

uniformly in i, i', j and j' . The joint probability on the left-hand side of (3.31) is zero if $j = j'$. Hence

$$(3.32) \quad E_Q(T_N^2 | Z(N) = \underline{n}) = \sum_{j=1}^N a_j^2 \sum_{i=1}^k \Delta_i^2 Q(Z_i^{(j)} = 1 | Z(N) = \underline{n}) \\ + \sum_{j \neq j'} \sum_{i \neq i'} a_j a_{j'} \Delta_i \Delta_{i'} Q(Z_i^{(j)} = 1, Z_{i'}^{(j')} = 1 | Z(N) = \underline{n}) \sim E_Q T_N^2.$$

Lemma 3.6 follows.

We can now prove Theorem 2.1. This will be done by substituting Lemmas 3.2-3.6 in Lemma 3.1. More specifically, let

(3.23) hold and let $h = h_N$ solve (3.24) which entails

$h_N \sim x_N / N^{1/2} \sigma_A$. From Lemmas 3.2 and 3.3,

$$(3.33) \quad \frac{Q(Z(N) = \underline{n})}{P(Z(N) = \underline{n})} \sim \exp\left[-\frac{N}{2} \sum_{i=1}^k (\lambda_i - q_{i.})^2 / \lambda_i\right] \\ = \exp[O(x_N^4/N)],$$

using (3.17). From Lemma 3.4, specifically (3.16), we have

$$(3.34) \quad \sum_{j=1}^N \log\left(\sum_{i=1}^k \lambda_i e^{h_N a_{j\Delta_i}}\right) \sim x_N^2/2.$$

By the definition of h_N ,

$$E_Q[(T_N - x_N \sigma_{\Delta} N^{\frac{1}{2}} - 2V_T)^2 | Z(N) = \underline{n}] = \text{var}_Q(T_N | Z(N) = \underline{n}) \sim v_T^2,$$

by Lemma 3.6. Hence, from Lemma 3.5,

$$(3.35) \quad \log A_N \sim -x_N^2.$$

Theorem 2.1 now follows by substituting (3.33), (3.34), and (3.35) in Lemma 3.1.

Proof of Theorem 2.2

The main difference between the proofs of Theorems 2.1 and 2.2 lies in the treatment of the sum A_N , defined in (3.9). Here,

we write A_N as $A_N = \exp[-hE_Q T_N] C_N$,

where

$$(3.36) \quad C_N = \int_B^\infty e^{-hVy} d\phi_N(y),$$

where

$$(3.37) \quad \bar{a} = \sum_{j=1}^N a_j q_{1j} (1-q_{1j}) / \sum_{j=1}^N q_{1j} (1-q_{1j}),$$

$$(3.38) \quad v^2 = \sum_{j=1}^N (a_j - \bar{a})^2 q_{1j} (1-q_{1j}),$$

$$(3.39) \quad \phi_N(y) = Q(T_N \leq yV + E_Q T_N | Z(N) = \underline{n}),$$

and

$$(3.40) \quad B = [x_N (\lambda \bar{\lambda}_N N)^{1/2} - E_Q T_N] / V.$$

Observe that $\phi_N(\cdot)$ is the conditional distribution referred to in the discussion following Theorem 2.2 (see equation (2.14)).

Now select p_1 and h . Choose $p_1 = p_N$ to be the solution of

$$(3.41) \quad n = \sum_{i=1}^N \{1 + (q_N/p_N) \exp[-x_N a_1 (\lambda_N \bar{\lambda}_N/N)^{1/2} / p_N q_N]\}^{-1}.$$

This is the p_N of (2.13). Choose

$$(3.42) \quad h = h_N = x_N (\lambda_N \bar{\lambda}_N/N)^{1/2} / p_N q_N.$$

Note that p_N is well-defined by (3.41) and we have $p_N \sim \lambda_N$ and $h_N \sim x_N / (\lambda_N \bar{\lambda}_N/N)^{1/2}$, so these choices are only slightly different from those in the proof of Theorem 2.1 -- but are more convenient for the proof of Theorem 2.2. The reasons for these choices will become apparent in Lemma 3.7 and display (3.45).

Lemmas 3.7 and 3.8 constitute the analysis of A_N .

Lemma 3.7. (Hájek (1964)). Let (3.41) hold. A necessary and sufficient condition for $\Delta_N \rightarrow 0$, where

$$\Delta_N = \sup_x |\phi_N(y) - \phi(y)|$$

and $\phi_N(\cdot)$ is defined in (3.39) is

$$(3.43) \quad v^{-2} \sum_{i \in A_\epsilon} (a_i - \bar{a})^2 \pi_i \bar{\pi}_i \rightarrow 0$$

for all $\epsilon > 0$, where $A_\epsilon = \{i: |a_i - \bar{a}| > \epsilon v\}$.

Proof. This is Theorem 7.1 of Hájek (1964).

Lemma 3.8. Let p_N and h_N be given by (3.41) and (3.42), respectively, and let (2.9) obtain. With C_N as in (3.36), we have, as $N \rightarrow \infty$, $C_N \rightarrow 0$.

Proof. We can write

$$(3.44) \quad C_N = \int_B^\infty e^{-h_N y v} d\phi(y) \\ + e^{-h_N v B} [\phi_N(B) - \phi(B)] + h v \int_B^\infty e^{-h_N y v} [\phi_N(y) - \phi(y)] dy.$$

Techniques similar to those of Lemma 3.4 yield $v^2 \sim N \lambda_N \bar{\lambda}_N$. Consider

$$(3.45) \quad B = v^{-1} [x_N (\lambda_N \bar{\lambda}_N h_N)^{1/2} - E_Q(T_N)] \\ = x_N - p q h (N / \lambda_N \bar{\lambda}_N)^{1/2} + o[(N h^2)^{-1/2}] \\ = o(x_N^{-1}),$$

with h given by (3.41). Hence

$$(3.46) \quad C_N = [1+o(1)][(2\pi x_N^2)^{-1/2} + \Delta_N].$$

It follows from (2.9) that A_ϵ is empty for large N ; hence $\Delta_N \rightarrow 0$ by Lemma 3.7. Since $x_N \rightarrow \infty$, Lemma 3.8 follows.

To complete the proof of Theorem 2.2, apply condition (2.11) and the selections of p_1 and h , (3.42) and (3.43), to Lemmas 3.1 - 3.4 with $k=2$ to obtain

$$(3.47) \quad P_N(x_N) \sim C_N \exp[-x_N^2/2 + J(x_N, \lambda_N)].$$

Proof of Lemmas 2.3 and 2.4

Lemma 2.4 is proved by expanding the logarithms.

In Lemma 2.3, the case $\lambda_N = \frac{1}{2} + O(N^{-1})$ is proved by substituting $p_N = \frac{1}{2} + O(N^{-1})$ to verify that it is a solution of (2.13). Otherwise, recall that

$$(3.48) \quad \lambda_N = p + (p - \frac{1}{2})\lambda_N \bar{\lambda}_N x_N^2 / pqN + o(N^{-1}),$$

using (3.17) and (3.42). Now, suppose $x_N^4 = o(N)$ and propose

$$(3.49) \quad p = \lambda_N + c_1 x_N^2 / N + o(N^{-1})$$

as a solution of (3.48). Substitute (3.49) in (3.48) and solve for

c_1 to obtain $c_1 = c_1(\lambda_N) = \lambda_N$, as in (2.20). Next, suppose $x_N^3 = o(N)$ and try

$$(3.50) \quad p = \lambda_N c_1(\lambda_N) x_N^2 / N + c_2 x_N^4 / N^2 + o(N^{-1})$$

as a solution of (3.48). Again, solve for c_2 to obtain $c_2 = c_2(\lambda_N)$, as in (2.21). The higher order coefficients $\{c_k(\lambda_N)\}$ can be found successively by continuing this iterative procedure. This is not done here because the algebra becomes very cumbersome. The proof of Lemma 2.3 is complete.

Proof of Corollaries 2.5 and 2.6

Corollary 2.5 is an immediate consequence of Lemmas 2.3 and 2.4, and Corollary 2.6 is proved in essentially the same manner as Lemma 2.3.

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