

RELIABILITY ASSESSMENT FOR HIGHLY RELIABLE SYSTEMS

BY

M. V. JOHNS, JR.

TECHNICAL REPORT NO. 1

NOVEMBER 15, 1975

U.S. ARMY RESEARCH OFFICE  
RESEARCH TRIANGLE PARK, NORTH CAROLINA  
CONTRACT NO. DAHCO4-75-G-0172

DEPARTMENT OF STATISTICS  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA



RELIABILITY ASSESSMENT FOR HIGHLY RELIABLE SYSTEMS

by

M. V. JOHNS, JR.

TECHNICAL REPORT NO. 1

NOVEMBER 15, 1975

U.S. ARMY RESEARCH OFFICE  
RESEARCH TRIANGLE PARK, NORTH CAROLINA  
CONTRACT NO. DAHCO4-75-G-0172

DEPARTMENT OF STATISTICS  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA

TABLE OF CONTENTS

|  | <u>Pages</u> |
|--|--------------|
| SECTION 1. Introduction .....                    | 1 - 5        |
| SECTION 2. Confidence Bounds .....               | 6 - 13       |
| SECTION 3. Approximations .....                  | 13 - 18      |
| APPENDIX 1. Proofs of Results of Section 2 ..... | 19 - 25      |
| APPENDIX 2. Proofs of Results of Section 3 ..... | 26 - 30      |
| APPENDIX 3. Remarks Concerning the Tables .....  | 31           |
| REFERENCES .....                                 | 32           |
| TABLES 1-4 .....                                 | i - xiv      |

## RELIABILITY ASSESSMENT FOR HIGHLY RELIABLE SYSTEMS

by M. V. Johns, Jr.

Introduction. The purpose of this study is to develop appropriate methods for the assessment of reliability for highly reliable systems, i.e., systems characterized by failure probabilities no larger than 5%. Particular emphasis is given to cases where the probability of system failure during any particular duty cycle is less than 1% and where substantial amounts of subsystem test data are available. Serial systems consisting of several component subsystems are considered and confidence bounds for the mean number of trials between failures based on subsystem failure data are developed. (The mean number of trials between failures (MTBF) is equal to  $1/(1-R)$  when  $R$  is the system reliability, i.e., the probability of completion of a failure-free duty cycle.) Tables needed for the implementation of the proposed procedure are provided. It is assumed that the failure data available for the component subsystems consists of the numbers of failures observed in a substantial number of independent trials for each subsystem where the numbers of trials may differ for different subsystems. Previously available methods for obtaining confidence bounds for series system reliability have typically required the assumption of exponentially distributed times to failure (see e.g. [3], [10]) or have involved ad hoc approximations (see e.g. [11]). Generally the data are assumed to have been obtained through inverse sampling (Type II censoring), i.e., subsystem testing continues until a predetermined number of failures has been observed—a procedure often difficult to implement in practice (see e.g. [3], [7]). The approach used in the present study is non-Bayesian, thereby avoiding the difficulties associated with subjective

probabilities. The underlying rationale is based on exact methods, although asymptotic approximations are proposed and evaluated.

Consider a system consisting of  $k$  subsystems operating independently in a series configuration. Let  $p_i$  be the probability that the  $i^{\text{th}}$  subsystem will fail on any given trial (duty cycle) and suppose that the subsystem is subjected to  $n_i$  trials,  $i = 1, 2, \dots, k$ . We envisage a situation where the  $p_i$ 's are substantially less than 1% and the  $n_i$ 's are of the order of several hundred to several thousand. For such cases, if  $X_i$  is the number of observed failures for the  $i^{\text{th}}$  subsystem, then the representation of the distribution of  $X_i$  by the Poisson approximation with parameter  $\lambda_i = n_i p_i$  is essentially exact. The system reliability is given by

$$R = \prod_{i=1}^k (1-p_i) = \prod_{i=1}^k \left(1 - \frac{\lambda_i}{n_i}\right).$$

For the values of the  $p_i$ 's considered here the approximation

$$R \doteq 1 - \sum_{i=1}^k \lambda_i/n_i$$

is essentially exact so that to find a lower confidence bound for the mean time between failures = MTBF =  $1/(1-R)$ , or for the system reliability,  $R$ , it suffices to obtain an upper confidence bound for  $\sum_{i=1}^k \lambda_i/n_i$ .

The problem thus reduces to that of finding an upper confidence bound for a linear function of Poisson parameters based on independent Poisson observations  $X_1, X_2, \dots, X_k$ . To treat this problem we introduce the parameter

$$(1) \quad \theta = \sum_{i=1}^k a_i \lambda_i,$$

where the  $a_i$ 's are normalized coefficients with  $\sum_{i=1}^k a_i = 1$  and  $a_i > 0$ ,  $i = 1, 2, \dots, k$ . The use of  $a_i$ 's which sum to one facilitates the construction of numerical tables. The connection with the original problem is as follows: For  $i = 1, 2, \dots, k$  let

$$(2) \quad a_i = \left[ n_i \left( \sum_{i=1}^k 1/n_i \right) \right]^{-1}.$$

If  $\theta^*$  is an upper confidence bound at confidence level  $1-\alpha$  for  $\theta$  given by (1), then the corresponding upper confidence bound for  $\sum_{i=1}^k \lambda_i/n_i$  is given by  $\theta^* \left( \sum_{i=1}^k 1/n_i \right)$ , and the  $1-\alpha$  level confidence statement for the quantity MTBF is

$$(3) \quad \text{MTBF} \geq \left[ \theta^* \left( \sum_{i=1}^k 1/n_i \right) \right]^{-1}.$$

Because of the additive property of the Poisson distribution, the data for subsystems for which the sample sizes  $n_i$  are the same may be combined so that  $k$  represents the number of different subsystem sample sizes or the number of distinct  $a_i$ 's. If all subsystems are subjected to the same number of trials we may take  $k$  equal to one and the problem reduces to the familiar one of finding an upper confidence bound for a single Poisson parameter.

The subsystem failure data may be developed through independent testing of the subsystems, or through testing of the complete system with the assignment of failures to the appropriate subsystems. Even in the latter case subsystem sample sizes may differ if subsystems are redesigned during the course of testing so that the trials and failures observed prior to redesign for such subsystems are not relevant to the reliability of the final version of the system.

It is worth noting that the same formal confidence bound problem arises from a somewhat different and more restrictive model involving continuous time sampling and exponentially distributed failure times. Thus, if we assume that the time to failure of the  $i^{\text{th}}$  subsystem has the exponential probability density function  $\mu_i e^{-\mu_i t}$ ,  $\mu_i > 0$ ,  $t > 0$ , and if the fixed aggregate testing time for the  $i^{\text{th}}$  subsystem is given by  $\tau_i$  and the total number of failures observed is  $N_i$ ,  $i = 1, 2, \dots, k$ , then the  $N_i$ 's which are sufficient statistics have independent Poisson distributions with corresponding parameters  $\lambda_i = \mu_i \tau_i$ . Such test data would be generated, for example, if a single unit of each subsystem were placed on test and repaired whenever a failure occurred until the total test time for each subsystem reached its corresponding limit,  $\tau_i$ . The system reliability for a single duty cycle (consisting of operation for a unit interval of time) is given by

$$R = \prod_{i=1}^k e^{-\mu_i} = \exp\left(-\sum_{i=1}^k \mu_i\right).$$

If the system is highly reliable the  $\mu_i$ 's will be very small and we have

$$1 - R \doteq \sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i / \tau_i$$

and the confidence bound problem becomes formally the same as the one previously introduced.

In Section 2 the general method for constructing the required confidence bounds is described and optimality properties are discussed. The method is based on the use of an appropriate linear ordering imposed upon the collection of possible sample vectors  $\underline{X} = (X_1, X_2, \dots, X_k)$  and exploits an idea first introduced by Buehler [2]. The use of the tables which have been computed for the cases  $k = 1$  and  $k = 2$  is illustrated in this section.

In Section 3 approximations are introduced which permit the calculation of the required confidence bounds for data beyond the range of the computed tables. These approximations, which are based on the maximum likelihood ratio (MLR) statistic and the asymptotically equivalent maximum likelihood estimate (MLE) confidence bound, are used to justify a procedure for reducing cases where  $k \geq 3$  to the  $k = 2$  case. The asymptotic expressions obtained are not, strictly speaking, "large sample" results since it is not the sample sizes (the  $n_i$ 's) which are assumed to become large but rather the largest of the  $\lambda_i$ 's.

The formal proofs of the optimality results of Section 2 and the asymptotic results of Section 3 are deferred to Appendices 1 and 2. Appendix 3 describes the construction of the tables.

2. Confidence Bounds. A frequent practice in dealing with confidence bound problems is to use procedures which are "optimal" in the sense that they correspond to a hypothesis testing procedure which has some desirable property such as being, say, uniformly most powerful unbiased. Such confidence bounds are said to be "uniformly most accurate unbiased" (see [5], pp. 176-180). This approach is relevant only when UMP tests exist and is somewhat questionable in any case, since the criteria of interest in confidence statements such as the expected size of the confidence region have no counterparts in the hypothesis testing context. For the present case, no UMP unbiased or invariant tests of hypotheses concerning linear functions of Poisson parameters exist so that another approach is required. As noted in Section 1, we have elected to exploit an idea introduced by Beuhler [2] which involves the construction of confidence bounds based on suitably generated ordering of the sample data vectors.

Before discussing the specific problem of constructing an upper confidence bound for the parameter  $\theta$  defined by (1), we consider the following more general situation: Let  $P_{\underline{\lambda}}$  represent a family of probability functions indexed by a (vector) parameter  $\underline{\lambda}$  lying in a set  $\Lambda$ , and suppose that for each  $\underline{\lambda}$ ,  $P_{\underline{\lambda}}$  is defined on the appropriate subsets of a sample space  $\mathcal{X}$  whose (possibly vector) generic point is designated by  $\underline{x}$  with  $\underline{X}$  denoting the corresponding random vector. Let  $\theta = \theta(\underline{\lambda})$  represent a real valued function of  $\underline{\lambda}$ . Suppose further that a linear ordering (denoted by  $\succeq$ ) of the elements of the sample space is specified and that this ordering may be thought of as being generated by a real valued function  $\phi(\underline{x})$ . I.e.,  $\underline{x} \succeq \underline{y}$  if and only if  $\phi(\underline{x}) \geq \phi(\underline{y})$ . We seek an upper  $1-\alpha$  level confidence bound  $\theta^*(\underline{X})$  for  $\theta$  which is monotone

in the specified ordering on  $\mathcal{X}$ . Thus we require

$$(4) \quad P_{\underline{\lambda}}\{\theta \leq \theta^*(\underline{x})\} \geq 1 - \alpha \quad \text{for all } \underline{\lambda} \in \Lambda, \quad \text{and}$$

$$(5) \quad \underline{x} \succeq \underline{y} \iff \theta^*(\underline{x}) \geq \theta^*(\underline{y}).$$

To achieve this we proceed as follows: For each real  $t$  in the range of  $\theta(\underline{\lambda})$  let  $S(t) = \{\underline{\lambda}; \theta(\underline{\lambda}) = t\}$ .

Assumption A. For each  $\underline{x} \in \mathcal{X}$  the function

$$h_{\underline{x}}(t) = \sup_{\underline{\lambda} \in S(t)} P_{\underline{\lambda}}\{\underline{X} \preceq \underline{x}\}$$

is monotone decreasing in  $t$ , and there exists a value  $t_{\alpha}(\underline{x})$  such that

$$h_{\underline{x}}(t_{\alpha}(\underline{x})) = \alpha,$$

and the collection of such values contains its greatest lower bound.

Def. For each  $\underline{x} \in \mathcal{X}$ ,  $\theta^*(\underline{x}) = \text{smallest } t_{\alpha}(\underline{x})$ .

The properties of  $\theta^*(\underline{x})$  are outlined in the following four propositions, the proofs of which are deferred to Appendix 1:

Proposition 1. Assumption A implies that  $\theta^*(\underline{x})$  is monotone in the ordering  $(\succeq)$ .

Proposition 2. Under Assumption A  $\theta^*(\underline{x})$  is a  $1-\alpha$  level confidence bound for  $\theta(\underline{\lambda})$ .

Proposition 3. Under Assumption A, if  $\tilde{\theta}(\underline{x})$  is any other  $1-\alpha$  level confidence bound for  $\theta(\underline{\lambda})$  which is monotone in  $(\succeq)$  then  $\tilde{\theta}(\underline{x}) \geq \theta^*(\underline{x})$  for all  $\underline{x} \in \mathcal{X}$ .

Proposition 4. Under Assumption A,  $\theta^*(\underline{x})$  is undominated in the sense that if  $\tilde{\theta}(\underline{x})$  is any  $1-\alpha$  confidence bound, then  $\sup_{\underline{y} \leq \underline{x}} \tilde{\theta}(\underline{y}) \geq \theta^*(\underline{x})$  for all  $\underline{x} \in \mathcal{X}$ .

Proposition 3 guarantees that  $\theta^*$  is optimal with respect to the given ordering and Proposition 4 insures weak admissibility with respect to any ordering, i.e., there does not exist a  $1-\alpha$  level confidence bound  $\tilde{\theta}$ , say, such that  $\tilde{\theta}(\underline{x}) < \theta^*(\underline{x})$  for all  $\underline{x} \in \mathcal{X}$ . Furthermore, if for every distinct  $\underline{x}_1, \underline{x}_2 \in \mathcal{X}$ ,  $\theta^*(\underline{x}_1) \neq \theta^*(\underline{x}_2)$ , then for any  $1-\alpha$  level confidence bound  $\tilde{\theta}$ , if  $\tilde{\theta}(\underline{x}') < \theta^*(\underline{x}')$  for some  $\underline{x}' \in \mathcal{X}$ , then  $\tilde{\theta}(\underline{x}'') > \theta^*(\underline{x}'')$  for some  $\underline{x}'' \in \mathcal{X}$ . The choice of the ordering ( $\geq$ ) or, equivalently, the order generating function  $\phi(\underline{x})$  remains to be determined. In the present application  $\phi$  will be chosen so as to insure that  $\theta^*(\underline{x})$  is reasonable for small true values of  $\theta(\underline{\lambda})$  and asymptotically optimal in the usual sense (uniformly most accurate) for large  $\theta(\underline{\lambda})$ .

Returning to the specific problem under consideration, we recall that  $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$ ,  $\theta(\underline{\lambda}) = \sum_{i=1}^k a_i \lambda_i$  and  $\underline{X} = (X_1, X_2, \dots, X_k)$  where the  $X_i$ 's are independent observations and  $X_i$  has a Poisson distribution with parameter  $\lambda_i$ ,  $i = 1, 2, \dots, k$ . The verification that Assumption A is satisfied for this case is deferred to Appendix 1.

To find  $\theta^*(\underline{x})$  for the case  $k=1$  (where all the subsystem sample sizes are equal) we use the natural ordering of  $x_1 = x = 0, 1, 2, \dots$  and observe that  $\theta(\underline{\lambda}) = \lambda_1$  so that  $\theta^*(x)$  is simply the value of  $\lambda_1$  for which  $P_{\lambda_1} \{X \leq x\} = \alpha$ . This corresponds to the standard UMP one-sided test concerning  $\lambda_1$ , and  $\theta^*$  coincides with the "classical"  $1-\alpha$  level confidence bound for  $\lambda_1$ . Tables for this bound have usually been based on the relationship between the Poisson and the chi-squared distributions. We have used the Poisson distribution directly to compute Tables 1 and 2, which give  $\theta^*(x)$  for  $\alpha = .10$  and  $\alpha = .05$  for values of  $x$  between 0 and 149.

For the case  $k \geq 2$  ( $k$  distinct subsystem sample sizes) the determination of  $\theta^*$  requires the simultaneous solution of  $k$  non-linear equations. To see this we note that for general  $k$

$$(6) \quad P_{\underline{\lambda}}\{X \leq \underline{x}\} = \sum_{\underline{y} \leq \underline{x}} P_{\underline{\lambda}}\{X = \underline{y}\},$$

where

$$(7) \quad P_{\underline{\lambda}}\{X = \underline{y}\} = \exp\left\{-\sum_{i=1}^k \lambda_i\right\} \prod_{i=1}^k (\lambda_i^{y_i}/y_i!).$$

We must maximize (6) under the constraint  $\sum_{i=1}^k a_i \lambda_i = t$  and obtain  $\theta^*(x)$  as the value of  $t$  for which the maximum assumes the value  $\alpha$ . The constrained maximization may be accomplished by setting

$$\lambda_k = \frac{1}{a_k} \left( t - \sum_{i=1}^{k-1} a_i \lambda_i \right)$$

in (6) and solving simultaneously the  $k-1$  equations obtained by setting the partial derivatives of (6) with respect to  $\lambda_i$ ,  $i = 1, 2, \dots, k-1$ , equal to zero, with the  $k^{\text{th}}$  equation obtained by setting (6) equal to  $\alpha$ . This is equivalent to finding the solution  $\underline{\lambda}^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_k^*)$  with  $\lambda_i^* \geq 0$ ,  $i = 1, 2, \dots, k$ , of the simultaneous equations

$$\frac{\partial}{\partial \lambda_j} P_{\underline{\lambda}}\{X \leq \underline{x}\} + \frac{\partial}{\partial \lambda_k} P_{\underline{\lambda}}\{X \leq \underline{x}\} \frac{\partial}{\partial \lambda_j} \frac{1}{a_k} \left( t - \sum_{i=1}^{k-1} a_i \lambda_i \right) = 0, \quad j = 1, 2, \dots, k-1$$

or

$$(8) \quad \frac{\partial}{\partial \lambda_j} P_{\underline{\lambda}}\{X \leq \underline{x}\} - \frac{a_j}{a_k} \frac{\partial}{\partial \lambda_k} P_{\underline{\lambda}}\{X \leq \underline{x}\} = 0, \quad j = 1, 2, \dots, k-1,$$

and

$$(9) \quad P_{\underline{\lambda}}\{X \leq \underline{x}\} = \alpha ,$$

and setting

$$(10) \quad \theta^*(\underline{x}) = \sum_{i=1}^k a_i \lambda_i^* .$$

Observing that

$$(11) \quad \frac{\partial}{\partial \lambda_j} P_{\underline{\lambda}}\{X = \underline{y}\} = \left( \frac{y_j}{\lambda_j} - 1 \right) P_{\underline{\lambda}}\{X = \underline{y}\} ,$$

we may write (8) and (9) more explicitly as

$$(12) \quad \sum_{\underline{y} \leq \underline{x}} \left\{ a_k \left( \frac{y_k}{\lambda_k} - 1 \right) - a_j \left( \frac{y_j}{\lambda_j} - 1 \right) \right\} P_{\underline{\lambda}}\{X = \underline{y}\} = 0 , \quad j = 1, 2, \dots, k-1 ,$$

and

$$(13) \quad \sum_{\underline{y} \leq \underline{x}} P_{\underline{\lambda}}\{X = \underline{y}\} = \alpha$$

where  $P_{\underline{\lambda}}\{X = \underline{y}\}$  is given by (7).

In Appendix 2 it is shown that for large  $\gamma = \max(\lambda_1, \lambda_2, \dots, \lambda_k)$  the asymptotically optimal upper confidence bound for  $\theta(\underline{\lambda})$  is given by

$$(14) \quad \hat{\theta}(\underline{x}) = \sum_{i=1}^n a_i X_i + c_{\alpha} \left( \sum_{i=1}^n a_i^2 X_i \right)^{1/2}$$

where  $c_{\alpha}$  is the upper  $(100\alpha)^{\text{th}}$  percentage point of the standard normal distribution. This bound is based on the asymptotic normal distribution of the maximum likelihood estimate of  $\theta(\lambda)$ . We therefore choose to generate the ordering of the sample points  $\underline{x}$  by means of the function  $\phi(\underline{x}) = \hat{\theta}(\underline{x})$ . This guarantees that (i) the bound  $\theta^*(\underline{x})$  will be asymptotically optimal and (ii) in the absence of strong prior information concerning relative subsystem reliabilities,  $\theta^*(\underline{x})$  will be "reasonable" for small values of  $\gamma$  (and hence for small  $X_i$ 's) in the sense that the subsystems are treated evenhandedly according to the principle of maximum likelihood. It should be born in mind that only the ordering of the sample points is determined by  $\phi(\underline{x})$  and that any monotone function of  $\phi$  would produce the same ordering. Small sample deficiencies in the maximum likelihood bound will not therefore be reflected in the performance of  $\theta^*(\underline{x})$ .

For the case  $k = 2$  (two distinct subsystem sample sizes), Tables 3 and 4 give  $\theta^*(\underline{x})$  for  $\alpha = .10$  and  $\alpha = .05$  for the first 100 ordered values of the sample point  $\underline{X} = (X_1, X_2)$  for six representative values of  $a_1$  ( $a_1 = .10, .20, .25, .30, 1/3, .40$ ). Note that  $a_1$  is necessarily less than  $1/2$  since  $a_1 \leq a_2 = 1 - a_1$  and the case  $a_1 = a_2 = 1/2$  reduces to the case  $k=1$ .

Example 1. Consider a weapons system consisting of five subsystems and suppose that the entire system is subjected to 10,000 trials (duty cycles) with necessary repairs being made after each subsystem failure. Suppose further that subsystem number five is redesigned after 5,000 trials so that only data for the last 5,000 trials of that subsystem are relevant to the reliability of the final system configuration. Suppose that the

observed data are as follows:

| <u>Subsystem</u> | <u>Number of Trials</u> | <u>Observed Failures</u> |
|------------------|-------------------------|--------------------------|
| 1                | 10,000                  | 1                        |
| 2                | 10,000                  | 10                       |
| 3                | 10,000                  | 5                        |
| 4                | 10,000                  | 0                        |
| 5                | 5,000                   | 1                        |

Since the first four subsystems were subjected to the same number of trials ( $n_1 = 10,000$ ) we combine their failure data, obtaining  $X_1 = 16$ . For the remaining subsystem we have  $n_2 = 5,000$  and  $X_2 = 1$ . To find a 90% lower confidence bound for MTBF for this system we first note that  $a_1 = 1/[10,000(1/10,000 + 1/5,000)] = 1/3$  by (2). From Table 3 we find that the upper bound  $\theta^*$  for  $\theta$  corresponding to  $a_1 = 1/3$  and  $(X_1, X_2) = (16, 1)$  is  $\theta^* = 8.422$ . By (3) we may then assert that at the 90% level of confidence

$$MTBF \geq 1/[8.422(1/10,000 + 1/5,000)] = 396 .$$

Example 2. For the system considered in Example 1, suppose that the failure data were the same as those given above except that the redesign of the fifth subsystem was not performed so that its sample size is also 10,000 trials. Since all the sample sizes are equal we are in the case  $k=1$  with the total number of failures being  $X = 17$ . For a 90% confidence bound we note that from Table 1, for  $X = 17$  the value of the upper bound  $\theta^*$  is 23.606. The 90% confidence statement for MTBF then becomes

$$MTBF \geq 1/[23.606(1/10,000)] = 424 .$$

It should be noted that had the number of trials in these examples been as small as 500 to 1000, the corresponding analysis would still be valid since the Poisson approximations for the binomial distributions would still be very close.

3. Approximations. In Appendix 2 the asymptotic theory (for large  $\gamma = \max(\lambda_1, \lambda_2, \dots, \lambda_k)$ ) of the maximum likelihood ratio procedure for testing hypotheses concerning  $\theta$  is used to obtain approximate  $1 - \alpha$  level confidence bounds for  $\theta(\underline{\lambda})$ . Specifically, if  $\hat{\mu}$  is the positive real root less than  $1/\max(a_1, a_2, \dots, a_k)$  of the equation

$$(15) \quad \chi_{1,2\alpha}^2 = 2 \sum_{i=1}^k X_i \left\{ \frac{a_i \hat{\mu}}{1 - a_i \hat{\mu}} + \log(1 - a_i \hat{\mu}) \right\},$$

where  $\chi_{1,2\alpha}^2$  is the upper  $100(2\alpha)^{\text{th}}$  percentage point of the chi-squared distribution with one degree of freedom, then the quantity

$$(16) \quad \tilde{\theta}(\underline{X}) = \sum_{i=1}^k a_i X_i / (1 - a_i \hat{\mu})$$

is an approximate upper  $1 - \alpha$  level confidence bound for  $\theta(\underline{\lambda})$ . This bound is asymptotically optimal and asymptotically equivalent to  $\hat{\theta}$  given by (14), but it is more precise than the latter for moderate values of the  $X_i$ 's. Equation (15) may easily be solved for  $\hat{\mu}$  by trial and error approximation using a hand calculator. For the case  $k = 2$ ,  $\tilde{\theta}$  is a good approximation to  $\theta^*$  for values of  $(X_1, X_2)$  beyond those tabulated. Comparisons of the M.L.R. approximations with the exact values of  $\theta^*$  from Tables 3 and 4 for the "largest" pairs  $(X_1, X_2)$  (corresponding to  $I = 100$ ) for each case are given in Tables A and B below:

Table A ( $\alpha=.10, k=2$ )

|                           | $a_1=.10$<br>$X_1=30, X_2=1$ | $a_1=.20$<br>$X_1=4, X_2=5$ | $a_1=.25$<br>$X_1=10, X_2=4$ | $a_1=.3$<br>$X_1=0, X_2=8$ | $a_1=1/3$<br>$X_1=15, X_2=2$ | $a_1=.4$<br>$X_1=7, X_2=6$ |
|---------------------------|------------------------------|-----------------------------|------------------------------|----------------------------|------------------------------|----------------------------|
| $\tilde{\theta}$ (M.L.R.) | 5.70                         | 7.59                        | 8.05                         | 8.53                       | 8.65                         | 9.02                       |
| $\theta^*$ (Table 3)      | 6.24                         | 7.88                        | 8.36                         | 9.10                       | 8.81                         | 9.18                       |

Table B ( $\alpha=.05, k=2$ )

|                           | $a_1=.10$<br>$X_1=17, X_2=2$ | $a_1=.20$<br>$X_1=3, X_2=5$ | $a_1=.25$<br>$X_1=10, X_2=4$ | $a_1=.30$<br>$X_1=16, X_2=2$ | $a_1=1/3$<br>$X_1=6, X_2=6$ | $a_1=.40$<br>$X_1=15, X_2=1$ |
|---------------------------|------------------------------|-----------------------------|------------------------------|------------------------------|-----------------------------|------------------------------|
| $\tilde{\theta}$ (M.L.R.) | 6.53                         | 8.34                        | 8.92                         | 9.23                         | 9.58                        | 9.73                         |
| $\theta^*$ (Table 4)      | 7.13                         | 8.70                        | 9.12                         | 9.60                         | 9.77                        | 10.10                        |

The M.L.R. bound always tends to underestimate the true bound, but the approximations are seen to be within 10% of the exact values in all cases and within 5% in most cases. The precision of  $\tilde{\theta}$  will, of course, improve further for larger  $X_i$ 's.

For  $k \geq 3$ , if some of the  $X_i$ 's are moderately large, it is reasonable to expect that the M.L.R. bound  $\tilde{\theta}$  will give satisfactory results. For  $k \geq 3$  and small  $X_i$ 's a better approximation is required. We observe that the asymptotic bound  $\hat{\theta}$  given by (14) depends only on the maximum likelihood estimate of  $\theta$ , i.e.,

$$(17) \quad \bar{\theta} = \sum_{i=1}^k a_i X_i,$$

and the maximum likelihood estimate of the variance of  $\bar{\theta}$ , i.e.,

$$(18) \quad \bar{v} = \sum_{i=1}^k a_i^2 X_i.$$

This suggests that cases involving  $k \geq 3$  may be reduced approximately to corresponding cases with  $k = 2$  by equating the corresponding values of  $\bar{\theta}$  and  $\bar{v}$ . A specific procedure for accomplishing this may be described as follows:

(i) Choose an integer  $k_1$ ,  $1 \leq k_1 < k$ , so that the two groups of constants  $a_1, a_2, \dots, a_{k_1}$  and  $a_{k_1+1}, a_{k_1+2}, \dots, a_k$  are as homogeneous as possible.

(ii) Compute the group averages

$$\bar{a}_1 = \frac{1}{k_1} \sum_{i=1}^{k_1} a_i, \quad \bar{a}_2 = \frac{1}{k-k_1} \sum_{i=k_1+1}^k a_i.$$

(iii) Determine the equivalent observations  $\bar{X}_1$  and  $\bar{X}_2$  for the  $k = 2$  case by setting

$$\bar{a}_1 \bar{X}_1 + \bar{a}_2 \bar{X}_2 = \bar{\theta} \quad \text{and} \quad \bar{a}_1^2 \bar{X}_1 + \bar{a}_2^2 \bar{X}_2 = \bar{v},$$

where  $\bar{\theta}$  and  $\bar{v}$  are given by (17) and (18), thereby obtaining

$$\bar{X}_1 = \frac{\bar{a}_2 \bar{\theta} - \bar{v}}{\bar{a}_1 (\bar{a}_2 - \bar{a}_1)} \quad \text{and} \quad \bar{X}_2 = \frac{\bar{v} - \bar{a}_1 \bar{\theta}}{\bar{a}_1 (\bar{a}_2 - \bar{a}_1)}.$$

(iv) Determine the normalized constants  $a_1^*$ ,  $a_2^*$  for the approximately equivalent  $k = 2$  case by

$$a_1^* = \bar{a}_1 / (\bar{a}_1 + \bar{a}_2), \quad a_2^* = \bar{a}_2 / (\bar{a}_1 + \bar{a}_2).$$

(v) Find the upper  $1 - \alpha$  confidence bound  $\theta^*$  for  $k = 2$  using  $a_1^*$ ,  $a_2^*$ ,  $\bar{X}_1$  and  $\bar{X}_2$  and Table 3 or Table 4, interpolating as necessary.

(vi) Compute the  $1 - \alpha$  upper confidence bound  $\theta^{**}$ , say, for the original ( $k \geq 3$ ) problem by the formula

$$\theta^{**} = (\bar{a}_1 + \bar{a}_2) \theta^*,$$

where  $\theta^*$  is the value found in (v).

The validity of this procedure was checked for a number of examples involving  $k = 3$  or  $k = 4$  using the M.L.R. bound  $\tilde{\theta}$  computed for the original example and the corresponding approximately equivalent  $k = 2$  case (using  $\bar{a}_1$  and  $\bar{a}_2$  instead of  $a_1^*$  and  $a_2^*$  for comparability). The

agreement in the corresponding values of  $\hat{\theta}$  was found to be extraordinarily close, usually to within at least three significant figures. The procedure was also found to be very insensitive to the choice of  $k_1$ . It should be noted that since the asymptotic bound depends on two quantities ( $\bar{\theta}$  and  $\bar{v}$ ) it is not reasonable to attempt to further reduce cases involving  $k \geq 2$  to the  $k = 1$  case.

The proposed procedure is illustrated in the following example:

Example. Suppose  $k = 3$ ,  $a_1 = 1/6$ ,  $a_2 = 1/3$ ,  $a_3 = 1/2$ , the observations are  $X_1 = 2$ ,  $X_2 = 3$ ,  $X_3 = 5$ , and we seek an upper 90% confidence bound for  $\theta$ . Then from (17) and (18) we have  $\bar{\theta} = 3.8333$  and  $\bar{v} = 1.6389$ . Choosing  $k_1 = 2$  we obtain  $\bar{a}_1 = \frac{1}{2} \left( \frac{1}{6} + \frac{1}{3} \right) = .25$  and  $\bar{a}_2 = a_2 = .50$ . From the formulas of (iii) we obtain  $\bar{X}_1 = 4.4444$  and  $\bar{X}_2 = 5.4444$ , and from (iv),  $a_1^* = 1/3$  and  $a_2^* = 2/3$ . Since  $\bar{X}_1$  and  $\bar{X}_2$  are not integers we must interpolate in Table 3 ( $\alpha = .10$ ). For  $a_1 = 1/3$  the relevant entries from Table 3 are as follows:

| <u>I</u> | <u>X<sub>1</sub></u> | <u>X<sub>2</sub></u> | <u><math>\theta^*</math></u> |
|----------|----------------------|----------------------|------------------------------|
| 63       | 4                    | 5                    | 7.339                        |
| 81       | 4                    | 6                    | 8.176                        |
| 71       | 5                    | 5                    | 7.592                        |
| 90       | 5                    | 6                    | 8.533                        |

Four point linear interpolation yields

$$\theta^* = (.5556)^2(7.339) + (.5556)(.4444)(8.176 + 7.592) + (.4444)^2(8.533) = 7.844 .$$

Thus from (vi) the 90% upper confidence bound for  $\theta$  is given by

$$\theta^{**} = (7.844)(.25 + .50) = 5.883 .$$

The values of the M.L.R. bound  $\tilde{\theta}$  for the original  $k = 3$  problem and the reduced  $k = 2$  problem are respectively 5.736 and 5.737. The very close agreement between these values (even though they both underestimate  $\theta^{**}$ ) strongly supports the validity of the suggested procedure.

## Appendix 1

This appendix contains the proofs of the four general propositions of Section 2 and the verification that Assumption A is satisfied for the Poisson confidence bound problem.

Proof of Proposition 1: Consider  $\underline{x}, \underline{y} \in \mathfrak{X}$  such that  $\underline{x} \succ \underline{y}$ . Then for all  $\underline{\lambda} \in \Lambda$ ,

$$P_{\underline{\lambda}}\{\underline{X} \leq \underline{x}\} \geq P_{\underline{\lambda}}\{\underline{X} \leq \underline{y}\},$$

and hence for all  $t$

$$\sup_{\underline{\lambda} \in S(t)} P_{\underline{\lambda}}\{\underline{X} \leq \underline{x}\} \geq \sup_{\underline{\lambda} \in S(t)} P_{\underline{\lambda}}\{\underline{X} \leq \underline{y}\}.$$

Thus, recalling the definition of  $\theta^*(\underline{x})$  and letting  $S^* = S(\theta^*(\underline{y})) = \{\underline{\lambda}: \theta(\underline{\lambda}) = \theta^*(\underline{y})\}$ , we have

$$\sup_{\underline{\lambda} \in S^*} P_{\underline{\lambda}}\{\underline{X} \leq \underline{x}\} \geq \alpha,$$

so that  $\theta^*(\underline{x}) \geq \theta^*(\underline{y})$  by the monotonicity part of Assumption A, which completes the proof.

Proof of Proposition 2: For fixed arbitrary  $\underline{\lambda} \in \Lambda$ , let  $t = \theta(\underline{\lambda})$ .

Then if  $\theta^{**}$  represents any inverse of  $\theta^*$  taking values in the equivalence classes of  $\mathfrak{X}$  generated by the ordering  $(\succ)$ , we have

$$\begin{aligned} P_{\underline{\lambda}}\{\theta(\underline{\lambda}) \geq \theta^*(\underline{X})\} &= P_{\underline{\lambda}}\{\underline{X} \leq \theta^{**}(t)\} \\ &\leq \sup_{\underline{\lambda} \in S(t)} P_{\underline{\lambda}}\{\underline{X} \leq \theta^{**}(t)\} \\ &\leq h(t) = \alpha, \end{aligned}$$

by the definition of  $\theta^*$ . Hence

$$P_{\underline{\lambda}}\{\theta(\underline{\lambda}) \leq \theta^*(\underline{x})\} \geq P_{\underline{\lambda}}\{\theta(\underline{\lambda}) < \theta^*(\underline{x})\} \geq 1 - \alpha,$$

and the proof is complete.

Proof of Proposition 3: Proceeding contrapositively we assume that there exists a point  $\underline{x}' \in \mathcal{X}$  such that  $\tilde{\theta}(\underline{x}') < \theta^*(\underline{x}')$ . Let  $S' = S(\tilde{\theta}(\underline{x}')) = \{\underline{\lambda}: \theta(\underline{\lambda}) = \tilde{\theta}(\underline{x}')\}$ . Then by Assumption A and the definition of  $\theta^*$  we have

$$\sup_{\underline{\lambda} \in S'} P_{\underline{\lambda}}\{\underline{X} \preceq \underline{x}'\} > \alpha,$$

or since  $\tilde{\theta}$  is monotone in  $(\succ)$ ,

$$\sup_{\underline{\lambda} \in S'} P_{\underline{\lambda}}\{\tilde{\theta}(\underline{X}) \leq \tilde{\theta}(\underline{x}')\} > \alpha.$$

Thus, there exists a  $\underline{\lambda}' \in S'$  such that

$$P_{\underline{\lambda}'}\{\tilde{\theta}(\underline{X}) \leq \theta(\underline{\lambda}')\} > \alpha,$$

which contradicts the assumption that  $\tilde{\theta}$  is a  $1-\alpha$  level confidence bound and completes the proof.

Proof of Proposition 4: For any  $1-\alpha$  level confidence bound  $\tilde{\theta}$  let  $\bar{\theta}(\underline{x}) = \sup_{\underline{y} \preceq \underline{x}} \tilde{\theta}(\underline{y})$ . Then  $\bar{\theta}$  is monotone in the ordering  $(\succeq)$  and if it is

also a  $1-\alpha$  level confidence bound, Proposition 3 guarantees that

$\bar{\theta}(\underline{x}) \geq \theta^*(\underline{x})$  for all  $\underline{x} \in \mathcal{X}$  which is the desired result. Suppose contrapositively that  $\bar{\theta}$  is not a  $1-\alpha$  level confidence bound. Then there exists a  $\underline{\lambda}^0 \in \Lambda$  such that

$$P_{\underline{\lambda}^0} \{ \bar{\theta}(\underline{X}) \leq \theta(\underline{\lambda}^0) \} > \alpha .$$

But  $\tilde{\theta}(\underline{x}) \leq \bar{\theta}(\underline{x})$  for all  $\underline{x} \in \mathcal{X}$ , so that

$$P_{\underline{\lambda}^0} \{ \tilde{\theta}(\underline{X}) \leq \theta(\underline{\lambda}^0) \} > \alpha ,$$

which contradicts the assumption that  $\tilde{\theta}$  is a  $1-\alpha$  level confidence bound and completes the proof.

We demonstrate that Assumption A is satisfied for the Poisson confidence bound problem under consideration by means of the following two lemmas:

For each  $t$  let  $\underline{\lambda}_t$  be a value of  $\underline{\lambda}$  for which the supremum of Assumption A is attained, i.e., for which (12) is satisfied. Let  $\bar{\theta}(\underline{x}) = \sum_{i=1}^k a_i X_i$ , and let  $\hat{\theta}(\underline{x})$  be as defined in (14).

Lemma 1. For  $t > 0$ ,  $\frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{ \underline{X} \leq \underline{x} \} < 0$  if and only if

$$E_{\underline{\lambda}_t} \{ \bar{\theta}(\underline{X}) | \hat{\theta}(\underline{X}) \leq \hat{\theta}(\underline{x}) \} < t .$$

Proof: Suppressing the dependence on  $t$  in the notation for the components of  $\underline{\lambda}_t = (\lambda_1, \lambda_2, \dots, \lambda_k)$  and recalling that  $\sum_{i=1}^k a_i \lambda_i = t$  we see that

$\frac{\partial \lambda_i}{\partial t} = 1/a_i$ ,  $i = 1, 2, \dots, k$ , which when combined with (11) yields

$$(A1) \quad \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{ \underline{X} \leq \underline{x} \} = \sum_{\underline{y} \leq \underline{x}} \sum_{j=1}^k \frac{1}{a_j} \left( \frac{y_j}{\lambda_j} - 1 \right) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} .$$

Now from (12), summing on  $j$  yields

$$(A2) \quad \sum_{\underline{y} \leq \underline{x}} \sum_{j=1}^{k-1} \frac{a_k}{a_j^2} \left( \frac{y_k}{\lambda_k} - 1 \right) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} = \sum_{\underline{y} < \underline{x}} \sum_{j=1}^{k-1} \frac{1}{a_j} \left( \frac{y_j}{\lambda_j} - 1 \right) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} .$$

From (A1) and (A2) we obtain

$$(A3) \quad \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{ \underline{X} \leq \underline{x} \} = \left( \sum_{j=1}^k \frac{1}{a_j^2} \right) \sum_{\underline{y} \leq \underline{x}} a_k \left( \frac{y_k}{\lambda_k} - 1 \right) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} .$$

By symmetry (A3) must also hold with  $a_i, y_i, \lambda_i$  replacing  $a_k, y_k, \lambda_k$  respectively for  $i = 1, 2, \dots, k-1$ . Hence multiplying each of these equations by the corresponding  $\lambda_i$  and summing we obtain

$$(A4) \quad \left( \sum_{i=1}^k \lambda_i \right) \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{ \underline{X} \leq \underline{x} \} = \left( \sum_{i=1}^k \frac{1}{a_i^2} \right) \sum_{\underline{y} \leq \underline{x}} (\bar{\theta}(\underline{y}) - t) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} ,$$

which yields the desired result upon observing that

$$E_{\underline{\lambda}_t} \{ \bar{\theta}(\underline{X}) | \underline{X} \leq \underline{y} \} = \left( P_{\underline{\lambda}_t} \{ \underline{X} \leq \underline{x} \} \right)^{-1} \sum_{\underline{y} \leq \underline{x}} \bar{\theta}(\underline{y}) P_{\underline{\lambda}_t} \{ \underline{X} = \underline{y} \} ,$$

and recalling that  $\underline{x} \leq \underline{y} \iff \hat{\theta}(\underline{x}) \leq \hat{\theta}(\underline{y})$ . Note that since  $t > 0$ ,  $\lambda_t \neq 0$  and hence  $\sum_{i=1}^k \lambda_i > 0$ .

Now consider a random vector  $\underline{Z} = (Z_1, Z_2, \dots, Z_n)$  where the  $Z_i$ 's are independent non-negative random variables. Let  $C$  be the hypercube in  $k$ -dimensional space defined for  $c \in (0, \infty)$  by

$$C = \{ \underline{z} = (z_1, z_2, \dots, z_k) : 0 \leq z_i \leq c, i = 1, 2, \dots, k \} .$$

For any measurable set  $R$  in the non-negative orthant of  $k$ -space, for  $\underline{v} \geq 0$  let

$$R_i(\underline{v}) = R \cap \{\underline{z} : z_i = v\}$$

for  $i = 1, 2, \dots, k$ .

Lemma 2.<sup>1</sup> If  $R \subset C$ ,  $P\{\underline{Z} \in R\} > 0$ , and if  $\underline{v} > \underline{v}'$  implies  $R_i(\underline{v}) \subset R_i(\underline{v}')$  for  $i = 1, 2, \dots, k$ , then

$$E\{Z_i | \underline{Z} \in R\} \leq E\{Z_i | Z_i \leq c\}$$

for  $i = 1, 2, \dots, k$ .

Proof: Let  $f_i(z)$ ,  $i = 1, 2, \dots, k$ , be the probability densities of the  $Z_i$ 's with respect to a dominating measure  $\mu = \mu_1 \times \mu_2 \times \dots \times \mu_k$ . Then taking  $i = 1$  and letting  $\mu^{(k-1)} = \mu_2 \times \dots \times \mu_k$ ,  $\underline{u} = (u_2, u_3, \dots, u_k)$ ,  $P_1(c) = P\{Z_1 \leq c\}$ , and  $f^{(k-1)}(\underline{u}) = \prod_{i=2}^k f_i(u_i)$ , we have

$$(A5) \quad E\{Z_1 | \underline{Z} \in R\} = \frac{1}{P\{\underline{Z} \in R\}} \int_0^c v f_1(v) \int_{R_1(v)} f^{(k-1)}(\underline{u}) d\mu^{(k-1)} d\mu_1.$$

But the inner integral on the right is decreasing in  $v$  so that by the well known inequality which states that if  $g_1$  and  $g_2$  are two functions monotone in opposite senses and  $Y$  is a random variable, then

$$E g_1(Y) g_2(Y) \leq E g_1(Y) E g_2(Y), \text{ we have}$$

---

<sup>1</sup>The author benefited from helpful discussions with Rupert G. Miller concerning the proof of this lemma.

$$\begin{aligned}
& \int_0^c \left( v \int_{R_1(v)} f^{(k-1)}(\underline{u}) d\mu^{(k-1)} \right) \frac{f_1(v)}{P_1(c)} d\mu_1 \\
(A6) \quad & \leq \int_0^c v \frac{f_1(v)}{P_1(c)} d\mu_1 \int_0^c \left( \int_{R_1(v)} f^{(k-1)}(\underline{u}) d\mu^{(k-1)} \right) \frac{f_1(v)}{P_1(c)} d\mu_1 \\
& = E\{Z_1 | Z_1 \leq c\} P\{Z \in R\} / P_1(c) .
\end{aligned}$$

Hence by (A5) and (A6)

$$E\{Z_1 | Z \in R\} \leq E\{Z_1 | Z_1 \leq c\} ,$$

and the proof is complete.

We observe that for any  $i$  ,  $E\{Z_i | Z_i \leq c\} < E Z_i$  , provided that  $P\{Z_i \leq c\} < 1$  , which implies that the conditional c.d.f. of  $Z_i$  , given  $\{Z_i \leq c\}$  , is strictly greater than the c.d.f. of  $Z_i$  on the interval  $(0, c)$  .

To verify that Assumption A is satisfied for the Poisson model under consideration, let  $Z_i = a_i X_i$  ,  $i = 1, 2, \dots, k$  , so that  $\hat{\theta}$  given by (14) becomes

$$\hat{\theta}(\underline{X}) = \sum_{i=1}^k Z_i + c_\alpha \left( \sum_{i=1}^n a_i Z_i \right)^{1/2} \stackrel{\text{def.}}{=} g(\underline{Z}) .$$

Let  $\underline{z} = (z_1, z_2, \dots, z_k)$  and for fixed arbitrary  $\underline{x} \in \mathcal{X}$  , let

$$R = \{ \underline{z} : \underline{z} \geq 0 , g(\underline{z}) \leq \hat{\theta}(\underline{x}) \} .$$

Since  $\frac{\partial}{\partial z_i} g(\underline{z}) > 0$  for all  $\underline{z} \geq 0$  , we conclude that for  $v > v'$  ,  $\underline{z} \in R_1(v)$  implies  $\underline{z} \in R_1(v')$  for  $i = 1, 2, \dots, k$  . Also, for sufficiently

large  $c > 0$ ,  $R \subset C$  and  $P_{\underline{\lambda}}\{\underline{Z} \in R\} > 0$ , so that the conditions of Lemma 2 are satisfied. Also, if  $\lambda_i > 0$ , then  $P\{Z_i \leq c\} < 1$  and by the remark following Lemma 2,  $E\{Z_i | Z_i \leq c\} < E Z_i$ . Hence noting that  $\bar{\theta}(\underline{x}) = \sum_{i=1}^k Z_i$  we have for  $\underline{\lambda} \neq 0$ ,

$$\begin{aligned} E_{\underline{\lambda}}\{\bar{\theta}(\underline{X}) | \hat{\theta}(\underline{X}) \leq \hat{\theta}(\underline{x})\} &= \sum_{i=1}^k E_{\underline{\lambda}}\{Z_i | \underline{Z} \in R\} \\ &< \sum_{i=1}^k E_{\underline{\lambda}} Z_i = E_{\underline{\lambda}} \bar{\theta}(\underline{x}). \end{aligned}$$

Thus, recalling that  $E_{\underline{\lambda}_t} \bar{\theta}(\underline{X}) = \sum_{i=1}^k a_i \lambda_i = t$ , we have by Lemma 1,

$$\frac{\partial}{\partial t} h_{\underline{x}}(t) = \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{\underline{X} \leq \underline{x}\} < 0,$$

and therefore  $h_{\underline{x}}(t)$  is decreasing in  $t$  for all  $\underline{x} \in \mathcal{X}$ . The existence of a smallest value of  $t_{\alpha}(\underline{x})$  follows from the continuity and form of the left-hand members of equations (12) and (13) and Assumption A is therefore satisfied.

Appendix 2

This appendix contains the detailed development of the approximations employed in Section 3. These approximations are based on the asymptotic properties of the maximum likelihood ratio (M.L.R.) test statistic.

As before, let  $\underline{X} = (X_1, X_2, \dots, X_k)$  and  $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$  where for each  $i$ ,  $X_i$  is a Poisson random variable with parameter  $\lambda_i$  and the  $X_i$ 's are independent. To obtain the M.L.R. test of  $H_0: \theta = \theta_0$  vs.  $H_1: \theta \neq \theta_0$ , we let  $\Lambda_0 = \{\underline{\lambda}: \theta_0 = \sum_{i=1}^k a_i \lambda_i\}$  and observe that the likelihood function is

$$L(\underline{\lambda}, \underline{X}) = \exp\left\{ \sum_{i=1}^k (X_i \log \lambda_i - \lambda_i - \log X_i!) \right\} .$$

Since the unconstrained maximum likelihood estimate (M.L.E.) of  $\lambda_i$  is  $X_i$ ,  $i = 1, 2, \dots, k$ , we have

$$\max_{\underline{\lambda}} \log L(\underline{\lambda}, \underline{X}) = \sum_{i=1}^k (X_i \log X_i - X_i - \log X_i!) .$$

To obtain the M.L.E.'s under the constraint that  $\underline{\lambda} \in \Lambda_0$  (i.e.,  $H_0$  is true) we use the Lagrange multiplier method and observe that for

$j = 1, 2, \dots, k$

$$\frac{\partial}{\partial \lambda_j} \left\{ \log L(\underline{X}, \underline{\lambda}) + \mu \left( \sum_{i=1}^k a_i \lambda_i - \theta_0 \right) \right\} = 0$$

if and only if

$$\frac{X_j}{\lambda_j} + a_j \mu = 1 .$$

Thus the restricted M.L.E.'s are

$$\hat{\lambda}_i = X_i / (1 - a_i \hat{\mu}),$$

for  $i = 1, 2, \dots, k$ , with  $\hat{\mu} = \hat{\mu}(\theta_0)$  determined by

$$(A7) \quad \sum_{i=1}^k a_i X_i / (1 - a_i \hat{\mu}) = \theta_0.$$

Thus, if  $S = \max_{\Lambda} L / \max_{\Lambda_0} L$ , the M.L.R. statistic is

$$(A8) \quad \begin{aligned} T &= -2 \log S \\ &= 2 \sum_{i=1}^k \left\{ X_i \log X_i - X_i - X_i \log \left( \frac{X_i}{1 - a_i \hat{\mu}} \right) + \frac{X_i}{1 - a_i \hat{\mu}} \right\} \\ &= 2 \sum_{i=1}^k X_i \left\{ \frac{a_i \hat{\mu}}{1 - a_i \hat{\mu}} + \log(1 - a_i \hat{\mu}) \right\}, \end{aligned}$$

or

$$(A9) \quad T = 2\theta_0 \hat{\mu} + 2 \sum_{i=1}^k X_i \log(1 - a_i \hat{\mu}).$$

If  $T$  were based on  $n$  i.i.d. observations we would expect  $T$  to be asymptotically chi-squared with one degree of freedom as  $n \rightarrow \infty$ . For the present case (since  $k$  is fixed) a slightly different analysis is needed. Let  $Z_i = \lambda_i^{-1/2} (X_i - \lambda_i)$ ,  $i = 1, 2, \dots, k$ . Then the  $Z_i$ 's are bounded in probability and if  $\lambda_i \rightarrow \infty$ , then  $Z_i$  is asymptotically a standard normal variable. Now  $X_i = \lambda_i^{1/2} Z_i + \lambda_i$  and from (A7)

$$\sum_{i=1}^k \frac{a_i \lambda_i^{1/2} Z_i + a_i \lambda_i}{1 - a_i \hat{\mu}} = \theta_0 = \sum_{i=1}^k a_i \lambda_i,$$

and hence

$$(A10) \quad \sum_{i=1}^k \frac{a_i (\lambda_i^{1/2} Z_i + a_i \lambda_i \hat{\mu})}{1 - a_i \hat{\mu}} = 0.$$

Let  $\gamma = \max(\lambda_1, \lambda_2, \dots, \lambda_k)$  and consider  $\gamma$  large. Then using the symbol  $O_p(g(\gamma))$  to represent a random quantity which when divided by  $g(\gamma)$  is bounded in probability as  $\gamma \rightarrow \infty$ , we observe that (A10) implies  $\hat{\mu} = O_p(\gamma^{-1/2})$  so that  $(1 - a_i \hat{\mu})^{-1} = 1 + O_p(\gamma^{-1/2})$  for each  $i$ , and hence

$$(A11) \quad \hat{\mu} = - \frac{\sum_{i=1}^k a_i \lambda_i^{1/2} Z_i}{\sum_{i=1}^k a_i^2 \lambda_i} (1 + O_p(\gamma^{-1/2})).$$

Now  $\log(1 - a_i \hat{\mu}) = -a_i \hat{\mu} - \frac{1}{2} a_i^2 \hat{\mu}^2 + O_p(\gamma^{-3/2})$ , so that (A9) becomes

$$\begin{aligned} T &= 2 \left\{ \hat{\mu} \sum_{i=1}^k a_i \lambda_i - \sum_{i=1}^k X_i \left( a_i \hat{\mu} + \frac{1}{2} a_i^2 \hat{\mu}^2 + O_p(\gamma^{-3/2}) \right) \right\} \\ &= 2 \left\{ -\hat{\mu} \sum_{i=1}^k a_i \lambda_i^{1/2} Z_i - \frac{1}{2} \hat{\mu}^2 \sum_{i=1}^k a_i^2 \lambda_i \right\} + O_p(\gamma^{-1/2}). \end{aligned}$$

Recalling (A11) we have

$$(A12) \quad \begin{aligned} T &= 2 \left\{ \hat{\mu}^2 (1 + O_p(\gamma^{-1/2})) \sum_{i=1}^k a_i^2 \lambda_i - \frac{1}{2} \hat{\mu}^2 \sum_{i=1}^k a_i^2 \lambda_i \right\} + O_p(\gamma^{-1/2}) \\ &= \hat{\mu}^2 \sum_{i=1}^k a_i^2 \lambda_i + O_p(\gamma^{-1/2}). \end{aligned}$$

Hence, letting  $\epsilon_\gamma$  represent a generic random quantity approaching zero in probability as  $\gamma \rightarrow \infty$ , we have by (A11) and (A12),

$$\begin{aligned}
(A13) \quad T &= \left[ \frac{\sum_{i=1}^k a_i \lambda_i^{1/2} z_i}{\left( \sum_{i=1}^k a_i^2 \lambda_i \right)^{1/2}} \right]^2 + \varepsilon_\gamma \\
&= \left[ \frac{\left( \sum_{i=1}^k a_i X_i - \theta_0 \right)}{\left( \sum_{i=1}^k a_i \lambda_i \right)^{1/2}} \right]^2 + \varepsilon_\gamma \\
&= \left[ \frac{\left( \sum_{i=1}^k a_i X_i - \theta_0 \right)}{\left( \sum_{i=1}^k a_i X_i \right)^{1/2}} \right]^2 + \varepsilon_\gamma,
\end{aligned}$$

since

$$\left( \sum_{i=1}^k a_i^2 X_i \right)^{1/2} = \left( \sum_{i=1}^k a_i^2 \lambda_i \right)^{1/2} (1 + \varepsilon_\gamma).$$

Thus, for large  $\gamma$ ,  $T$  is approximately the square of a standard normal variate and the approximate two-sided test of  $H_0: \theta = \theta_0$  at the  $\alpha$  significance level is to reject if  $T > \chi_{1, \alpha}^2$ , where  $\chi_{1, \alpha}^2$  is the upper  $\alpha^{\text{th}}$  quantile of the chi-squared distribution with one degree of freedom. The corresponding asymptotic two-sided confidence statement for  $\theta$  is

$$\sum_{i=1}^k a_i X_i - c_{\alpha/2} \left( \sum_{i=1}^k a_i^2 X_i \right)^{1/2} \leq \theta \leq \sum_{i=1}^k a_i X_i + c_{\alpha/2} \left( \sum_{i=1}^k a_i^2 X_i \right)^{1/2}$$

with probability  $1 - \alpha$ , where  $c_{\alpha/2}$  is the upper  $(\frac{1}{2}\alpha)^{\text{th}}$  quantile of the standard normal distribution. The corresponding one-sided confidence statement at level  $1 - \alpha$  is  $\theta \leq \hat{\theta}$  where  $\hat{\theta}$  is given by (14). To obtain an asymptotic upper (one-sided) level  $1 - \alpha$  confidence bound for  $\theta$  using the M.L.R. statistic directly, one first finds the root  $\hat{\mu}_0$  lying in the interval  $(0, 1/\max(a_1, a_2, \dots, a_k))$  of the equation

(A14)

$$T = \chi_{1,2\alpha}^2,$$

where  $T$  is given by (A8). Substituting  $\hat{\mu}_0$  for  $\hat{\mu}$  in (A7) then yields  $\theta_0 = \tilde{\theta}$ , the required upper  $1 - \alpha$  confidence bound for  $\theta$ . The negative root of (A14) will yield a lower confidence bound for  $\theta$ .

### Appendix 3

Tables 3 and 4 for the case  $k = 2$  were produced by solving (12) and (13) simultaneously for  $\lambda_1^*$  and  $\lambda_2^*$  by numerical methods and computing  $\theta^*$  from (10). The numerical methods used involved a finitized analogue of Newton's method for two dimensions. The calculations were done in part on a self-contained mini-computer system (Wang 2200) and in part on an IBM 360/60 system. In a small proportion of the cases considered the procedure failed to converge in a reasonable number of iterations and the tabulated values were obtained by interpolation according to the values of  $\hat{\theta}$  given by (14) subject to the requirement that  $\theta^*$  be monotone in the given ordering.

An examination of Tables 3 and 4 shows that often the same or very nearly the same value of  $\theta^*$  occurs for several successive sample points. This reflects the fact that at certain stages an additional pair  $(x_1, x_2) = \underline{x}$  may contribute very little to the quantity  $P_{\underline{\lambda}}\{\underline{X} \leq \underline{x}\}$  in the vicinity of the solution pair  $(\lambda_1^*, \lambda_2^*) = \underline{\lambda}^*$ . In higher dimensions ( $k > 2$ ) this phenomenon might be exploited to condense the corresponding tables considerably.

Tables 1 and 2 were computed by directly generating the required Poisson probabilities rather than by the traditional method utilizing the corresponding chi-squared probabilities.

The author wishes to acknowledge the assistance rendered by Mr. David Pasta in the performance of the computations described above.

## References and Related Literature

- [1] Abraham, J. K. (1962), "A Confidence Interval for the Reliability of Multicomponent Systems," Proceedings of the Seventh Conference on the Design of Experiments in Army Research, Development and Testing, ARODR 62-2, pp. 483-518.
- [2] Buehler, R. J. (1957), "Confidence Intervals for the Product of Two Binomial Parameters," Journal of the American Statistical Association, Vol. 52, pp. 482-493.
- [3] El Mawaziny, A. H. and Buehler, R. J. (1967), "Confidence Limits for the Reliability of a Series Systems," Journal of the American Statistical Association, Vol. 62, pp. 1452-1459.
- [4] Harris, B. (1971), "Hypothesis Testing and Confidence Intervals for Products and Quotients of Poisson Parameters with Applications to Reliability," Journal of the American Statistical Association, Vol. 66, pp. 609-613.
- [5] Lehmann, E. L. (1959), Testing Statistical Hypotheses, John Wiley and Sons, New York, N.Y.
- [6] Lenter, M. M. and Buehler, R. J. (1963), "Some inferences about Gamma Parameters with an Application to a Reliability Problem," Journal of the American Statistical Association, Vol. 58, pp. 670-677.
- [7] Lieberman, G. J. and Ross, S. M. (1971), "Confidence Intervals for Independent Exponential Series Systems," Journal of the American Statistical Association, Vol. 66, pp. 837-840.
- [8] Lipow, M. and Riley, J. (1959), Tables of Upper Confidence Bounds on Failure Probability of 1, 2, and 3 Component Serial Systems, Vols. I and II, Space Technology Laboratories AD-609-100 and AD-636-718.
- [9] Madansky, A. (1965), "Approximate Confidence Limits for the Reliability of Series and Parallel Systems," Technometrics, Vol. 7, pp. 495-503.
- [10] Mann, N. R. and Grubbs, F. E. (1972), "Approximately Optimum Confidence Bounds on Series-System Reliability for Exponential Time to Fail Data," Biometrika, Vol. 59, pp. 191-204.
- [11] Mann, N. R. and Grubbs, F. E. (1974), "Approximately Optimum Confidence Bounds for System Reliability Based on Component Test Data," Technometrics, Vol. 16, pp. 335-348.
- [12] Steck, G. P. (1957), Upper Confidence Limits for the Failure Probability of Complex Networks, SC-433 (TR), Sandia Corporation.

TABLE 1

| ALPHA=.10 |        | K=1 |         |     |         |
|-----------|--------|-----|---------|-----|---------|
| X         | THETA* | X   | THETA*  | X   | THETA*  |
| 0         | 2.303  | 50  | 60.339  | 100 | 114.075 |
| 1         | 3.890  | 51  | 61.429  | 101 | 115.138 |
| 2         | 5.322  | 52  | 62.518  | 102 | 116.202 |
| 3         | 6.681  | 53  | 63.605  | 103 | 117.265 |
| 4         | 7.994  | 54  | 64.692  | 104 | 118.327 |
| 5         | 9.275  | 55  | 65.779  | 105 | 119.390 |
| 6         | 10.532 | 56  | 66.864  | 106 | 120.452 |
| 7         | 11.771 | 57  | 67.949  | 107 | 121.514 |
| 8         | 12.995 | 58  | 69.033  | 108 | 122.576 |
| 9         | 14.206 | 59  | 70.116  | 109 | 123.637 |
| 10        | 15.407 | 60  | 71.199  | 110 | 124.698 |
| 11        | 16.598 | 61  | 72.281  | 111 | 125.759 |
| 12        | 17.782 | 62  | 73.362  | 112 | 126.819 |
| 13        | 18.958 | 63  | 74.442  | 113 | 127.879 |
| 14        | 20.128 | 64  | 75.523  | 114 | 128.939 |
| 15        | 21.292 | 65  | 76.602  | 115 | 129.999 |
| 16        | 22.452 | 66  | 77.680  | 116 | 131.059 |
| 17        | 23.606 | 67  | 78.759  | 117 | 132.118 |
| 18        | 24.756 | 68  | 79.836  | 118 | 133.177 |
| 19        | 25.902 | 69  | 80.913  | 119 | 134.235 |
| 20        | 27.045 | 70  | 81.990  | 120 | 135.294 |
| 21        | 28.184 | 71  | 83.066  | 121 | 136.352 |
| 22        | 29.320 | 72  | 84.141  | 122 | 137.410 |
| 23        | 30.453 | 73  | 85.216  | 123 | 138.468 |
| 24        | 31.583 | 74  | 86.291  | 124 | 139.525 |
| 25        | 32.711 | 75  | 87.364  | 125 | 140.582 |
| 26        | 33.836 | 76  | 88.438  | 126 | 141.640 |
| 27        | 34.959 | 77  | 89.511  | 127 | 142.696 |
| 28        | 36.080 | 78  | 90.583  | 128 | 143.753 |
| 29        | 37.198 | 79  | 91.655  | 129 | 144.809 |
| 30        | 38.315 | 80  | 92.727  | 130 | 145.865 |
| 31        | 39.430 | 81  | 93.798  | 131 | 146.921 |
| 32        | 40.543 | 82  | 94.869  | 132 | 147.977 |
| 33        | 41.654 | 83  | 95.939  | 133 | 149.033 |
| 34        | 42.764 | 84  | 97.009  | 134 | 150.088 |
| 35        | 43.872 | 85  | 98.078  | 135 | 151.143 |
| 36        | 44.978 | 86  | 99.147  | 136 | 152.198 |
| 37        | 46.083 | 87  | 100.216 | 137 | 153.253 |
| 38        | 47.187 | 88  | 101.284 | 138 | 154.307 |
| 39        | 48.289 | 89  | 102.352 | 139 | 155.361 |
| 40        | 49.390 | 90  | 103.419 | 140 | 156.416 |
| 41        | 50.490 | 91  | 104.486 | 141 | 157.470 |
| 42        | 51.589 | 92  | 105.553 | 142 | 158.523 |
| 43        | 52.686 | 93  | 106.620 | 143 | 159.577 |
| 44        | 53.782 | 94  | 107.686 | 144 | 160.630 |
| 45        | 54.878 | 95  | 108.751 | 145 | 161.683 |
| 46        | 55.972 | 96  | 109.817 | 146 | 162.736 |
| 47        | 57.065 | 97  | 110.882 | 147 | 163.789 |
| 48        | 58.158 | 98  | 111.946 | 148 | 164.842 |
| 49        | 59.249 | 99  | 113.011 | 149 | 165.894 |

TABLE 2

| ALPHA=.05 |        | K=1 |         |     |         |
|-----------|--------|-----|---------|-----|---------|
| X         | THETA* | X   | THETA*  | X   | THETA*  |
| 0         | 2.996  | 50  | 63.287  | 100 | 118.078 |
| 1         | 4.744  | 51  | 64.402  | 101 | 119.160 |
| 2         | 6.296  | 52  | 65.516  | 102 | 120.241 |
| 3         | 7.754  | 53  | 66.628  | 103 | 121.322 |
| 4         | 9.153  | 54  | 67.740  | 104 | 122.403 |
| 5         | 10.513 | 55  | 68.850  | 105 | 123.484 |
| 6         | 11.842 | 56  | 69.960  | 106 | 124.564 |
| 7         | 13.148 | 57  | 71.069  | 107 | 125.643 |
| 8         | 14.434 | 58  | 72.176  | 108 | 126.722 |
| 9         | 15.705 | 59  | 73.283  | 109 | 127.801 |
| 10        | 16.962 | 60  | 74.389  | 110 | 128.879 |
| 11        | 18.207 | 61  | 75.494  | 111 | 129.957 |
| 12        | 19.443 | 62  | 76.599  | 112 | 131.035 |
| 13        | 20.669 | 63  | 77.702  | 113 | 132.112 |
| 14        | 21.886 | 64  | 78.805  | 114 | 133.189 |
| 15        | 23.097 | 65  | 79.906  | 115 | 134.265 |
| 16        | 24.301 | 66  | 81.007  | 116 | 135.342 |
| 17        | 25.499 | 67  | 82.108  | 117 | 136.418 |
| 18        | 26.692 | 68  | 83.207  | 118 | 137.493 |
| 19        | 27.879 | 69  | 84.306  | 119 | 138.568 |
| 20        | 29.062 | 70  | 85.404  | 120 | 139.643 |
| 21        | 30.240 | 71  | 86.501  | 121 | 140.718 |
| 22        | 31.415 | 72  | 87.598  | 122 | 141.792 |
| 23        | 32.585 | 73  | 88.694  | 123 | 142.866 |
| 24        | 33.752 | 74  | 89.790  | 124 | 143.940 |
| 25        | 34.916 | 75  | 90.884  | 125 | 145.014 |
| 26        | 36.076 | 76  | 91.979  | 126 | 146.087 |
| 27        | 37.234 | 77  | 93.072  | 127 | 147.160 |
| 28        | 38.389 | 78  | 94.165  | 128 | 148.232 |
| 29        | 39.541 | 79  | 95.257  | 129 | 149.305 |
| 30        | 40.690 | 80  | 96.349  | 130 | 150.376 |
| 31        | 41.837 | 81  | 97.441  | 131 | 151.449 |
| 32        | 42.982 | 82  | 98.531  | 132 | 152.520 |
| 33        | 44.125 | 83  | 99.621  | 133 | 153.592 |
| 34        | 45.266 | 84  | 100.711 | 134 | 154.663 |
| 35        | 46.404 | 85  | 101.800 | 135 | 155.733 |
| 36        | 47.541 | 86  | 102.888 | 136 | 156.804 |
| 37        | 48.675 | 87  | 103.977 | 137 | 157.873 |
| 38        | 49.808 | 88  | 105.064 | 138 | 158.944 |
| 39        | 50.940 | 89  | 106.151 | 139 | 160.014 |
| 40        | 52.069 | 90  | 107.238 | 140 | 161.083 |
| 41        | 53.197 | 91  | 108.324 | 141 | 162.152 |
| 42        | 54.324 | 92  | 109.409 | 142 | 163.221 |
| 43        | 55.449 | 93  | 110.494 | 143 | 164.290 |
| 44        | 56.572 | 94  | 111.579 | 144 | 165.359 |
| 45        | 57.695 | 95  | 112.663 | 145 | 166.427 |
| 46        | 58.816 | 96  | 113.747 | 146 | 167.495 |
| 47        | 59.935 | 97  | 114.831 | 147 | 168.563 |
| 48        | 61.054 | 98  | 115.914 | 148 | 169.630 |
| 49        | 62.171 | 99  | 116.996 | 149 | 170.698 |

TABLE 3

| ALPHA= .10 |    |    |        | A1=.10 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 2.072  | 51     | 3  | 2  | 4.881  |     |    |    |        |
| 2          | 1  | 0  | 2.079  | 52     | 16 | 1  | 4.881  |     |    |    |        |
| 3          | 2  | 0  | 2.111  | 53     | 31 | 0  | 4.881  |     |    |    |        |
| 4          | 3  | 0  | 2.163  | 54     | 4  | 2  | 4.943  |     |    |    |        |
| 5          | 4  | 0  | 2.225  | 55     | 17 | 1  | 4.943  |     |    |    |        |
| 6          | 5  | 0  | 2.293  | 56     | 32 | 0  | 4.943  |     |    |    |        |
| 7          | 6  | 0  | 2.367  | 57     | 5  | 2  | 5.013  |     |    |    |        |
| 8          | 7  | 0  | 2.444  | 58     | 18 | 1  | 5.013  |     |    |    |        |
| 9          | 8  | 0  | 2.524  | 59     | 33 | 0  | 5.013  |     |    |    |        |
| 10         | 9  | 0  | 2.606  | 60     | 6  | 2  | 5.087  |     |    |    |        |
| 11         | 10 | 0  | 2.690  | 61     | 19 | 1  | 5.087  |     |    |    |        |
| 12         | 11 | 0  | 2.776  | 62     | 34 | 0  | 5.087  |     |    |    |        |
| 13         | 12 | 0  | 2.863  | 63     | 7  | 2  | 5.165  |     |    |    |        |
| 14         | 13 | 0  | 2.951  | 64     | 20 | 1  | 5.165  |     |    |    |        |
| 15         | 14 | 0  | 3.040  | 65     | 35 | 0  | 5.165  |     |    |    |        |
| 16         | 15 | 0  | 3.130  | 66     | 8  | 2  | 5.246  |     |    |    |        |
| 17         | 0  | 1  | 3.501  | 67     | 21 | 1  | 5.246  |     |    |    |        |
| 18         | 16 | 0  | 3.501  | 68     | 36 | 0  | 5.246  |     |    |    |        |
| 19         | 1  | 1  | 3.507  | 69     | 9  | 2  | 5.329  |     |    |    |        |
| 20         | 17 | 0  | 3.507  | 70     | 22 | 1  | 5.329  |     |    |    |        |
| 21         | 2  | 1  | 3.540  | 71     | 37 | 0  | 5.329  |     |    |    |        |
| 22         | 18 | 0  | 3.540  | 72     | 10 | 2  | 5.414  |     |    |    |        |
| 23         | 3  | 1  | 3.592  | 73     | 23 | 1  | 5.414  |     |    |    |        |
| 24         | 19 | 0  | 3.592  | 74     | 11 | 2  | 5.501  |     |    |    |        |
| 25         | 4  | 1  | 3.654  | 75     | 38 | 0  | 5.501  |     |    |    |        |
| 26         | 20 | 0  | 3.654  | 76     | 24 | 1  | 5.501  |     |    |    |        |
| 27         | 5  | 1  | 3.723  | 77     | 12 | 2  | 5.589  |     |    |    |        |
| 28         | 21 | 0  | 3.723  | 78     | 0  | 3  | 6.013  |     |    |    |        |
| 29         | 6  | 1  | 3.797  | 79     | 39 | 0  | 6.013  |     |    |    |        |
| 30         | 22 | 0  | 3.797  | 80     | 25 | 1  | 6.013  |     |    |    |        |
| 31         | 7  | 1  | 3.875  | 81     | 13 | 2  | 6.013  |     |    |    |        |
| 32         | 8  | 1  | 3.955  | 82     | 1  | 3  | 6.019  |     |    |    |        |
| 33         | 23 | 0  | 3.955  | 83     | 40 | 0  | 6.019  |     |    |    |        |
| 34         | 9  | 1  | 4.038  | 84     | 26 | 1  | 6.019  |     |    |    |        |
| 35         | 24 | 0  | 4.038  | 85     | 14 | 2  | 6.019  |     |    |    |        |
| 36         | 10 | 1  | 4.123  | 86     | 2  | 3  | 6.052  |     |    |    |        |
| 37         | 25 | 0  | 4.123  | 87     | 41 | 0  | 6.052  |     |    |    |        |
| 38         | 11 | 1  | 4.209  | 88     | 27 | 1  | 6.052  |     |    |    |        |
| 39         | 26 | 0  | 4.209  | 89     | 15 | 2  | 6.052  |     |    |    |        |
| 40         | 12 | 1  | 4.297  | 90     | 3  | 3  | 6.104  |     |    |    |        |
| 41         | 27 | 0  | 4.297  | 91     | 42 | 0  | 6.104  |     |    |    |        |
| 42         | 0  | 2  | 4.790  | 92     | 28 | 1  | 6.104  |     |    |    |        |
| 43         | 13 | 1  | 4.790  | 93     | 16 | 2  | 6.104  |     |    |    |        |
| 44         | 28 | 0  | 4.790  | 94     | 4  | 3  | 6.166  |     |    |    |        |
| 45         | 1  | 2  | 4.796  | 95     | 43 | 0  | 6.166  |     |    |    |        |
| 46         | 14 | 1  | 4.796  | 96     | 29 | 1  | 6.166  |     |    |    |        |
| 47         | 29 | 0  | 4.796  | 97     | 17 | 2  | 6.166  |     |    |    |        |
| 48         | 2  | 2  | 4.829  | 98     | 5  | 3  | 6.236  |     |    |    |        |
| 49         | 15 | 1  | 4.829  | 99     | 44 | 0  | 6.236  |     |    |    |        |
| 50         | 30 | 0  | 4.829  | 100    | 30 | 1  | 6.236  |     |    |    |        |

TABLE 3 (CONT.)

| ALPHA= .10 |    |    |        | A1=.20 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.842  | 51     | 20 | 0  | 5.810  |     |    |    |        |
| 2          | 1  | 0  | 1.872  | 52     | 15 | 1  | 5.810  |     |    |    |        |
| 3          | 2  | 0  | 1.979  | 53     | 0  | 4  | 6.395  |     |    |    |        |
| 4          | 3  | 0  | 2.123  | 54     | 10 | 2  | 6.395  |     |    |    |        |
| 5          | 4  | 0  | 2.286  | 55     | 5  | 3  | 6.395  |     |    |    |        |
| 6          | 5  | 0  | 2.460  | 56     | 21 | 0  | 6.395  |     |    |    |        |
| 7          | 0  | 1  | 3.111  | 57     | 16 | 1  | 6.395  |     |    |    |        |
| 8          | 6  | 0  | 3.111  | 58     | 1  | 4  | 6.414  |     |    |    |        |
| 9          | 1  | 1  | 3.142  | 59     | 11 | 2  | 6.431  |     |    |    |        |
| 10         | 7  | 0  | 3.142  | 60     | 6  | 3  | 6.434  |     |    |    |        |
| 11         | 2  | 1  | 3.251  | 61     | 22 | 0  | 6.536  |     |    |    |        |
| 12         | 8  | 0  | 3.251  | 62     | 17 | 1  | 6.536  |     |    |    |        |
| 13         | 3  | 1  | 3.398  | 63     | 2  | 4  | 6.536  |     |    |    |        |
| 14         | 9  | 0  | 3.399  | 64     | 12 | 2  | 6.536  |     |    |    |        |
| 15         | 4  | 1  | 3.567  | 65     | 7  | 3  | 6.536  |     |    |    |        |
| 16         | 10 | 0  | 3.567  | 66     | 23 | 0  | 6.536  |     |    |    |        |
| 17         | 5  | 1  | 3.748  | 67     | 18 | 1  | 6.536  |     |    |    |        |
| 18         | 11 | 0  | 3.751  | 68     | 3  | 4  | 6.686  |     |    |    |        |
| 19         | 0  | 2  | 4.258  | 69     | 8  | 3  | 6.688  |     |    |    |        |
| 20         | 6  | 1  | 4.258  | 70     | 13 | 2  | 6.688  |     |    |    |        |
| 21         | 1  | 2  | 4.288  | 71     | 24 | 0  | 6.688  |     |    |    |        |
| 22         | 12 | 0  | 4.288  | 72     | 4  | 4  | 6.857  |     |    |    |        |
| 23         | 7  | 1  | 4.288  | 73     | 19 | 1  | 6.857  |     |    |    |        |
| 24         | 2  | 2  | 4.398  | 74     | 9  | 3  | 6.864  |     |    |    |        |
| 25         | 13 | 0  | 4.398  | 75     | 14 | 2  | 6.867  |     |    |    |        |
| 26         | 8  | 1  | 4.398  | 76     | 25 | 0  | 7.027  |     |    |    |        |
| 27         | 3  | 2  | 4.548  | 77     | 0  | 5  | 7.420  |     |    |    |        |
| 28         | 14 | 0  | 4.548  | 78     | 5  | 4  | 7.420  |     |    |    |        |
| 29         | 9  | 1  | 4.548  | 79     | 20 | 1  | 7.420  |     |    |    |        |
| 30         | 4  | 2  | 4.719  | 80     | 10 | 3  | 7.420  |     |    |    |        |
| 31         | 15 | 0  | 4.719  | 81     | 15 | 2  | 7.420  |     |    |    |        |
| 32         | 10 | 1  | 4.721  | 82     | 26 | 0  | 7.420  |     |    |    |        |
| 33         | 5  | 2  | 4.904  | 83     | 1  | 5  | 7.420  |     |    |    |        |
| 34         | 0  | 3  | 5.344  | 84     | 6  | 4  | 7.463  |     |    |    |        |
| 35         | 16 | 0  | 5.344  | 85     | 21 | 1  | 7.479  |     |    |    |        |
| 36         | 11 | 1  | 5.344  | 86     | 11 | 3  | 7.490  |     |    |    |        |
| 37         | 6  | 2  | 5.344  | 87     | 16 | 2  | 7.498  |     |    |    |        |
| 38         | 1  | 3  | 5.362  | 88     | 2  | 5  | 7.561  |     |    |    |        |
| 39         | 17 | 0  | 5.438  | 89     | 27 | 0  | 7.561  |     |    |    |        |
| 40         | 12 | 1  | 5.486  | 90     | 7  | 4  | 7.562  |     |    |    |        |
| 41         | 7  | 2  | 5.486  | 91     | 22 | 1  | 7.562  |     |    |    |        |
| 42         | 2  | 3  | 5.486  | 92     | 12 | 3  | 7.562  |     |    |    |        |
| 43         | 18 | 0  | 5.486  | 93     | 17 | 2  | 7.562  |     |    |    |        |
| 44         | 13 | 1  | 5.486  | 94     | 3  | 5  | 7.712  |     |    |    |        |
| 45         | 8  | 2  | 5.486  | 95     | 28 | 0  | 7.712  |     |    |    |        |
| 46         | 3  | 3  | 5.636  | 96     | 8  | 4  | 7.714  |     |    |    |        |
| 47         | 19 | 0  | 5.636  | 97     | 23 | 1  | 7.714  |     |    |    |        |
| 48         | 14 | 1  | 5.636  | 98     | 13 | 3  | 7.714  |     |    |    |        |
| 49         | 9  | 2  | 5.637  | 99     | 18 | 2  | 7.714  |     |    |    |        |
| 50         | 4  | 3  | 5.810  | 100    | 4  | 5  | 7.884  |     |    |    |        |

TABLE 3 (CONT.)

| ALPHA= .10 |    |    |        | A1=.25 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.727  | 51     | 17 | 0  | 6.224  |     |    |    |        |
| 2          | 1  | 0  | 1.781  | 52     | 6  | 3  | 6.230  |     |    |    |        |
| 3          | 2  | 0  | 1.944  | 53     | 10 | 2  | 6.230  |     |    |    |        |
| 4          | 3  | 0  | 2.151  | 54     | 3  | 4  | 6.443  |     |    |    |        |
| 5          | 4  | 0  | 2.381  | 55     | 14 | 1  | 6.443  |     |    |    |        |
| 6          | 0  | 1  | 2.917  | 56     | 18 | 0  | 6.443  |     |    |    |        |
| 7          | 5  | 0  | 2.917  | 57     | 7  | 3  | 6.498  |     |    |    |        |
| 8          | 1  | 1  | 2.972  | 58     | 0  | 5  | 6.956  |     |    |    |        |
| 9          | 6  | 0  | 2.972  | 59     | 11 | 2  | 6.956  |     |    |    |        |
| 10         | 2  | 1  | 3.142  | 60     | 4  | 4  | 6.956  |     |    |    |        |
| 11         | 7  | 0  | 3.142  | 61     | 15 | 1  | 6.956  |     |    |    |        |
| 12         | 3  | 1  | 3.362  | 62     | 8  | 3  | 6.956  |     |    |    |        |
| 13         | 0  | 2  | 3.991  | 63     | 19 | 0  | 6.956  |     |    |    |        |
| 14         | 4  | 1  | 3.991  | 64     | 1  | 5  | 6.986  |     |    |    |        |
| 15         | 8  | 0  | 3.991  | 65     | 12 | 2  | 7.028  |     |    |    |        |
| 16         | 1  | 2  | 4.047  | 66     | 5  | 4  | 7.028  |     |    |    |        |
| 17         | 5  | 1  | 4.047  | 67     | 16 | 1  | 7.028  |     |    |    |        |
| 18         | 9  | 0  | 4.047  | 68     | 9  | 3  | 7.028  |     |    |    |        |
| 19         | 2  | 2  | 4.217  | 69     | 20 | 0  | 7.028  |     |    |    |        |
| 20         | 6  | 1  | 4.220  | 70     | 2  | 5  | 7.044  |     |    |    |        |
| 21         | 10 | 0  | 4.220  | 71     | 13 | 2  | 7.139  |     |    |    |        |
| 22         | 3  | 2  | 4.433  | 72     | 6  | 4  | 7.215  |     |    |    |        |
| 23         | 7  | 1  | 4.450  | 73     | 17 | 1  | 7.215  |     |    |    |        |
| 24         | 11 | 0  | 4.450  | 74     | 10 | 3  | 7.286  |     |    |    |        |
| 25         | 0  | 3  | 5.011  | 75     | 21 | 0  | 7.294  |     |    |    |        |
| 26         | 4  | 2  | 5.011  | 76     | 3  | 5  | 7.407  |     |    |    |        |
| 27         | 8  | 1  | 5.011  | 77     | 14 | 2  | 7.407  |     |    |    |        |
| 28         | 12 | 0  | 5.011  | 78     | 7  | 4  | 7.449  |     |    |    |        |
| 29         | 1  | 3  | 5.066  | 79     | 0  | 6  | 7.899  |     |    |    |        |
| 30         | 5  | 2  | 5.066  | 80     | 18 | 1  | 7.899  |     |    |    |        |
| 31         | 9  | 1  | 5.066  | 81     | 11 | 3  | 7.899  |     |    |    |        |
| 32         | 13 | 0  | 5.066  | 82     | 4  | 5  | 7.899  |     |    |    |        |
| 33         | 2  | 3  | 5.238  | 83     | 22 | 0  | 7.899  |     |    |    |        |
| 34         | 6  | 2  | 5.242  | 84     | 15 | 2  | 7.899  |     |    |    |        |
| 35         | 10 | 1  | 5.242  | 85     | 8  | 4  | 7.899  |     |    |    |        |
| 36         | 14 | 0  | 5.242  | 86     | 1  | 6  | 7.900  |     |    |    |        |
| 37         | 3  | 3  | 5.456  | 87     | 19 | 1  | 7.914  |     |    |    |        |
| 38         | 7  | 2  | 5.489  | 88     | 12 | 3  | 7.914  |     |    |    |        |
| 39         | 0  | 4  | 5.995  | 89     | 5  | 5  | 7.914  |     |    |    |        |
| 40         | 11 | 1  | 5.995  | 90     | 23 | 0  | 7.914  |     |    |    |        |
| 41         | 15 | 0  | 5.995  | 91     | 16 | 2  | 7.914  |     |    |    |        |
| 42         | 4  | 3  | 5.995  | 92     | 2  | 6  | 8.130  |     |    |    |        |
| 43         | 8  | 2  | 5.995  | 93     | 9  | 4  | 8.130  |     |    |    |        |
| 44         | 1  | 4  | 6.051  | 94     | 20 | 1  | 8.130  |     |    |    |        |
| 45         | 12 | 1  | 6.051  | 95     | 13 | 3  | 8.130  |     |    |    |        |
| 46         | 16 | 0  | 6.051  | 96     | 6  | 5  | 8.136  |     |    |    |        |
| 47         | 5  | 3  | 6.051  | 97     | 24 | 0  | 8.136  |     |    |    |        |
| 48         | 9  | 2  | 6.051  | 98     | 17 | 2  | 8.136  |     |    |    |        |
| 49         | 2  | 4  | 6.224  | 99     | 3  | 6  | 8.351  |     |    |    |        |
| 50         | 13 | 1  | 6.224  | 100    | 10 | 4  | 8.355  |     |    |    |        |

TABLE 3 (CONT.)

| ALPHA= .10 |    |    |        | A1=.30 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.612  | 51     | 12 | 1  | 6.589  |     |    |    |        |
| 2          | 1  | 0  | 1.703  | 52     | 4  | 4  | 6.594  |     |    |    |        |
| 3          | 2  | 0  | 1.941  | 53     | 15 | 0  | 6.594  |     |    |    |        |
| 4          | 3  | 0  | 2.229  | 54     | 7  | 3  | 6.594  |     |    |    |        |
| 5          | 0  | 1  | 2.723  | 55     | 10 | 2  | 6.594  |     |    |    |        |
| 6          | 4  | 0  | 2.723  | 56     | 2  | 5  | 6.830  |     |    |    |        |
| 7          | 1  | 1  | 2.819  | 57     | 13 | 1  | 6.830  |     |    |    |        |
| 8          | 2  | 1  | 3.054  | 58     | 5  | 4  | 6.898  |     |    |    |        |
| 9          | 5  | 0  | 3.074  | 59     | 16 | 0  | 6.898  |     |    |    |        |
| 10         | 0  | 2  | 3.726  | 60     | 8  | 3  | 6.962  |     |    |    |        |
| 11         | 3  | 1  | 3.726  | 61     | 0  | 6  | 7.372  |     |    |    |        |
| 12         | 6  | 0  | 3.726  | 62     | 11 | 2  | 7.372  |     |    |    |        |
| 13         | 1  | 2  | 3.820  | 63     | 3  | 5  | 7.372  |     |    |    |        |
| 14         | 4  | 1  | 3.823  | 64     | 14 | 1  | 7.372  |     |    |    |        |
| 15         | 7  | 0  | 3.823  | 65     | 6  | 4  | 7.372  |     |    |    |        |
| 16         | 2  | 2  | 4.059  | 66     | 17 | 0  | 7.372  |     |    |    |        |
| 17         | 5  | 1  | 4.094  | 67     | 1  | 6  | 7.469  |     |    |    |        |
| 18         | 8  | 0  | 4.096  | 68     | 9  | 3  | 7.469  |     |    |    |        |
| 19         | 0  | 3  | 4.676  | 69     | 12 | 2  | 7.469  |     |    |    |        |
| 20         | 3  | 2  | 4.676  | 70     | 4  | 5  | 7.475  |     |    |    |        |
| 21         | 6  | 1  | 4.676  | 71     | 15 | 1  | 7.475  |     |    |    |        |
| 22         | 9  | 0  | 4.676  | 72     | 7  | 4  | 7.475  |     |    |    |        |
| 23         | 1  | 3  | 4.772  | 73     | 18 | 0  | 7.475  |     |    |    |        |
| 24         | 4  | 2  | 4.776  | 74     | 2  | 6  | 7.711  |     |    |    |        |
| 25         | 7  | 1  | 4.776  | 75     | 10 | 3  | 7.711  |     |    |    |        |
| 26         | 10 | 0  | 4.776  | 76     | 13 | 2  | 7.711  |     |    |    |        |
| 27         | 2  | 3  | 5.012  | 77     | 5  | 5  | 7.787  |     |    |    |        |
| 28         | 5  | 2  | 5.060  | 78     | 0  | 7  | 8.239  |     |    |    |        |
| 29         | 8  | 1  | 5.087  | 79     | 16 | 1  | 8.239  |     |    |    |        |
| 30         | 11 | 0  | 5.101  | 80     | 8  | 4  | 8.239  |     |    |    |        |
| 31         | 0  | 4  | 5.597  | 81     | 19 | 0  | 8.239  |     |    |    |        |
| 32         | 3  | 3  | 5.597  | 82     | 3  | 6  | 8.239  |     |    |    |        |
| 33         | 6  | 2  | 5.597  | 83     | 11 | 3  | 8.239  |     |    |    |        |
| 34         | 9  | 1  | 5.597  | 84     | 6  | 5  | 8.239  |     |    |    |        |
| 35         | 12 | 0  | 5.597  | 85     | 14 | 2  | 8.239  |     |    |    |        |
| 36         | 1  | 4  | 5.691  | 86     | 1  | 7  | 8.337  |     |    |    |        |
| 37         | 4  | 3  | 5.696  | 87     | 17 | 1  | 8.337  |     |    |    |        |
| 38         | 7  | 2  | 5.696  | 88     | 9  | 4  | 8.337  |     |    |    |        |
| 39         | 10 | 1  | 5.696  | 89     | 20 | 0  | 8.337  |     |    |    |        |
| 40         | 2  | 4  | 5.932  | 90     | 4  | 6  | 8.344  |     |    |    |        |
| 41         | 13 | 0  | 5.932  | 91     | 12 | 3  | 8.344  |     |    |    |        |
| 42         | 5  | 3  | 5.991  | 92     | 7  | 5  | 8.455  |     |    |    |        |
| 43         | 8  | 2  | 6.040  | 93     | 15 | 2  | 8.462  |     |    |    |        |
| 44         | 0  | 5  | 6.492  | 94     | 2  | 7  | 8.579  |     |    |    |        |
| 45         | 11 | 1  | 6.492  | 95     | 18 | 1  | 8.579  |     |    |    |        |
| 46         | 3  | 4  | 6.492  | 96     | 10 | 4  | 8.579  |     |    |    |        |
| 47         | 14 | 0  | 6.492  | 97     | 5  | 6  | 8.661  |     |    |    |        |
| 48         | 6  | 3  | 6.492  | 98     | 21 | 0  | 8.661  |     |    |    |        |
| 49         | 9  | 2  | 6.492  | 99     | 13 | 3  | 8.661  |     |    |    |        |
| 50         | 1  | 5  | 6.589  | 100    | 0  | 8  | 9.096  |     |    |    |        |

TABLE 3 (CONT.)

| ALPHA= .10 |    |    |        | A1=1/3 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.534  | 51     | 0  | 6  | 6.668  |     |    |    |        |
| 2          | 1  | 0  | 1.663  | 52     | 9  | 2  | 6.668  |     |    |    |        |
| 3          | 2  | 0  | 1.965  | 53     | 7  | 3  | 6.668  |     |    |    |        |
| 4          | 0  | 1  | 2.593  | 54     | 14 | 0  | 6.720  |     |    |    |        |
| 5          | 3  | 0  | 2.593  | 55     | 5  | 4  | 6.755  |     |    |    |        |
| 6          | 1  | 1  | 2.727  | 56     | 12 | 1  | 6.755  |     |    |    |        |
| 7          | 4  | 0  | 2.734  | 57     | 3  | 5  | 6.984  |     |    |    |        |
| 8          | 2  | 1  | 3.031  | 58     | 1  | 6  | 7.164  |     |    |    |        |
| 9          | 0  | 2  | 3.548  | 59     | 10 | 2  | 7.164  |     |    |    |        |
| 10         | 5  | 0  | 3.548  | 60     | 8  | 3  | 7.164  |     |    |    |        |
| 11         | 3  | 1  | 3.548  | 61     | 15 | 0  | 7.164  |     |    |    |        |
| 12         | 1  | 2  | 3.685  | 62     | 6  | 4  | 7.164  |     |    |    |        |
| 13         | 6  | 0  | 3.685  | 63     | 4  | 5  | 7.339  |     |    |    |        |
| 14         | 4  | 1  | 3.775  | 64     | 13 | 1  | 7.339  |     |    |    |        |
| 15         | 2  | 2  | 3.991  | 65     | 2  | 6  | 7.474  |     |    |    |        |
| 16         | 7  | 0  | 3.994  | 66     | 11 | 2  | 7.474  |     |    |    |        |
| 17         | 0  | 3  | 4.454  | 67     | 0  | 7  | 7.515  |     |    |    |        |
| 18         | 5  | 1  | 4.454  | 68     | 9  | 3  | 7.515  |     |    |    |        |
| 19         | 3  | 2  | 4.454  | 69     | 16 | 0  | 7.515  |     |    |    |        |
| 20         | 1  | 3  | 4.593  | 70     | 7  | 4  | 7.515  |     |    |    |        |
| 21         | 8  | 0  | 4.593  | 71     | 5  | 5  | 7.592  |     |    |    |        |
| 22         | 6  | 1  | 4.593  | 72     | 14 | 1  | 7.592  |     |    |    |        |
| 23         | 4  | 2  | 4.718  | 73     | 3  | 6  | 7.671  |     |    |    |        |
| 24         | 2  | 3  | 4.900  | 74     | 12 | 2  | 7.702  |     |    |    |        |
| 25         | 9  | 0  | 4.901  | 75     | 1  | 7  | 7.990  |     |    |    |        |
| 26         | 0  | 4  | 5.329  | 76     | 10 | 3  | 7.990  |     |    |    |        |
| 27         | 7  | 1  | 5.329  | 77     | 8  | 4  | 7.990  |     |    |    |        |
| 28         | 5  | 2  | 5.329  | 78     | 17 | 0  | 7.990  |     |    |    |        |
| 29         | 3  | 3  | 5.329  | 79     | 6  | 5  | 7.991  |     |    |    |        |
| 30         | 10 | 0  | 5.329  | 80     | 15 | 1  | 7.991  |     |    |    |        |
| 31         | 1  | 4  | 5.469  | 81     | 4  | 6  | 8.176  |     |    |    |        |
| 32         | 8  | 1  | 5.469  | 82     | 13 | 2  | 8.176  |     |    |    |        |
| 33         | 6  | 2  | 5.470  | 83     | 2  | 7  | 8.302  |     |    |    |        |
| 34         | 4  | 3  | 5.592  | 84     | 11 | 3  | 8.302  |     |    |    |        |
| 35         | 11 | 0  | 5.618  | 85     | 0  | 8  | 8.310  |     |    |    |        |
| 36         | 2  | 4  | 5.778  | 86     | 9  | 4  | 8.310  |     |    |    |        |
| 37         | 9  | 1  | 5.779  | 87     | 18 | 0  | 8.320  |     |    |    |        |
| 38         | 0  | 5  | 5.820  | 88     | 7  | 5  | 8.347  |     |    |    |        |
| 39         | 7  | 2  | 5.834  | 89     | 16 | 1  | 8.422  |     |    |    |        |
| 40         | 5  | 3  | 5.902  | 90     | 5  | 6  | 8.533  |     |    |    |        |
| 41         | 12 | 0  | 5.902  | 91     | 14 | 2  | 8.533  |     |    |    |        |
| 42         | 3  | 4  | 6.130  | 92     | 3  | 7  | 8.648  |     |    |    |        |
| 43         | 10 | 1  | 6.131  | 93     | 12 | 3  | 8.648  |     |    |    |        |
| 44         | 1  | 5  | 6.324  | 94     | 1  | 8  | 8.807  |     |    |    |        |
| 45         | 8  | 2  | 6.324  | 95     | 10 | 4  | 8.807  |     |    |    |        |
| 46         | 6  | 3  | 6.325  | 96     | 19 | 0  | 8.807  |     |    |    |        |
| 47         | 13 | 0  | 6.351  | 97     | 8  | 5  | 8.807  |     |    |    |        |
| 48         | 4  | 4  | 6.488  | 98     | 17 | 1  | 8.807  |     |    |    |        |
| 49         | 11 | 1  | 6.490  | 99     | 6  | 6  | 8.808  |     |    |    |        |
| 50         | 2  | 5  | 6.634  | 100    | 15 | 2  | 8.808  |     |    |    |        |

TABLE 3 (CONT.)

| ALPHA= .10 |    |    |        | A1=.40 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.381  | 51     | 5  | 4  | 6.767  |     |    |    |        |
| 2          | 1  | 0  | 1.640  | 52     | 10 | 1  | 6.814  |     |    |    |        |
| 3          | 0  | 1  | 2.334  | 53     | 2  | 6  | 6.986  |     |    |    |        |
| 4          | 2  | 0  | 2.334  | 54     | 7  | 3  | 6.986  |     |    |    |        |
| 5          | 1  | 1  | 2.563  | 55     | 12 | 0  | 6.986  |     |    |    |        |
| 6          | 3  | 0  | 2.681  | 56     | 4  | 5  | 6.987  |     |    |    |        |
| 7          | 0  | 2  | 2.938  | 57     | 9  | 2  | 7.134  |     |    |    |        |
| 8          | 2  | 1  | 3.079  | 58     | 1  | 7  | 7.134  |     |    |    |        |
| 9          | 4  | 0  | 3.079  | 59     | 6  | 4  | 7.278  |     |    |    |        |
| 10         | 1  | 2  | 3.416  | 60     | 11 | 1  | 7.278  |     |    |    |        |
| 11         | 3  | 1  | 3.532  | 61     | 3  | 6  | 7.418  |     |    |    |        |
| 12         | 0  | 3  | 3.615  | 62     | 8  | 3  | 7.421  |     |    |    |        |
| 13         | 5  | 0  | 3.615  | 63     | 0  | 8  | 7.432  |     |    |    |        |
| 14         | 2  | 2  | 3.799  | 64     | 13 | 0  | 7.436  |     |    |    |        |
| 15         | 4  | 1  | 3.911  | 65     | 5  | 5  | 7.511  |     |    |    |        |
| 16         | 1  | 3  | 4.228  | 66     | 10 | 2  | 7.511  |     |    |    |        |
| 17         | 6  | 0  | 4.228  | 67     | 2  | 7  | 7.724  |     |    |    |        |
| 18         | 3  | 2  | 4.352  | 68     | 7  | 4  | 7.725  |     |    |    |        |
| 19         | 0  | 4  | 4.480  | 69     | 12 | 1  | 7.725  |     |    |    |        |
| 20         | 5  | 1  | 4.480  | 70     | 4  | 6  | 7.859  |     |    |    |        |
| 21         | 2  | 3  | 4.598  | 71     | 9  | 3  | 7.867  |     |    |    |        |
| 22         | 7  | 0  | 4.598  | 72     | 1  | 8  | 7.889  |     |    |    |        |
| 23         | 4  | 2  | 4.814  | 73     | 14 | 0  | 7.895  |     |    |    |        |
| 24         | 1  | 4  | 5.014  | 74     | 6  | 5  | 8.012  |     |    |    |        |
| 25         | 6  | 1  | 5.014  | 75     | 11 | 2  | 8.012  |     |    |    |        |
| 26         | 3  | 3  | 5.144  | 76     | 3  | 7  | 8.154  |     |    |    |        |
| 27         | 8  | 0  | 5.144  | 77     | 0  | 9  | 8.179  |     |    |    |        |
| 28         | 0  | 5  | 5.221  | 78     | 8  | 4  | 8.179  |     |    |    |        |
| 29         | 5  | 2  | 5.239  | 79     | 13 | 1  | 8.246  |     |    |    |        |
| 30         | 2  | 4  | 5.477  | 80     | 5  | 6  | 8.246  |     |    |    |        |
| 31         | 7  | 1  | 5.477  | 81     | 2  | 8  | 8.454  |     |    |    |        |
| 32         | 4  | 3  | 5.598  | 82     | 10 | 3  | 8.454  |     |    |    |        |
| 33         | 9  | 0  | 5.599  | 83     | 15 | 0  | 8.454  |     |    |    |        |
| 34         | 1  | 5  | 5.781  | 84     | 7  | 5  | 8.455  |     |    |    |        |
| 35         | 6  | 2  | 5.782  | 85     | 4  | 7  | 8.592  |     |    |    |        |
| 36         | 3  | 4  | 5.915  | 86     | 12 | 2  | 8.596  |     |    |    |        |
| 37         | 8  | 1  | 5.917  | 87     | 1  | 9  | 8.737  |     |    |    |        |
| 38         | 0  | 6  | 5.963  | 88     | 9  | 4  | 8.750  |     |    |    |        |
| 39         | 5  | 3  | 6.056  | 89     | 6  | 6  | 8.751  |     |    |    |        |
| 40         | 10 | 0  | 6.058  | 90     | 14 | 1  | 8.752  |     |    |    |        |
| 41         | 2  | 5  | 6.238  | 91     | 3  | 8  | 8.882  |     |    |    |        |
| 42         | 7  | 2  | 6.238  | 92     | 11 | 3  | 8.882  |     |    |    |        |
| 43         | 4  | 4  | 6.364  | 93     | 0  | 10 | 8.891  |     |    |    |        |
| 44         | 9  | 1  | 6.366  | 94     | 8  | 5  | 8.891  |     |    |    |        |
| 45         | 1  | 6  | 6.535  | 95     | 16 | 0  | 8.891  |     |    |    |        |
| 46         | 6  | 3  | 6.535  | 96     | 5  | 7  | 8.973  |     |    |    |        |
| 47         | 11 | 0  | 6.535  | 97     | 13 | 2  | 8.973  |     |    |    |        |
| 48         | 3  | 5  | 6.672  | 98     | 2  | 9  | 9.178  |     |    |    |        |
| 49         | 8  | 2  | 6.674  | 99     | 10 | 4  | 9.178  |     |    |    |        |
| 50         | 0  | 7  | 6.698  | 100    | 7  | 6  | 9.178  |     |    |    |        |

TABLE 4

| ALPHA= .05 |    |    |        | A1=.10 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 2.695  | 51     | 2  | 2  | 5.705  |     |    |    |        |
| 2          | 1  | 0  | 2.700  | 52     | 16 | 1  | 5.705  |     |    |    |        |
| 3          | 2  | 0  | 2.735  | 53     | 32 | 0  | 5.705  |     |    |    |        |
| 4          | 3  | 0  | 2.787  | 54     | 3  | 2  | 5.757  |     |    |    |        |
| 5          | 4  | 0  | 2.849  | 55     | 17 | 1  | 5.757  |     |    |    |        |
| 6          | 5  | 0  | 2.917  | 56     | 33 | 0  | 5.757  |     |    |    |        |
| 7          | 6  | 0  | 2.991  | 57     | 4  | 2  | 5.819  |     |    |    |        |
| 8          | 7  | 0  | 3.068  | 58     | 18 | 1  | 5.819  |     |    |    |        |
| 9          | 8  | 0  | 3.148  | 59     | 34 | 0  | 5.819  |     |    |    |        |
| 10         | 9  | 0  | 3.230  | 60     | 5  | 2  | 5.889  |     |    |    |        |
| 11         | 10 | 0  | 3.314  | 61     | 19 | 1  | 5.889  |     |    |    |        |
| 12         | 11 | 0  | 3.400  | 62     | 35 | 0  | 5.889  |     |    |    |        |
| 13         | 12 | 0  | 3.487  | 63     | 6  | 2  | 5.963  |     |    |    |        |
| 14         | 13 | 0  | 3.575  | 64     | 20 | 1  | 5.963  |     |    |    |        |
| 15         | 14 | 0  | 3.664  | 65     | 36 | 0  | 5.963  |     |    |    |        |
| 16         | 15 | 0  | 3.754  | 66     | 7  | 2  | 6.041  |     |    |    |        |
| 17         | 16 | 0  | 3.844  | 67     | 21 | 1  | 6.041  |     |    |    |        |
| 18         | 17 | 0  | 3.936  | 68     | 37 | 0  | 6.041  |     |    |    |        |
| 19         | 0  | 1  | 4.269  | 69     | 8  | 2  | 6.121  |     |    |    |        |
| 20         | 1  | 1  | 4.275  | 70     | 22 | 1  | 6.121  |     |    |    |        |
| 21         | 18 | 0  | 4.275  | 71     | 38 | 0  | 6.121  |     |    |    |        |
| 22         | 2  | 1  | 4.309  | 72     | 9  | 2  | 6.204  |     |    |    |        |
| 23         | 19 | 0  | 4.309  | 73     | 23 | 1  | 6.204  |     |    |    |        |
| 24         | 3  | 1  | 4.360  | 74     | 39 | 0  | 6.204  |     |    |    |        |
| 25         | 20 | 0  | 4.360  | 75     | 10 | 2  | 6.289  |     |    |    |        |
| 26         | 4  | 1  | 4.422  | 76     | 24 | 1  | 6.289  |     |    |    |        |
| 27         | 21 | 0  | 4.422  | 77     | 40 | 0  | 6.289  |     |    |    |        |
| 28         | 5  | 1  | 4.491  | 78     | 11 | 2  | 6.376  |     |    |    |        |
| 29         | 22 | 0  | 4.491  | 79     | 25 | 1  | 6.376  |     |    |    |        |
| 30         | 6  | 1  | 4.565  | 80     | 41 | 0  | 6.376  |     |    |    |        |
| 31         | 23 | 0  | 4.565  | 81     | 12 | 2  | 6.464  |     |    |    |        |
| 32         | 7  | 1  | 4.643  | 82     | 26 | 1  | 6.464  |     |    |    |        |
| 33         | 24 | 0  | 4.643  | 83     | 0  | 3  | 6.978  |     |    |    |        |
| 34         | 8  | 1  | 4.723  | 84     | 42 | 0  | 6.978  |     |    |    |        |
| 35         | 25 | 0  | 4.723  | 85     | 13 | 2  | 6.978  |     |    |    |        |
| 36         | 9  | 1  | 4.806  | 86     | 27 | 1  | 6.978  |     |    |    |        |
| 37         | 26 | 0  | 4.806  | 87     | 1  | 3  | 6.984  |     |    |    |        |
| 38         | 10 | 1  | 4.891  | 88     | 43 | 0  | 6.984  |     |    |    |        |
| 39         | 27 | 0  | 4.891  | 89     | 14 | 2  | 6.984  |     |    |    |        |
| 40         | 11 | 1  | 4.977  | 90     | 28 | 1  | 6.984  |     |    |    |        |
| 41         | 28 | 0  | 4.977  | 91     | 2  | 3  | 7.018  |     |    |    |        |
| 42         | 12 | 1  | 5.065  | 92     | 15 | 2  | 7.018  |     |    |    |        |
| 43         | 29 | 0  | 5.065  | 93     | 44 | 0  | 7.018  |     |    |    |        |
| 44         | 13 | 1  | 5.154  | 94     | 29 | 1  | 7.018  |     |    |    |        |
| 45         | 0  | 2  | 5.666  | 95     | 3  | 3  | 7.069  |     |    |    |        |
| 46         | 30 | 0  | 5.666  | 96     | 16 | 2  | 7.069  |     |    |    |        |
| 47         | 14 | 1  | 5.666  | 97     | 45 | 0  | 7.069  |     |    |    |        |
| 48         | 1  | 2  | 5.672  | 98     | 30 | 1  | 7.069  |     |    |    |        |
| 49         | 15 | 1  | 5.672  | 99     | 4  | 3  | 7.132  |     |    |    |        |
| 50         | 31 | 0  | 5.672  | 100    | 17 | 2  | 7.132  |     |    |    |        |

TABLE 4 (CONT.)

| ALPHA= .05 |    |    |        | A1=.20 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 2.396  | 51     | 4  | 3  | 6.664  |     |    |    |        |
| 2          | 1  | 0  | 2.425  | 52     | 15 | 1  | 6.664  |     |    |    |        |
| 3          | 2  | 0  | 2.534  | 53     | 21 | 0  | 6.664  |     |    |    |        |
| 4          | 3  | 0  | 2.677  | 54     | 10 | 2  | 6.666  |     |    |    |        |
| 5          | 4  | 0  | 2.840  | 55     | 5  | 3  | 6.850  |     |    |    |        |
| 6          | 5  | 0  | 3.015  | 56     | 0  | 4  | 7.323  |     |    |    |        |
| 7          | 6  | 0  | 3.198  | 57     | 16 | 1  | 7.323  |     |    |    |        |
| 8          | 0  | 1  | 3.795  | 58     | 22 | 0  | 7.323  |     |    |    |        |
| 9          | 7  | 0  | 3.795  | 59     | 11 | 2  | 7.323  |     |    |    |        |
| 10         | 1  | 1  | 3.825  | 60     | 6  | 3  | 7.323  |     |    |    |        |
| 11         | 8  | 0  | 3.825  | 61     | 1  | 4  | 7.365  |     |    |    |        |
| 12         | 2  | 1  | 3.934  | 62     | 17 | 1  | 7.411  |     |    |    |        |
| 13         | 9  | 0  | 3.934  | 63     | 23 | 0  | 7.464  |     |    |    |        |
| 14         | 3  | 1  | 4.081  | 64     | 12 | 2  | 7.464  |     |    |    |        |
| 15         | 10 | 0  | 4.081  | 65     | 7  | 3  | 7.464  |     |    |    |        |
| 16         | 4  | 1  | 4.248  | 66     | 2  | 4  | 7.464  |     |    |    |        |
| 17         | 11 | 0  | 4.248  | 67     | 18 | 1  | 7.464  |     |    |    |        |
| 18         | 5  | 1  | 4.429  | 68     | 13 | 2  | 7.464  |     |    |    |        |
| 19         | 0  | 2  | 5.037  | 69     | 24 | 0  | 7.464  |     |    |    |        |
| 20         | 12 | 0  | 5.037  | 70     | 8  | 3  | 7.464  |     |    |    |        |
| 21         | 6  | 1  | 5.037  | 71     | 3  | 4  | 7.614  |     |    |    |        |
| 22         | 1  | 2  | 5.067  | 72     | 19 | 1  | 7.614  |     |    |    |        |
| 23         | 7  | 1  | 5.067  | 73     | 14 | 2  | 7.614  |     |    |    |        |
| 24         | 13 | 0  | 5.067  | 74     | 25 | 0  | 7.614  |     |    |    |        |
| 25         | 2  | 2  | 5.176  | 75     | 9  | 3  | 7.614  |     |    |    |        |
| 26         | 8  | 1  | 5.176  | 76     | 4  | 4  | 7.787  |     |    |    |        |
| 27         | 14 | 0  | 5.176  | 77     | 20 | 1  | 7.787  |     |    |    |        |
| 28         | 3  | 2  | 5.325  | 78     | 15 | 2  | 7.787  |     |    |    |        |
| 29         | 9  | 1  | 5.325  | 79     | 26 | 0  | 7.787  |     |    |    |        |
| 30         | 15 | 0  | 5.325  | 80     | 10 | 3  | 7.790  |     |    |    |        |
| 31         | 4  | 2  | 5.494  | 81     | 5  | 4  | 7.975  |     |    |    |        |
| 32         | 10 | 1  | 5.495  | 82     | 0  | 5  | 8.410  |     |    |    |        |
| 33         | 16 | 0  | 5.495  | 83     | 21 | 1  | 8.410  |     |    |    |        |
| 34         | 5  | 2  | 5.677  | 84     | 16 | 2  | 8.410  |     |    |    |        |
| 35         | 0  | 3  | 6.202  | 85     | 27 | 0  | 8.410  |     |    |    |        |
| 36         | 11 | 1  | 6.202  | 86     | 11 | 3  | 8.410  |     |    |    |        |
| 37         | 17 | 0  | 6.202  | 87     | 6  | 4  | 8.410  |     |    |    |        |
| 38         | 6  | 2  | 6.202  | 88     | 1  | 5  | 8.420  |     |    |    |        |
| 39         | 1  | 3  | 6.298  | 89     | 22 | 1  | 8.499  |     |    |    |        |
| 40         | 12 | 1  | 6.343  | 90     | 17 | 2  | 8.552  |     |    |    |        |
| 41         | 18 | 0  | 6.343  | 91     | 28 | 0  | 8.552  |     |    |    |        |
| 42         | 7  | 2  | 6.343  | 92     | 12 | 3  | 8.552  |     |    |    |        |
| 43         | 2  | 3  | 6.343  | 93     | 7  | 4  | 8.552  |     |    |    |        |
| 44         | 13 | 1  | 6.343  | 94     | 2  | 5  | 8.552  |     |    |    |        |
| 45         | 19 | 0  | 6.343  | 95     | 23 | 1  | 8.552  |     |    |    |        |
| 46         | 8  | 2  | 6.343  | 96     | 18 | 2  | 8.552  |     |    |    |        |
| 47         | 3  | 3  | 6.493  | 97     | 13 | 3  | 8.552  |     |    |    |        |
| 48         | 14 | 1  | 6.493  | 98     | 29 | 0  | 8.552  |     |    |    |        |
| 49         | 20 | 0  | 6.493  | 99     | 8  | 4  | 8.552  |     |    |    |        |
| 50         | 9  | 2  | 6.493  | 100    | 3  | 5  | 8.703  |     |    |    |        |

TABLE 4 (CONT.)

| ALPHA= .05 |    |    |        | A1=.25 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 2.247  | 51     | 2  | 4  | 7.092  |     |    |    |        |
| 2          | 1  | 0  | 2.301  | 52     | 6  | 3  | 7.096  |     |    |    |        |
| 3          | 2  | 0  | 2.464  | 53     | 10 | 2  | 7.096  |     |    |    |        |
| 4          | 3  | 0  | 2.671  | 54     | 14 | 1  | 7.096  |     |    |    |        |
| 5          | 4  | 0  | 2.901  | 55     | 18 | 0  | 7.096  |     |    |    |        |
| 6          | 0  | 1  | 3.557  | 56     | 3  | 4  | 7.309  |     |    |    |        |
| 7          | 5  | 0  | 3.557  | 57     | 7  | 3  | 7.332  |     |    |    |        |
| 8          | 1  | 1  | 3.613  | 58     | 11 | 2  | 7.409  |     |    |    |        |
| 9          | 6  | 0  | 3.613  | 59     | 0  | 5  | 7.884  |     |    |    |        |
| 10         | 2  | 1  | 3.781  | 60     | 15 | 1  | 7.884  |     |    |    |        |
| 11         | 7  | 0  | 3.781  | 61     | 19 | 0  | 7.884  |     |    |    |        |
| 12         | 3  | 1  | 3.997  | 62     | 4  | 4  | 7.884  |     |    |    |        |
| 13         | 8  | 0  | 4.001  | 63     | 8  | 3  | 7.884  |     |    |    |        |
| 14         | 4  | 1  | 4.239  | 64     | 12 | 2  | 7.884  |     |    |    |        |
| 15         | 0  | 2  | 4.720  | 65     | 1  | 5  | 7.941  |     |    |    |        |
| 16         | 9  | 0  | 4.720  | 66     | 16 | 1  | 7.941  |     |    |    |        |
| 17         | 5  | 1  | 4.720  | 67     | 20 | 0  | 7.941  |     |    |    |        |
| 18         | 1  | 2  | 4.777  | 68     | 5  | 4  | 7.941  |     |    |    |        |
| 19         | 10 | 0  | 4.777  | 69     | 9  | 3  | 7.941  |     |    |    |        |
| 20         | 2  | 2  | 4.946  | 70     | 13 | 2  | 7.941  |     |    |    |        |
| 21         | 6  | 1  | 4.948  | 71     | 2  | 5  | 8.113  |     |    |    |        |
| 22         | 11 | 0  | 4.948  | 72     | 17 | 1  | 8.113  |     |    |    |        |
| 23         | 3  | 2  | 5.160  | 73     | 21 | 0  | 8.113  |     |    |    |        |
| 24         | 7  | 1  | 5.172  | 74     | 6  | 4  | 8.118  |     |    |    |        |
| 25         | 0  | 3  | 5.815  | 75     | 10 | 3  | 8.140  |     |    |    |        |
| 26         | 12 | 0  | 5.815  | 76     | 14 | 2  | 8.175  |     |    |    |        |
| 27         | 4  | 2  | 5.815  | 77     | 3  | 5  | 8.175  |     |    |    |        |
| 28         | 8  | 1  | 5.815  | 78     | 18 | 1  | 8.175  |     |    |    |        |
| 29         | 1  | 3  | 5.871  | 79     | 22 | 0  | 8.221  |     |    |    |        |
| 30         | 5  | 2  | 5.871  | 80     | 7  | 4  | 8.367  |     |    |    |        |
| 31         | 13 | 0  | 5.871  | 81     | 0  | 6  | 8.882  |     |    |    |        |
| 32         | 9  | 1  | 5.871  | 82     | 11 | 3  | 8.882  |     |    |    |        |
| 33         | 2  | 3  | 6.041  | 83     | 15 | 2  | 8.882  |     |    |    |        |
| 34         | 6  | 2  | 6.044  | 84     | 4  | 5  | 8.882  |     |    |    |        |
| 35         | 14 | 0  | 6.044  | 85     | 19 | 1  | 8.882  |     |    |    |        |
| 36         | 10 | 1  | 6.044  | 86     | 23 | 0  | 8.882  |     |    |    |        |
| 37         | 3  | 3  | 6.257  | 87     | 8  | 4  | 8.882  |     |    |    |        |
| 38         | 7  | 2  | 6.274  | 88     | 1  | 6  | 8.938  |     |    |    |        |
| 39         | 11 | 1  | 6.274  | 89     | 12 | 3  | 8.938  |     |    |    |        |
| 40         | 15 | 0  | 6.274  | 90     | 16 | 2  | 8.938  |     |    |    |        |
| 41         | 0  | 4  | 6.864  | 91     | 5  | 5  | 8.938  |     |    |    |        |
| 42         | 4  | 3  | 6.864  | 92     | 20 | 1  | 8.938  |     |    |    |        |
| 43         | 3  | 2  | 6.864  | 93     | 9  | 4  | 8.938  |     |    |    |        |
| 44         | 12 | 1  | 6.864  | 94     | 24 | 0  | 8.938  |     |    |    |        |
| 45         | 16 | 0  | 6.864  | 95     | 2  | 6  | 9.111  |     |    |    |        |
| 46         | 1  | 4  | 6.921  | 96     | 13 | 3  | 9.111  |     |    |    |        |
| 47         | 5  | 3  | 6.921  | 97     | 17 | 2  | 9.111  |     |    |    |        |
| 48         | 9  | 2  | 6.921  | 98     | 6  | 5  | 9.116  |     |    |    |        |
| 49         | 13 | 1  | 6.921  | 99     | 21 | 1  | 9.116  |     |    |    |        |
| 50         | 17 | 0  | 6.921  | 100    | 10 | 4  | 9.116  |     |    |    |        |

TABLE 4 (CONT.)

| ALPHA= .05 |    |    |        | A1=.30 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 2.097  | 51     | 15 | 0  | 7.358  |     |    |    |        |
| 2          | 1  | 0  | 2.188  | 52     | 1  | 5  | 7.455  |     |    |    |        |
| 3          | 2  | 0  | 2.426  | 53     | 4  | 4  | 7.459  |     |    |    |        |
| 4          | 3  | 0  | 2.714  | 54     | 7  | 3  | 7.459  |     |    |    |        |
| 5          | 0  | 1  | 3.321  | 55     | 10 | 2  | 7.459  |     |    |    |        |
| 6          | 4  | 0  | 3.321  | 56     | 13 | 1  | 7.459  |     |    |    |        |
| 7          | 1  | 1  | 3.416  | 57     | 2  | 5  | 7.695  |     |    |    |        |
| 8          | 5  | 0  | 3.416  | 58     | 16 | 0  | 7.695  |     |    |    |        |
| 9          | 2  | 1  | 3.666  | 59     | 5  | 4  | 7.745  |     |    |    |        |
| 10         | 6  | 0  | 3.742  | 60     | 8  | 3  | 7.777  |     |    |    |        |
| 11         | 0  | 2  | 4.406  | 61     | 11 | 2  | 7.804  |     |    |    |        |
| 12         | 3  | 1  | 4.406  | 62     | 0  | 6  | 8.289  |     |    |    |        |
| 13         | 1  | 2  | 4.501  | 63     | 14 | 1  | 8.289  |     |    |    |        |
| 14         | 7  | 0  | 4.501  | 64     | 3  | 5  | 8.289  |     |    |    |        |
| 15         | 4  | 1  | 4.504  | 65     | 17 | 0  | 8.289  |     |    |    |        |
| 16         | 2  | 2  | 4.740  | 66     | 6  | 4  | 8.289  |     |    |    |        |
| 17         | 5  | 1  | 4.765  | 67     | 9  | 3  | 8.289  |     |    |    |        |
| 18         | 8  | 0  | 4.765  | 68     | 12 | 2  | 8.289  |     |    |    |        |
| 19         | 0  | 3  | 5.427  | 69     | 1  | 6  | 8.386  |     |    |    |        |
| 20         | 3  | 2  | 5.427  | 70     | 15 | 1  | 8.386  |     |    |    |        |
| 21         | 6  | 1  | 5.427  | 71     | 4  | 5  | 8.390  |     |    |    |        |
| 22         | 9  | 0  | 5.427  | 72     | 18 | 0  | 8.390  |     |    |    |        |
| 23         | 1  | 3  | 5.522  | 73     | 7  | 4  | 8.390  |     |    |    |        |
| 24         | 4  | 2  | 5.525  | 74     | 10 | 3  | 8.390  |     |    |    |        |
| 25         | 7  | 1  | 5.525  | 75     | 2  | 6  | 8.626  |     |    |    |        |
| 26         | 10 | 0  | 5.525  | 76     | 13 | 2  | 8.626  |     |    |    |        |
| 27         | 2  | 3  | 5.761  | 77     | 16 | 1  | 8.626  |     |    |    |        |
| 28         | 5  | 2  | 5.796  | 78     | 5  | 5  | 8.683  |     |    |    |        |
| 29         | 8  | 1  | 5.798  | 79     | 19 | 0  | 8.683  |     |    |    |        |
| 30         | 11 | 0  | 5.798  | 80     | 8  | 4  | 8.728  |     |    |    |        |
| 31         | 0  | 4  | 6.407  | 81     | 0  | 7  | 9.203  |     |    |    |        |
| 32         | 3  | 3  | 6.407  | 82     | 11 | 3  | 9.203  |     |    |    |        |
| 33         | 6  | 2  | 6.407  | 83     | 3  | 6  | 9.203  |     |    |    |        |
| 34         | 9  | 1  | 6.407  | 84     | 14 | 2  | 9.203  |     |    |    |        |
| 35         | 12 | 0  | 6.407  | 85     | 17 | 1  | 9.203  |     |    |    |        |
| 36         | 1  | 4  | 6.503  | 86     | 6  | 5  | 9.203  |     |    |    |        |
| 37         | 4  | 3  | 6.506  | 87     | 20 | 0  | 9.203  |     |    |    |        |
| 38         | 7  | 2  | 6.506  | 88     | 9  | 4  | 9.203  |     |    |    |        |
| 39         | 10 | 1  | 6.506  | 89     | 1  | 7  | 9.300  |     |    |    |        |
| 40         | 13 | 0  | 6.506  | 90     | 12 | 3  | 9.300  |     |    |    |        |
| 41         | 2  | 4  | 6.742  | 91     | 4  | 6  | 9.305  |     |    |    |        |
| 42         | 5  | 3  | 6.785  | 92     | 15 | 2  | 9.305  |     |    |    |        |
| 43         | 8  | 2  | 6.800  | 93     | 7  | 5  | 9.305  |     |    |    |        |
| 44         | 11 | 1  | 6.809  | 94     | 18 | 1  | 9.305  |     |    |    |        |
| 45         | 14 | 0  | 6.810  | 95     | 21 | 0  | 9.305  |     |    |    |        |
| 46         | 0  | 5  | 7.358  | 96     | 10 | 4  | 9.305  |     |    |    |        |
| 47         | 3  | 4  | 7.358  | 97     | 2  | 7  | 9.541  |     |    |    |        |
| 48         | 6  | 3  | 7.358  | 98     | 13 | 3  | 9.541  |     |    |    |        |
| 49         | 9  | 2  | 7.358  | 99     | 5  | 6  | 9.604  |     |    |    |        |
| 50         | 12 | 1  | 7.358  | 100    | 16 | 2  | 9.604  |     |    |    |        |

TABLE 4 (CONT.)

| ALPHA= .05 |    |    |        | A1=1/3 |    |    |        | K=2 |    |    |        |
|------------|----|----|--------|--------|----|----|--------|-----|----|----|--------|
| I          | X1 | X2 | THETA* | I      | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |
| 1          | 0  | 0  | 1.997  | 51     | 9  | 2  | 7.457  |     |    |    |        |
| 2          | 1  | 0  | 2.125  | 52     | 0  | 6  | 7.894  |     |    |    |        |
| 3          | 2  | 0  | 2.427  | 53     | 14 | 0  | 7.894  |     |    |    |        |
| 4          | 0  | 1  | 3.162  | 54     | 7  | 3  | 7.894  |     |    |    |        |
| 5          | 3  | 0  | 3.162  | 55     | 5  | 4  | 7.894  |     |    |    |        |
| 6          | 1  | 1  | 3.295  | 56     | 12 | 1  | 7.894  |     |    |    |        |
| 7          | 4  | 0  | 3.300  | 57     | 3  | 5  | 7.894  |     |    |    |        |
| 8          | 2  | 1  | 3.599  | 58     | 10 | 2  | 7.894  |     |    |    |        |
| 9          | 0  | 2  | 4.197  | 59     | 1  | 6  | 8.035  |     |    |    |        |
| 10         | 5  | 0  | 4.197  | 60     | 8  | 3  | 8.035  |     |    |    |        |
| 11         | 3  | 1  | 4.197  | 61     | 15 | 0  | 8.035  |     |    |    |        |
| 12         | 1  | 2  | 4.333  | 62     | 6  | 4  | 8.036  |     |    |    |        |
| 13         | 6  | 0  | 4.333  | 63     | 13 | 1  | 8.036  |     |    |    |        |
| 14         | 4  | 1  | 4.341  | 64     | 4  | 5  | 8.175  |     |    |    |        |
| 15         | 2  | 2  | 4.638  | 65     | 11 | 2  | 8.177  |     |    |    |        |
| 16         | 7  | 0  | 4.639  | 66     | 2  | 6  | 8.344  |     |    |    |        |
| 17         | 0  | 3  | 5.168  | 67     | 9  | 3  | 8.345  |     |    |    |        |
| 18         | 5  | 1  | 5.168  | 68     | 16 | 0  | 8.345  |     |    |    |        |
| 19         | 3  | 2  | 5.168  | 69     | 0  | 7  | 8.765  |     |    |    |        |
| 20         | 8  | 0  | 5.168  | 70     | 7  | 4  | 8.765  |     |    |    |        |
| 21         | 1  | 3  | 5.306  | 71     | 14 | 1  | 8.765  |     |    |    |        |
| 22         | 6  | 1  | 5.306  | 72     | 5  | 5  | 8.765  |     |    |    |        |
| 23         | 4  | 2  | 5.385  | 73     | 12 | 2  | 8.765  |     |    |    |        |
| 24         | 9  | 0  | 5.389  | 74     | 3  | 6  | 8.765  |     |    |    |        |
| 25         | 2  | 3  | 5.613  | 75     | 10 | 3  | 8.765  |     |    |    |        |
| 26         | 7  | 1  | 5.616  | 76     | 17 | 0  | 8.765  |     |    |    |        |
| 27         | 0  | 4  | 6.102  | 77     | 1  | 7  | 8.906  |     |    |    |        |
| 28         | 5  | 2  | 6.102  | 78     | 8  | 4  | 8.906  |     |    |    |        |
| 29         | 10 | 0  | 6.102  | 79     | 15 | 1  | 8.906  |     |    |    |        |
| 30         | 3  | 3  | 6.102  | 80     | 6  | 5  | 8.907  |     |    |    |        |
| 31         | 8  | 1  | 6.102  | 81     | 13 | 2  | 8.915  |     |    |    |        |
| 32         | 1  | 4  | 6.241  | 82     | 4  | 6  | 9.059  |     |    |    |        |
| 33         | 6  | 2  | 6.241  | 83     | 11 | 3  | 9.061  |     |    |    |        |
| 34         | 11 | 0  | 6.241  | 84     | 18 | 0  | 9.061  |     |    |    |        |
| 35         | 4  | 3  | 6.288  | 85     | 2  | 7  | 9.134  |     |    |    |        |
| 36         | 9  | 1  | 6.359  | 86     | 9  | 4  | 9.134  |     |    |    |        |
| 37         | 2  | 4  | 6.548  | 87     | 0  | 8  | 9.142  |     |    |    |        |
| 38         | 0  | 5  | 7.009  | 88     | 16 | 1  | 9.144  |     |    |    |        |
| 39         | 7  | 2  | 7.009  | 89     | 7  | 5  | 9.249  |     |    |    |        |
| 40         | 12 | 0  | 7.009  | 90     | 14 | 2  | 9.287  |     |    |    |        |
| 41         | 5  | 3  | 7.009  | 91     | 5  | 6  | 9.344  |     |    |    |        |
| 42         | 10 | 1  | 7.009  | 92     | 12 | 3  | 9.344  |     |    |    |        |
| 43         | 3  | 4  | 7.009  | 93     | 3  | 7  | 9.568  |     |    |    |        |
| 44         | 1  | 5  | 7.148  | 94     | 19 | 0  | 9.568  |     |    |    |        |
| 45         | 8  | 2  | 7.148  | 95     | 10 | 4  | 9.631  |     |    |    |        |
| 46         | 13 | 0  | 7.148  | 96     | 1  | 8  | 9.631  |     |    |    |        |
| 47         | 6  | 3  | 7.148  | 97     | 17 | 1  | 9.631  |     |    |    |        |
| 48         | 4  | 4  | 7.274  | 98     | 8  | 5  | 9.631  |     |    |    |        |
| 49         | 11 | 1  | 7.274  | 99     | 15 | 2  | 9.765  |     |    |    |        |
| 50         | 2  | 5  | 7.457  | 100    | 6  | 6  | 9.765  |     |    |    |        |

TABLE 4 (CONT.)

| ALPHA= .05 |    |    | A1=.40 | K=2 |    |    |        |  |
|------------|----|----|--------|-----|----|----|--------|--|
| I          | X1 | X2 | THETA* | I   | X1 | X2 | THETA* |  |
| 1          | 0  | 0  | 1.797  | 51  | 5  | 4  | 7.556  |  |
| 2          | 1  | 0  | 2.056  | 52  | 10 | 1  | 7.556  |  |
| 3          | 0  | 1  | 2.846  | 53  | 2  | 6  | 7.651  |  |
| 4          | 2  | 0  | 2.846  | 54  | 7  | 3  | 7.724  |  |
| 5          | 1  | 1  | 3.080  | 55  | 12 | 0  | 7.724  |  |
| 6          | 3  | 0  | 3.158  | 56  | 4  | 5  | 7.908  |  |
| 7          | 0  | 2  | 3.777  | 57  | 9  | 2  | 7.911  |  |
| 8          | 2  | 1  | 3.777  | 58  | 1  | 7  | 8.106  |  |
| 9          | 4  | 0  | 3.777  | 59  | 6  | 4  | 8.107  |  |
| 10         | 1  | 2  | 4.004  | 60  | 11 | 1  | 8.107  |  |
| 11         | 3  | 1  | 4.103  | 61  | 3  | 6  | 8.237  |  |
| 12         | 5  | 0  | 4.209  | 62  | 8  | 3  | 8.238  |  |
| 13         | 0  | 3  | 4.209  | 63  | 13 | 0  | 8.238  |  |
| 14         | 2  | 2  | 4.410  | 64  | 0  | 8  | 8.259  |  |
| 15         | 4  | 1  | 4.537  | 65  | 5  | 5  | 8.371  |  |
| 16         | 6  | 0  | 4.537  | 66  | 10 | 2  | 8.375  |  |
| 17         | 1  | 3  | 4.876  | 67  | 2  | 7  | 8.572  |  |
| 18         | 3  | 2  | 4.985  | 68  | 7  | 4  | 8.573  |  |
| 19         | 5  | 1  | 5.086  | 69  | 12 | 1  | 8.573  |  |
| 20         | 0  | 4  | 5.091  | 70  | 4  | 6  | 8.579  |  |
| 21         | 7  | 0  | 5.206  | 71  | 9  | 3  | 8.579  |  |
| 22         | 2  | 3  | 5.206  | 72  | 14 | 0  | 8.579  |  |
| 23         | 4  | 2  | 5.464  | 73  | 1  | 8  | 8.878  |  |
| 24         | 1  | 4  | 5.713  | 74  | 6  | 5  | 8.878  |  |
| 25         | 6  | 1  | 5.713  | 75  | 11 | 2  | 8.878  |  |
| 26         | 3  | 3  | 5.831  | 76  | 3  | 7  | 9.010  |  |
| 27         | 8  | 0  | 5.831  | 77  | 8  | 4  | 9.013  |  |
| 28         | 0  | 5  | 5.933  | 78  | 0  | 9  | 9.065  |  |
| 29         | 5  | 2  | 5.933  | 79  | 13 | 1  | 9.074  |  |
| 30         | 2  | 4  | 6.108  | 80  | 5  | 6  | 9.114  |  |
| 31         | 7  | 1  | 6.130  | 81  | 10 | 3  | 9.155  |  |
| 32         | 4  | 3  | 6.213  | 82  | 2  | 8  | 9.339  |  |
| 33         | 9  | 0  | 6.213  | 83  | 15 | 0  | 9.339  |  |
| 34         | 1  | 5  | 6.527  | 84  | 7  | 5  | 9.339  |  |
| 35         | 6  | 2  | 6.528  | 85  | 12 | 2  | 9.339  |  |
| 36         | 3  | 4  | 6.650  | 86  | 4  | 7  | 9.344  |  |
| 37         | 8  | 1  | 6.651  | 87  | 9  | 4  | 9.457  |  |
| 38         | 0  | 6  | 6.774  | 88  | 1  | 9  | 9.461  |  |
| 39         | 5  | 3  | 6.774  | 89  | 14 | 1  | 9.461  |  |
| 40         | 10 | 0  | 6.775  | 90  | 6  | 6  | 9.591  |  |
| 41         | 2  | 5  | 6.903  | 91  | 11 | 3  | 9.640  |  |
| 42         | 7  | 2  | 6.914  | 92  | 3  | 8  | 9.775  |  |
| 43         | 4  | 4  | 7.113  | 93  | 16 | 0  | 9.775  |  |
| 44         | 9  | 1  | 7.151  | 94  | 8  | 5  | 9.778  |  |
| 45         | 1  | 6  | 7.324  | 95  | 0  | 10 | 9.814  |  |
| 46         | 6  | 3  | 7.324  | 96  | 13 | 2  | 9.848  |  |
| 47         | 11 | 0  | 7.324  | 97  | 5  | 7  | 9.877  |  |
| 48         | 3  | 5  | 7.451  | 98  | 10 | 4  | 9.877  |  |
| 49         | 8  | 2  | 7.498  | 99  | 2  | 9  | 10.095 |  |
| 50         | 0  | 7  | 7.556  | 100 | 15 | 1  | 10.096 |  |