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KILL PROBABILITY OF A GAUSSIAN DISTRIBUTED
COOKIE-CUTTER WEAPON AGAINST A RANDOM UNIFORMLY
DISTRIBUTED POINT TARGET WITHIN AN ELLIPSE

A. R. DiDonato, et al

Naval Surface Weapons Center
Dahlgren Laboratory, Virginia

February 1976

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR-3453	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) KILL PROBABILITY OF A GAUSSIAN DISTRIBUTED COOKIE-CUTTER WEAPON AGAINST A RANDOM UNIFORMLY DISTRIBUTED POINT TARGET WITHIN AN ELLIPSE		5. TYPE OF REPORT & PERIOD COVERED Final
7. AUTHOR(s) A. R. DiDonato M. P. Jarnagin, Jr. R. K. Jageman		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Surface Weapons Center Dahlgren Laboratory Dahlgren, Virginia 22448		8. CONTRACT OR GRANT NUMBER(s)
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Surface Weapons Center Dahlgren Laboratory Dahlgren, Virginia 22448		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE February 1976
		13. NUMBER OF PAGES 137
		15. SECURITY CLASS. (of this Report) UNCLASSIFIED
		16a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A solution by deterministic methods is described of the problem of computing the single-shot kill probability of a point target at a random point from a uniform distribution over the interior of an arbitrary ellipse in the plane, given that the distribution of shots is uncorrelated bivariate normal with respect to a rectangular coordinate system in the plane, and that the weapon has a cookie-cutter damage function with prescribed lethal radius R. This solution has been programmed at NSWC, Dahlgren Laboratory. The numerical evaluation of a double integral, whose integrand contains the so-called elliptic coverage function, is required. Computer results clearly show the superiority of this solution over a non-deterministic Monte Carlo method of Weidman and Brunner.		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 68 IS OBSOLETE
S/N 0102-LF-014-6601

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Technical Report No. TR-3453
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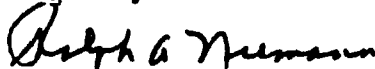
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FOREWORD

This technical report was written in the Applied Mathematics Section, Science and Mathematics Research Group, Warfare Analysis Department. Most of the work and all of the machine computations were done in 1974 and 1975. However, the essential mathematical steps of the present method were developed in 1968 shortly after the publication by D.R. Weidman and M.P. Brunner of their model for the solution of the same problem. See NWL Technical Note TN-K/21-68.

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ABSTRACT

A solution by deterministic methods is described of the problem of computing the single-shot kill probability of a point target at a random point from a uniform distribution over the interior of an arbitrary ellipse in the plane, given that the distribution of shots is uncorrelated bivariate normal with respect to a rectangular coordinate system in the plane, and that the weapon has a cookie-cutter damage function with prescribed lethal radius R . This solution has been programmed at NSWC, Dahlgren Laboratory. The numerical evaluation of a double integral, whose integrand contains the so-called elliptic coverage function, is required. Computer results clearly show the superiority of this solution over a non-deterministic, Monte Carlo method of Weidman and Brunner.

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1. INTRODUCTION

The problem treated in this report is as follows. A bomb falls in an uncorrelated bivariate normal distribution, with standard deviations σ_h and σ_k and mean point at the origin, with respect to a rectangular Cartesian coordinate system Chk. The bomb is assumed to have a known lethal radius R and a cookie-cutter lethality function; that is, everything is destroyed in a circle of radius R with the impact point as center, and nothing is damaged outside this lethal circle. This lethality or damage function has been used in many weapons effectiveness studies.

An ellipse of semi-axes a and b, $a \geq b$, is given, with center at a known point (u, v) in the Chk system and the major axis making a known angle ϕ (phi) with the h-axis (see Figure 1). A point target is at an unknown random position in the interior of the ellipse, governed by a uniform distribution over the area of the ellipse. It is required to find the probability P_K that the point target is destroyed by a single bomb.

An expression for P_K is derived in terms of a double integral, (2-12) or (2-13), which must be evaluated numerically. The expression is a function of eight variables, u, v, ϕ , a, b, R, σ_h , σ_k , of which seven are independent. We could normalize with respect to R, or σ_h , or in other ways, and thus reduce the number of variables by one.

The variables of integration in (2-12) and (2-13) are r and θ , parametric coordinates in the ellipse. In each element of area, the function $P(h, k, \sigma_h, \sigma_k, R)$ is evaluated by NSWC/DL subroutine PKILL, the coordinates (h, k) of the impact point of the bomb being expressed in terms of u, v, c (= b/a), ϕ , r and θ (2-11). This subroutine is based on [2], and computes the probability that a single bomb, falling in an elliptical normal distribution of the type described above, will fall in a circle of center (h, k) and radius R.

In special cases, namely when and only when $\sigma_h = \sigma_k$, we can use NSWC subroutine CIRCVC, computing the circular coverage function [5], instead of PKILL. CIRCVC is at least ten times faster than PKILL. Subroutine CIRCVC computes the probability $P(R, d)$ that a single bomb, falling in a standard circular normal distribution, i.e., $\sigma_h = \sigma_k = 1$, will fall in a circle of radius R with center offset a distance d from the mean point of the distribution. Further details on this special case are given in Appendix A.

This method, referred to hereafter as the DJH method, in contrast to a different approach discussed below, and in [1], is entirely deterministic. The accuracy of the computed P_K is limited only by that of the numerical evaluation of the double integral. If an iterative numerical double integration routine is used with successive halving of the step sizes in each direction, P_K can be computed to any reasonable accuracy level by continuing the iterations until successive iterates agree with each other within a preassigned tolerance. A FORTRAN listing of the program is given in Appendix F.

A non-deterministic, Monte Carlo method is used in [1]. A simulated bomb impact point (h, k) is determined by the use of random normal numbers. One then computes the area common to the ellipse and the lethal circle, i.e., the circle of center (h, k) and radius R . The conditional probability of destruction of the point target, given that the bomb hits at (h, k) is then given by the ratio of this common area to the total area of the ellipse. This follows from the uniform distribution of the target positions throughout the interior of the ellipse. By replicating this procedure a large number of times and taking the mean of the conditional probabilities, an approximation $\bar{p}$ is obtained to the overall kill probability P_K .

The method used in [1] for estimating the error in computing P_K is to use the probable error (P.E.). However, their expression for this quantity is not correct. The correct expression is given by

$$\text{P.E.} = 0.6745 \sqrt{\frac{\sum (p_i - \bar{p})^2}{n(n-1)}} \quad (1-1)$$

where p_i ($i = 1, 2, \dots, n$) is a conditional probability as described above, and $\bar{p}$ is the mean of these n conditional probabilities. This mean, $\bar{p}$, is the estimate obtained for the kill probability, P_K , by the particular simulation involving the n conditional probabilities.

The validity of (1-1) is established in Appendix C. Although this corrected formula for the P.E., being smaller than the incorrect formula by a factor $1/\sqrt{n-1}$ (see (C6)), puts their method in a better light, the computer results in Section 3 show, not surprisingly, the superiority of the method (DJH) of this report for computing P_K .

See Appendix C for a detailed discussion of the P.E. in the Weidman-Brunner (WB) method.

2. DESCRIPTION OF DJH METHOD

The geometry of the problem is shown in Figure 1. A bomb falls in an uncorrelated bivariate normal distribution centered at C, the origin of the rectangular Cartesian coordinate system Chk, with standard deviations σ_h and σ_k along the h- and k-axes, respectively. In general $\sigma_h > \sigma_k$. If $\sigma_h = \sigma_k$, we have a special case (Appendix A). The probability density function of the impact point of the bomb is

$$f(h, k) = \frac{1}{2\pi\sigma_h\sigma_k} \exp \left[-\frac{1}{2} \left(\frac{h^2}{\sigma_h^2} + \frac{k^2}{\sigma_k^2} \right) \right].$$

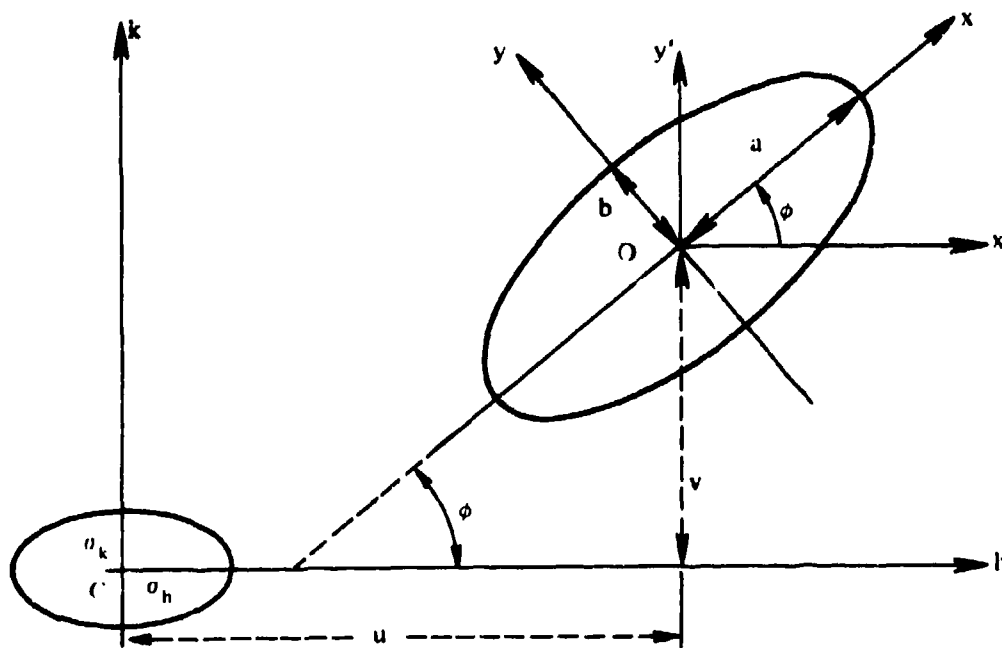


Figure 1. Geometry of the Problem

It is assumed that the bomb has a cookie-cutter lethality function with known lethal radius R (see first paragraph of Section 1).

A point target is located at a random point from a uniform distribution over the area of a given ellipse, centered at a point O with known coordinates (u, v) in the Chk system. The semi-axes are a, b, with $a > b$ in general. If $a = b$, the ellipse is a circle, and this is a special case (Appendix A). The major axis of the ellipse is inclined at a known angle ϕ to the h-axis, measured positive counterclockwise from the h-axis and in the range $-90 < \phi < 90$ degrees.

A rectangular coordinate system Oxy is set up with origin at O, the center of the ellipse, and x- and y-axes along the major and minor axes, respectively.

We shall need the equations of transformation from the Oxy to the Chk coordinate system. We set up an intermediate system Ox'y' (Figure 1), with the x'- and y'-axes parallel to the h- and k-axes, respectively. Then the equations of translation between the Chk and Ox'y' systems, and of rotation through the given angle ϕ from the Ox'y' to the Oxy system, are the following:

$$\left. \begin{aligned} h &= x' + u \\ k &= y' + v \end{aligned} \right\} \quad (2-1)$$

$$\left. \begin{aligned} x' &= x \cos \phi - y \sin \phi \\ y' &= x \sin \phi + y \cos \phi \end{aligned} \right\} \quad (2-2)$$

Combining (2-1) with (2-2), we get the following equations of transformation from the Oxy to the Chk system, in terms of the constants u, v, and ϕ .

$$\left. \begin{aligned} h &= u + x \cos \phi - y \sin \phi \\ k &= v + x \sin \phi + y \cos \phi \end{aligned} \right\} \quad (2-3)$$

The equation of the ellipse (Figure 1) in the Oxy system is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (2-4)$$

Let $b/a = c$, with $0 < c < 1$ except in the special case in which the ellipse is a circle (Appendix A), when $c = 1$.

We represent points on or inside the ellipse by curvilinear coordinates, r, θ , where

$$\left. \begin{aligned} x &= r \cos \theta \\ y &= cr \sin \theta \end{aligned} \right\} \quad (2-5)$$

The curvilinear coordinates, r, θ can be given geometrical significance in the following way. Since $b = ac$, (2-4) is equivalent to

$$x^2 + \frac{y^2}{c^2} = a^2 \quad (2-6)$$

and if we put

$$\left. \begin{aligned} \xi &= x \\ \eta &= y/c \end{aligned} \right\} \quad (2-7)$$

we have

$$\xi^2 + \eta^2 = a^2 \quad (2-8)$$

or, the ellipse corresponds to a circle of radius a in the $\xi\eta$ -plane; while for interior points in the ellipse, (2-5) and (2-7) give

$$\left. \begin{aligned} \xi &= r \cos \theta \\ \eta &= r \sin \theta \end{aligned} \right\} \quad (2-9)$$

or, r and θ are polar coordinates on and inside the auxiliary circle represented by (2-8).

By a standard theorem in analysis (for example, see [3, p. 348]), the element of area dA or $dx \, dy$ transforms in the following way

$$\begin{aligned} dA &= \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr \, d\theta = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr \, d\theta \\ &= \begin{vmatrix} \cos \theta & c \sin \theta \\ -r \sin \theta & cr \cos \theta \end{vmatrix} dr \, d\theta = cr \, dr \, d\theta \end{aligned} \quad (2-10)$$

$|\partial(x,y)/\partial(r,\theta)|$ being the Jacobian determinant.

Subroutine PKILL, or the elliptical coverage routine, [2], does the following calculation. Suppose we are given a rectangular coordinate system Chk and a bomb falling in an uncorrelated bivariate normal distribution with mean point C and standard deviations σ_h and σ_k as in the present problem. Suppose we are also given a circle with the center at a given point (h, k) and a given radius R. This subroutine computes the probability $P(h, k, \sigma_h, \sigma_k, R)$ that a single bomb falls in the given circle. This probability, computed by subroutine PKILL, is also designated as P_{KILL} , but P_{KILL} should not be confused with the overall single shot kill probability P_K computed by the DJH or the WB method of this report. Suppose that the lethal radius of the bomb, with the so-called cookie-cutter damage function, is also R, say 50 feet (this of course is the reason for taking this constant R as one of the input parameters in computing $P(h, k, \sigma_h, \sigma_k, R)$). Then a point target at (h, k) is destroyed if, and only if, the impact point of the bomb is in the given circle centered at (h, k). It is obvious from a simple geometrical figure that the point (h, k) is in the lethal circle of the bomb, if, and only if, the bomb's impact point is in the circle of radius R centered at (h, k). Thus, the single shot probability of destruction of the point target at (h, k) is given by $P(h, k, \sigma_h, \sigma_k, R)$ as computed by subroutine PKILL.

We proceed to set up the double integral whose value, designated as P_K , is the probability of destruction of the point target at a random point from a uniform distribution over the area of the given ellipse. Given any element of area $cr dr d\theta$ of the ellipse (2-6), the probability that the point target lies in this element of area is $cr dr d\theta / (\pi ab)$, the ratio of the area of the element to the entire area of the ellipse, since the distribution of the target position is uniform over the ellipse. If the target does lie in this element of area, and if a representative point in the element has coordinates (h, k) in the Chk system (all points of the element have the same coordinates, neglecting infinitesimals of higher order), then the conditional probability of destruction of the target is $P(h, k, \sigma_h, \sigma_k, R) \cdot cr dr d\theta / (\pi ab)$. The coordinates of the point being (x, y) in the Oxy system, or (r, θ) in curvilinear coordinates, the (h, k) coordinates of the point, by (2-3) and (2-5) are given by

$$\left. \begin{aligned} h &= u + r \cos \theta \cos \phi - cr \sin \theta \sin \phi \\ k &= v + r \cos \theta \sin \phi + cr \sin \theta \cos \phi \end{aligned} \right\} \quad (2-11)$$

u, v, and ϕ being constants. Hence the conditional probability of destruction of the target, given that it lies in the element of area $cr dr d\theta$, is

$$P(u + r \cos \theta \cos \phi - cr \sin \theta \sin \phi, v + r \cos \theta \sin \phi + cr \sin \theta \cos \phi, \sigma_h, \sigma_k, R) cr dr d\theta / (\pi ab),$$

which could be written more briefly as $P(h, k, \sigma_h, \sigma_k, R)$ or $dr d\theta/(\pi ab)$, with h, k given by (2-11).

The overall kill probability P_K is obtained by integrating these conditional probabilities over the entire ellipse, since the target must be somewhere in the ellipse. The curvilinear coordinate r goes from 0 to a (every point on the ellipse having this coordinate $r = a$), and θ from θ to 2π . There is no symmetry here, in general, because of the arbitrary position of the ellipse in the Chk system (Figure 1), so that θ must be integrated through all four quadrants, 0 to 2π . The result is

$$P_K = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a P(u + r \cos \theta \cos \phi - cr \sin \theta \sin \phi, v + r \cos \theta \sin \phi + cr \sin \theta \cos \phi, \sigma_h, \sigma_k, R) \cdot r dr d\theta \quad (2-12)$$

Here we have replaced $c/(\pi ab)$ by $c/(\pi a \cdot ac)$ or $(1/\pi a^2)$. This can be written more compactly as

$$P_K = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a P(h, k, \sigma_h, \sigma_k, R) \cdot r dr d\theta \quad (2-13)$$

with h, k given by (2-11). The P functions on the right are computed by subroutine PKILL. The double integral as written in (2-12) or (2-13) is now ready for evaluation by a numerical double integration subroutine, which will treat the variables r and θ as dummy variables like the variables of integration in any double integral, so that their geometrical significance as curvilinear coordinates in the ellipse, or polar coordinates in an auxiliary circle, is no longer relevant to the calculation.

One fact that we can deduce from (2-12) or (2-13), and which is of some theoretical significance, and is useful in designing test cases and criteria for determining cases in which $P_K \sim 1$ or $P_K \sim 0$, is that the computed value of the kill probability, P_K , is simply the space average, over the ellipse of integration, of the values of the elliptical coverage function $P(h, k, \sigma_h, \sigma_k, R)$, appearing in the integrand in (2-13) and computed by subroutine PKILL, [2].

We can prove this in the following way. Let $\bar{P}$ denote the average value of $P(h, k, \sigma_h, \sigma_k, R)$. Then

$$\bar{P} = \iint P(h, k, \sigma_h, \sigma_k, R) dA / \iint dA \quad (2-14)$$

where each double integration, in the parametric variables r and θ , is over the ellipse of integration. Since $dA = cr dr d\theta$ by (2-10), we then have

$$\bar{P} = \frac{\iint P(h, k, \sigma_h, \sigma_k, R) cr dr d\theta}{\iint cr dr d\theta} \quad (2-15)$$

Now we move the two factors of c outside their respective integral signs and cancel them, and trivially evaluate the integral

$$\int_0^{2\pi} \int_0^a r dr d\theta \text{ as } \pi a^2,$$

so that we have

$$\bar{P} = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a P(h, k, \sigma_h, \sigma_k, R) r dr d\theta \quad (2-16)$$

which is the same as the expression for the kill probability P_K in (2-13), so that we have

$$\bar{P} = P_K \quad (2-17)$$

as stated.

3. NUMERICAL RESULTS

The basic equation for the computation of the kill probability P_K by the DJH method is (2-12), or equivalently, (2-13), with h, k given by (2-11). We have to evaluate numerically a double integral, and we have to call the subroutine PKILL, computing the elliptical coverage function $P(h, k, \sigma_h, \sigma_k, R)$, [2], for each point used in the integration.

The double integration routine used is a CDC software routine, using Simpson's rule in each dimension, provided by Control Data Corporation, manufacturer of the CDC 6700 machine. This routine allows for the selection of different tolerances for the horizontal and vertical integrations. However, the authors of the present report did not give systematic study to the selection of these tolerances until several of the cases in the testing program required an excessive amount of computing time, more than one minute in some cases (page 16). The error analysis which was then worked out, as well as a detailed description of the integration method, is given later in this section, pages 21-29. Here we give a brief general description of the method, assuming that the positive tolerances, denoted as ϵ_0 and ϵ_1 , have been appropriately chosen.

We follow here the notation of (3-1) and Figure 3 (page 22), the integrand $F(x, y)$ being unspecified and the integral being regarded as an iterated one. For each integration point $y = s$ used in the vertical integration (with respect to y), the single integral $\int_a^b F(x, s) dx$ or $f(s)$ is approximated by Simpson's rule.

We give first an overall summary of the two integrations, followed by a few more details, but for fuller details see pages 21-29. For a given $y = s$, where $c \leq s \leq d$, the exact value of the integral $\int_a^b F(x, s) dx$ is denoted by I_s , and the Simpson approximation by I_s . The calculation of I_s involves a tolerance which is denoted here as ϵ_0 . A function of y , namely I_s or I_y , is thereby defined on $[c, d]$. This function, in general, has discontinuities at values of y for which the number of subintervals of $[a, b]$ changes, say from 16 to 32 subintervals (or from 17 to 33 points), but because of the smooth and slowly changing integrand here, (2-12), we can assume that the points of discontinuity are finite in number and, therefore, the integral $\int_c^d I_y dy$ is approximated by Simpson sums $[(d - c)/3n] \sum k_i I_{s_i}$ as the number of integration points becomes infinite, the multipliers k_i being the familiar Simpson factors 1, 4, 2, 4, ..., 2, 4, 1.

An overall tolerance ϵ , say $\epsilon = 0.005$, is specified. The objective is to use sufficient points in both the horizontal and the vertical integrations so that we have a high degree of confidence, on heuristic grounds at least, that the final approximation I is within ϵ of the true value of the double integral; or, $|I - \bar{I}| < \epsilon$. The tolerance ϵ_0 referred to above, and a tolerance ϵ_1 used in the vertical integration (with respect to y), depend on ϵ in a manner indicated below.

For a given y , or s , sufficient points on $[a, b]$ are taken to give a high degree of confidence that the approximation I_s is within ϵ_0 of the exact value $\bar{I}_s$ or $\int_a^b F(x, s) dx$. The usual rule in practice is that two successive Simpson sums $\frac{b-a}{3n_1} \sum_{i=0}^{n_1} k_i F(x_i, s)$, and $\frac{b-a}{3n_2} \sum_{j=0}^{n_2} k_j F(x_j, s)$ (where $n_2 = 2n_1$, the x_i 's, $i = 0, 1, \dots, n_1$ are equally spaced on $[a, b]$, with the x_j 's, $j = 0, 1, \dots, n_2$ also equally spaced at half the mesh length of the x_i 's) agree with each other within ϵ_0 . A stricter criterion could be used if desired, for example, that of three successive Simpson sums, the largest and smallest must be within ϵ_0 of each other. But, once a criterion has been selected, it will be applicable for every value of y or s on $[c, d]$, and a function I_y is defined on $[c, d]$ as described above. Note that different numbers of points on $[a, b]$, in general, are used on different horizontal lines, i.e., for different values of y or s . This is already implied by the remarks above regarding discontinuities in I_y at values of y for which the number of points on $[a, b]$ changes.

It is shown that the integral $\int_c^d I_y dy$ differs by less than $(d - c) \epsilon_0$ from the exact value of the double integral, $\int_c^d \bar{I}_y dy$ or $\bar{I}$ (3-8). But a sufficient number of points on $[c, d]$ are taken so that (to a high degree of confidence at least) the approximating Simpson sum $\frac{d-c}{3n} \sum k_i I_{y_i}$ (or $\frac{d-c}{3n} \sum k_i I_{y_i}$) is within ϵ_1 of the value of the integral $\int_c^d I_y dy$ (3-10), where ϵ_1 , like ϵ_0 , is one of the tolerances depending on the overall tolerance ϵ . Therefore, the final Simpson approximation, $\frac{d-c}{3n} \sum k_i I_{y_i}$, in the vertical integration, is within $(d - c) \epsilon_0 + \epsilon_1$ of the exact value $\bar{I}$ of the double integral (3-13).

This sum, $(d - c) \epsilon_0 + \epsilon_1$, is equated with the overall tolerance ϵ (3-15). The tolerances ϵ_0 and ϵ_1 may be chosen in any manner such that this equality is satisfied.

The most obvious choice is $\epsilon_0 = \epsilon_1$, in which case each of these tolerances has the value $\epsilon/[1 + (d - c)]$ (3-17).

We now consider in slightly more detail the computation by Simpson's rule of a typical one of the horizontal integrals (i.e., with respect to x) $\int_a^b F(x, s) dx$, where s is some fixed value of y on $[c, d]$. If n subintervals of $[a, b]$ are used, or $n + 1$ points, n being an even integer, the Simpson's rule approximation is

$$I_{s,n} = \frac{b-a}{3n} \sum_{i=0}^n k_i F(x_i, s) = \frac{b-a}{3n} \left\{ F(a, s) + 4F\left(a + \frac{b-a}{n}, s\right) + 2F\left(a + 2\frac{b-a}{n}, s\right) + \dots + 4F\left[a + (n-1)\frac{b-a}{n}, s\right] + F(b, s) \right\}$$

The step size $(b-a)/n$ is repeatedly reduced by successive doubling of n until we reach a value such that $|I_{s,2n} - I_{s,n}| < \epsilon_0$, in case the usual criterion described above is used. A similar method is used in the vertical integration, with tolerance ϵ_1 .

First we give the numerical values taken by the input parameters $\sigma_h, \sigma_k, R, u, v, a, b$, and ϕ in the Table E3 in final form as they appear in this technical report, pages 59 through 109. Actually, values are assigned to the ratio b/a rather than to b itself. Later in this section, we discuss the testing program involving both the DJH and WB methods, and such topics as simple tests for detecting cases in which $P_K \approx 0$ or 1 without performing the double integration of (2-12) or (2-13), and the error analysis for the method of double integration as used in the DJH program, in the general case where P_K is not near 0 or 1.

The standard deviation σ_h takes the value 1 in every case because of the normalization which is used, a takes the values 0.1, 0.5, 1, and 2 on every page; the ratio b/a always takes the values 0.2, 0.5, 0.8, and 1; and the angle ϕ takes the values $0^\circ, 45^\circ, 90^\circ$, and -45° on every page. Therefore, the only parameters whose values can vary from page to page are R, σ_k, u , and v (see Table 1), and these are the parameters whose values are listed in the appended table. The total number of cases in the main table (Table E3) is 59,904, on 51 pages, 59 to 109.

Table 1. Values of R , σ_k , u , and v for Main Table (E3)

Pages	Values of R	Values of σ_k	Values of u	Values of v
59-62	0.25	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
63-66	0.50	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
67-70	0.75	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
71-78	1	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
79-86	1.5	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
87-94	2	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1 (See Note)
95-98	2.5	0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
99-106	3	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
107- 109	4	1	0,0.5,1,2,2.5,3,3.5,4, 4.5,5,5.5,6	0,0.25,0.50,1

Note: For $R = 2$, $\sigma_k = 0.2$: $v = 0, 0.25, 0.50, 1.50$

But for $R = 2$, $\sigma_k = 0.5, 0.8, 1$: $v = 0, 0.25, 0.50, 1$

The tolerance, or ϵ in the error analysis, is generally 0.005, indicating that, although the main table gives values of P_k to three decimal digits, the third digit may be incorrect by several units, and we should round off to two decimal digits to obtain reliable results. The tolerance was taken as 0.001 in two pages of the table (pages 59 and 63) where σ_k takes its smallest value, 0.2, and R its two smallest values, 0.25 and 0.50. But a tolerance of 0.001 entailed the use of too much computer time in general.

At the top of each page are the values of σ_h , σ_k and R for all of the cases on that page, but since $\sigma_h = 1$ always, the significant values are those of σ_k and R . The values of u , v and ϕ appear at or near the left end of the lines, and those of a and b as column headings.

The angle ϕ , on every page, takes the four values listed above (-45° , 0° , 45° , 90°), and similarly a and the ratio b/a take the four values for each which are listed. The most common values for u and v are $u = 0, 0.5, 1, 1.5, 2$, and $v = 0, 0.25, 0.5, 1$. The larger values of u (greater than 2.0) are used with the larger values of R , since for these values of R and small to moderate values of u , the values of P_K are generally near 1. The kill probability P_K increases strictly monotonically with R for fixed values of the other input parameters, and we always have $P_K \rightarrow 1$ as $R \rightarrow \infty$. On page 107 of the table, where $\sigma_k = 1$, $R = 4$, the smallest value of P_K for $u \leq 1$ is 0.941, and for $u \leq 2$ it is 0.838, the smallest value on the page. Hence, in order to cover a substantial part of the range (0,1) for P_K , an additional page (page 109) was computed for $\sigma_k = 1$, $R = 4$, with u taking values up to 6.

Also, on pages 87-88 of the tables, the values of v are 0, 0.25, 0.5, 1.5, since on these pages, where $\sigma_k = 0.2$, $R = 2$, the values of P_K are generally insensitive to the value of v , and we get a wider range of values of P_K with these values of v than with the usual values 0, 0.25, 0.5, 1.

Further comments on the selection of values for the input parameters are given later in this section, pages 18-20.

In checking out the DJH method of this report (see (2-12) and (2-13)), 22 cases were run on the CDC 6700 computer. These cases are listed in Table 2. Actually, one case was inadvertently assigned two case numbers, 14 and 15, so that there were only 21 different cases, but we follow the original numbering here. Some of these cases were checked by three different methods, (a) double integration using subroutine PKILL (i.e., the general DJH method), (b) double integration using subroutine CIRCIV instead of PKILL (see Appendix A, page 31), and (c) single integration using CIRCIV (see (A2) and (A3) in Appendix A). Methods (b) and (c) can be used if, and only if $\sigma_h = \sigma_k$, as stated in Appendix A.

The results were uniformly consistent, justifying the conclusion that the DJH program computes P_K with an absolute error not exceeding a few units in the third decimal digit. Although this level of accuracy is adequate for most applications, five to six decimal digit accuracy can be attained for P_K by a simple change in the PKILL subroutine to yield six decimal digit accuracy instead of three as it now stands, and a willingness to pay the costs of significantly increased computer time. The time will, in general, be more than doubled, in obtaining results correct to six rather than three decimal digits. Subroutine PKILL, [2], on the CDC 6700 machine, requires approximately 5 milliseconds for three-digit accuracy and 10 milliseconds for six digits. In

Table 2. Test Cases for DJH and WB Methods

Case	σ_h	σ_k	R	u	v	a	b	ϕ
1	1	2	1	0	2	0.10	0.30	0
2	2	1	1	0	2	0.30	0.10	0
3	2	1	1	0	2	0.10	0.30	0
4	1	3	4	0	2	2	3	0
5	1	2	1	0	2	4	12	0
6	2	1	1	0	2	12	4	0
7	2	1	1	0	2	4	12	0
8	3	2	1	0	2.5	0.0625	0.03125	0
9	3	2	1	0	2.5	0.125	0.0625	0
10	3	2	1	0	2.5	0.25	0.125	0
11	3	2	1	0	2.5	0.5	0.25	0
12	3	2	1	0	2.5	1	0.5	0
13	1	1	1	0	4	0.10	0.30	0
14	1	3	4	0	4	1	1	0
15	1	3	4	0	4	1	1	0
16	1	1	1	0	4	1	3	0
17	1	6	1	0	6	0.10	0.30	0
18	6	1	1	0	6	0.30	0.10	0
19	6	1	1	0	6	0.10	0.30	0
20	1	6	1	0	12	4	12	0
21	6	1	1	0	12	12	4	0
22	6	1	1	0	12	4	12	0

addition, there will be a finer tolerance, say $\epsilon = 10^{-6}$, in the error analysis later in this section, pages 21-29, in computing to six digits, than the tolerance of 10^{-3} in the three-digit calculation, and this will entail more calls to the PKILL subroutine.

In case 16, computed by the three methods mentioned above, with the objective of obtaining six-digit agreement, the smallest and largest of the three results for P_K were 0.02251899 and 0.02251922 respectively. We note that method (c), the third method used, is quite different from the other two in setting up the integration.

All except cases 8 through 12 were run by the method of [1], which will be referred to as the WB method (Weidman and Brunner). These five cases were designed to test the DJH method by using a sequence of decreasing ellipses closing down on the point (0,4), the smallest one having a major axis of 0.0625.

The results obtained by the WB method were of variable quality. In some cases they were surprisingly good, as in Case 6 in which the DJH and WB methods gave results of 0.019962 and 0.019922 respectively. An isolated case of good agreement could be due to chance, but in several other cases the agreement was quite good.

But in some other cases the WB method, when applied more than once, generating different simulated bomb impact points each time, gave results that differed significantly from one another. Thus, in Case 2, which was run three times with the WB and once with the deterministic DJH method, the WB method, using 25, 50, and 100 simulated impact points, gave results of 0.011872, 0.005936 and 0.0217015 respectively. Yet the apparently correct P_K , as given by the deterministic DJH method, was 0.044321. Moreover, the 50% confidence interval for the WB method, computed by the method described in Appendix C (see also a brief description in the next paragraph of this section), in none of the three cases contained the correct value, 0.044321. For the calculation in which 100 impact points were used, the probable error was 0.008998, and so the 50% confidence interval was 0.0217015 ± 0.008998 or (0.012704, 0.030700).

The 50% confidence interval is $(\bar{p} - P.E., \bar{p} + P.E.)$, where $\bar{p}$ is the mean of the n conditional probabilities $\{p_i\}$ in a particular simulation, and P.E. is the probable error computed by (1-1) or (C5). For a justification of this expression for the 50% confidence interval, see the latter part of Appendix C, pages 45-46 of this report. As for the sample of size $n = 100$, see the passage from [9] quoted in connection with (C7) in

this report. The point involved here is that, although the conditional probabilities p_i in the WB method cannot be assumed to be normally distributed, the means $\bar{p}$ of samples of size n are approximately normally distributed, by the central limit theorem, and the larger the value of n , the better is the approximation to normality. Reference [9, p. 111] states that this normality assumption is quite accurate in most cases for $n > 10$. Hence, we can assume that, in this application, it is very accurate for $n = 100$, and that the failure of the 50% confidence interval to include the apparently correct value $P_K = 0.044321$ cannot be attributed to the fact that the means $\bar{p}$ in the WB method are not from an exactly normal distribution. It should be borne in mind that, no matter how large the sample size n may be, the probability that the 50% confidence interval will cover the true population mean, P_K or p_0 , is, by definition, only 0.50. For higher confidence levels such as 90%, the confidence intervals are longer than for the 50% level.

This last-mentioned calculation (WB method with $n = 100$) used something of the order of 36 seconds on the CDC 6700 computer, since this and a similar run, using 100 impact points on a different case, took a total of 73 seconds. If fewer simulated impact points are used, the machine time is proportionately shorter. But since the P.E. (probable error) is approximately proportional to $1/n$, where n is the number of simulated bomb impact points, the smaller values of n are associated with relatively large P.E.'s.

The computing time per case by the DJH method, using double integration (2-12) or (2-13) and subroutine PKILL, was between 1.0 and 1.5 seconds for those cases which were timed, of the 22 cases referred to above, in comparing the DJH and WB methods. However, the computing time per case was much larger, in some cases of the order of one minute or more, in some of the extreme cases which were computed in selecting a set of input values for the tables in this technical report. This led to some changes in the double integration subroutine used, in an effort to increase computing speed while maintaining acceptable accuracy. This point is discussed in some detail below.

Simple tests were included in the program for detecting some, but not all, of the cases in which the result will be $P_K < 0.005$ and, therefore, P_K will be given as 0.00 in a two-place table, and for detecting some of the cases resulting in $P_K > 0.995$ (and, therefore, $P_K = 1.00$ correct to two decimal places). We will give a discussion in terms of a small but unspecified ϵ , although the actual tolerance used in computing Table E3 was $\epsilon = 0.005$, so that this table may be regarded as correct to two decimal digits. If the values in a table are subject to errors not exceeding 0.005, the table

is conventionally regarded as correct to two decimal digits, although there are exceptions, as for instance when the computed value is 0.824, rounding off to 0.82, with an error of 0.003, making the true value 0.827, which rounds off to 0.83 rather than 0.82. More generally, if the errors do not exceed $5 \times 10^{-n-1}$ or $(1/2) \times 10^{-n}$, the table is conventionally regarded as correct to n decimal digits, although it would be more correct to state that the error, after rounding, does not exceed one unit in the n th decimal place.

For the coordinates of the points $(h_{\min}, k_{\min})$ and $(h_{\max}, k_{\max})$ involved in these tests, and further details, see Appendix D.

In the first of these tests, for the detection of cases in which $P_K < \epsilon$ and P_K will be given as zero to the appropriate number of digits, a point $(h_{\min}, k_{\min})$ with non-negative coordinates is determined, such that, for every point (h, k) on or inside the ellipse, we have $|h| \geq h_{\min}$, $|k| \geq k_{\min}$ and, therefore, $P(h, k, \sigma_h, \sigma_k, R) \leq P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R)$, since the elliptical coverage function decreases with increasing $|h|$ and also with increasing $|k|$. Further, since P_K as given by (2-12) or (2-13) is simply an average value of $P(h, k, \sigma_h, \sigma_k, R)$ over the interior of the ellipse, by (2-17), it will be true in such a case that $P_K < P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R)$ with strict inequality, since we always have strict inequality at interior points of the ellipse. Hence, if $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R) \leq \epsilon$, requiring only one call to the PKILL subroutine for the selected values of h and k , we must have $P_K < \epsilon$. This test will fail in some cases in its objective of saving computing time, since it is possible to have $P_K < \epsilon < P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R)$; but it cannot lead to an incorrect decision that $P_K < \epsilon$, since, if $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R) \leq \epsilon$, we must have $P_K \leq \epsilon$.

We remark here that, in the preliminary computation of some of the pages of the present main table with the coarse tolerance of $\epsilon = 0.050$, the criterion $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R) \leq \epsilon$ was replaced by $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R) \leq \epsilon/2$, but this is "on the safe side" and certainly not less rigorous than the standard criterion involving ϵ . The objective was to get a smoother decrease in the values of P_K with increasing u and v , as illustrated by the following example. With $\epsilon = 0.050$, suppose that $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R) = 0.045$ and $P_K = 0.040$ if computed by the regular DJH program. If the regular ϵ criterion is applied, $P_K = 0.0$ (with accuracy to only one decimal digit if $\epsilon = 0.050$). But $P(h_{\min}, k_{\min}, \sigma_h, \sigma_k, R)$ and the computed P_K are both greater than $\epsilon/2 = 0.025$, so that P_K is not printed as 0.0 under the $\epsilon/2$ criterion. The value $P_K = 0.040$, though probably not accurate to three digits, may have a substantially smaller actual error than the maximum error indicated by the analysis, and the value 0.040 contributes more than the alternative value 0.0 to a smooth decrease in P_K with increasing u and v .

In the analogous test for $P_K > 1 - \epsilon$, a point $(h_{\max}, k_{\max})$ is found, such that, for every point (h,k) on or inside the ellipse, we have $|h| \leq h_{\max}$, $|k| \leq k_{\max}$. Then $P_K > P(h_{\max}, k_{\max}, \sigma_h, \sigma_k, R)$ (by reasoning similar to that in the corresponding step in discussing the criterion for $P_K < \epsilon$). If a call to the PKILL subroutine determines that $P(h_{\max}, k_{\max}, \sigma_h, \sigma_k, R) \geq 1 - \epsilon$, we must have $P_K > 1 - \epsilon$. Here it is possible to have $P_K > 1 - \epsilon > P(h_{\max}, k_{\max}, \sigma_h, \sigma_k, R)$, in which case the test fails to save computing time; but if $P(h_{\max}, k_{\max}, \sigma_h, \sigma_k, R) \geq 1 - \epsilon$, we must have $P_K > 1 - \epsilon$ and, therefore, the result to be printed in the main table will be $P_K = 1.00 \dots$ (to the appropriate number of decimal digits) without the necessity of evaluating the double integral by the regular DJH program.

Originally, it was planned to compute a table of 16,384 values of P_K , with R normalized at 1 and the remaining input parameters as follows:

$$\begin{array}{ll} a = 1,2,5,10 & \sigma_k/\sigma_h = 0.1,0.5,0.8,1 \\ b/a = 0.1,0.2,0.5,1 & u = 0,2,5,10 \\ \phi = 45^\circ, 0^\circ, 45^\circ, 90^\circ & v = 0,2,4,6 \\ \sigma_h = 0.5,1,2,5 & R = 1 \end{array}$$

If all combinations are taken, the resulting table has 4^7 or 16,384 values.

However, it was found from theoretical considerations involving somewhat more than the test described above for $P_K > 0.9995$, that such a table would contain no values exceeding this number (an unrealistically fine tolerance of $\epsilon = 0.0005$ was under consideration at this time). A key fact here is that the ratio of $R (=1)$ to the smallest value of σ_h (0.5) is only 2.

Several skeleton tables, each of 128 values, were computed with the object of selecting a set of input parameters which would result in a better balanced table in the sense of having a greater proportion of values close to 1.000, and also a reasonable proportion close to 0.000. The following is a typical set resulting in 128 cases, with $R = 1$ in all cases:

$$\begin{array}{ll} \sigma_h = 0.1,2 & u = 0,10 \\ \sigma_k/\sigma_h = 0.1,1 & v = 0,4 \\ a = 0.1,8 & \phi = 45^\circ, 90^\circ \\ b/a = 0.1,1 & R = 1 \end{array}$$

The short tests described above for detecting cases in which $P_k < 0.005$ or $P_k > 0.995$ were used, with positive results in a substantial proportion of cases in these skeleton tables, saving computer time.

Later it was decided to normalize by taking $\sigma_h = 1$ (rather than $R = 1$) with $\sigma_k \leq \sigma_h$ always. A number of sets of 128 test cases were computed with this normalization, a typical set being:

$$\begin{array}{ll} \sigma_h = 1 & u = 2.0, 3.0 \\ \sigma_k = 0.2, 0.5 & v = 1.0, 1.5 \\ a = 0.5, 2.0 & \phi = 0^\circ, 45^\circ \\ b/a = 0.5, 1.0 & R = 0.5, 1.0 \end{array}$$

However, in a number of these test cases, the computing time on the CDC 6700 machine was unexpectedly long, sometimes more than one minute per case. It was concluded that the numerical double integration subroutine was probably using an unnecessarily fine mesh in many of these cases, and study was given to methods of avoiding this. The PKILL subroutine is a relatively slow one, requiring something of the order of 5 milliseconds even for 3-place accuracy on the 6700 machine. The CIRCV subroutine (circular coverage function) is an order of magnitude faster, but can be used if, and only if, $\sigma_h = \sigma_k$, as has been stated. At 5 milliseconds per case, only about 200 elliptical coverage functions per second can be computed to three-place accuracy. Hence, if the double integration subroutine should use a mesh of 128 subintervals or 129 points on each dimension of the rectangle of integration, there would be 129^2 or 16,641 calls to the PKILL subroutine, requiring more than 83 seconds in calls to this subroutine at 200 calls per second.

The error analysis, resulting in improved speed in computing the double integral of (2-12) or (2-13), while maintaining acceptable accuracy, is given later in this section.

A number of test runs of 128 cases at a time, with σ_h always having the value 1, were made. Also, curves were computed and plotted in the hk -plane for $P(h, k, \sigma_h, \sigma_k, R) = 0.005$, and 0.995 for $\sigma_h = 1$, and various values of σ_k and R . This facilitates the determination of ranges of values of the input variables in the double integral of (2-12) which would result in values of $P_k = 0.000$ or 1.000 to three decimal digits, by the tests described earlier in this section. On the basis of all these tests, it was tentatively decided to compute a table of 16 pages with 1,024 values to a page, making a total of 16,384 ($= 2^{14}$) cases, with the following input values:

$\sigma_h = 1$	(1 value)
$\sigma_k = 0.2, 0.5, 0.8, 1.0$	(4 values)
$u = 0, 0.5, 1.0, 3.0$	(4 values)
$v = 0, 0.25, 0.50, 1.50$	(4 values)
$\phi = 0^\circ, 45^\circ, 90^\circ, -45^\circ$	(4 values)
$a = 0.1, 0.5, 1.0, 2.0$	(4 values)
$b/a = 0.2, 0.5, 0.8, 1.0$	(4 values)
$R = 0.5, 1.0, 2.0, 4.0$	(4 values)

Taking all combinations, the total number of cases is $4^7 = 2^{14} = 16,384$ as stated above.

Later, it was decided to include the value $R = 0.25$ on the ground that this might be a more realistic value of the lethal radius R than the larger values, from the practical military point of view. This would make a total of 20 pages of tables, or $5 \times 4^6 = 20,480$ cases.

When the computing of the pages was started, it was found that some of the pages had an excessive number of values near 0 or 1, and additional pages were computed by modifying and augmenting the set of values of the input variables, especially u , on some of the pages. The resulting total for Table E3 was 51 pages, containing 59,904 values of P_k , as stated earlier in this section, page 11.

A tolerance $\epsilon = 0.005$ was used on all pages except 59 and 63, on which the tolerance was 0.001 (see page 12 for further comments on this). The computing time per case varied widely, being 0.10 second or less on some pages, and 0.85 second or more for other pages. The overall average, from available recorded times, was about 0.4 second per case.

In using the DJH program, based on (2-12), arbitrary positive values can be given to σ_h , σ_k , a , b , and R , and arbitrary real values to u , v , and $|\phi| \leq 90^\circ$. There is nothing in the analysis which rules out the case $\sigma_k > \sigma_h$. If u or v or both are negative, the corresponding positive value or values should be taken, with appropriate change in ϕ ; for example, $u = -4$, $v = 2$, $\phi = 30^\circ$ is equivalent to $u = 4$, $v = 2$, $\phi = -30^\circ$, as is clear if a figure is drawn, the elliptical coverage function being symmetric with respect to both axes.

But if the table in this report is used, with or without interpolation, it must be noted that, in the table, $\sigma_h = 1$ and $\sigma_h \geq \sigma_k$. Hence, it may be necessary to normalize the input variables (except ϕ) by making $\sigma_h = 1$, and if $\sigma_k > \sigma_h$ originally, the roles of h and k must be interchanged. If we have the case $\sigma_h = 5$, $\sigma_k = 2$, $u = 3$, $v = 2$, $a = 2$, $b = 1.5$, $\phi = 30^\circ$, $R = 3$, we simply divide all distances by 5 and obtain as an equivalent case $\sigma_h = 1$, $\sigma_k = 0.4$, $u = 0.6$, $v = 0.4$, $a = 0.4$, $b = 0.3$, $\phi = 30^\circ$ (no change in ϕ), $R = 0.6$.

Now, suppose we have a case with $\sigma_k > \sigma_h$, say $\sigma_h = 2$, $\sigma_k = 5$, $u = 3$, $v = 2$, $a = 2$, $b = 1.5$, $\phi = 30^\circ$, $R = 3$. We first divide all distances by the larger σ , σ_k in this case, and get the equivalent case $\sigma_h = 0.4$, $\sigma_k = 1$, $u = 0.6$, $v = 0.4$, $a = 0.4$, $b = 0.3$, $\phi = 30^\circ$, $R = 0.6$. Then, reversing the roles of h and k , we will also have to reverse those of u and v and make an appropriate change in ϕ , obtaining the equivalent case $\sigma_h = 1$, $\sigma_k = 0.4$, $u = 0.4$, $v = 0.6$, $a = 0.4$, $b = 0.3$, $\phi = 60^\circ$, $R = 0.6$. This should be clear from a study of Figure 2, bearing in mind the symmetry of the elliptical coverage function with respect to both axes.

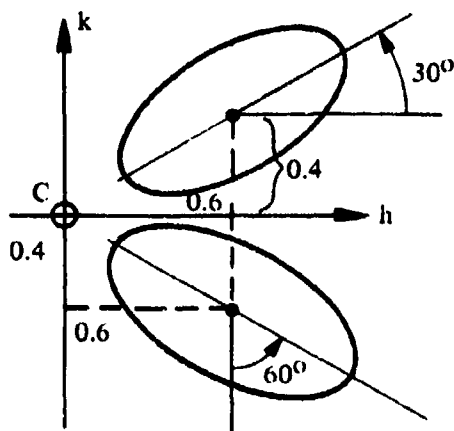


Figure 2. Interchange of σ_h , σ_k in Example

Here we discuss the numerical evaluation of the double integral (Figure 3)

$$\bar{I} = \int_c^d \left[\int_a^b F(x, y) dx \right] dy \quad (3-1)$$

by a method which has been chosen as an efficient one, using two tolerances ϵ_0 and ϵ_1 and an overall tolerance ϵ

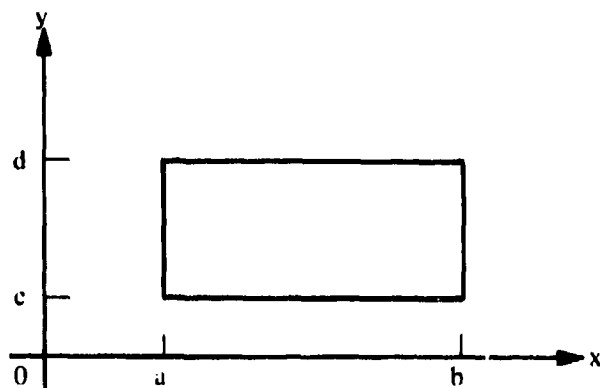


Figure 3. Area of Integration for Double Integral

These three ϵ 's may be given appropriate positive values depending on the accuracy required; for example, the overall tolerance ϵ , on which ϵ_0 and ϵ_1 depend in the present analysis (see (3-15)), is sometimes given a value of 0.01 in crude calculations, and sometimes a value of 0.00001 if much more accurate results are desired.

In (2-12) and (2-13), the variables of integration are r and θ . We leave open the possibility of identifying r with x and θ with y , or the reverse.

We regard the integral as an iterated one,

$$\bar{I} = \int_c^d \left[\int_a^b F(x, y) dx \right] dy \equiv \int_c^d f(y) dy \quad (3-2)$$

For a fixed y , the integral $f(y) = \int_a^b F(x, y) dx$ is approximated by Simpson's rule, but the number of subintervals for two different ordinates or values of y is not necessarily the same. Hence, we do not have a two-dimensional grid of the less efficient type sometimes considered in the literature, for example in [7, p. 186, problem 54].

For $y = s$ for arbitrary s on (c, d) , let

$$\bar{I}_s = \text{exact value of } f(y) = f(s) = \int_a^b F(x, s) dx \quad (3-3)$$

$$I_s = \text{Simpson's rule approximation for } \bar{I}_s \quad (3-4)$$

In general, a bar will indicate an exact value, as in $\bar{I}_s$. I_s in this case indicates an approximation.

$$\bar{I}_s = I_s + E_s \quad (3-5)$$

where E_s is the error committed by using the Simpson's rule approximation with the chosen number of subintervals of $[a, b]$.

For a given s , i.e., a given value of y , let a sufficiently large number of subintervals of $[a, b]$ be chosen so that

$$|\bar{I}_s - I_s| = |E_s| < \epsilon_0 \quad (3-6)$$

where ϵ_0 is one of our fundamental tolerances.

We are really comparing successive approximations, say $(I_s)_n$ and $(I_s)_{2n}$, with each other, where n or $2n$ denotes the number of subintervals into which $[a, b]$ is divided, rather than comparing any approximation with the exact value $\bar{I}_s$, since in practice we do not know the value of the latter. We do not in this analysis derive rigorous error bounds for integration by Simpson's rule. Numerical analysts have developed several such rigorous bounds, using the fourth derivative of the integrand [7, p. 75, (3.5.23)], or evaluating a line integral in the complex plane, or using other methods. The calculation of any of these rigorous bounds would require a relatively large amount of machine time, and, moreover, such bounds are often ultra-conservative, i.e., much larger than the actual error committed. It is felt by the present authors that sufficient accuracy for practical computations can be attained by developing simple tests involving convergence of successive Simpson approximations to a common limit. Such tests can give a high degree of confidence in the reliability of the results, though not absolute certainty. We may decide that (3-6) is satisfied if two successive iterates agree with each other within ϵ_0 ; or, if we are more conservative, within $\epsilon_0/2$; or, of three successive iterates, say $(I_s)_n$, $(I_s)_{2n}$ and $(I_s)_{4n}$, the largest and smallest are within ϵ_0 of each other. We assume here that we have some such criterion, but we do not specify at this stage what it must be. We state in (3-17) below, how the tolerances are handled in the present NSWC program, but the program is flexible in that changes in the tolerances are easily made.

Now by definition,

$$\int_c^d I_s dy = \int_c^d I_y dy = I$$

= exact value of double integral (3-7)

Also, I_s is a well-defined function of s or y once a numerical value of ϵ_0 has been assigned and a criterion for the satisfaction of (3-6) has been chosen as discussed above.

We assume also that $\int_c^d I_s dy$ or $\int_c^d I_y dy$ exists. I_y or $\int_a^b F(x, y) dx$ (3-3), the exact value of the integral from a to b , is a continuous function of y , by the continuity of the integrand of (2-12) as a function of two variables. A consideration of the manner of forming the Simpson sums shows that the approximation I_y is also a continuous function of y in any interval on the y -axis in which the number of subdivisions of $[a, b]$ in forming the Simpson approximation to $\int_a^b F(x, y) dx$ does not change. But I_y does have a discontinuity, in general, at a point where we switch from 16 to 32 subintervals, for example, in approximating $\int_a^b F(x, y) dx$. Suppose $I_{y_0} = \int_a^b F(x, y_0) dx$ uses 16 subintervals for some $y = y_0$, but $\int_a^b F(x, y) dx$ uses 32 subintervals for some neighboring values of y , say for all y in the open interval $(y_0, y_0 + k)$ for some positive k . Then I_y , as a function of y , will in general have a discontinuity for $y = y_0$. Nevertheless, the integrand in (2-12) is such a smooth slowly changing function of its arguments that we can be confident on heuristic grounds that the integrand in $\int_c^d I_y dy$ has a small finite number of discontinuities. But the Riemann integral $\int_c^d I_y dy$ exists if, and only if, the set of discontinuities is of Lebesgue measure zero [8, p. 230]. This includes the present case, in which the discontinuities are assumed to be finite in number. Hence, the integral $\int_c^d I_y dy$ or $\int_c^d I_s dy$ exists. It is shown below in (3-8) that it differs by less than $(d - c) \epsilon_0$ from the exact value of the double integral, $\int_c^d I_s dy$ or I , i.e., it approximates I arbitrarily closely for small ϵ_0 . Also, the Simpson sums $\frac{d-c}{3n} \sum_{i=0}^n k_i I_{y_i}$, which are used in approximating the integral with respect to y , the k_i 's being the Simpson multipliers 1, 4, 2, ..., 4, 1, converge to the value of the integral $\int_c^d I_y dy$ as the number

of subdivisions of $[c, d]$ becomes infinite (see (3-10)). It is not difficult to prove that the Simpson sums, assuming that they converge to the true value of the integral in the integration of a continuous function over a finite interval, also converge to the value of the integral if the function has a finite number of discontinuities with a finite jump at each such point.

Assuming then that $\int I_s dy$ exists, and assuming that (3-6) holds everywhere on $[c, d]$, i.e., that we take a sufficient number of subintervals of $[a, b]$ so that (3-6) holds for all y on $[c, d]$, we have

$$\begin{aligned} \left| \int \bar{I}_s dy - \int I_s dy \right| &= \left| \int (\bar{I}_s - I_s) dy \right| \\ &\leq \int |\bar{I}_s - I_s| dy < \int_c^d \epsilon_0 dy = (d - c) \epsilon_0 \end{aligned} \quad (3-8)$$

or

$$\left| \bar{I} - \int I_s dy \right| < (d - c) \epsilon_0 \quad (3-9)$$

since

$$\int \bar{I}_s dy = \bar{I}$$

Now, suppose that we take a sufficiently large number of subintervals on $[c, d]$ so that

$$\left| \int I_s dy - \frac{d - c}{3n} \sum k_i I_{s_i} \right| < \epsilon_1 \quad (3-10)$$

where the sum uses the I_s 's for the finite number of points taken on $[c, d]$ ($= n + 1$ points where n is the number of subintervals of $[c, d]$). Here, ϵ_1 is another fundamental tolerance, like ϵ_0 . As in the case of ϵ_0 , we can require two successive iterates $\frac{d - c}{3n} \sum k_i I_{s_i}$ to agree with each other within ϵ_1 , or within $\epsilon_1/2$, or choose some other criterion. We do not know the value of the integral $\int I_s dy$, so we are forced to adopt some arbitrary criterion for the satisfaction of (3-10).

With (3-6) and (3-10) satisfied, we now have

$$\left| T - \frac{d-c}{3n} \sum k_i I_{s_i} \right| = \left| T - \int I_s dy + \int I_s dy - \frac{d-c}{3n} \sum k_i I_{s_i} \right| \quad (3-11)$$

$$\left| T - \frac{d-c}{3n} \sum k_i I_{s_i} \right| \leq \left| T - \int I_s dy \right| + \left| \int I_s dy - \frac{d-c}{3n} \sum k_i I_{s_i} \right| \quad (3-12)$$

$$\left| T - \frac{d-c}{3n} \sum k_i I_{s_i} \right| < (d-c) \epsilon_0 + \epsilon_1 \quad (3-13)$$

Now suppose that ϵ is some overall tolerance, and we wish to choose ϵ_0 and ϵ_1 in such a way that

$$\left| T - \frac{d-c}{3n} \sum k_i I_{s_i} \right| < \epsilon; \quad (3-14)$$

that is, we wish the computed Simpson sum approximating the double integral to be correct within ϵ . From (3-13) and (3-14), we equate ϵ with $(d-c) \epsilon_0 + \epsilon_1$, so that we have

$$\left| T - \frac{d-c}{3n} \sum k_i I_{s_i} \right| < (d-c) \epsilon_0 + \epsilon_1 = \epsilon \quad (3-15)$$

Equation (3-15) clearly allows some latitude in deciding on the individual values of ϵ_0 and ϵ_1 . The present NSWC/DL program takes $\epsilon_0 = \epsilon_1$, from which follows

$$[1 + (d-c)] \epsilon_0 = \epsilon \quad (3-16)$$

$$\epsilon_0 = \epsilon_1 = \frac{\epsilon}{1 + (d-c)} \quad (3-17)$$

If we wish to experiment with various values of the ratio $k = \epsilon_1/\epsilon_0$, a simple generalization would be, from (3-15),

$$(d-c) \epsilon_0 + k \epsilon_0 = \epsilon \quad (3-18)$$

$$\epsilon_0 = \frac{\epsilon}{k + (d-c)} \quad (3-19a)$$

and

$$\epsilon_1 = k \epsilon_0 = \frac{k \epsilon}{k + (d - c)} \quad (3-19a)$$

where k is any positive real number, less than, equal to, or greater than 1. The case $k = 1$ reduces to (3-17). It is possible that computational experience would show that some value such as $k = 2$ or $1/2$ is generally more efficient than $k = 1$. The determination of such an optimal value, or approximately optimal value, of k , would be purely an empirical matter. We could try various values of k for a fixed value of ϵ , and regard as optimal that value of k which resulted in the smallest expenditure of machine time while maintaining the specified accuracy.

A simple modification of this error analysis is necessary in case the constant factor $1/(\pi a^2)$ outside the integral signs in (2-12) and (2-13) is not included as part of the integrand at every step of the analysis, but only once at the end in computing the true value of P_K . This is the case in the DJH program.

Let the integrand in (2-12) and (2-13), not including the factor $1/(\pi a^2)$, be denoted by $F^*(\theta, r)$, and, in general, let the asterisk or "star" denote this omission of the constant factor. Then (2-13) can obviously be written in either of the forms

$$\begin{aligned} P_K &= \int_0^a \left[\int_0^{2\pi} F(\theta, r) d\theta \right] dr \\ &= \frac{1}{\pi a^2} \int_0^a \left[\int_0^{2\pi} F^*(\theta, r) d\theta \right] dr \end{aligned} \quad (3-20)$$

where

$$F^*(\theta, r) = P(h, k, \sigma_h, \sigma_k, R) r \quad (3-21)$$

$$F(\theta, r) = \frac{1}{\pi a^2} F^*(\theta, r) \quad (3-22)$$

and h, k are given by (2-11).

Also,

$$I^* = \iint F^* d\theta dr \quad (3-23)$$

and

$$P_K = I = \frac{1}{\pi a^2} I^* \quad (3-24)$$

Now, following the foregoing analysis but using the "star" in the sense which has been described, we evaluate the double integral I^* to an accuracy such that

$$\left| I^* - \frac{d-c}{3n} \sum k_i I_{s_i}^* \right| < \epsilon^* \quad (3-25)$$

(with ϵ^* to be determined in terms of the overall tolerance ϵ), with

$$\epsilon_0^* = \epsilon_1^* = \frac{\epsilon^*}{1 + (d - c)} \quad (3-26)$$

as in (3-17). We will then have

$$\begin{aligned} \left| I - \frac{d-c}{3n} \sum k_i I_{s_i} \right| &= \left| \frac{1}{\pi a^2} I^* - \frac{1}{\pi a^2} \frac{d-c}{3n} \sum k_i I_{s_i}^* \right| \\ &= \frac{1}{\pi a^2} \left| I^* - \frac{d-c}{3n} \sum k_i I_{s_i}^* \right| < \frac{1}{\pi a^2} \epsilon^* \end{aligned} \quad (3-27)$$

But we wish to have

$$\left| I - \frac{d-c}{3n} \sum k_i I_{s_i} \right| < \epsilon, \quad (3-28)$$

and so we put $\epsilon^*/(\pi a^2) = \epsilon$ or

$$\epsilon^* = \pi a^2 \cdot \epsilon \quad (3-29)$$

and therefore, from (3-26),

$$\epsilon_0^* = \epsilon_1^* = \frac{\epsilon^*}{1 + (d - c)} = \pi a^2 \cdot \frac{\epsilon}{1 + (d - c)}$$

The tolerances are handled in this way in the DJH program. The constant ϵ is set at 0.005 in the present program, but can easily be changed to some other desired value such as 0.001 or 0.05.

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APPENDIX A
SPECIAL CASES

SPECIAL CASES

Two special cases are worthy of mention here: (a) $\sigma_h = \sigma_k$; (b) $a = b$, i.e., the ellipse is a circle.

(a) We have assumed that $\sigma_h \geq \sigma_k$, with strict inequality in general. But if $\sigma_h = \sigma_k = \sigma$, the distribution is circular normal, and if we divide all coordinates by σ , we have a standard circular normal distribution, i.e., with $\sigma = 1$. We would write $\tilde{h} = h/\sigma$, $\tilde{k} = k/\sigma$ or replace h by $\sigma\tilde{h}$, and similarly replace k by $\sigma\tilde{k}$. Also, we would divide, u , v , a , b , and R by σ . The angle ϕ would be unchanged by the transformation. If $\sigma_h = \sigma_k = \sigma = 1$, the case is simplified and the original symbols, h , k , u , v , a , b , and R can be retained.

We would then evaluate a double integral as in (2-12) or (2-13), but call the subroutine CIRCVC, computing the circular coverage function $P(R, d)$ instead of subroutine PKILL, which computes the elliptical coverage function $P(h, k, \sigma_h, \sigma_k, R)$. The former subroutine is at least ten times as fast as the latter subroutine.

This substitution of subroutine CIRCVC for PKILL, and the reduction discussed below of the double integral to a single integral, also using CIRCVC, can be done if and only if $\sigma_h = \sigma_k$. The reason is that if $\sigma_h \neq \sigma_k$ a transformation must be made, in order to produce a circular normal distribution, in which unequal factors of stretching or compression along the h - and k -axes are used. This would convert the lethal circle of radius R into a lethal ellipse and this is contrary to the assumption that the bomb's lethal area is a circle.

If $\sigma_h = \sigma_k = \sigma$, involving an initial similarity transformation if $\sigma \neq 1$, as discussed above, we can even express P_K in terms of a single rather than a double integral, but at the cost of solving a fourth degree equation in general at every integration step to find the points of intersection, possibly four in number, of the circle $h^2 + k^2 = r^2$ (where r is the variable of integration) with the ellipse.

Figure A1 shows a case in which a circle with center at C and radius r (also a circle of radius $r + dr$) intersects the ellipse in four points. The arcs of the first circle (radius r) subtend central angles θ_1 and θ_2 . Let $\theta(r) = \theta_1 + \theta_2$. Then the sum of the two shaded areas is $(\theta_1 + \theta_2) r dr$ or $\theta(r) r dr$ neglecting infinitesimals of higher

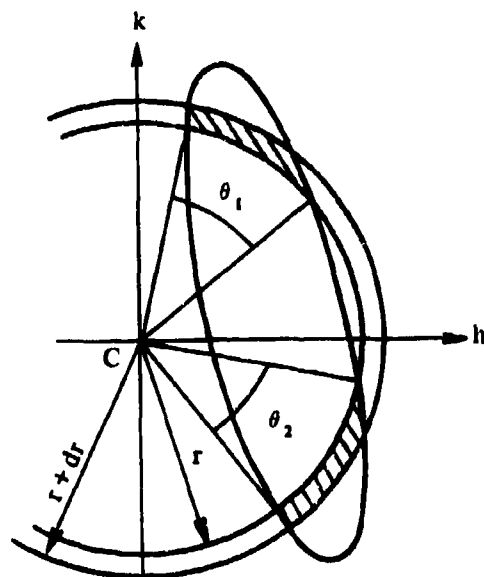


Figure A1. Case $\sigma_h = \sigma_k$, Ellipse Intersected at Four Points

order than the first. This area can be described as the area occupied by points that are in the interior of the ellipse and at a distance between r and $r + dr$ from the center C of the distribution. Because of the uniform distribution in the ellipse of possible positions of the point target, the probability that the target lies in the shaded area is $\theta(r) r dr / (\pi ab)$, i.e., the ratio of the shaded area to the total area of the ellipse. All of these points in the shaded area are at distance r from C , neglecting infinitesimals of higher order. Hence, given that the target lies somewhere in the shaded area, the conditional probability of its destruction is $P(R, r)$, the circular coverage function as computed by subroutine CIRC.V. Hence, the total probability that the target lies in the shaded area (for a given r) and is destroyed is $\theta(r) r dr P(R, r) / (\pi ab)$, and the overall kill probability P_K is computed by integrating this expression from r_1 to r_2 , which are respectively the minimum and maximum of values of r such that a circle of radius r , center at C , contains points that are on or inside the ellipse. If the point C is on or inside the ellipse, $r_2 > r_1 = 0$, but if C is exterior to the ellipse, $r_2 > r_1 > 0$. Thus, we evaluate the single integral:

$$P_K = \frac{1}{\pi ab} \int_{r_1}^{r_2} \theta(r) P(R, r) \cdot r dr \quad (A1)$$

The determination of r_1 , r_2 and $\theta(r)$ in general involves the numerical solution of fourth degree algebraic equations, but is simpler in special cases as discussed below. If C is in the interior of the ellipse, so that $r_1 = 0$, we take $\theta(r) = 2\pi$ for all values of r such that the circle of radius r (center at C) lies entirely in the interior of the ellipse. If C is on the ellipse, we also have $r_1 = 0$, but for all circles of radius $r > 0$, $\theta(r) < 2\pi$.

A special case which was programmed and used in checking out the general program of this report is shown in Figure A2. In this case, the limits r_1 and r_2 of (A1) are obvious by inspection, and the computation of the function $\theta(r)$ of (A1) involves only square roots and not the solution of more general fourth degree equations.

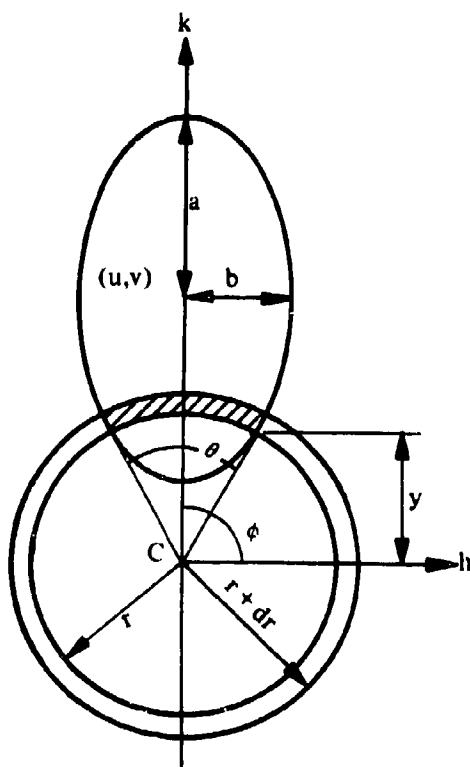


Figure A2. Special Case Programmed

Here, we have $\sigma_h = \sigma_k = 1$, $u = 0$, $\phi = 90^\circ$, $v > a > b$, so that the point C is exterior to the ellipse. From the figure we clearly have $r_1 = v - a$, $r_2 = v + a$. It is intuitively clear that for $v - a < r < v + a$, the circle of radius r , center at C, will intersect the ellipse in two, and only two points, with equal ordinates by symmetry, and this is confirmed by analysis. The common ordinate y of these two points of intersection is given by:

$$y = \frac{a \sqrt{(a^2 - b^2)(r^2 - b^2) + b^2 v^2} - b^2 v}{a^2 - b^2} \quad (\text{A2})$$

and, when y has been computed, the abscissas of the two points of intersection (circle and ellipse) are $\pm \sqrt{r^2 - y^2}$. Hence, the angle θ (Figure A2) is $\arctan(\sqrt{r^2 - y^2}/y)$, and the entire central angle corresponding to the shaded area is twice this or $2 \arctan(\sqrt{r^2 - y^2}/y)$. Therefore, using (A1), the kill probability P_K is given by the integral:

$$P_K = \frac{2}{\pi ab} \int_{v-a}^{v+a} \arctan \frac{\sqrt{r^2 - y^2}}{y} P(R, r) r dr \quad (\text{A3})$$

with y at each integration step, or each value of r , given by (A2).

Although a program based on Figure A2 and (A2) and (A3), with $a > b$, was used in checking out the general method of this report, we would have a still simpler special case if the ellipse were a circle of radius a , with $a < v$ (see Figure A2 and imagine that $a = b$). In this case, the ordinate y of the points of intersection of the two circles, $h^2 + k^2 = r^2$ and $h^2 + (k - v)^2 = a^2$, is given by:

$$y = \frac{r^2 + v^2 - a^2}{2v} \quad (\text{A4})$$

and P_K would be given by (A3), with y computed by (A4) instead of (A2).

(b) If $a = b$, implying that $c = 1$ and the ellipse is a circle (in the general case where σ_h and σ_k are not necessarily equal), the positions of the axes of the ellipse are indeterminate, and for convenience we can take $\phi = 0$ so that the x - and y -axes are parallel to the h - and k -axes respectively. This would simplify (2-3) which would become

$$\left. \begin{array}{l} h = u + x \\ k = v + y \end{array} \right\}$$

Also, c would be 1 in (2-5). If $\sigma_h \neq \sigma_k$, we would still have to compute a double integral as in (2-12), using subroutine PKILL, but with the simplifications which have been noted in the coordinate transformations, (2-3) and (2-11).

APPENDIX B
ALTERNATIVE METHOD OF COMPUTING P_k

ALTERNATIVE METHOD OF COMPUTING P_K

We outline here an alternative method of computing the kill probability P_K , this time using true polar coordinates r, θ in the ellipse. The curvilinear coordinates introduced in (2-5) and also denoted as r, θ are not polar coordinates in the ellipse if $a > b$.

Figure B1 shows a quadrant of the ellipse (see also Figure 1 in Section 2). First, we wish to find the coordinate r for a point such as E on the ellipse, with given polar angle θ , to establish limits on r in terms of θ . The equations of the line OE, and of the ellipse are:

$$y = x \tan \theta \quad (B1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (B2)$$

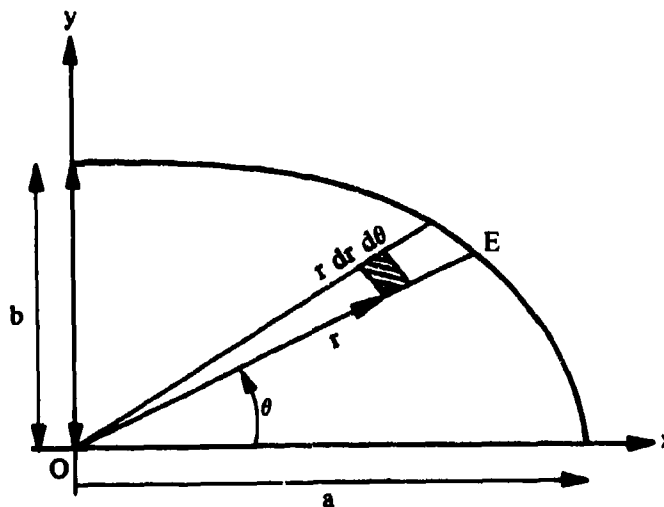


Figure B1. Polar Coordinates in the Ellipse

Solving these equations simultaneously, we find that the rectangular coordinates (x, y) of the point E are:

$$x = (ab/S) \cos \theta, \quad y = (ab/S) \sin \theta \quad (B3)$$

and, therefore

$$r = \sqrt{x^2 + y^2} = ab/S \quad (B4)$$

where

$$S = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \quad (B5)$$

Equation (B4) provides the upper limit on the variable r in the subsequent double integration.

Now, considering a polar element of area $r dr d\theta$ (Figure B1) at a point (r, θ) inside the ellipse, we must, as before convert its rectangular coordinates $(r \cos \theta, r \sin \theta)$ to the Chk system, by (2-3). The result is:

$$\begin{cases} h = u + r \cos \theta \cos \phi - r \sin \theta \sin \phi = u + r \cos(\theta + \phi) \\ k = v + r \cos \theta \sin \phi + r \sin \theta \cos \phi = v + r \sin(\theta + \phi) \end{cases} \quad (B6)$$

By reasoning similar to that used in setting up (2-12), we obtain the following double integral in terms of the present true polar coordinates r, θ

$$P_K = \frac{1}{\pi ab} \int_0^{2\pi} \int_0^{ab/S} P[u + r \cos(\theta + \phi), v + r \sin(\theta + \phi), \sigma_h, \sigma_k, R] r dr d\theta \quad (B7)$$

The P functions, as in the case of (2-12), are computed by subroutine PKILL.

APPENDIX C

PROBABLE ERROR IN THE WEIDMAN-BRUNNER METHOD

PROBABLE ERROR IN THE WEIDMAN-BRUNNER METHOD

In the WB (Weidman-Brunner) method as described in [1] and in the present technical report, the standard statistical method for determining the probable error and confidence intervals is one using the Student t distribution, [4, p. 224], [6, p. 178], and [10, p. 189]. For a given problem, σ_h , σ_k , R , u , v , a , b , and ϕ specified, there exists a true kill probability, P_K or p_0 , which we are estimating or approximating by the sample mean, $\bar{p}$, of a sample of size n (the number of simulated bombs). For a fixed and sufficiently large n , these sample means $\{\bar{p}\}$ are from an approximately normal population with mean P_K or p_0 and standard deviation σ (for individual probabilities p_i), the standard deviation of the sample mean $\bar{p}$ being $\sigma/\sqrt{n}$. The point which makes the t distribution the appropriate one to use in computing confidence intervals is that σ (as well as P_K or p_0) is unknown. The method cited from [4], [6], [10], and given in many other statistical texts, is designed for this situation, finding confidence intervals for the mean of a normal population with unknown standard deviation σ .

However, in the present technical report, we do not develop the above-mentioned method in detail, since it is not used in [1] or in the Weidman-Brunner computer program. Instead, we follow this computer program and [1] more closely, and, with each sample, estimate the population standard deviation σ by means of the sample standard deviation. It is believed that this method if followed up correctly will give results that are roughly equivalent to those given by the method which uses the t distribution.

The method used in [1] for computing the probable error (P.E.) is open to criticism in several respects. A study of the FORTRAN program given in this reference shows that the P.E. is given by the formula:

$$\text{P.E.} = 0.6745 \left[\sum_{i=1}^n (p_i - \bar{p})^2 / n \right]^{1/2} \quad (\text{C1})$$

where n is the number of simulated bombs, p_i is the i -th conditional probability determined as explained in Section 1, and $\bar{p}$ is the mean of the n p_i 's, i.e., $\bar{p}$ is the approximation to the kill probability P_K . The integer n is usually taken as 25 in the

WB method, although the values $n = 50$ and $n = 100$ were used in some of the cases discussed in Section 3 of the present report. The radical in (C1) represents the standard deviation σ of the sample of size n , and the constant 0.6745 is a familiar one in statistics, where $P.E. = 0.6745\sigma$, since, for a normal distribution with mean 0 and standard deviation σ , we have:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-0.6745\sigma}^{0.6745\sigma} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) dx = 1/2 \quad (C2)$$

or, half of the items lie between the limits $\pm 0.6745\sigma$.

The denominator under the radical in (C1) should be $n - 1$ instead of n in order to get an unbiased estimate of the standard deviation [6, p. 154]; actually, the use of $n - 1$ gives an unbiased estimate of the variance σ^2 , and a very small bias in the value of σ itself. For large n , there is no significant difference between using n and $n - 1$ in (C1).

A second and more serious criticism is that even if we substitute $n - 1$ for n in (C1), the result is the P.E. for an individual item p_i and not for the sample mean $\bar{p}$. To compute the P.E. for the latter, an extra factor of $\sqrt{n}$ is needed in the denominator of (C1) as shown by the following discussion.

The conditional probabilities $p_1, p_2, \dots, p_n$ in the WB method are independent random variables with identical (though in general unknown) distributions. The distributions are identical since, for each of the n simulated bomb drops, the procedure is exactly the same. But we cannot assume that this common distribution is normal, since it involves a number of factors including the uniform distribution of the point target in the ellipse.

In this situation, we can get a good approximation for large n , for the variance (hence, for the probable error P.E.) of the sample mean $\bar{p}$ by means of the central limit theorem [4, p. 136] and [6, p. 191]. Reference [6] states (changing the notation slightly to agree with our present notation) that if $p_1, p_2, \dots, p_n$ are independent, identically distributed random variables, each with mean p_0 and variance σ^2 , and if $y_n = p_1 + p_2 + \dots + p_n = n\bar{p}$, and

$$Z_n = \frac{y_n - np_0}{\sigma\sqrt{n}} = \frac{n(\bar{p} - p_0)}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{p} - p_0)}{\sigma} \quad (C3)$$

then Z_n is "asymptotically normal" with mean 0 and variance 1, in the sense that the distribution function $F_n(p)$ of Z_n satisfies the relation

$$\lim_{n \rightarrow \infty} F_n(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^p \exp(-u^2/2) du \quad (C4)$$

Here, p_0 is the true kill probability for a given problem of which $\bar{p}$ is an estimate on the basis of one sample of n bombs. We think of a large number of samples of size n . The random variable p_1 takes a large number of values in these successive samples, with mean p_0 and variance σ^2 and similarly with $p_2, p_3, \dots$

Since $\sqrt{n}(\bar{p} - p_0)/\sigma$ has approximately mean 0 and variance 1, as stated in the formulation of the central limit theorem in [6, p. 191], from which (C3) and (C4) are taken, $\bar{p} - p_0$ must have mean 0 and variance σ^2/n and, therefore, $\bar{p}$, the sample mean, must have, asymptotically, mean p_0 (the true kill probability) and variance σ^2/n , establishing the fact that the variance of the mean is $1/n$ times the variance of an individual p_i .

The facts that the mean of all these sample means is p_0 , and the variance is σ^2/n , could have been proved by simpler theorems, [6, pp. 135 and 137] but to prove that the sample means $\bar{p}$ are asymptotically normally distributed (and this fact is needed in computing the probable error) requires the central limit theorem. Also, if the individual random variables p_i were known to be normally distributed, we could get very simple and exact, not merely asymptotic, results for the distribution of the sample means $\bar{p}$. But under the present circumstances, with the p_i 's not normally distributed, we apparently cannot do better than the present asymptotic analysis.

Since the variance of the sample means $\bar{p}$ is asymptotically σ^2/n , as has been shown, the corresponding standard deviation is $\sigma/\sqrt{n}$, where σ is the standard deviation for an individual value p_i . For this σ we have to use the estimate $[\Sigma(p_i - \bar{p})^2/(n-1)]^{1/2}$, square root of an unbiased estimate of the variance as stated above. We multiply this by $1/\sqrt{n}$ to get an estimate for the standard deviation of the sample mean $\bar{p}$, and by 0.6745 to get the probable error, P.E. (see (C2)). Hence, we have

$$P.E. = 0.6745 [\Sigma(p_i - \bar{p})^2 / n(n-1)]^{1/2} \quad (C5)$$

Let the P.E. of [1] be denoted by $P.E._{WB}$ (see (C1)). Comparing (C1) and (C5), we see that $P.E./P.E._{WB} = 1/\sqrt{n-1}$, or

$$P.E. = \frac{P.E._{WB}}{\sqrt{n-1}} \quad (C6)$$

$P.E._{WB}$ is the value printed out for the probable error if the FORTRAN program of [1] is run on a computer. These printed values were corrected as indicated in (C6) in interpreting the results of computed cases.

Although the distribution of the sample mean $\bar{p}$ in the WB method approaches normality as the sample size n becomes infinite, there appears to be no simple means of determining how fast its approach to normality is, or how large the sample size should be to obtain a meaningful estimate of the probable error in the computed approximation $\bar{p}$ to the true kill probability p_0 . In [9, p. 111], in a discussion of this application of the central limit theorem, the following statements are made: "In practical terms, a normality assumption for the sampling distribution of $\bar{x}$ becomes reasonable in many cases for $N > 4$ and quite accurate in most cases for $N > 10$. Hence, for reasonably large sample sizes (4.34) applies to the sampling distribution of $\bar{x}$ computed for any random variable x , regardless of its probability distribution function." Equation (4.34) [9] is

$$\frac{(x - \mu_x) \sqrt{N}}{\sigma_x} = Z \quad (C7)$$

which is stated to have a standardized normal distribution if x is normally distributed. The normality is only asymptotic if x has some other distribution. The expression on the left-hand side is equivalent to the expression $\sqrt{n}(\bar{p} - p_0)/\sigma$ of (C3), with appropriate changes in notation. Thus, the authors of [9] regard samples of more than 10 items as adequately large in order for the use of the central limit theorem to give meaningful and even accurate results in most cases. The regular sample size in the WB method is $n = 25$ as stated earlier in this Appendix, and this sample size, therefore, appears to be adequate.

The Chebyshev inequality rigorously assures us, as shown below, that the 50% confidence interval determining the probable error will never be larger than the interval $(\bar{p} - \sqrt{2} \sigma_{\bar{p}}, \bar{p} + \sqrt{2} \sigma_{\bar{p}})$, where $\sigma_{\bar{p}} (= \sigma/\sqrt{n})$ is the standard deviation of the mean $\bar{p}$ and σ is the standard deviation of an individual item p_i . This is true, however

small the sample size n may be. This implies an upper bound of $\sqrt{2} \sigma_{\bar{p}}$ or $\sqrt{2} \sigma / \sqrt{n}$ for the probable error, which of course, is substantially larger than the asymptotic value $0.6745 \sigma / \sqrt{n}$, for large n , implied by (C5), and based on the use of the central limit theorem.

The result stated is derived from the Chebyshev inequality in the following way. This inequality, [4, p. 135, (10)], with the notation changed to be compatible with the present situation, may be stated as follows:

$$\text{Prob } (|\bar{p} - p_0| > b) < \frac{\sigma^2}{nb^2} = \frac{\sigma_{\bar{p}}^2}{b^2} \quad (\text{C8})$$

where b is an arbitrary real number. Now, we ask what value b must have in order that the stated probability shall be $1/2$, in order to find 50% confidence limits. Hence, we put

$$\frac{\sigma_{\bar{p}}^2}{b^2} = 1/2 \quad (\text{C9})$$

of which the solution is:

$$b = \sqrt{2} \sigma_{\bar{p}} = \sqrt{2} \sigma / \sqrt{n} \quad (\text{C10})$$

so that the 50% confidence interval, as stated, is shorter than the interval $(\bar{p} - \sqrt{2} \sigma_{\bar{p}}, \bar{p} + \sqrt{2} \sigma_{\bar{p}})$. Although this result is perfectly rigorous and independent of the sample size n , it provides only an upper bound for the length of the 50% confidence interval or for the magnitude of P.E.

For a normal population, the probable error (P.E.), is simply the half-length of the 50% confidence interval. More precisely, suppose the population mean is $\bar{x}$, in the notation of [10], and the standard deviation (for individual items) is σ . Let a random sample of size n be drawn, with sample mean $\bar{x}_1$. Then, the 50% confidence interval for that sample is (C_1, C_2) , where $C_1 = \bar{x}_1 - 0.6745\sigma/\sqrt{n}$, $C_2 = \bar{x}_1 + 0.6745\sigma/\sqrt{n}$, or $C_1 = \bar{x}_1 - \text{P.E.}$, $C_2 = \bar{x}_1 + \text{P.E.}$, since the probable error for the sample mean in this case is $0.6745\sigma/\sqrt{n}$. Thus, the length of the 50% confidence interval is $C_2 - C_1$ or 2 P.E., or, the P.E. is the half-length of the confidence interval. See the derivation of (C5) above, where the sample standard deviation $[\sum(p_i - \bar{p})^2 / (n - 1)]^{1/2}$ (for individual items) is multiplied by $1/\sqrt{n}$ to get the standard deviation for the mean $\bar{p}$, and then by 0.6745 to get the probable error, P.E., for the sample mean.

We show here how this relation between the P.E. and the 50% confidence interval follows from [10, pp. 188-189], and we refer the interested reader to this standard statistical handbook for further details.

Let the confidence coefficient be β , in the notation of [10], and here we are interested in the value $\beta = 1/2$. By interpolating in Table IX in [10], we find that λ_β or $\lambda_{1/2} = 0.6745$. This well-known constant is the value λ_β such that

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\sigma\lambda_\beta}^{\sigma\lambda_\beta} \exp \left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right] dx = 1/2$$

(see (C2)). Then [10, p. 189], $C_1 = \bar{x}_1 - \lambda_\beta \sigma / \sqrt{n} = \bar{x}_1 - 0.6745\sigma / \sqrt{n}$, $C_2 = \bar{x}_1 + \lambda_\beta \sigma / \sqrt{n} = \bar{x}_1 + 0.6745\sigma / \sqrt{n}$, and the 50% confidence interval is $[C_1, C_2]$. The significance of the term "confidence interval" itself is brought out in the following statement from [10]: "If it is asserted, whenever a sample is drawn, that the unknown population mean x lies between the limits C_1 and C_2 calculated for that particular sample, then the probability that the assertion is correct is β ", with $\beta = 1/2$ here.

In the application made in this report to the WB method, the unknown population mean $\bar{x}$ corresponds to p_0 or the true but not exactly known P_K , the sample mean $\bar{x}_1$ is $\bar{p}$ or the average of the n conditional probabilities p_i for that sample and the population standard deviation σ in each case is estimated or approximated by the sample standard deviation $[\Sigma(p_i - \bar{p})^2 / (n - 1)]^{1/2}$.

Also, the results quoted from [10] assume a normal population. The means $\bar{p}$ here are not exactly normally distributed, but they are asymptotically normal for large sample size n as we have shown, and the normality assumption can be assumed to be good for $n > 10$.

APPENDIX D
DETERMINATION OF POINTS $(h_{min}, k_{min}), (h_{max}, k_{max})$
OF SECTION 3

DETERMINATION OF POINTS $(h_{\min}, k_{\min})$, $(h_{\max}, k_{\max})$

OF SECTION 3

In Section 3, pages 16-18, we used points $(h_{\min}, k_{\min})$, $(h_{\max}, k_{\max})$, which were useful in simple tests, using very little computer time, for detecting cases in which the kill probability, P_K is approximately 0 or 1, within a prescribed tolerance. For instance, if the general tolerance is 0.005, so that only two decimal digits can be assumed to be correct in the values of P_K , then, for any case in which $P_K > 0.995$, the true value is closer to 1.00 than to 0.99.

In this appendix, we give without proofs, the equations for the coordinates of these points $(h_{\min}, k_{\min})$, $(h_{\max}, k_{\max})$ (see page 52).

Suppose we are given (see Figure D1) an ellipse such as those considered in this report, with center at (u, v) relative to the Chk coordinate system, semi-major and semi-minor axes a and b , respectively and major axis making an angle ϕ with the h -axis, with $-90^\circ < \phi \leq 90^\circ$.

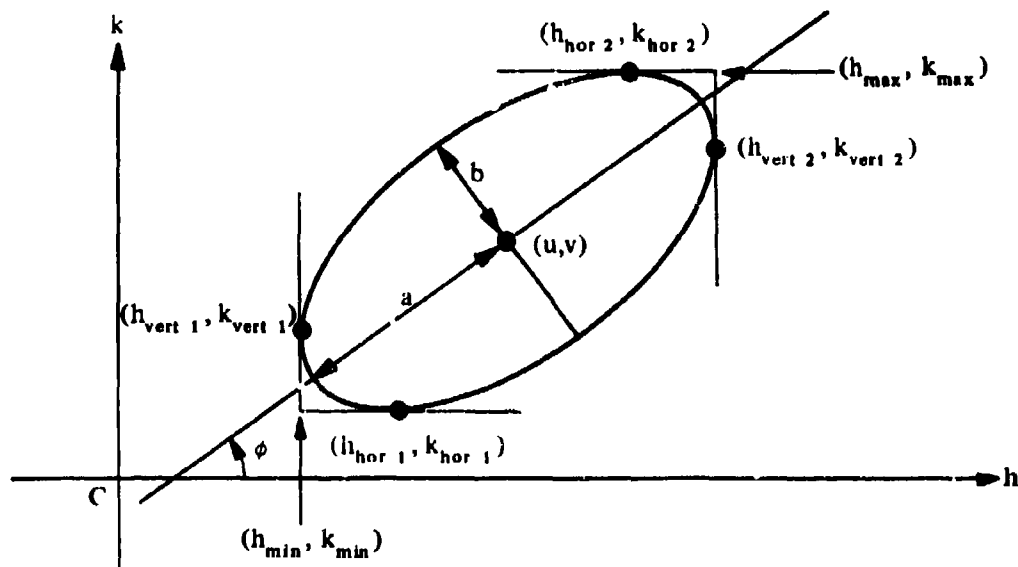


Figure D1. Values of $h_{\min}, k_{\min}, h_{\max}, k_{\max}$ for $-90^\circ < \phi \leq 90^\circ$

There are obviously two horizontal tangents and two vertical tangents to the ellipse. We denote the points of tangency of the horizontal tangents, as in Figure D1, as $(h_{hor\ 1}, k_{hor\ 1})$ and $(h_{hor\ 2}, k_{hor\ 2})$ with $k_{hor\ 2} > k_{hor\ 1}$ always, i.e., $(h_{hor\ 2}, k_{hor\ 2})$ is the point of tangency of the upper horizontal tangent. The relative magnitudes of $h_{hor\ 1}$ and $h_{hor\ 2}$ depends on whether ϕ is a first-quadrant or fourth-quadrant angle.

The points of tangency of the vertical tangents are denoted as $(h_{vert\ 1}, k_{vert\ 1})$, $(h_{vert\ 2}, k_{vert\ 2})$ with $h_{vert\ 2} > h_{vert\ 1}$, always, i.e., $(h_{vert\ 2}, k_{vert\ 2})$ is the point of tangency of the rightmost vertical tangent (Figure D1).

It can be shown by analytic geometry that the coordinates of these four points of tangency are given by the following equations. We do not give the proof here.

$$h_{hor\ 1} = u - \frac{(a^2 - b^2) \sin \phi \cos \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \quad (D1a)$$

$$k_{hor\ 1} = v - \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi} \quad (D1b)$$

$$h_{hor\ 2} = u + \frac{(a^2 - b^2) \sin \phi \cos \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \quad (D1c)$$

$$k_{hor\ 2} = v + \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi} > 0 \quad (D1d)$$

$$h_{vert\ 1} = u - \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} \quad (D1e)$$

$$k_{vert\ 1} = v - \frac{(a^2 - b^2) \sin \phi \cos \phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \quad (D1f)$$

$$h_{vert\ 2} = u + \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} > 0 \quad (D1g)$$

$$k_{vert\ 2} = v + \frac{(a^2 - b^2) \sin \phi \cos \phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \quad (D1h)$$

It is seen from these equations that $k_{\text{hor } 2}$ and $h_{\text{vert } 2}$ are necessarily positive, since u and v are assumed non-negative and the radicals $+\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}$ and $+\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}$ denote the positive root in each case.

We consider further the particular case shown in Figure D1, in which the ellipse lies entirely in the first quadrant. The situation is not so simple if the ellipse extends into other quadrants. In Figure D1, we take $h_{\text{min}} = h_{\text{vert } 1}$, $k_{\text{min}} = k_{\text{hor } 1}$. The point $(h_{\text{min}}, k_{\text{min}})$ so defined has the following property. If (h, k) is any point on or inside the ellipse, then:

$$h \geq h_{\text{min}} \quad (\text{D2a})$$

$$k \geq k_{\text{min}} \quad (\text{D2b})$$

Hence, because the elliptical coverage function $P(h, k, \sigma_h, \sigma_k, R)$ decreases with increasing $|h|$, and also with increasing $|k|$, we must have

$$P(h, k, \sigma_h, \sigma_k, R) < P(h_{\text{min}}, k_{\text{min}}, \sigma_h, \sigma_k, R) \quad (\text{D3})$$

for any point (h, k) on or inside the ellipse, with strict inequality in (D3), since we must have strict inequality in (D2a) or (D2b) or both (the point $(h_{\text{min}}, k_{\text{min}})$ itself not being on or inside the ellipse).

Since (D3) holds for every point on or inside the ellipse, it must hold for the average value $\bar{P}$ of $P(h, k, \sigma_h, \sigma_k, R)$ throughout the ellipse. But the kill probability, P_K is equal to the average value, $\bar{P}$, by (2-17). Hence, we must have

$$P_K = \bar{P} < P(h_{\text{min}}, k_{\text{min}}, \sigma_h, \sigma_k, R) \quad (\text{D4})$$

But the calculation of P_K by (2-12) or (2-13) in the general case, for the input values used in Table E3 in the present report, requires something of the order of 0.4 second or 400 milliseconds, whereas one call to the subroutine PKILL, for computing $P(h_{\text{min}}, k_{\text{min}}, \sigma_h, \sigma_k, R)$, requires only about 5 or 10 milliseconds for three-digit or six-digit accuracy, respectively, on the CDC 6700 machine.

Similarly, in Figure D2, we take $h_{\text{max}} = h_{\text{vert } 2}$, $k_{\text{max}} = k_{\text{hor } 2}$ and by an argument similar to the one given above for $(h_{\text{min}}, k_{\text{min}})$, we conclude that:

$$P_K = \bar{P} > P(h_{\text{max}}, k_{\text{max}}, \sigma_h, \sigma_k, R) \quad (\text{D5})$$

The calculation of $P(h_{\max}, k_{\max}, \sigma_h, \sigma_k, R)$ requires only one call to the PKILL sub-routine.

The complications which are introduced when the ellipse extends into other quadrants than the first are sufficiently illustrated by the case shown in Figure D2.

The relevant analogs of (D2a and b) here are

$$|h| \geq |h_{\min}| \quad (D6a)$$

$$|k| \geq |k_{\min}| \quad (D6b)$$

for arbitrary points (h, k) on or inside the ellipse, and it is seen from the figure that (D6a) is not satisfied for all points (h, k) of the ellipse if we take $h_{\min} = h_{\text{vert } 1}$ as before. We must in this case take $h_{\min} = 0$, since in this case we clearly have $h = 0$ for infinitely many points inside the ellipse.

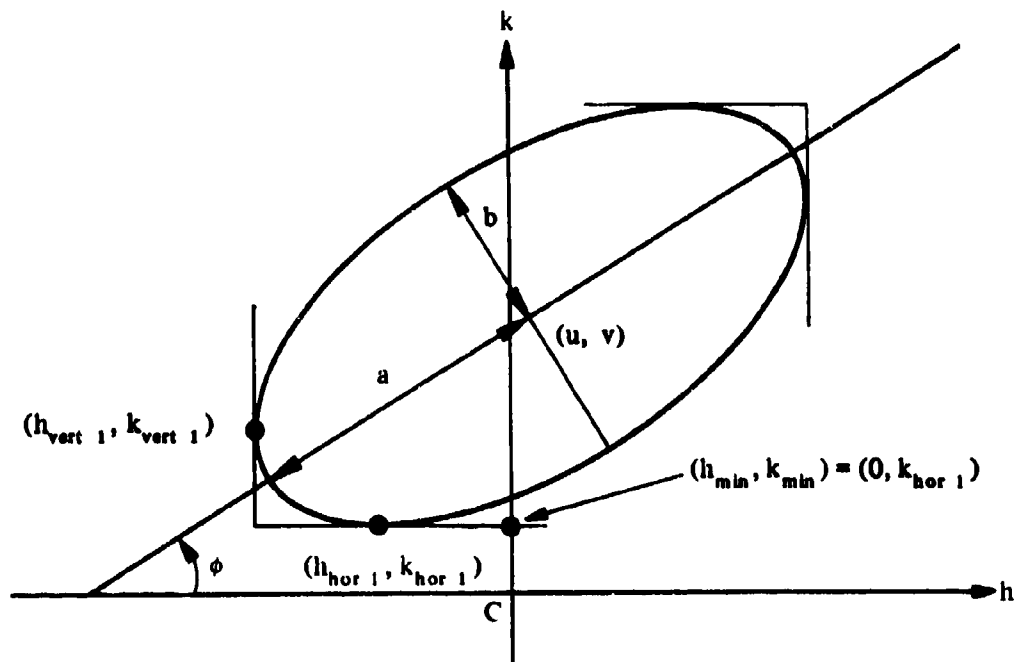


Figure D2. Ellipse in First and Second Quadrants

The general equations for $h_{\min}$, $k_{\min}$, $h_{\max}$, $k_{\max}$, regardless of the quadrants into which the ellipse extends, are

$$h_{\min} = \max(h_{\text{vert } 1}, 0) \quad (\text{D7a})$$

$$k_{\min} = \max(k_{\text{hor } 1}, 0) \quad (\text{D7b})$$

$$h_{\max} = h_{\text{vert } 2} > 0 \quad (\text{D7c})$$

$$k_{\max} = k_{\text{hor } 2} > 0 \quad (\text{D7d})$$

with $h_{\text{vert } 1}$, etc., in (D7a, b, c and d) given by (D1e, b, g and d), respectively. Equations (D7a and b) refer to the algebraic maxima; thus, for example, $\max(-3, 0) = 0$.

Equations (D7a, b, c and d) are the general equations for all values of the angle ϕ . Only four values of ϕ , namely, 0° , 45° , 90° , and -45° , are used in the table given in this report. We give in tabular form below, the values of $h_{\min}$, $k_{\min}$, $h_{\max}$, $k_{\max}$ for these values of ϕ .

	$\phi=0^\circ$	$\phi=45^\circ$	$\phi=90^\circ$	$\phi=-45^\circ$
$h_{\min}$	$\max(u-a, 0)$	$\max(u-\rho, 0)$	$\max(u-b, 0)$	$\max(u-\rho, 0)$
$k_{\min}$	$\max(v-b, 0)$	$\max(v-\rho, 0)$	$\max(v-a, 0)$	$\max(v-\rho, 0)$
$h_{\max}$	$u+a$	$u+\rho$	$u+b$	$u+\rho$
$k_{\max}$	$v+b$	$v+\rho$	$v+a$	$v+\rho$

Here, $\rho = \sqrt{(a^2 + b^2)}/2$. It will be noted that the entries here for $\phi = 45^\circ$ and $\phi = -45^\circ$ are the same, as they should be. From (D1e, b, g and d), it is seen that $h_{\text{vert } 1}$, $k_{\text{hor } 1}$, $h_{\text{vert } 2}$, $k_{\text{hor } 2}$, which occur in (D7a, b, c and d) are the same for these two values of ϕ .

APPENDIX E
IDENTIFICATIONS FOR TABLE E3
TABLE E3

IDENTIFICATIONS FOR TABLE E3

Weapon falls in uncorrelated elliptical normal distribution with mean point of distribution at C, the origin of Chk rectangular coordinate system. Lethal radius of weapon is R, with cookie-cutter lethality function. Point target is at random point from uniform distribution in ellipse. Center of ellipse is at (u, v) in Chk system. Semi-major and semi-minor axes are a and b respectively. Major axis makes angle ϕ with h-axis, positive counterclockwise, $-90^\circ < \phi \leq 90^\circ$. $\sigma_h = 1$ throughout the Table E3; $0 < \sigma_k \leq \sigma_h$; $a \geq b > 0$; $u \geq 0$, $v \geq 0$; $R > 0$. This is a direct table giving kill probability P_K as a function of σ_h ($= 1$), σ_k , R, u, v, a, b, ϕ .

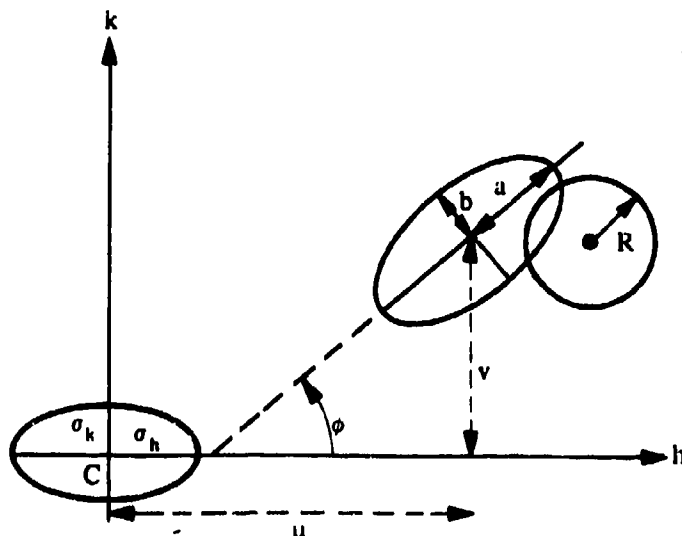


Figure E1. Input Parameters σ_h , σ_k , u, v, a, b, ϕ , R

Ranges for the variables are as given in Table E1 (also given on page 12). σ_h always has the value 1, a takes the values 0.1, 0.5, 1, 2, and ratio b/a takes the values 0.2, 0.5, 0.8, and 1 on every page. Angle ϕ takes the values 0° , 45° , 90° , -45° on every page. Therefore, the only variables whose values are given in Table E1 are R, σ_k , u, and v. The total number of cases in the table is 59,904, on pages 59-109. Three decimal digits are given for the values of P_K , but since the tolerance ϵ , as discussed in Section 3, was generally 0.005, the third digit may be in error by several units, and the values of P_K are reliable to only two decimal digits.

Table E1. Values of R , σ_k , u , v for Main Table (E3)

Pages	Values of R	Values of σ_k	Values of u	Values of v
59-62	0.25	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
63-66	0.50	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
67-70	0.75	0.2,0.5,0.8,1	0,0.5,1,1.5,2	0,0.25,0.50,1
71-78	1	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
79-86	1.5	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
87-94	2	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1 (See Note)
95-98	2.5	0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
99-106	3	0.2,0.5,0.8,1	0,0.5,1,1.5,2,2.5,3,4,5	0,0.25,0.50,1
107- 109	4	1	0,0.5,1,2,2.5,3,3.5,4, 4.5,5,5.5,6	0,0.25,0.50,1

Note: For $R = 2$, $\sigma_k = 0.2$: $v = 0, 0.25, 0.50, 1.50$;

But for $R = 2$, $\sigma_h = 0.5, 0.8, 1$: $v = 0, 0.25, 0.50, 1$

Examples

We give two examples of the use of the table. In Example 1 there is only a trivial scaling problem. In Example 2 we must interchange the roles of h and k and make corresponding changes in u and v , and interpolate with respect to v . See also several examples in Section 3, pages 20-21.

Example 1. A bomber flies east along the h -axis (see Figure E1) and aims a bomb at the origin C . The bomb falls in an uncorrelated bivariate normal distribution with dispersions $\sigma_h = 250$ ft in range, $\sigma_k = 125$ ft in deflection. The lethal radius of the bomb is 500 feet. A point target is at a random point from a uniform distribution in the ellipse shown, with center at $u = 375$ ft, $v = 250$ ft, semi-axes, $a = 125$ ft, $b = 62.5$ ft, with the major axis inclined at 45° as shown. Find the single shot kill probability P_K .

Solution. We must normalize by dividing all distances by $\sigma_h = 250$. The unnormalized and normalized values of the input variables are as follows:

	σ_h	σ_k	R	u	v	a	b	ϕ
unnormalized	250	125	500	375	250	125	62.5	45°
normalized	1	0.5	2	1.5	1	0.5	0.25	45°

On page 89 of the table, entering the table with the normalized values of the variables, we find that $P_k = 0.540$. Since the tolerance ϵ is 0.005, the last digit is in doubt, but the figures 0.54 can be regarded as reliable.

Example 2. Suppose we are given the values $\sigma_h = 160$, $\sigma_k = 200$, etc., as on the first line of Table E2.

Table E2. Data for Example 2

	σ_h	σ_k	R	u	v	a	b	ϕ	P_k
1. Original Data	160	200	300	140	400	200	100	90°	
2. Normalized (divide by 200)	0.8	1	1.5	0.7	2	1	0.5	90°	
3. Rearranged with $\sigma_h > \sigma_k$	1	0.8	1.5	2	0.7	1	0.5	0°	
4. Tabulated Value (page 84)	1	0.8	1.5	2	0.5	1	0.5	0°	0.234
5. Tabulated Value	1	0.8	1.5	2	1	1	0.5	0°	0.149

By interpolation, $P_k = 0.200$ for data on line 3.

We first normalize by dividing distances (all variables except ϕ) by the larger standard deviation, $\sigma_k = 200$, producing the data on line 2. Then we must rearrange with $\sigma_h \geq \sigma_k$, in order to use the table in this report, producing the values on line 3. The values of σ_h and σ_k are interchanged, and, as a consequence, also the values of u and v are interchanged from the values on line 2.

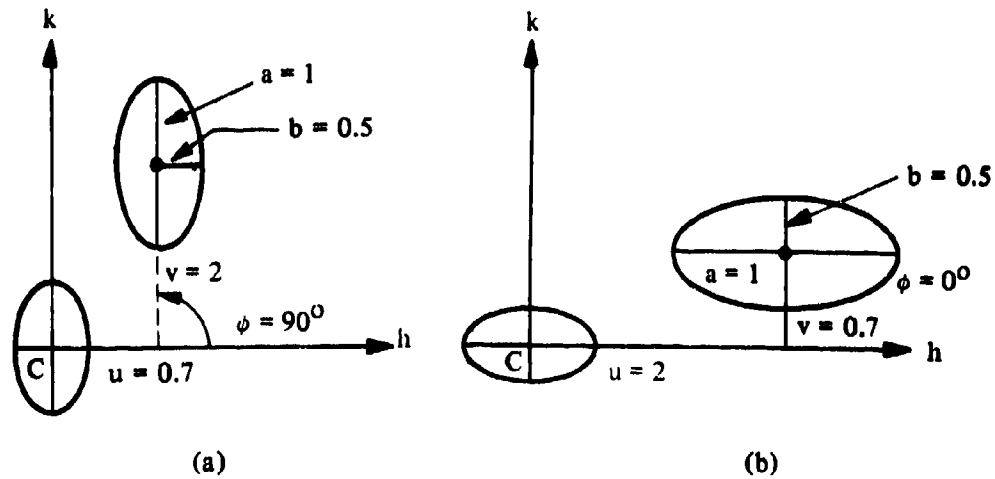


Figure E2. Normalized Data for Example 2
(a) Line 3, (b) Line 3 of Table E2

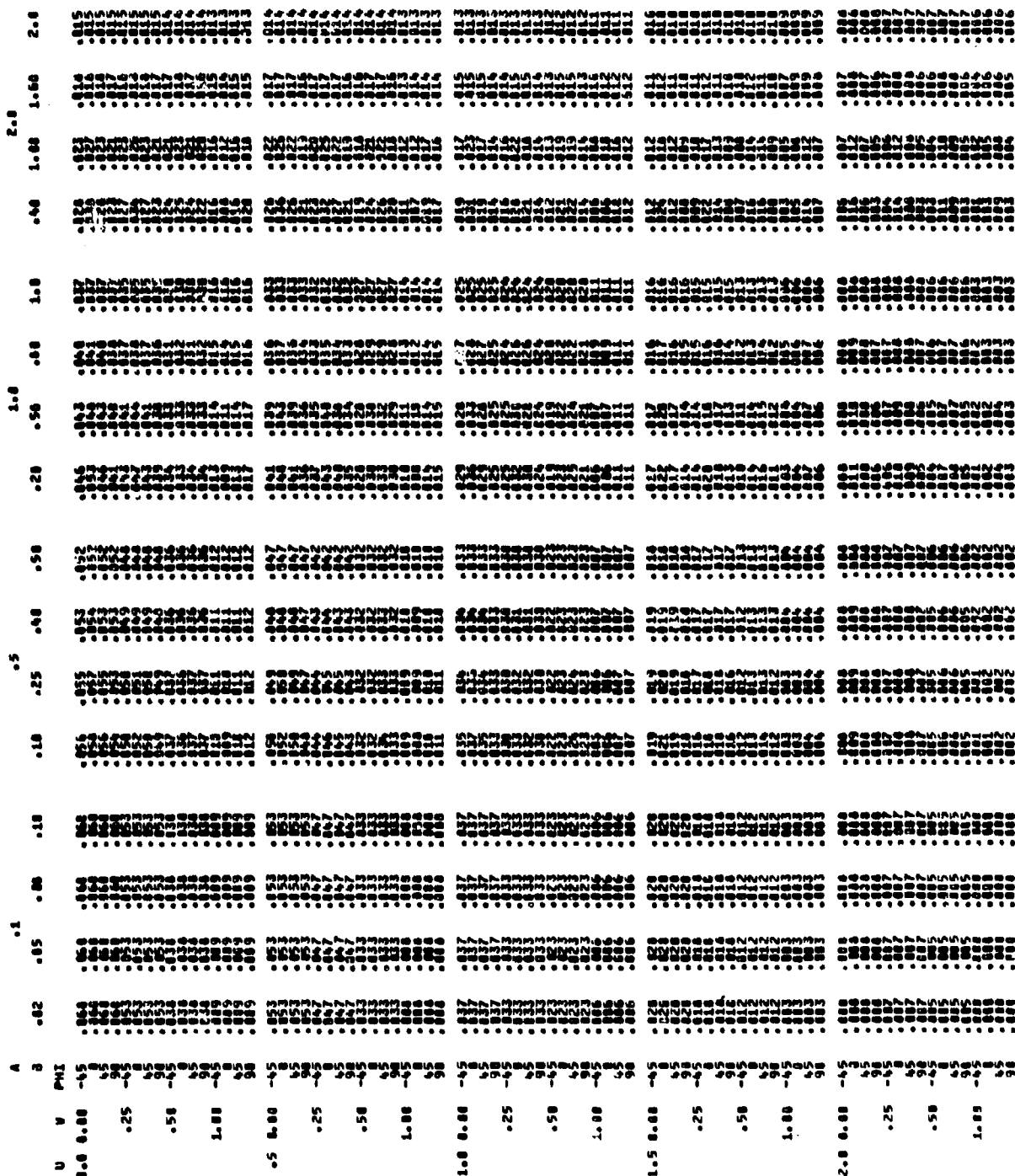
However, a and b are still the semi-major and semi-minor axes respectively, of the ellipse, so are not interchanged, but a consequence of the rearrangement is that ϕ is changed from 90° to 0° . A consideration of (a) and (b) of Figure E2 will convince the reader that these figures represent the data on lines 2 and 3 respectively, and that the cases are equivalent.

However, the value $v = 0.7$ is not given in the table and so we must interpolate between the values $v = 0.5$ and $v = 1$ which are given (see lines 4 and 5). The results from the table, page 84, are $P_K = 0.234$ for $v = 0.5$ (line 4) and $P_K = 0.149$ for the data on line 5, with $v = 1$. Interpolating linearly, we find that for the data on line 3, $P_K = 0.200$. Here the linear interpolation over a relatively large span, $v = 0.5$ to $v = 1$, introduces an additional error which is difficult to assess on theoretical grounds. However, this case was computed directly by the DJH program, using the original data (line 1, Table E2). In the computer program, σ_h is not required to have the

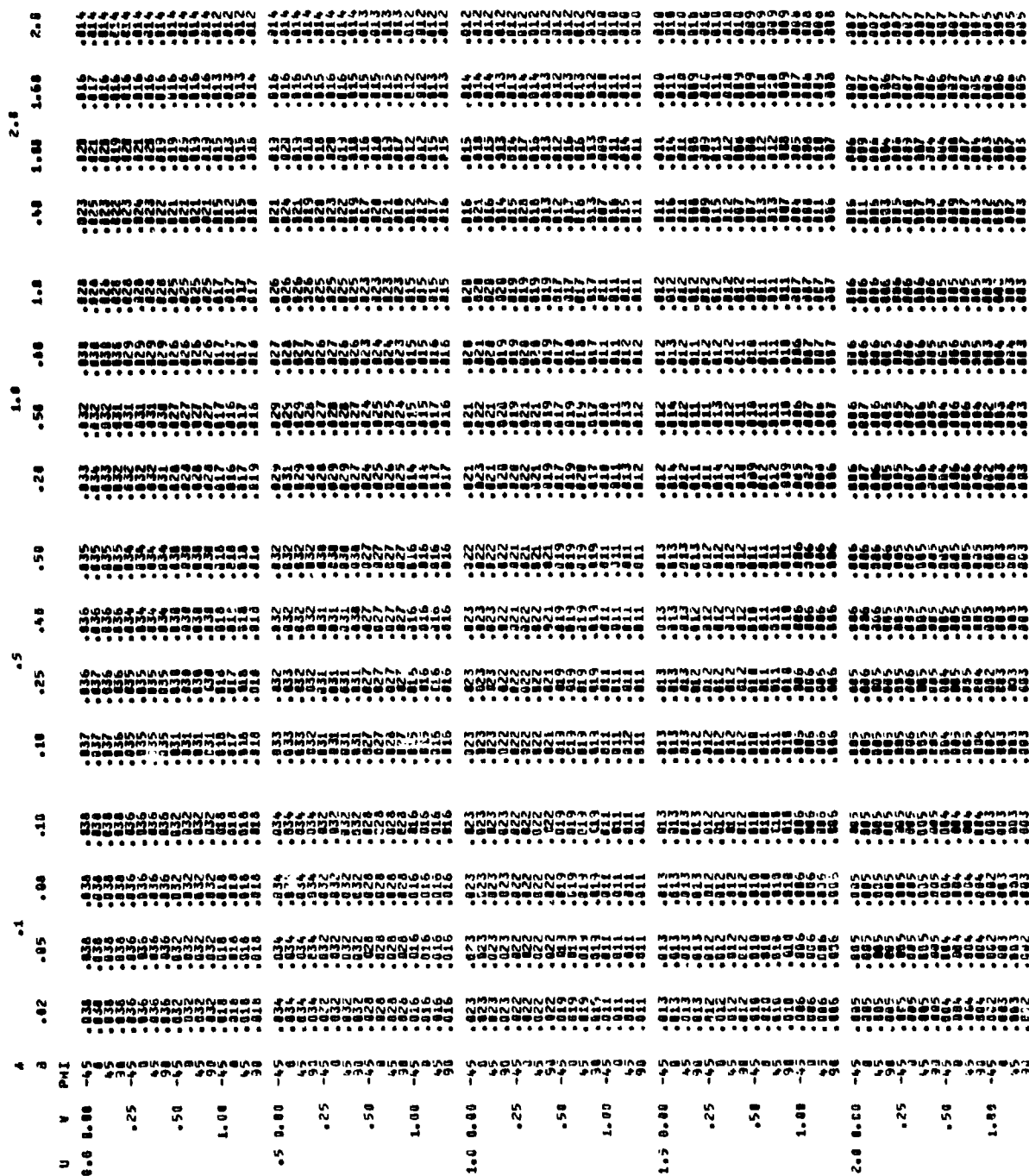
value 1 or to be not less than σ_k , although such restrictions were imposed in computing Table E3 for this report. The direct computation, with a tolerance $\epsilon = 0.005$, gave a result of $P_k = 0.210897 \dots$, rounding to 0.211 and thus differing by slightly more than one unit in the second decimal digit from the result obtained by interpolating in Table E3. Since the overall tolerance was $\epsilon = 0.005$, the interpolation introduced only a small additional error in this case.

[illegible]

SIGMA N= 1 SIGMA K= -5 R= .25

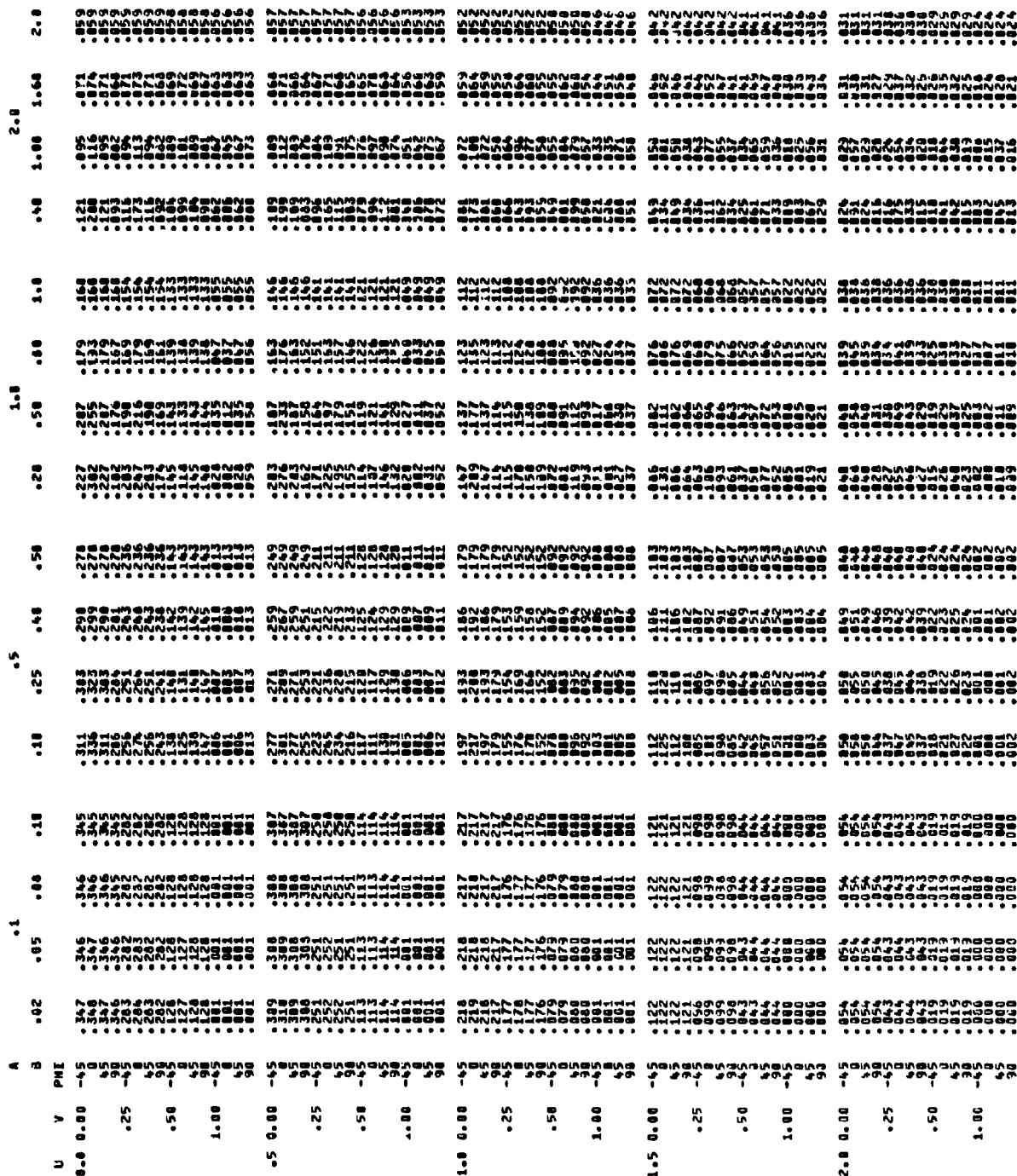


SIGMA H = 1 SIGMA K = 0.5 R = .25



[illegible]

SIGMA W = 1 SIGMA K = .2 R = .50



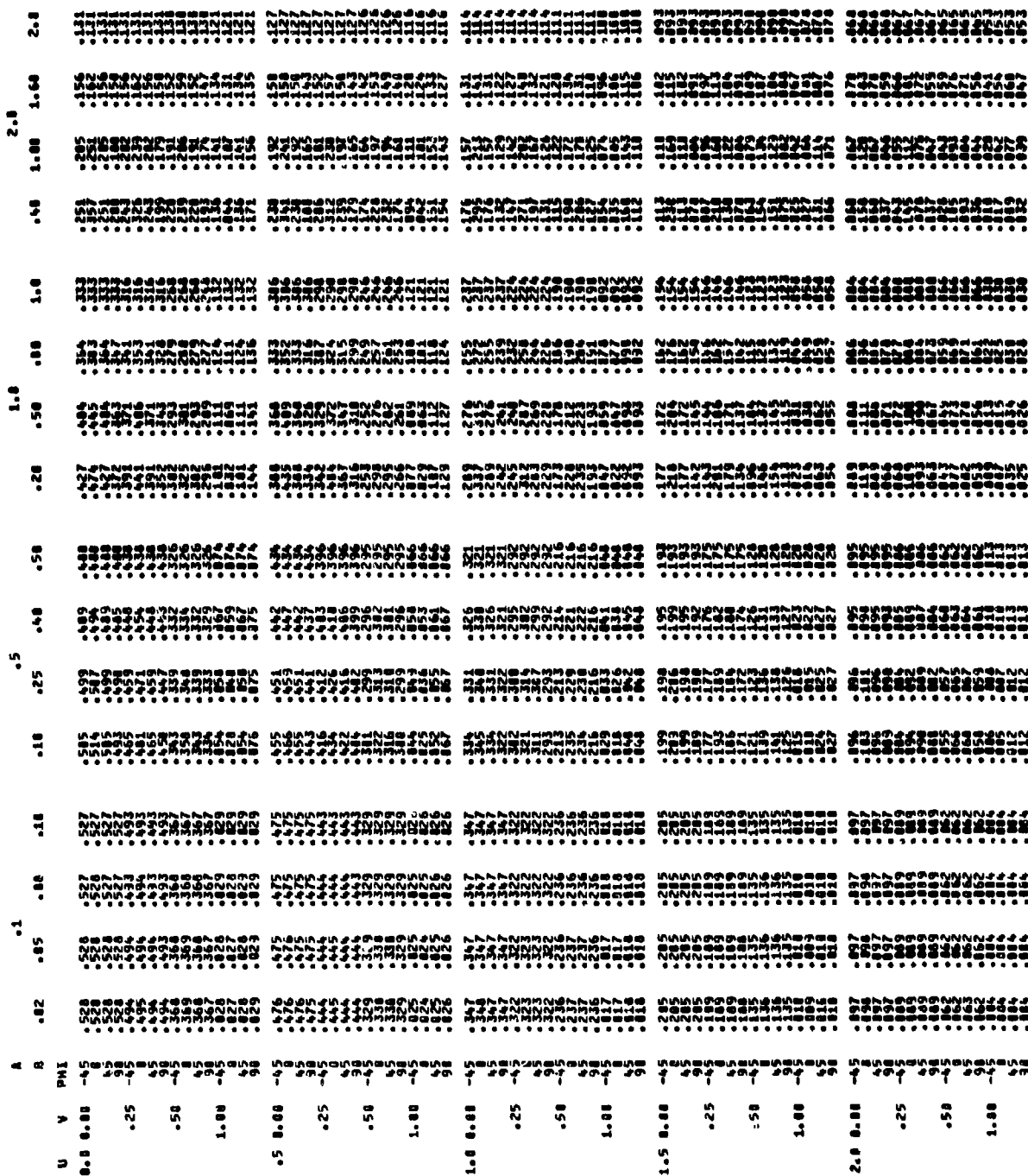
[illegible]

[illegible]

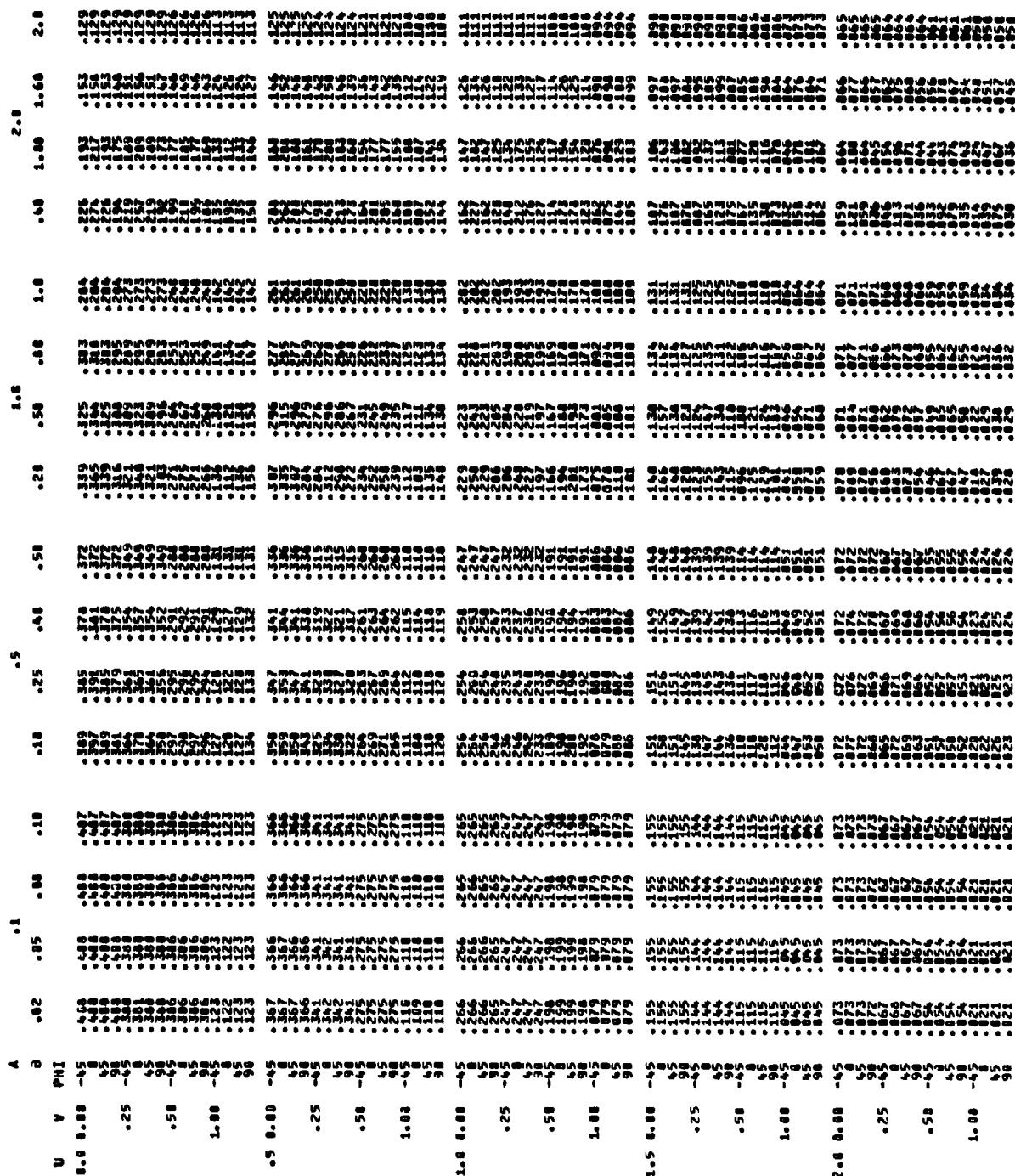
SIGMA M= 1 SIGMA K=1.0 R= .50

U	V	A	B	PMI										
					.02	.05	.10	.15	.20	.25	.30	.40	.50	.60
0.0	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
.5	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
1.0	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
1.5	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
2.0	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117
	0.00	-5	-5	117	117	117	117	117	117	117	117	117	117	117

SIGMA H= 1 SIGMA K= .2 R= .75

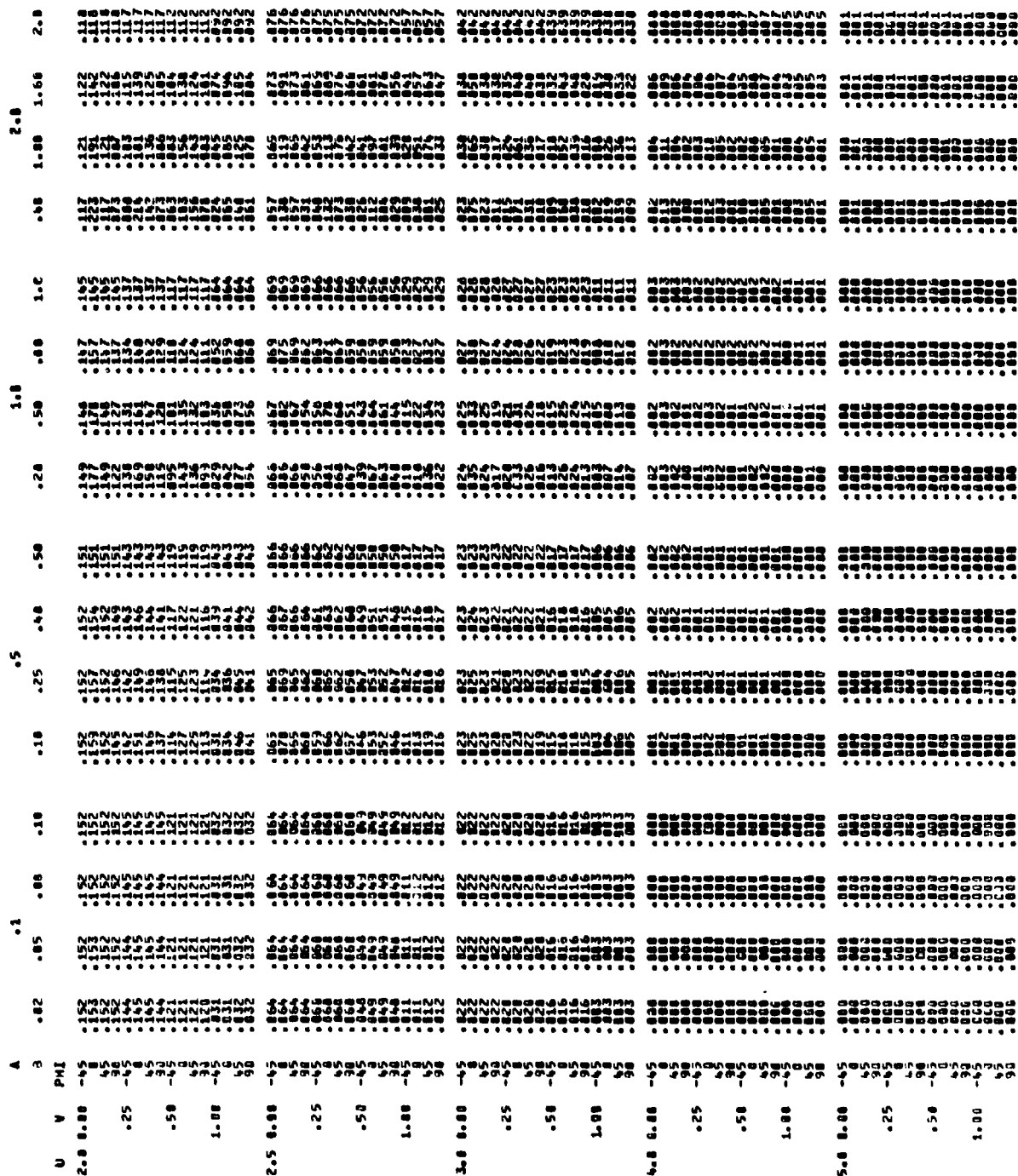


SIGMA H= 1 SIGMA K= .5 R= .75



[illegible]

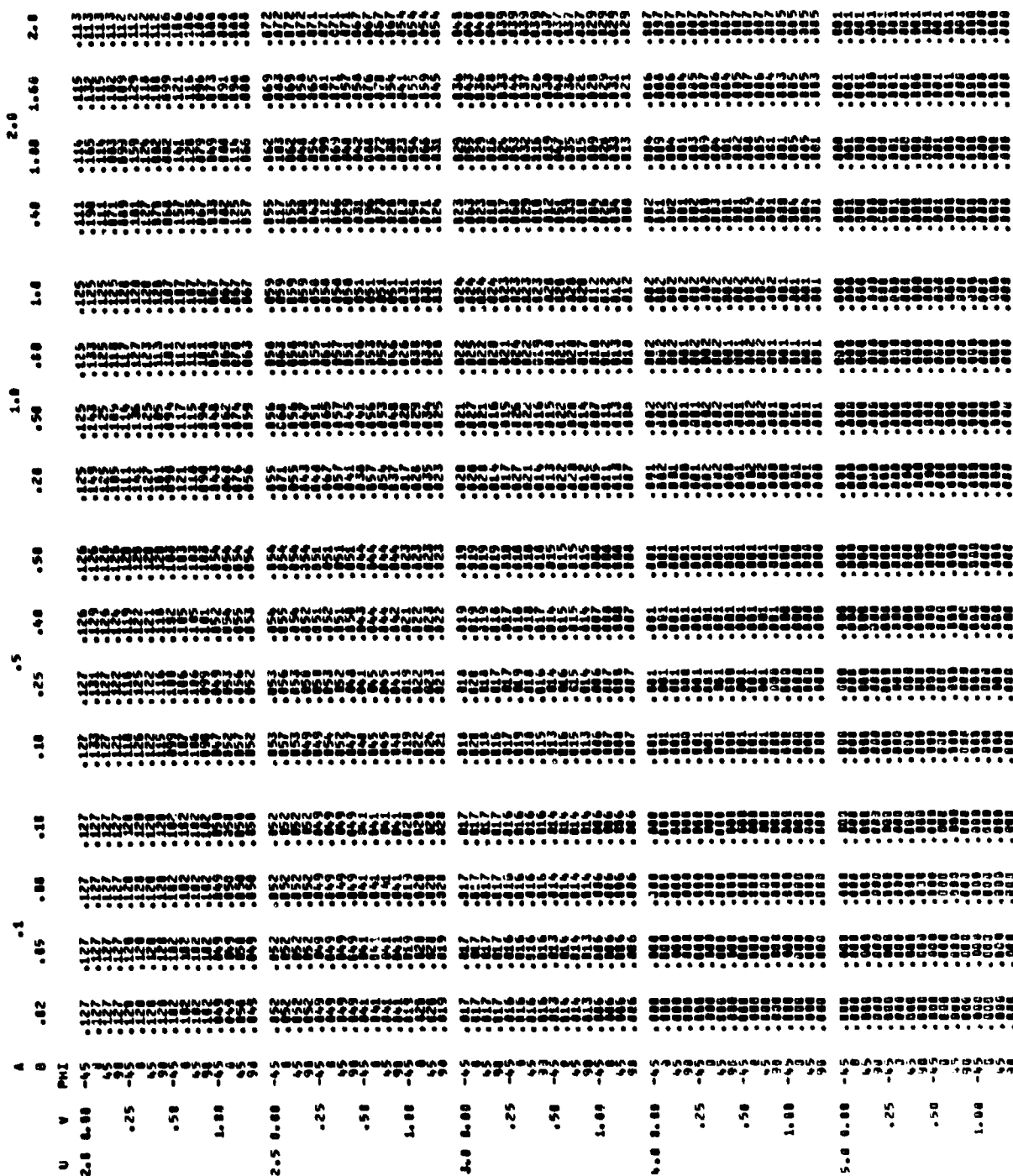
SIGMA H= 1 SIGMA K= .2 R=1.00



R=1.00

[illegible]

SIGMA M= 1 SIGMA C= .5 R=1.00

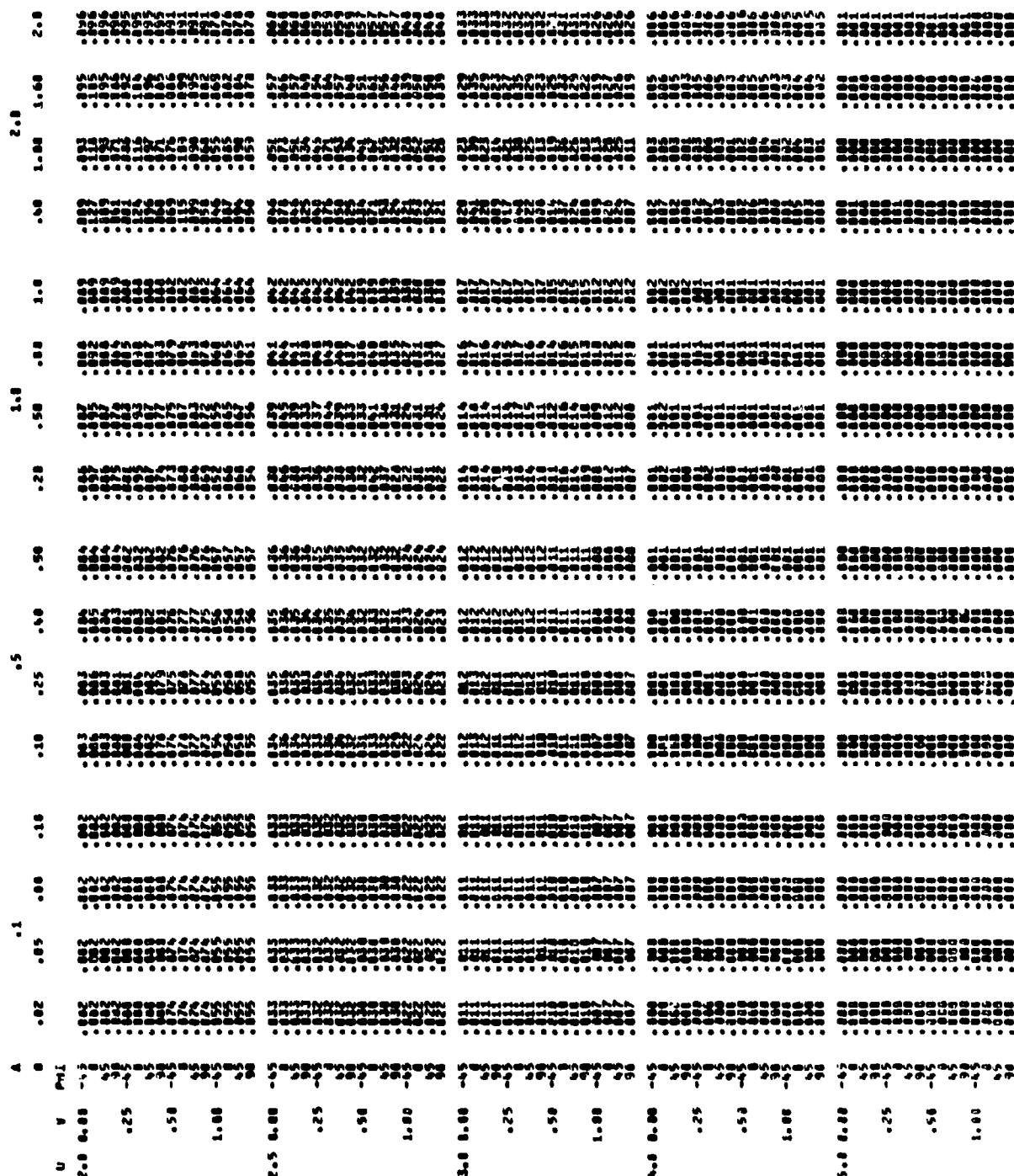


	4	3	2	1	0	1	2	3	4
U	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
V	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
P	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
PHI	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0

SIGMA H= 1 SIGMA K=1.0 R=1.00

U	V	PHI	A										2.0									
			.02	.05	.1	.25	.5	.60	.50	.20	.50	.50	.50	.50	.50	.50	.50	.50	.50	.50	.50	.50
0.0	0.00	-																				
	.25	-																				
	.50	-																				
.5	0.00	-																				
	.25	-																				
	.50	-																				
1.0	0.00	-																				
	.25	-																				
	.50	-																				
1.5	0.00	-																				
	.25	-																				
	.50	-																				

SIGMA H= 1 SIGMA K=1.0 R=1.00



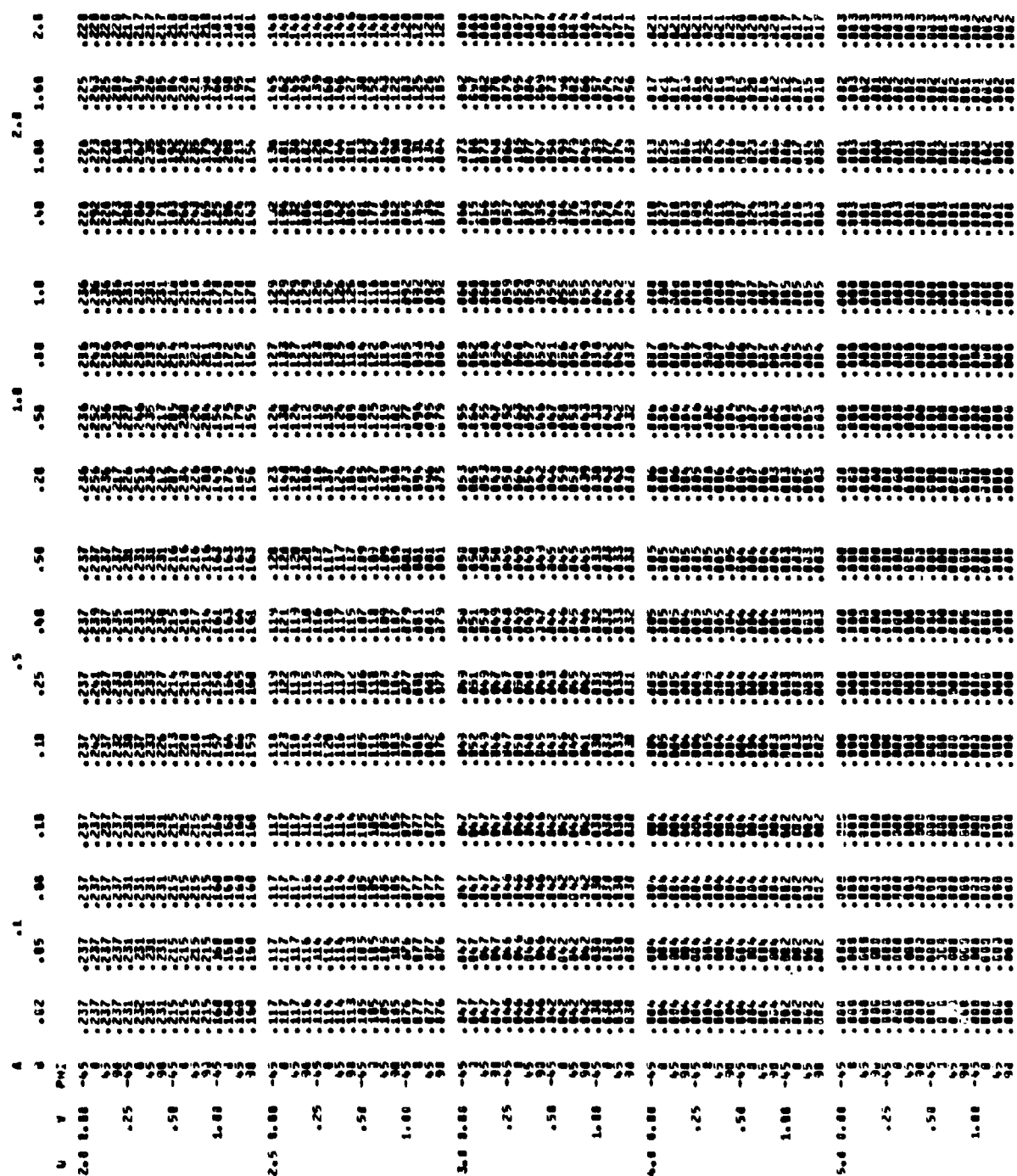
[illegible]

SIGMA M= 1 SIGMA K= .5 R=1.50

A	B	PHI	U	V	-1										.5										1																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																													
					.02	.05	.10	.25	.40	.50	.70	.80	.90	1.00	.02	.05	.10	.25	.40	.50	.70	.80	.90	1.00	.02	.05	.10	.25	.40	.50	.70	.80	.90	1.00																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																				
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$R=1.59$ [illegible]

SIGMA M= 1 SIGMA R= .6 R=1.50



SIGMA M= 1 SIGMA K=1.0 R=1.50

A	V	U	Sigma															
			M= 1 SIGMA K=1.0 R=1.50															
3	0.00	0.00	2.0															
			1.60															
			1.00															
			0.00															
3	0.25	0.00	2.0															
			1.60															
			1.00															
			0.00															
3	0.50	0.00	2.0															
			1.60															
			1.00															
			0.00															
3	1.00	0.00	2.0															
			1.60															
			1.00															
			0.00															
3	0.00	0.25	2.0															
			1.60															
			1.00															
			0.00															
3	0.25	0.25	2.0															
			1.60															
			1.00															
			0.00															
3	0.50	0.50	2.0															
			1.60															
			1.00															
			0.00															
3	1.00	1.00	2.0															
			1.60															
			1.00															
			0.00															

[illegible]

SIGMA H= 1 SIGMA K= .2 R=2.00

U	V	PHI	A										2.0									
			.02	.05	.1	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
0.0	0.00	-																				
	.25	-																				
	.50	-																				
	1.00	-																				
5	0.00	-																				
	.25	-																				
	.50	-																				
	1.00	-																				
10	0.00	-																				
	.25	-																				
	.50	-																				
	1.00	-																				

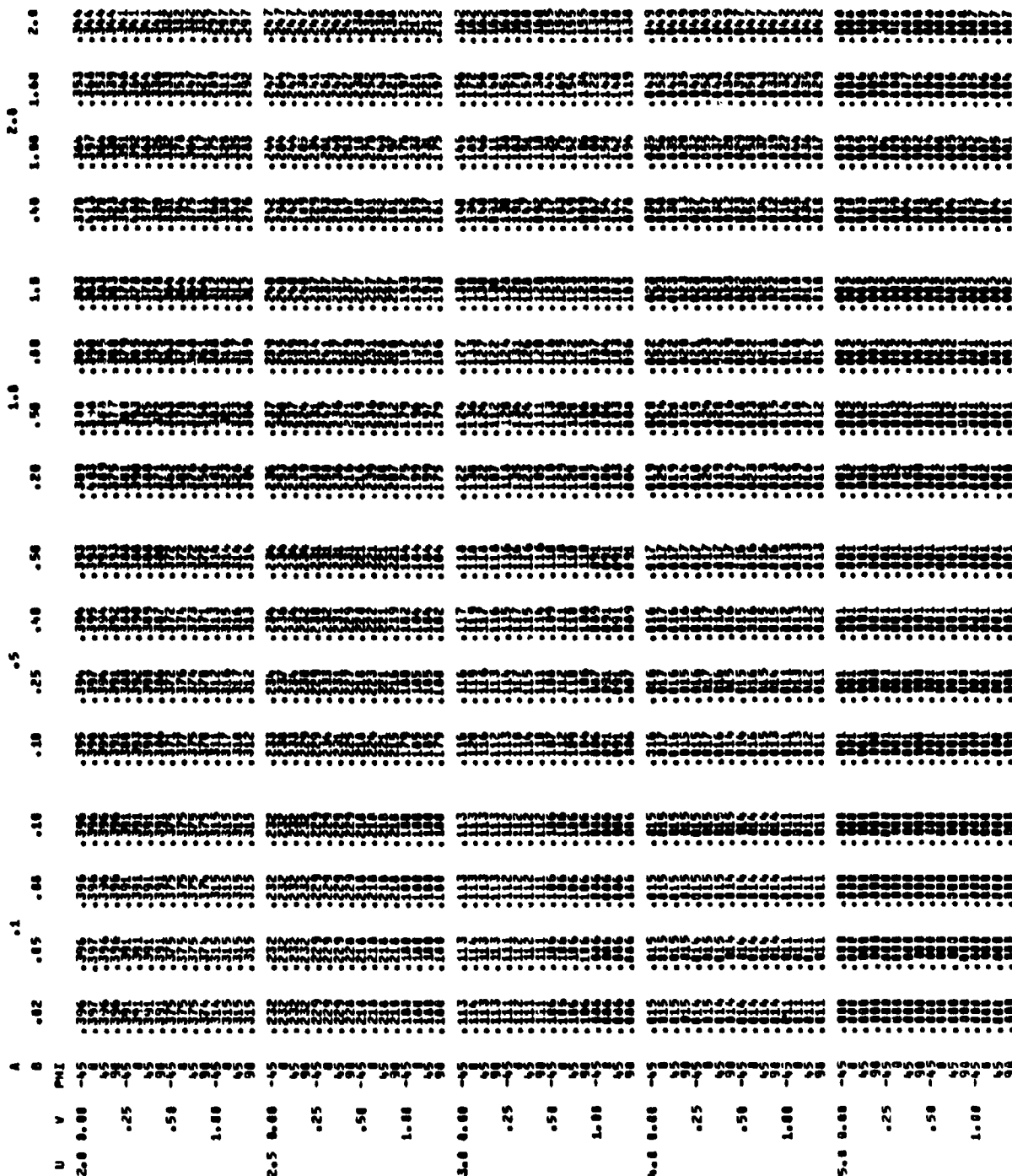
SIGMA M=1 SIGMA K=0.8 R=2.50

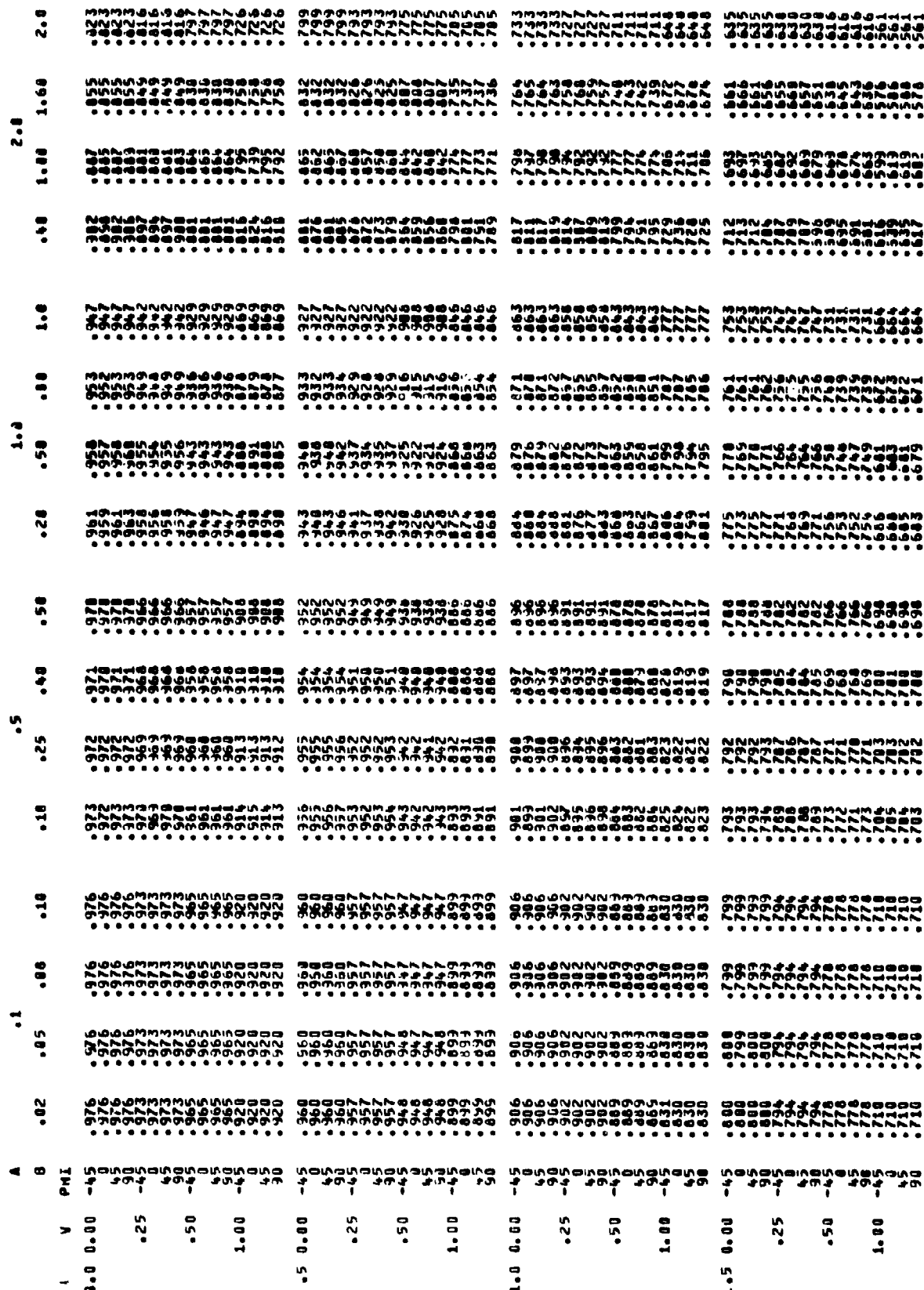
2.8	2.0	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	1.50	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	1.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.50	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
1.0	0.50	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
0.5	0.50	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
0.1	0.50	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0.00	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
4	3	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	2	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	1	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				
	0	[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]					[REDACTED]				

SIGMA H= 1 SIGMA K=1.0

[illegible]

SIGMA W= 1 SIGMA K=1.0 R=2.10





5169A M2 1

U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL	AM	AN	AO	AP	AQ	AR	AS	AT	AU	AV	AW	AX	AY	AZ	BA	BB	BC	BD	BE	BF	BG	BH	BI	BJ	BK	BL	BM	BN	BO	BP	BQ	BR	BS	BT	BU	BV	BW	BX	BY	BZ	CA	CB	CC	CD	CE	CF	CG	CH	CI	CJ	CK	CL	CM	CN	CO	CP	CQ	CR	CS	CT	CU	CV	CW	CX	CY	CZ	DA	DB	DC	DD	DE	DF	DG	DH	DI	DJ	DK	DL	DM	DN	DO	DP	DQ	DR	DS	DT	DU	DV	DW	DX	DY	DZ	EA	EB	EC	ED	EE	EF	EG	EH	EI	EJ	EK	EL	EM	EN	EO	EP	EQ	ER	ES	ET	EU	EV	EW	EX	EY	EZ	FA	FB	FC	FD	FE	FF	FG	FH	FI	FJ	FK	FL	FM	FN	FO	FP	FQ	FR	FS	FT	FU	FV	FW	FX	FY	FZ	GA	GB	GC	GD	GE	GF	GG	GH	GI	GJ	GK	GL	GM	GN	GO	GP	GQ	GR	GS	GT	GU	GV	GW	GX	GY	GZ	HA	HB	HC	HD	HE	HF	HG	HH	HI	HJ	HK	HL	HM	HN	HO	HP	HQ	HR	HS	HT	HU	HV	HW	HX	HY	HZ	IA	IB	IC	ID	IE	IF	IG	IH	II	IJ	IK	IL	IM	IN	IO	IP	IQ	IR	IS	IT	IU	IV	IW	IX	IY	IZ	JA	JB	JC	JD	JE	JF	JG	JH	JI	IJ	JK	KL	JM	JN	JO	JP	JQ	JR	JS	JT	JU	JV	JW	JX	JY	JZ	KA	KB	KC	KD	KE	KF	KG	KH	KI	KJ	KK	KL	KM	KN	KO	KP	KQ	KR	KS	KT	KU	KV	KW	KX	KY	KZ	LA	LB	LC	LD	LE	LF	LG	LH	LI	LJ	LK	LL	LM	LN	LO	LP	LQ	LR	LS	LT	LU	LV	LW	LX	LY	LZ	MA	MB	MC	MD	ME	MF	MG	MH	MI	MJ	MK	ML	MM	MN	MO	MP	MQ	MR	MS	MT	MU	MV	MW	MX	MY	MZ	NA	NB	NC	ND	NE	NF	NG	NH	NI	NJ	NK	NL	NM	NN	NO	NP	NQ	NR	NS	NT	NU	NV	NW	NX	NY	NZ	OA	OB	OC	OD	OE	OF	OG	OH	OI	OJ	OK	OL	OM	ON	OO	OP	OQ	OR	OS	OT	OU	OV	OW	OX	OY	OZ	PA	PB	PC	PD	PE	PF	PG	PH	PI	PJ	PK	PL	PM	PN	PO	PP	PQ	PR	PS	PT	PU	PV	PW	PX	PY	PZ	QA	QB	QC	QD	QE	QF	QG	QH	QI	QJ	QK	QL	QM	QN	QO	QP	QQ	QR	QS	QT	QU	QV	QW	QX	QY	QZ	RA	RB	RC	RD	RE	RF	RG	RH	RI	RJ	RK	RL	RM	RN	RO	RP	RQ	RR	RS	RT	RU	RV	RW	RX	RY	RZ	SA	SB	SC	SD	SE	SF	SG	SH	SI	SJ	SK	SL	SM	SN	SO	SP	SQ	SR	SS	ST	SU	SV	SW	SX	SY	SZ	TA	TB	TC	TD	TE	TF	TG	TH	TI	TJ	TK	TL	TM	TN	TO	TP	TQ	TR	TS	TT	TU	TV	TW	TX	TY	TZ	UA	UB	UC	UD	UE	UF	UG	UH	UI	UJ	UK	UL	UM	UN	UO	UP	UQ	UR	US	UT	UU	UV	UW	UX	UY	UZ	VA	VB	VC	VD	VE	VF	VG	VH	VI	VJ	VK	VL	VM	VN	VO	VP	VQ	VR	VS	VT	VU	VV	VW	VX	VY	VZ	WA	WB	WC	WD	WE	WF	WG	WH	WI	WJ	WK	WL	WM	WN	WO	WP	WQ	WR	WS	WT	WU	WV	WW	WX	WY	WZ	XA	XB	XC	XD	XE	XF	XG	XH	XI	XJ	XK	XL	XM	XN	XO	XP	XQ	XR	XS	XT	XU	XV	XW	XX	XY	XZ	YA	YB	YC	YD	YE	YF	YG	YH	YI	YJ	YK	YL	YM	YN	YO	YP	YQ	YR	YS	YT	YU	YV	YW	YX	YY	YZ	ZA	ZB	ZC	ZD	ZE	ZF	ZG	ZH	ZI	ZJ	ZK	ZL	ZM	ZN	ZO	ZP	ZQ	ZR	ZS	ZT	ZU	ZV	ZW	ZX	ZY	ZZ
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SIGMA W= 1 SIGMA K=1.0 R=2.50

			.1										.5										1.0										2.0																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																		
A	B	PHI	.02	.05	.08	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.1

[illegible]

3.10

[illegible]

A	B	ID	U	V	S	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0																							
						0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0																								
						0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0																									
						0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0																										
1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0																			
																					0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0									
																					0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0										
																					0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0											
0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0										
																														0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
																														0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	
																														0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0		
0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0											
																													0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	
																													0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0		
																													0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0			
0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0												
																												0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0		
																												0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0			
																												0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0				
0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0													
																											0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0			
																											0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0				
																											0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0					
0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0														
																										0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0				
																										0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0					
																										0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0						
0.4	0.3	0.2	0.1	0.0	0.9	0.8	0.7																																

[illegible]

A	3	PHI	U	V	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5	22.0	22.5	23.0	23.5	24.0	24.5	25.0	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5	30.0	30.5	31.0	31.5	32.0	32.5	33.0	33.5	34.0	34.5	35.0	35.5	36.0	36.5	37.0	37.5	38.0	38.5	39.0	39.5	40.0	40.5	41.0	41.5	42.0	42.5	43.0	43.5	44.0	44.5	45.0	45.5	46.0	46.5	47.0	47.5	48.0	48.5	49.0	49.5	50.0	50.5	51.0	51.5	52.0	52.5	53.0	53.5	54.0	54.5	55.0	55.5	56.0	56.5	57.0	57.5	58.0	58.5	59.0	59.5	60.0	60.5	61.0	61.5	62.0	62.5	63.0	63.5	64.0	64.5	65.0	65.5	66.0	66.5	67.0	67.5	68.0	68.5	69.0	69.5	70.0	70.5	71.0	71.5	72.0	72.5	73.0	73.5	74.0	74.5	75.0	75.5	76.0	76.5	77.0	77.5	78.0	78.5	79.0	79.5	80.0	80.5	81.0	81.5	82.0	82.5	83.0	83.5	84.0	84.5	85.0	85.5	86.0	86.5	87.0	87.5	88.0	88.5	89.0	89.5	90.0	90.5	91.0	91.5	92.0	92.5	93.0	93.5	94.0	94.5	95.0	95.5	96.0	96.5	97.0	97.5	98.0	98.5	99.0	99.5	100.0	100.5	101.0	101.5	102.0	102.5	103.0	103.5	104.0	104.5	105.0	105.5	106.0	106.5	107.0	107.5	108.0	108.5	109.0	109.5	110.0	110.5	111.0	111.5	112.0	112.5	113.0	113.5	114.0	114.5	115.0	115.5	116.0	116.5	117.0	117.5	118.0	118.5	119.0	119.5	120.0	120.5	121.0	121.5	122.0	122.5	123.0	123.5	124.0	124.5	125.0	125.5	126.0	126.5	127.0	127.5	128.0	128.5	129.0	129.5	130.0	130.5	131.0	131.5	132.0	132.5	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0	137.5	138.0	138.5	139.0	139.5	140.0	140.5	141.0	141.5	142.0	142.5	143.0	143.5	144.0	144.5	145.0	145.5	146.0	146.5	147.0	147.5	148.0	148.5	149.0	149.5	150.0	150.5	151.0	151.5	152.
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SIGMA N= 1 SIGMA K= .8 R=3.00

A	U	V	PHI																
2.6	2.6	2.00	-45	.25	.50	1.00	.02	.05	.08	.10	.15	.25	.40	.50	.80	1.00	1.50	2.00	
2.5	2.5	2.00	-45	.25	.50	1.00	.02	.05	.08	.10	.15	.25	.40	.50	.80	1.00	1.50	2.00	
3.0	3.0	2.00	-45	.25	.50	1.00	.02	.05	.08	.10	.15	.25	.40	.50	.80	1.00	1.50	2.00	
4.0	4.0	2.00	-45	.25	.50	1.00	.02	.05	.08	.10	.15	.25	.40	.50	.80	1.00	1.50	2.00	
5.0	5.0	2.00	-45	.25	.50	1.00	.02	.05	.08	.10	.15	.25	.40	.50	.80	1.00	1.50	2.00	

[illegible]

[illegible]

[illegible]

[illegible]

SIGMA H= 1 SIGMA K=1.0 R=4.00

A		.1										.5										1.0										2.0									
U	V	PHI										PHI										PHI										PHI									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
+	0.00	+										+										+										+									
-	.25	-										-										-										-									
+	.50	+										+										+										+									
-	1.00	-										-										-										-									
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APPENDIX F

FORTRAN LISTING OF DJH PROGRAM

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FORTRAN LISTING OF DJH PROGRAM

EXPLANATION OF FORTRAN LISTING FOR P_K

Routine LDBSMP evaluates the integral

$$P_K = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a P(h, k, \sigma_h, \sigma_k, R) r dr d\theta$$

with

$$P(h, k, \sigma_h, \sigma_k, R) = \int_{h-R}^{h+R} \int_{k-\sqrt{R^2-(x-h)^2}}^{k+\sqrt{R^2-(x-h)^2}} \exp \left[-\frac{1}{2} \left(\frac{x^2}{\sigma_h^2} + \frac{y^2}{\sigma_k^2} \right) \right] dy dx$$

(See page 6)

evaluated by subroutine PKILL, where

$$\begin{aligned} h &= u + r \cos \theta \cos \phi - cr \sin \theta \sin \phi \\ k &= v + r \cos \theta \sin \phi + cr \sin \theta \cos \phi \quad (\text{See (2-11)}) \end{aligned}$$

Identifying the other variables

- a is major axis of the ellipse
- b is minor axis of the ellipse
- c is b/a
- u is the abscissa of the center of the ellipse along h axis
- v is the ordinate of the center of the ellipse along k axis
- σ_h is the standard deviation in the h direction
- σ_k is the standard deviation in the k direction
- R is the lethal radius of the weapon
- ϕ is the inclination of the major axis of the ellipse to the h axis measured positive counterclockwise from h-axis, $-90 < \phi \leq 90$.

Routine LDBSMP calls subroutine ARIEL in the following manner

CALL ARIEL (u, v, a, b, σ_h , σ_k , ϕ , R, P_K, tolerance)

with tolerance set to 0.005. Its basic function is to set up the input and locate the output.

Subroutine ARIEL calls the double integration routine DBSMP to compute P_K . When $\sigma_h = \sigma_k$ the routine is much faster because the circular coverage subroutine CIRCV can be used instead of the elliptic coverage function subroutine PKILL, (see page 32).

Subroutine DBSMP is called by

CALL DBSMP (xu, xl, yu, yl, eps, N, g, PBXY),

where,

xu is the upper limit of integration of x
 xl is the lower limit of integration of x
 yu is the upper limit of integration of y
 yl is the lower limit of integration of y
 eps is a tolerance which is used to terminate the integration, when

$$|I_{n_2} - I_{n_1}| \leq \text{eps}/(1 + y_u - y_l) \text{ (See page 23.)}$$

where

I_{n_1} approximates P_K with partitionings of the (x, y) (or r, θ) intervals of integration and n_2 is a refined partition of n_1 .

N is the maximum number of subdivisions of an interval of integration in either the x or y direction.

g is the location of the value of P_K .

PBXY is a subroutine which evaluates the integrand at any point in the interval of integration.

Subroutine INLINE is used by DBSMP.

Subroutine PBXY-3 replaces PBXY in DBSMP when $\sigma_h = \sigma_k$, i.e., when CIRCV is used instead of PKILL.

Subroutine CIRCV is called in the following way

CALL CIRCV (R, d, l, P, IR),

where

R is defined above

d is the distance from the origin to a point of integration (h, k), i.e.,

$$d = \sqrt{h^2 + k^2}.$$

P is the location where the value of P is stored

IR is an error indicator which is not applicable here.

CIRCV uses the special functions ERF and ERFD in tabulated form. They are defined as

$$\text{ERF}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-y^2} dy$$

$$\text{ERFD} = \frac{2}{\sqrt{\pi}} e^{-t^2}$$

```

      PROGRAM LDPSMP (OUTPUT )
      COMMON/ONLY1/FR(1025),EPSL,NPTS ,N3,N5,N7,IOP5
      IOP5=1
5      91  FORMAT ( 1H0,6E22.15 )
      97  FORMAT ( 1H1 )
      PRINT 97
      PI=3.1415 92653 58979
      U1=140.
      V1=400.
10     A1=200.
      B3=100.
      SX1=160.
      SY3=200.
      P3=(90.*PI)/180.
15     R3=300.
      EPS1=.0001
      CALL ARIEL ( U1,V1,A1,B3,SX1,SY3,P3,R3,GS,EPS1)
      PRINT 91,U1,V1,A1,B3,SX1,SY3,P3,R3,GS
      CALL EXIT
20     END

      SUBROUTINE ARIEL(UU,VV,A,R,SIGX,SIGY,ANGLE,ARKILL,GS,EPS1 )
      EXTERNAL PBXY ,PBXY3
      COMMON /T101/U,V,PHI,CEE,SIGMAX,SIGMAY,RKILL,SNPHI,CSPHI
      U=UU
      V=VV
      RKILL=ARKILL
      SIGMAY=SIGY
      SIGMAX=SIGX
      PI=3.14159265358979
10     CEE=R/A
      TWOPI=2.*PI
      PHI=ANGLE
      SNPHI=SIN(PHI)
      CSPHI=COS(PHI)
15     XU=TWOPI
      EP=EPS1*PI*A*A
      E1=EP
      XL=0
      YU=A
20     YL=0
      IF ( SIGX.EQ.SIGY) GO TO 1101
      CALL DBSMP ( XU,XL,YU,YL,E1,NMAX,GS,PBXY )
      GS=(2.*GS)/(TWOPI*A*A )
      GO TO 1151
25     1101 CONTINUE
      U=U/SIGX
      V=V/SIGX
      SIGMAX=1.
      SIGMAY=1.
30     RKILL=RKILL/SIGX
      YU=A/SIGX
      E1=EPS1*PI*YU*YU
      CALL DBSMP ( XU,XL,YU,YL,E1,NMAX,GS,PBXY3 )
      GS=GS/( PI*( A*A)/(SIGX*SIGX) )
35     1151 CONTINUE
      RETURN
      END

```

```

SUBROUTINE DBSMP(XUP,XLO,YUP,YLO,EPSR,NMAX,EVALFN,PBXY )
COMMON/ONLY1/FR(1025),EPSL,NPTS ,N3,N5,N7,IOP5
PI=3.1415 9265 3589 79
NMAX=1025
5      N3=1
      N5=1
      N7=0
      EPSL=EPSR/(1.+(YUP-YLO))
      H=(XUP-XLO)/2.
10      C
      C      FOR 3 LINES
      C
      IY=0
      N=3
15      HK=(YUP-YLO)/2.
      DO 100 JJ=1,3
      IY=IY+1
      FY=YLO+FLOAT(JJ-1)*HK
200      CALL INLINE( FY, JJ, IY, PBXY, H , XLO, NMAX )
20      CONS2=FR(1)+FR(3)
      OLVAL=(HK*(CONS2+4.*FR(2)))/3.
      C
      C      SAVE LINES ALREADY INTEGRATED AND MULTIPLY BY 2
      C
25      150 PLO=0.
      DO 200 J=3,N
      L=N-J+2
      NUM=2*L-1
      FR(NUM)=FR(L)
30      200 PLO=PLO+FR(L)
      PLO=2.*PLO
      C
      C      COMPUTE ADDITIONAL LINES AND MULTIPLY BY 4
      C
35      PLN=0.
      N=2*N-1
      HK=HK/2.
      DO 300 JJ=2,N,2
      IY=IY+1
40      FY=YLO+FLOAT(JJ-1)*HK
      CALL INLINE( FY, JJ, IY, PBXY, H , XLO, NMAX )
300      PLN=PLN+FR(JJ)
      PLN=4.*PLN
45      C
      C      INTEGRATE RECTANGLE
      C
      EVALFN=(HK*(CONS2+PLO+PLN))/3.
      C
      C      TEST FOR DESIRED PRECISION
50      C      IF (ABS(EVALFN-OLVAL)-(EPSR*ABS(EVALFN))) 500,500,400
      IF (ABS(EVALFN-OLVAL)-(EPSL
      ) ) 500,500,400
400      OLVAL=EVALFN
      IF ( N.LT.NMAX) GO TO 150
55      C 500 RETURN
      500 CONTINUE

      IF ( IOP5.EQ.1 ) EVALFN=EVALFN+(1./15.)*(EVALFN-OLVAL )
      RETURN
      END

```

REGISTER ALLOCATION

4 REGISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 26 26

```

SUBROUTINE INLINE( Y,JJ, IZ, FBXY, OH0, XLO, HMAX )
COMMON/ONLY1/FR(1025),EPSL,NPTS ,N3,N5,N7,IOP5

C
C      INITIALIZATION FOR FUNCTION EVALUATION AT 3 POINTS
5      C
      IY=IZ
      H=OH0
      N=3
      IX=1
10     X=XLO
      CALL PBXY( X, Y, IY, IX, ANS)
      CONS1=ANS
      IX=2
      X=XLO+H
15     CALL PBXY( X, Y, IY, IX, ANS)
      SUM2=ANS
      IX=3
      X=X+H
20     CALL PBXY( X, Y, IY, IX, ANS)
      CONS1=CONS1+ANS
      OFR=H*(CONS1+4.*SUM2)/3.

C
C      SAVE PREVIOUSLY COMPUTED VALUES AND MULTIPLY BY 2
25     C
      PFO=2.*SUM2

C
C      COMPUTE NEW VALUES AND MULTIPLY BY 4
30     C
      SUM4=0.
      M=2*M-1
      H=H/2.
      DO 30 J=2,M,2
      IX=IX+1
      X=XLO+FLOAT(J-1)*H
35     CALL PBXY( X, Y, IY, IX, ANS)
      30 SUM4=SUM4+ANS
      PFN=4.*SUM4
      SUM2=SUM2+SUM4

C
C      INTEGRATE ALONG LINE AND TEST FOR DESIRED PRECISION
40     C
      FR(JJ)=(H*(CONS1+PFN+PFO))/3.
      IF ( M.EQ.5 ) GO TO 50
C      IF (ABS(FR(JJ)-OFR)-EPSL*ABS(FR(JJ))) 40,40,50
45     IF (ABS(FR(JJ)-OFR)-EPSL ) 40,40,50
      50 OFR=FR(JJ)
      IF (M.GE.HMAX) RETURN
      IF (IY.NE.1.AND.IX.EQ.NPTS) IY=1
      GO TO 25
50     40 IF (IY.EQ.1) NPTS=IX
      N7=N7+M
      IF ( N3.LT.M ) N3=M
      N5=N5+1
      IF ( IOP5.EQ.1 ) FR(JJ)=FR(JJ)+(1./15.)*(FR(JJ)-OFR )
55     RETURN

      END

```

SUBROUTINE PKILL (RRR,SXR,SYR,HR,RKR,RESULT)
 DIMENSION MN(6),XI(24,6),WI(24,6)
 THIS PKILL IS FOR 3 DIGIT ACCURACY.

C
C

```

5      REAL K
      INTEGER OG
      DATA MNME4/.99999975/,TONME4/1.9999995/,A/5.452/,ASTAR/5.574/,A1/4 EF001
A.892/,A5/5.253/,SQRT2/1.4142136/,MN/6,8,12,16,20,24/,TMSQPI/1.1283 EF002
8792/,SQ202/.70710678/,TOL/.1E-07/, EF003
10     CXI/-.93246951,-.66120939,-.23861919,.23861919,.86120939,.93246951, EF004
D18*0.0,-.96028986,-.79666648,-.52553241,-.18343484,.18343484,.5255 EF005
E3241,.79666648,.96028986,16*0.0,-.98156063,-.90411726,-.76990267,- EF006
F.58731795,-.36783150,-.12523341,.12523341,.36783150,.58731795,.769 EF007
G90267,-.90411726,.98156063,12*0.0,-.98940093,-.94457502,-.86563120, EF008
15     H-.75540441,-.61787624,-.45801678,-.28160355,-.95012510E-01,.950125 EF009
I10E-01,.28160355,.45801678,.61787624,.75540441,.86563120,.94457502 EF010
J,.98940093,8*0.0,-.99312860,-.96397193,-.91223443,-.83911697,-.746 EF011
K33191,-.63605368,-.51086780,-.37370609,-.22778585,-.76526521E-01,. EF012
L76526521E-01,.22778585,.37370609,.51086780,.63605368,.74633191,.83 EF013
20     M911697,.91223443,.96397193,.99312860,4*0.0,-.99518722,-.97472856,- EF014
N.93827455,-.88641553,-.82000199,-.74012419,-.64809365,-.54542147,- EF015
O.437379351,-.31504268,-.19111887,-.64056893E-01,.64056893E-01,.1911 EF016
P1887,.31504268,.43379351,.54542147,.64809365,.74012419,.82000199,. EF017
Q88641553,.93827455,.97472856,.99518722/ EF018
25     DATA WI/.17132449,.36076157,.46791393,.46791393,.36076157,.1713244 EF019
A9,18*0.0,.10122854,.22238103,.31370665,.36268378,.36268378,.3137 EF020
B0665,.22238103,.10122854,16*0.0,.47175336E-01,.10693933,.16807833, EF021
C.20316743,.23349254,.24914705,.24914705,.23349254,.20316743,.16807 EF022
D833,.10693933,.47175336E-01,12*0.0,.27152459E-01,.62253574E-01,.95 EF023
30     E158512E-01,.12462897,.14959599,.16915652,.18260342,.18945061,.1894 EF024
F5061,.18260342,.16915652,.14959599,.12462897,.95158512E-01,.622535 EF025
G24E-01,.27152459E-01,8*0.0,.17614007E-1,.40601430E-1,.62672048E-1, EF026
H.83276742E-1,.10193012,.11819453,.13168864,.14209611,.14917249,.15 EF027
I275339,.15275339,.14917249,.14209611,.13168864,.11819453,.10193012 EF028
35     J,.83276742E-1,.62672048E-1,.40601430E-1,.17614007E-1,4*0.0,.123412 EF029
K30E-01,.28531389E-01,.44277439E-01,.59298985E-01,.73746481E-01,.86 EF030
L190162E-01,.97618652E-01,.10744427,.11550567,.12167047,.12583746,. EF031
M12793820,.12793820,.12583746,.12167047,.11550567,.10744427,.976186 EF032
N52E-01,.86190162E-01,.73746481E-01,.59298985E-01,.44277439E-01,.28 EF033
40     O531389E-01,.12341230E-01/ EF034
      R=RRR
      SX=SXR
      SY=SYR
      H=HHR
45     K=RKR
      H2=H*H+K*K
      IF (SX-SY) 1,2,3
1      S=SY
      GO TO 4
50     2 H = SQRT(H2)
      K=0.
      3 S=SX
      4 TEMP=R-A5*S
      IF (TEMP) 5,6,6
55     5 TEMP=R+A1*S

```

```

        IF(TEMP*TEMP-H*H-K*K) 8,8,9
6 IF(TEMP*TEMP-H*H-K*K) 5,7,7
7 P=1.
        GO TO 75
60 P=0.
        GO TO 75
        9 IF(R-H+A1*SX) 8,8,10
10 IF(R-K+A1*SY) 8,8,11
11 SX2=SX*SQRT2
65 SY2=SY*SQRT2
        T=H*SY+K*SX-A1*SX2*SY
        IF(T) 13,13,12
12 IF(R*(SX*SY+SY*SY)-T*T) 8,8,13
13 ASX=A*SX
70 ASY=A*SY
        TEMP=R-H-ASX
        IF(TEMP) 14,14,15
14 G0=0.
        GO TO 16
75 G0=TEMP/R
15 RARY=R-K-ASY
        IF(RARY) 17,17,18
17 G0P=0.
        GO TO 19
80 G0P=RARY/R
18 IF(H-ASX) 20,20,21
20 G1=1.
        GO TO 22
85 G1=(R-H+ASX)/R
21 TEMP=K-ASY
        IF(TEMP) 23,23,24
23 G1P=1.
        GO TO 25
90 G1P=(R-TEMP)/R
24 YI=G1-G0-G1P+G0P
        IF(YI) 27,27,26
25 RARY=SX
        TEMP=H
        T=G1
95 U=G1
        SX=SY
        H=K
        G1=G1P
        G0=G0P
100 SY=RARY
        K=TEMP
        G1P=T
        G0P=U
26 G1G0P=G1*G0P
105 G1PG0=G1P*G0
        IF(G1G0P-G1PG0) 32,32,28
27 IF(ABS(YI) - TOL) 31,31,29
28 U=G0+G1+G0P+G1P
        f=2.*(G0*G0+G1*G1+G0P*G0P+G1P*G1P)-U*U
110 IF(T) 30,32,32

```

```

30 SS=64.*G1GOP*G1PG0
  IF(SS-T*T) 31,32,32
115 31 RARY=SX
    TEMP=H
    T=G1
    U=G0
    SX=SY
    H=K
    G1=G1P
120  G0=G0P
    SY=RARY
    K=TEMP
    G1P=T
    GOP=U
125 32 E1 = SQRT(G1)
    E0 = SQRT(G0)
    EN=(E1-E0)*(.17*R/SX+(R/SY)/(1.05* ABS(R-K)/SY+10.))
    SX2=SX*SQRT2
    SY2=SY*SQRT2
130  ASY=A*SY
    IF ( EN-2.0 ) 41,40,40
    40 OG=4
    GO TO 50
    41 IF ( EN-.7 ) 43,42,42
135  42 OG=3
    GO TO 50
    43 IF ( EN-.5 ) 45,44,44
    44 OG=2
    GO TO 50
140  45 OG=1
    50 IA=1
    IB=1
    IF(H) 52,51,52
145  51 IH=2
    GO TO 53
    52 IH=1
    53 IF(K) 54,55,54
    54 IK=1
    GO TO 56
150  55 IK=2
    56 P=0.
    N=NN(OG)
    DO 74 I=1,N
    TIJ=(E1-E0)*(XI(I,OG)+1.)/2.+E0
155  V=TIJ*TIJ
    U=R*(1.-V)
    XL=V*(2.-V)
    XQ = SQRT(XL)
    XL=XQ-(ASY-K)/R
160  Q=XQ-(ASY+K)/R
    XIJ=(H-U)/SX2
    X=-XIJ*XIJ
    F1 = EXP(X)*TMSQPI
    GO TO (57,58),IH
165  57 YIJ=(H+U)/SX2

```

```

X=-YIJ*YIJ
F2 = EXP(X)*TWSQPI
FIJ=F1+F2
GO TO (59,72,69),IB
170 58 FIJ=2.*F1
GO TO (59,72,69),IA
59 IF(XL) 60,71,71
60 D11=(K+R*XQ)/SY2
IF(D11-A*SQ2D2) 62,61,61
175 61 D11=ASTAR*SQ2D2
62 P1 = ERFNWL(D11)
63 GO TO (64,70),IK
64 D01=(K-R*XQ)/SY2
IF(D01+A*SQ2D2) 65,65.66
180 65 D01=-ASTAR*SQ2D2
66 P2 = ERFNWL(D01)
IF(D01) 67,67,68
67 Z = P1 + ABS(P2)
GO TO 69
185 68 Z=P1-P2
69 P=P+MI(I,OG)*FIJ*Z*TIJ
GO TO 74
70 P2=P1
GO TO 67
190 71 P1=WNME4
IA=2
IB=2
72 IF(Q) 63,73,73
73 Z=TONME4
195 IA=3
IB=3
GO TO 69
74 CONTINUE
P=P*R*(E1-E0)/(SX2*4.)
200 75 RESULT=P
RETURN
END

```

```

SUBROUTINE CIRCV (RK, DC, KR, PV, IR)
  IR = 0
  IF (RK) 10, 20, 30
5   10 IR = 7
   15 PV = -1.
      RETURN
   20 PV = 0.
      RETURN
   40 IR = 5
10   GO TO 15
   100 PV = 1.
      RETURN

C
   30 A = RK
15   B = ABS(DC)
      C = 0.
      IF (KR) 40, 220, 50

C
   50 WLOC = A-B
20   IF (WLOC + 6.1094103) 20, 20, 90
   90 IF (WLOC - 6.5446793) 110, 100, 100
110  3A = A*B
      KFLAG = 0
      X = SIGN(1., WLOC)
25   IF (BA - 10.) 120, 120, 240

C
220  IF (A - 6.5446793) 222, 100, 100
222  IF (3) 290, 290, 224
224  OFF = A*A
30   SA = B*B
      IF (OFF/SA - OFF - 40.) 230, 230, 290
230  OFF = A/B*.5
      A = (1.-B)*OFF
      B = (1.+B)*OFF
35   KFLAG = -1
      BA = A*B
      WLOC = A-B

C
120  S = ERFD(.70710678*A)*ERFD(.70710678*B)
40   DE = A*A + B*B
      DD = (2./DE*BA)**2
      Q = ABS(WLOC*(A+B)/DE)
      Z0 = .78539816*S
      W0 = Q-Q*Z0
45   Z2NM2 = Z0
      W2NM2 = W0
      SIG0 = 0.
      IF (KFLAG) 130, 160, 160

C
50   130 C = C+1.
      IF (C-1000.) 140, 150, 150
   140 Z2NM2 = (.5/C*BA)**2 + Z2NM2
      OFF = DD*W2NM2
      W2NM2 = (OFF - .5/C*OFF) - (.4./DE*C + 1.) *Z2NM2*Q
55   SIGB = W2NM2+SIG0

```

```

        IF (ABS(W2NM2) - .0000005) 150, 150, 130
150 PV = W0 + SIG8
    RETURN
C
60 160 SIGA = 0.
    170 C = C+1.
        IF (C-1000.) 180, 200, 200
180 Z2NM2 = (8A/C*.5)**2 + Z2NM2
    SIGA = Z2NM2+SIGA
65 180 OFF = DD*W2NM2
    W2NM2 = (OFF - .5/C*OFF) - (4./DE*C + 1.) *Z2NM2*Q
    SIG8 = W2NM2 + SIG8
        IF (ABS(W2NM2) - .0000005) 190, 190, 170
190 IF (ABS(Z2NM2) - .0000005) 200,200,170
70 200 OFF = 1.-Z0-SIGA
    PV = ((W0 + SIG8)*X + OFF)*.5
    RETU= N
C
75 240 OFF = ABS(.70710678*WLOC)
    SA = ERFD(OFF)
    SB = -ERF(OFF) + 1.
    S = 4.*BA
    SQ = SQRT(2.*BA)
    SIGA = SQ*SA
80 240 W2NM2 = .25*SA/SQ
    Q = (A+B)/SQ*SB*.70710678
    SIG8 = Q
    DE = ABS((A+B)/S*WLOC)
    DD = WLOC/S*WLOC
85 250 C = C+1.
    IF (C-1000.) 260, 280, 280
260 WLOC = Q*DD
    Q = DE*W2NM2 + .5/C*WLOC - WLOC
    SIG8 = Q + SIG8
90 260 Z2NM2 = (C*2. - 1.)*W2NM2
    SIGA = Z2NM2 + SIGA
    W2NM2 = (.5+C)/(1.+C)*Z2NM2/S
    IF (ABS(Q) - .0000005) 270, 270, 250
270 IF (ABS(Z2NM2) - .0000005) 280, 280, 250
95 280 PV = ((1.-SIG8)*X - SIGA/S + 1.) * .5
    RETURN
C
100 290 OFF = .70710678*A
    SA = ERFD(OFF)
    SB = 1.-ERF(OFF)
    OFF = 1.-B*B
    WLOC = SQRT(OFF)
    W2NM2 = .35355339/A*B/WLOC
    Q = SB/WLOC
105 290 SIG8 = Q
    SA = SA*B/OFF
    SB = B*B/OFF
    OFF = SB/A**2
110 300 C = C+1.
    IF (C-1000.) 310, 320, 320

```

115

```
310 WLOC = Q*SB  
    Q = SA*W2NM2 + WLOC*.5/C - WLOC  
    SIGB = Q+SIGB  
    W2NM2 = (2.*C-1.)*W2NM2*OFF*(C+.5) / (C+1.)  
    IF (ABS(Q) - .0000005) 320, 320, 300  
320 PV = 1.-SIGB  
    RETURN  
    END
```

FUNCTION ERF(X)

DIMENSION TERA(415), TERO(451)

DATA (TERA(I), I=1,125) / 1., 13*.99999999, 6*.99999998,

5

1 5*.99999997, 3*.99999996, 2*.99999995, 2*.99999994,
2 2*.99999993, 2*.99999992, .99999991, .99999990, 2*.99999989,

10

3 .99999988, .99999987, .99999986, .99999985, .99999983,
4 .99999982, .99999981, .99999979, .99999977, .99999976,
5 .99999974, .99999972, .99999969, .99999967, .99999964,
6 .99999962, .99999959, .99999956, .99999952, .99999948,
7 .99999945, .99999940, .99999936, .99999931, .99999926,
8 .99999920, .99999914, .99999908, .99999901, .99999893,
9 .99999885, .99999877, .99999868, .99999858, .99999848,
1 .99999837, .99999825, .99999812, .99999798, .99999784,
2 .99999768, .99999751, .99999734, .99999715, .99999694,
3 .99999672, .99999649, .99999624, .99999598, .99999570,
4 .99999548, .99999517, .99999473, .99999436, .99999397,
5 .99999356, .99999311, .99999264, .99999214, .99999168,
6 .99999103, .99999042, .99998977, .99998908, .99998835,
7 .99998757, .99998674, .99998586, .99998492, .99998392,
8 .99998286, .99998173, .99998053, .99997926, .99997791,
9 .99997647, .99997495, .99997333, .99997162, .99996980,
1 .99996787, .99996582, .99996365, .99996134, .99995890 /

15

DATA (TERA(I), I=126,223) / .99995632, .99995358, .99995067,

20

1 .99994768, .99994434, .99994090, .99993725, .99993339,
2 .99992931, .99992499, .99992042, .99991559, .99991048,
3 .99990508, .99989938, .99989335, .99988699, .99988026,
4 .99987316, .99986567, .99985776, .99984941, .99984060,
5 .99983131, .99982151, .99981118, .99980029, .99978881,
6 .99977671, .99976397, .99975054, .99973648, .99972151,
7 .99970584, .99968934, .99967198, .99965371, .99963450,
8 .99961429, .99959305, .99957071, .99954724, .99952257,
9 .99949665, .99946942, .99944083, .99941080, .99937928,
1 .99934620, .99931149, .99927506, .99923686, .99919679,
2 .99915478, .99911073, .99906457, .99901620, .99896551,
3 .99891242, .99885682, .99879861, .99873766, .99867387,
4 .99860712, .99853728, .99846423, .99838783, .99830795,
5 .99822444, .99813715, .99804594, .99795065, .99785111,
6 .99774715, .99763861, .99752529, .99740702, .99728361,
7 .99715485, .99702053, .99688046, .99673441, .99658215,
8 .99642346, .99625818, .99608581, .99590635, .99571945,
9 .99552485, .99532227, .99511141, .99489200, .99466372,
1 .99442628, .99417933, .99392257, .99365565, .99337823 /

25

DATA (TERA(I), I=224,321) / .99308994, .99279043, .99247932,

30

1 .99215622, .99182075, .99147249, .99111103, .99073595,
2 .99034681, .98994316, .98952454, .98909050, .98864055,
3 .98817420, .98769094, .98719028, .98667167, .98613459,
4 .98557850, .98508283, .98440701, .98379046, .98315259,
5 .98249279, .98181044, .98110492, .98037559, .97962178,
6 .97884284, .97803809, .97720684, .97634838, .97546202,
7 .97454701, .97368263, .97262812, .97162273, .97058569,
8 .96951621, .96841350, .96727675, .96610515, .96489786,
9 .96365407, .96237290, .96105351, .95969503, .95829657,
1 .95685725, .95537618, .95385244, .95228512, .95067330,
2 .94901604, .94731240, .94556144, .94376220, .94191372,
3 .94001503, .93806516, .93606312, .93400794, .93189863,

35

40

45

50

55

4 .92973419, .92751363, .92523594, .92290013, .92050518,
 5 .91805010, .91553388, .91295551, .91031398, .90760829,
 6 .90483743, .90280048, .89909620, .89612384, .89308233,
 7 .88997867, .88678789, .88353301, .88020507, .87680310,
 8 .87332616, .86977330, .86614359, .86243611, .85864995,
 9 .85478421, .85083802, .84681050, .84270079, .83850807,
 1 .83423150, .82987029, .82542365, .82089081, .81627102 /
 DATA (TERA(I), I=322,415) / .81156356, .80676772, .80188283,
 1 .79690821, .79184325, .78668732, .78143985, .77610027,
 2 .77066806, .76514271, .75952376, .75381075, .74800328,
 3 .74210096, .73610345, .73001043, .72382161, .71753675,
 4 .71115563, .70467808, .69810394, .69143312, .68466555,
 5 .67788119, .67084006, .66378220, .65662770, .64937669,
 6 .64202933, .63458583, .62704644, .61941146, .61168127,
 7 .60385609, .59593650, .58792290, .57981581, .57161576,
 8 .56332337, .55493925, .54646410, .53789863, .52924362,
 9 .52049988, .51166826, .50274967, .49374505, .48465579,
 1 .47548172, .46622512, .45688669, .44746762, .43796909,
 2 .42839236, .41873870, .40900945, .39920598, .38932970,
 3 .37938205, .36936453, .35927865, .34912599, .33890815,
 4 .32862676, .31828350, .30788807, .29741822, .28689972,
 5 .27632639, .26570006, .25502260, .24429591, .23352192,
 6 .22270259, .21183989, .20093584, .18999246, .17901181,
 7 .16799597, .15694703, .14586711, .13475835, .12362290,
 8 .11246292, .10128059, .090078126, .078857720, .067621594,
 9 .056371978, .045111106, .033841222, .022564575, .011283416, 0. /
 DATA (TERD(I), I=1, 78) / .18113059E-08, .19816462E-08,
 1 .21676596E-08, .23706118E-08, .25920474E-08, .28136002E-08,
 2 .30970439E-08, .33843033E-08, .36974673E-08, .40388018E-08,
 3 .44107647E-08, .48160210E-08, .52574603E-08, .57382144E-08,
 4 .62616772E-08, .68315260E-08, .74517438E-08, .81266442E-08,
 5 .88608977E-08, .96595598E-08, .10528102E-07, .11472445E-07,
 6 .12498993E-07, .13614673E-07, .14826974E-07, .16143994E-07,
 7 .17574404E-07, .19127901E-07, .20814463E-07, .22645204E-07,
 8 .24632041E-07, .26787841E-07, .29126490E-07, .31662977E-07,
 9 .34413471E-07, .37395414E-07, .40627618E-07, .44138364E-07,
 1 .47925517E-07, .52036639E-07, .56489121E-07, .61310313E-07,
 2 .66529674E-07, .72178923E-07, .78292207E-07, .84906281E-07,
 3 .92060694E-07, .99797993E-07, .10814394E-06, .11720776E-06,
 4 .12698235E-06, .13754458E-06, .14955557E-06, .16128098E-06,
 5 .17459135E-06, .18896240E-06, .20447548E-06, .22121787E-06,
 6 .23928327E-06, .25877218E-06, .27979245E-06, .30245971E-06,
 7 .32689796E-06, .35324013E-06, .38162867E-06, .41221624E-06,
 8 .44516637E-06, .48065419E-06, .51486725E-06, .56000632E-06,
 9 .60428629E-06, .65193709E-06, .70320473E-06, .75835232E-06,
 1 .81766120E-06, .88143219E-06, .94998679E-06, .10236686E-05 /
 DATA (TERD(I), I=79,156) / .11028445E-05, .11879068E-05,
 1 .12792741E-05, .13773933E-05, .14827416E-05, .15958281E-05,
 2 .17171961E-05, .18474250E-05, .19971328E-05, .21369782E-05,
 3 .22976636E-05, .24699374E-05, .26545968E-05, .28524914E-05,
 4 .30645257E-05, .32916626E-05, .35349275E-05, .37954113E-05,
 5 .40742749E-05, .43727530E-05, .46921589E-05, .50338887E-05,
 6 .53994268E-05, .57903503E-05, .62083353E-05, .66551620E-05,
 7 .71327211E-05, .76430199E-05, .81881894E-05, .87704910E-05,
 8 .93923243E-05, .10056235E-04, .10764921E-04, .11521246E-04.

115

9 .12328243E-04, .13189127E-04, .14107306E-04, .15086387E-04,
 1 .16130192E-04, .17242768E-04, .18428397E-04, .19691613E-04,
 2 .21037210E-04, .22470263E-04, .23996135E-04, .25620500E-04,
 3 .27349351E-04, .29189025E-04, .31146217E-04, .33227997E-04,
 4 .35441831E-04, .37795684E-04, .40297636E-04, .42956707E-04,
 5 .45782082E-04, .48783532E-04, .51971360E-04, .55356429E-04,
 6 .58950187E-04, .62764699E-04, .66812674E-04, .71107499E-04,
 7 .75663267E-04, .80494817E-04, .85617765E-04, .91048542E-04,
 8 .96804434E-04, .10290362E-03, .10936521E-03, .11620929E-03,
 9 .12345698E-03, .13113047E-03, .13925305E-03, .14784919E-03,
 1 .15694459E-03, .16656619E-03, .17674231E-03, .18750262E-03 /

120

DATA (TERD(I), I=157,234) / .19887824E-03, .21090184E-03,

125

1 .22360761E-03, .23703144E-03, .25121089E-03, .26618533E-03,
 2 .28199597E-03, .29868598E-03, .31630853E-03, .33488688E-03,
 3 .35449449E-03, .37517508E-03, .39698274E-03, .41997400E-03,
 4 .44420794E-03, .46974632E-03, .49665360E-03, .52499713E-03,
 5 .55484722E-03, .58627725E-03, .61936378E-03, .65418671E-03,
 6 .69082932E-03, .72937850E-03, .76992476E-03, .81256247E-03,
 7 .85738992E-03, .90450949E-03, .95402778E-03, .10060558E-02,
 8 .10607090E-02, .11181075E-02, .11783764E-02, .12416455E-02,
 9 .13080500E-02, .13777304E-02, .14508325E-02, .15275078E-02,
 1 .16079137E-02, .16922136E-02, .17805771E-02, .18741800E-02,
 2 .19702048E-02, .20718409E-02, .21782842E-02, .22897383E-02,
 3 .24064136E-02, .25285285E-02, .26963889E-02, .27899886E-02,
 4 .29298098E-02, .30760230E-02, .32288871E-02, .33886708E-02,
 5 .35556487E-02, .37301092E-02, .39123473E-02, .41026681E-02,
 6 .43013869E-02, .45088292E-02, .47253306E-02, .49512374E-02,
 7 .51869067E-02, .54327069E-02, .56890172E-02, .5956287E-02,
 8 .62347440E-02, .65249776E-02, .68273562E-02, .71423198E-02,
 9 .74703176E-02, .78118164E-02, .81672930E-02, .85372378E-02,
 1 .89221551E-02, .93225623E-02, .97389910E-02, .10171986E-01 /

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DATA (TERD(I), I=235,312) / .10622108E-01, .11089930E-01,

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1 .11576041E-01, .12081043E-01, .12605554E-01, .13150207E-01,
 2 .13715650E-01, .14302545E-01, .14911571E-01, .15543422E-01,
 3 .16198806E-01, .16878448E-01, .17583888E-01, .18313482E-01,
 4 .19070402E-01, .19854636E-01, .20666985E-01, .21508278E-01,
 5 .22379324E-01, .23280999E-01, .24214158E-01, .25179685E-01,
 6 .26178475E-01, .27211441E-01, .28279510E-01, .29383624E-01,
 7 .30524740E-01, .31703831E-01, .32921881E-01, .34179892E-01,
 8 .35478877E-01, .36819865E-01, .38203897E-01, .39632028E-01,
 9 .41105318E-01, .42624854E-01, .44191723E-01, .45807027E-01,
 1 .47471879E-01, .49187401E-01, .50954726E-01, .52774996E-01,
 2 .54649361E-01, .56578979E-01, .58565016E-01, .60608644E-01,
 3 .62711040E-01, .64873398E-01, .67096878E-01, .69382697E-01,
 4 .71732046E-01, .74146103E-01, .76626882E-01, .79173173E-01,
 5 .81788571E-01, .84473470E-01, .87229059E-01, .90056524E-01,
 6 .92957846E-01, .95931799E-01, .98981951E-01, .10218866E+00,
 7 .10531307E+00, .10859632E+00, .11195954E+00, .11540383E+00,
 8 .11893829E+00, .12254088E+00, .12623402E+00, .13001348E+00,
 9 .13387915E+00, .13783227E+00, .14187374E+00, .14608451E+00,
 1 .15022558E+00, .15453761E+00, .15894171E+00, .16343862E+00 /

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DATA (TERD(I), I=313,390) / .16802916E+00, .17271408E+00,

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1 .17749413E+00, .18236999E+00, .18734232E+00, .19241173E+00,
 2 .19757880E+00, .20284406E+00, .20820799E+00, .21367101E+00,
 3 .21923353E+00, .22489587E+00, .23069833E+00, .23652112E+00,

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170 4 .24248443E+00, .24854838E+00, .25471382E+00, .26097837E+00,
5 .26734435E+00, .27381084E+00, .28037767E+00, .28704457E+00,
6 .29381124E+00, .30067728E+00, .30764223E+00, .31470557E+00,
7 .32186671E+00, .32912497E+00, .33647960E+00, .34392978E+00,
8 .35147462E+00, .35911315E+00, .36684438E+00, .37466696E+00,
9 .38257990E+00, .39058184E+00, .39867140E+00, .40684713E+00,
1 .41510750E+00, .42345088E+00, .43187557E+00, .44037979E+00,
2 .44896167E+00, .45761925E+00, .46635891E+00, .47515331E+00,
3 .48402546E+00, .49296467E+00, .50196857E+00, .51103471E+00,
175 4 .52016059E+00, .52934348E+00, .53858079E+00, .54786972E+00,
5 .55720740E+00, .56659089E+00, .57601720E+00, .58548322E+00,
6 .59498579E+00, .60452167E+00, .61408756E+00, .62368006E+00,
7 .63329574E+00, .64293107E+00, .65258247E+00, .66224628E+00,
8 .67191881E+00, .68159628E+00, .69127486E+00, .70095067E+00,
180 9 .71061978E+00, .72027819E+00, .72992188E+00, .73954676E+00,
1 .74914872E+00, .75872358E+00, .76826714E+00, .77777518E+00 /
DATA (TERD(I), I=391,451) / .78724343E+00, .79666759E+00,
1 .80604334E+00, .81536635E+00, .82463224E+00, .83383665E+00,
2 .84297518E+00, .85204344E+00, .86103703E+00, .86995155E+00,
185 3 .87878258E+00, .88752573E+00, .89617662E+00, .90473087E+00,
4 .91318411E+00, .92153281E+00, .92977025E+00, .93789454E+00,
5 .94590063E+00, .95378427E+00, .96154138E+00, .96916756E+00,
6 .97665895E+00, .98401143E+00, .99122108E+00, .99828371E+00,
7 .10051957E+01, .10119531E+01, .10185522E+01, .10249894E+01,
190 8 .10312609E+01, .10373633E+01, .10432932E+01, .10490471E+01,
9 .10546218E+01, .10600141E+01, .10652209E+01, .10702393E+01,
1 .10750662E+01, .10796989E+01, .10841348E+01, .10883712E+01,
2 .10924056E+01, .10962357E+01, .10998593E+01, .11032741E+01,
3 .11064783E+01, .11094698E+01, .11122469E+01, .11148801E+01,
195 4 .11171516E+01, .11192762E+01, .11211806E+01, .11229636E+01,
5 .11243243E+01, .11255617E+01, .11265752E+01, .11273641E+01,
6 .11279279E+01, .11282663E+01, .11283792E+01 /
ERF = 1.
ARGMX = 4.14
200 GO TO 10
ENTRY ERFD
ERF = 0.
ARGMX = 4.5
205 10 V = ABS(X)
SIGN = 1.
IF (X.LT.0.) SIGN = -1.
IF (ARGMX .LE. V) RETURN
L = 100.*V + .5
X1 = .01*FLOAT(L)
210 DX = V-X1
J = 451-L
IF (ERF) 20, 30, 20
20 ERF = (TERA(J-36) + TERD(J)*DX*(1.-DX*X1))*SIGN
RETURN
215 30 ERF = (TERD(J) * (1. - DX*(DX + 2.*X1*(1.-DX*X1) ) ) ) *SIGN
RETURN
END

```

```

SUBROUTINE P9XY (X,Y,IX,IY,ANS)
COMMON /T101/U,V,PHI,CEE,SIGMAX,SIGMAY,RKILL,SNPHI,CSPHI
R=Y
5   T1=COS(X)
    T3=SIN(X)
    A=U+R*T1*CSPHI-CEE*R*T3*SNPHI
    B=V+R T1*SNPHI+CEE*R*T3*CSPHI
    A=ABS(A)
    B=ABS(B)
10  CALL RKILL ( RKILL,SIGMAX,SIGMAY,A,B,ANS )
    ANS=R*ANS
    RETURN
END

```

```

SUBROUTINE PBXY3(X,Y,IX,IY,ANS)
COMMON /T101/U,V,PHI,CEE,SIGMAX,SIGMAY,RKILL,SNPHI,CSPHI
91  FORMAT ( 1H0,5H IR=,I7, 29H CIRC V HAS BAD INPUT VALUES )
R=Y
5   T1=COS(X)
    T3=SIN(X)
    A=U+R*T1*CSPHI-CEE*R*T3*SNPHI
    B=V+R*T1*SNPHI+CEE*R*T3*CSPHI
    O1=SQRT ( A*A+B*B )
10  CALL CIRC V ( RKILL,O1,1,ANS,IR )
    IF ( IR.NE.0 ) PRINT 91,IR
    ANS=ANS*R
    RETURN
END

```