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COMPUTATIONAL EXPERIENCE WITH OPTIMAL VALUE FUNCTION AND LAGRANGE MULTIPLIER SENSITIVITY IN NLP

by

Robert L. Armacost

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10 May 1976

The George Washington University
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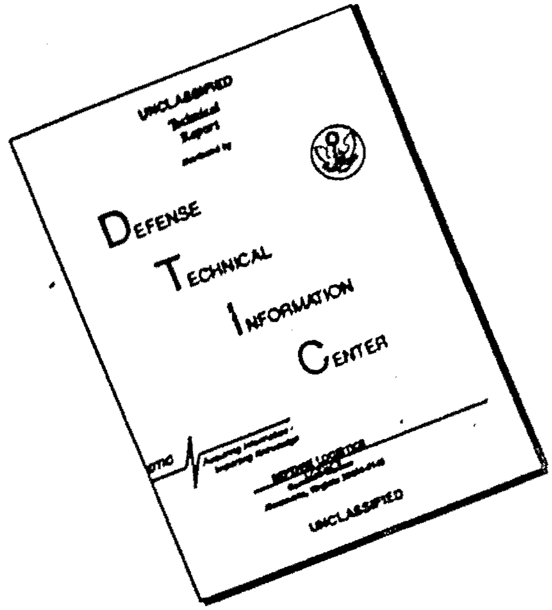

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*Presently assigned to Coast Guard Headquarters, Washington, D.C. The opinions or assertions contained herein are the private ones of the author and are not to be construed as official or reflecting the views of the Commandant or the Coast Guard at large.

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1. Introduction

Several aspects of sensitivity analysis in nonlinear programming have been examined from a computational viewpoint by Armacost and Fiacco (1974). That work was based on the theory developed by Fiacco (1973) and used the computational procedures implemented by Armacost and Mylander (1973) using the SUMT-Version 4 computer code with the logarithmic-quadratic loss penalty function modified to estimate the partial derivatives of the solution point and the objective function taken with respect to certain specified problem parameters. Fiacco (1973) developed the necessary formulas to provide estimates of the partial derivatives of the Lagrange multipliers taken with respect to the problem parameters using the logarithmic-quadratic loss penalty function. Armacost and Fiacco (1975) developed the formulas to obtain first and second order sensitivity information for the optimal value function (a function of the parameters defined by the objective function evaluated at the solution point). Additionally, Armacost and Fiacco (1976) have shown that when the parameters are the right-hand side components of the constraints, the partial derivatives of the Lagrange multipliers are the components of the Hessian of the optimal value function. The supporting theory is addressed in Section 2.

Armacost and Fiacco (1974) focus on the computational experience with four example problems discussing various aspects of the sensitivity analysis procedure and results. They note that for large problems with a large number of parameters, a very large number of partial derivatives will be estimated. This is not only time consuming, but is also burdensome to the user who must evaluate all of them, many of which are zero or very close to zero in value. In addition, it is unlikely that the value of the solution (the value of the objective function evaluated at the solution point) will be sensitive to more than a relatively few parameters. Because of this, the method developed by Armacost and Fiacco (1975) to estimate the first order sensitivity of the optimal value function is incorporated in the computer program here to provide an option for preliminary screening of the parameters to determine which ones affect the optimal value function. Using the formulas developed by Fiacco (1973), a second option is included which permits the calculation of the sensitivity estimates for the Lagrange multipliers. The computer code and options used to accomplish these calculations are discussed in Section 3.

In Section 4, the new computational experience with Lagrange multiplier and optimal value function sensitivity using these options is presented for the four examples of Armacost and Fiacco (1974).

In Section 5, a sensitivity analysis is conducted for a large scale, multi-item inventory model developed by Schrady and Choe (1971) for the U. S. Navy. While the example used is the same small one used by Schrady and Choe, it nonetheless exhibits the value of performing such a sensitivity analysis in real world situations and illustrates the care that must be taken in interpreting the sensitivity results.

2. Supporting Theory

The problems considered here are of the form of Problem $P(\epsilon)$.

$$\begin{array}{lll} \text{minimize} & f(x,\epsilon) & \\ \text{subject to} & g_i(x,\epsilon) \geq 0, \quad i=1,\ldots,m, & P(\epsilon) \\ & h_j(x,\epsilon) = 0, \quad j=1,\ldots,p. & \end{array}$$

When certain assumptions are satisfied, Fiacco (1973) and Armacost and Fiacco (1975) have shown the existence of the first order sensitivity of a Kuhn-Tucker triple and the first and second order sensitivity of the optimal value function. Additionally, they provide the means of estimating this sensitivity by way of the logarithmic-quadratic loss penalty function. The following four assumptions are sufficient to establish these results and are assumed to hold throughout this Section.

A1 --The functions defining Problem $P(\epsilon)$ are twice continuously differentiable in (x,ϵ) in a neighborhood of $(x^*,0)$.

A2 --The second order sufficient conditions for a local minimum of Problem $P(0)$ hold at x^* with associated Lagrange multipliers u^* and w^*.

A3 --The gradients $\nabla_x g_i(x^*,0)$ for all i such that $g_i(x^*,0) = 0$, and $\nabla_x h_j(x^*,0)$, $j=1,\ldots,p$ are linearly independent.

A4 --Strict complementary slackness holds at $(x^*,0)$ (i.e., $u_i^* > 0$ for all i such that $g_i(x^*,0) = 0$).

The main results are presented without proof and are stated here for completeness. The portion of the theory used in the computational algorithm is made specific. The Lagrangian for Problem $P(\epsilon)$ is

$$L(x,u,w,\epsilon) = f(x,\epsilon) - \sum_{i=1}^{m} u_i g_i(x,\epsilon) + \sum_{j=1}^{p} w_j h_j(x,\epsilon)$$

where u_i, $i=1,\ldots m$ and w_j, $j=1,\ldots,p$ are the Lagrange multipliers associated with the inequality and equality constraints respectively.

The first result was proved as Theorem 2.1 by Fiacco (1973).

THEOREM 1: (First order sensitivity of a Kuhn-Tucker triple)

If assumptions A1, A2, A3 and A4 hold for Problem $P(\epsilon)$ at $(x^*, 0)$, then

(a) x^* is a local isolated minimizing point of Problem $P(0)$ and the associated Lagrange multipliers u^* and w^* are unique;

(b) for ϵ in a neighborhood of 0, there exists a unique, once continuously differentiable vector function $y(\epsilon) = (x(\epsilon), u(\epsilon), w(\epsilon))^T$ satisfying the second order sufficient conditions for a local minimum of Problem $P(\epsilon)$ such that $(x(0), u(0), w(0)) = (x^*, u^*, w^*)$ and hence, $x(\epsilon)$ is a locally unique, local minimum of Problem $P(\epsilon)$ with associated unique Lagrange multipliers $u(\epsilon)$ and $x(\epsilon)$; and

(c) for ϵ near 0, the set of binding inequalities is unchanged, strict complementary slackness holds for $u_i(\epsilon)$ for i such that $g_i(x(\epsilon), \epsilon) = 0$, and the binding constraint gradients are linearly independent at $x(\epsilon)$.

Let $y(\epsilon) = (x(\epsilon), u(\epsilon), w(\epsilon))^T$ be a Kuhn-Tucker triple where $x(\epsilon)$ solves Problem $P(\epsilon)$, then the optimal value function is defined as $f^*(\epsilon) = f(x(\epsilon), \epsilon)$ and the optimal value Lagrangian is $L^*(\epsilon) = L(x(\epsilon), u(\epsilon), w(\epsilon), \epsilon)$.

The second result was recently established by Armacost and Fiacco (197[illegible]) in their Theorem [illegible], stated here as Theorem 2.

THEOREM 2: (First and second order changes in the optimal value function)

If assumptions A1, A2, A3 and A4 hold for Problem $P(\epsilon)$ at $(x^*, 0)$, then for ϵ near 0,

(a) $f^*(\epsilon) = L^*(\epsilon)$;

(b) $\nabla_\epsilon f^*(\epsilon) = \nabla_\epsilon L(x, u, w, \epsilon)\Big|_{(x,u,w)=(x(\epsilon), u(\epsilon), w(\epsilon))}$

$$= \nabla_\epsilon f(x,\epsilon) - \sum_{i=1}^{m} u_i \nabla_\epsilon g_i(x,\epsilon) + \sum_{j=1}^{p} w_j \nabla_\epsilon h_j(x,\epsilon) \Big|_{(x,u,w)=(x(\epsilon),u(\epsilon),w(\epsilon))} ;$$

(c) $\nabla_\epsilon^2 f^*(\epsilon) = \nabla_\epsilon(\nabla_\epsilon L(x(\epsilon),u(\epsilon),w(\epsilon),\epsilon)^T)$.

The problems in subsequent Sections are solved using the logarithmic-quadratic loss penalty function $W(x,\epsilon,r)$ defined as

$$W(x,\epsilon,r) \equiv f(x,\epsilon) - r\sum_{i=1}^{m} \ln g_i(x,\epsilon) + (1/2r)\sum_{j=1}^{p} h_j(x,\epsilon)^2. \quad (1)$$

The following result was obtained by Fiacco (1973) as Theorem 3.1.

THEOREM 3: (Approximation of first order sensitivity results and determination of estimates from $W(x,\epsilon,r)$)

If assumptions A1, A2, A3 and A4 hold for Problem $P(\epsilon)$, then for (ϵ,r) near $(0,0)$, there exists a locally unique, once continuously differentiable vector function $y(\epsilon,r) = (x(\epsilon,r),u(\epsilon,r),w(\epsilon,r))^T$ satisfying

$$\nabla_x L(x,u,w,\epsilon) = 0,$$
$$u_i g_i(x,\epsilon) = r, \quad i=1,\ldots,m,$$
$$h_j(x,\epsilon) = w_j r, \quad j=1,\ldots,p,$$

with $(x(0,0),u(0,0),w(0,0)) = (x^*,u^*,w^*)$, and such that for any (ϵ,r) near $(0,0)$ and $r > 0$, $x(\epsilon,r)$ is a locally unique unconstrained minimizing point of $W(x,\epsilon,r)$, with $g_i(x(\epsilon,r),\epsilon) > 0$, $i=1,\ldots,m$, and $\nabla_x^2 W(x(\epsilon,r),\epsilon,r)$ is positive definite.

Since the system of equations in Theorem 3 is identically equal to zero at $r = 0$, it follows that $\nabla_\epsilon y(\epsilon,r)$ can be calculated for (ϵ,r) near $(0,0)$. The following result was shown by Fiacco (1973) following his Theorem 3.1.

COROLLARY 3.1: (Convergence of estimates using $W(x,\epsilon,r)$)

If assumptions A1, A2, A3 and A4 hold for Problem $P(\epsilon)$, then for any ϵ

near 0,

(a) $\lim_{r \to 0^+} y(\epsilon,r) = y(\epsilon,0) = y(\epsilon)$, the Kuhn-Tucker triple characterized in Theorem 1; and

(b) $\lim_{r \to 0^+} \nabla_\epsilon y(\epsilon,r) = \nabla_\epsilon y(\epsilon,0) = \nabla_\epsilon y(\epsilon)$.

Armacost and Fiacco (1974) reported computational experience with sensitivity analysis in four sample nonlinear programming problems. The algorithm uses the fact that the Hessian of the penalty function is positive definite for r small enough and that the gradient of the penalty function is identically zero in a neighborhood of the solution point. Thus, the gradient of the solution point taken with respect to the parameter vector ϵ is estimated as

$$\nabla_\epsilon x(\epsilon,r) = -\nabla_x{}^2 W(x,\epsilon,r)^{-1} \nabla_{\epsilon x}{}^2 W(x,\epsilon,r)\Big|_{x=x(\epsilon,r)}. \qquad (2)$$

Using the fact that $u_i(\epsilon,r) = r/g_i(x(\epsilon,r),\epsilon)$ and $w_j(\epsilon,r) = (1/r)h_j(x(\epsilon,r),\epsilon)$, the chain rule can be applied to obtain $\nabla_\epsilon u_i(\epsilon,r)$ and $\nabla_\epsilon w_j(\epsilon,r)$ as shown by Fiacco (1973). Convergence was shown by Fiacco (1973) following his Corollary 3.1. The above approach is equivalent to calculating $\nabla_\epsilon y(\epsilon,r)$ directly from the system of equations in Theorem 3.

The logarithmic-quadratic loss penalty function can also be used to provide estimates of the first and second order sensitivity of the optimal value function. Let the optimal value penalty function be defined as $W^*(\epsilon,r) = W(x(\epsilon,r),\epsilon,r)$. The first order portion of the sensitivity results developed by Armacost and Fiacco (1975) in their Theorem 4 and Corollary 4.1 follow.

THEOREM 4: (First order sensitivity of $W^*(\epsilon,r)$ and estimates for $f^*(\epsilon)$) If assumptions A1, A2, A3 and A4 hold for Problem $P(\epsilon)$, then for (ϵ,r) near $(0,0)$ and $r > 0$,

(a) $\lim_{r \to 0^+} W^*(\epsilon,r) = L^*(\epsilon) = f^*(\epsilon)$;

(b) $\nabla_\epsilon W^*(\epsilon,r) = \nabla_\epsilon L(x,u,w,\epsilon)\Big|_{(x,u,w)=(x(\epsilon,r),u(\epsilon,r),w(\epsilon,r))}$; and (3)

(c) $\lim_{r \to 0^+} \nabla_\epsilon W^*(\epsilon,r) = \nabla_\epsilon L(x(\epsilon),u(\epsilon),w(\epsilon),\epsilon) = f^*(\epsilon)$.

Another estimate of the optimal value function is obtained as $f^{\#}(\epsilon,r) \equiv f(x(\epsilon,r),\epsilon)$. Direct application of the chain rule for differentiation then yields an estimate of the first order sensitivity of the optimal value function as

$$\nabla_\epsilon f^{\#}(\epsilon,r) = \nabla_x f(x,\epsilon)\, \nabla_\epsilon x(\epsilon,r) + \nabla_\epsilon f(x,\epsilon). \tag{4}$$

Under the given assumptions, continuity assures that $f^{\#}(\epsilon,r) \to f^*(\epsilon)$ and $\nabla_\epsilon f^{\#}(\epsilon,r) \to \nabla_\epsilon f^*(\epsilon)$ as $r \to 0^+$. Thus, both $\nabla_\epsilon f^{\#}(\epsilon,r)$ and $\nabla_\epsilon W^*(\epsilon,r)$ are estimates of $\nabla_\epsilon f^*(\epsilon)$ for r sufficiently small. It is beyond the scope of this Section to explore the relationship between these estimates. It is easily shown, however, that

$$\nabla_\epsilon W^*(\epsilon,r) = \nabla_\epsilon f^{\#}(\epsilon,r) - \sum_{i=1}^{m} u_i(\nabla_x g_i \nabla_\epsilon x(\epsilon,r) + \nabla_\epsilon g_i) + \sum_{j=1}^{p} w_j(\nabla_x h_j \nabla_\epsilon x(\epsilon,r) + \nabla_\epsilon h_j)\Big|_{x=x(\epsilon,r)} .$$

It is easily shown that the terms in the summations on the right approach zero as r approaches zero. Armacost and Fiacco (1974) used equations (2) and (4) to examine the trajectory and convergence properties of the gradients of the solution point and the optimal value function from a computational point of view. Here, equation (3) is also used to estimate the first order sensitivity of the optimal value function and has the advantage that $\nabla_\epsilon x(\epsilon,r)$ need not be calculated.

3. User Options and Computer Codes

The basic SUMT-Version 4 computer program and instructions for its use are described in Mylander, Holmes and McCormick (1971). The basic sensitivity analysis subroutines, user instructions, and instructions for integrating the sensitivity package with the SUMT-Version 4 code are described in Armacost and Mylander (1973). Briefly, the conduct of a sensitivity analysis is controlled by the variable NEXOP3 which is given a value on the "Second Option Card" in the SUMT input data deck. There are four choices: no sensitivity analysis, a sensitivity analysis at the final subproblem, a sensitivity analysis at each subproblem along the penalty function minimizing trajectory, or a sensitivity analysis at the final subproblem for a range of differencing increments. In conjunction with this option, two additional options are added here and come into play whenever a sensitivity analysis is conducted. The first option (technically Option 4) is controlled by the variable NEXOP4 and determines whether the partial derivatives of the Lagrange multipliers will be calculated. When the calculation is done, the formulas described by Fiacco (1973) are used. The second option added here (Option 5) permits a screening of the parameters to reduce the number of partial derivatives which are estimated by limiting further analysis to those parameters which will affect the optimal value of the objective function by an amount exceeding 0.1 percent of its current value. This option is controlled by the variable NEXOP5. The estimate of sensitivity of the optimal value function with respect to a particular parameter under this option is calculated using the Armacost and Fiacco (1975) result which involves the partial derivative of the Lagrangian taken with respect to the parameter under consideration. Subroutines LMULT and

PRESEN and related coding in Subroutine SENS implement Option 4 and Option 5, respectively. Subroutines SENS, LMULT and PRESEN are listed in Appendix A. Specific instructions for using these two options in conjunction with the "Second Option Card" are given below in Table 1. This information should be added to Table 5 in Mylander, Holmes and McCormick (1971).

TABLE 1

THE SECOND OPTION CARD

Option	Column	Value	Meaning
4	28	=0	Do not estimate the partial derivatives of the estimates of the Lagrange multipliers.
		=1	Estimate the partial derivatives of the estimates of the Lagrange multipliers whenever a sensitivity analysis of the solution point is conducted.
5	35	=0	Estimate the partial derivatives of the optimal value function and eliminate those parameters which do not affect the optimal value function from subsequent sensitivity calculations.
		=1	Estimate the partial derivatives of the optimal value function with respect to all parameters, but continue all subsequent sensitivity calculations with respect to all parameters.
		=2	Do not estimate the partial derivatives of the optimal value function first. Conduct the sensitivity analysis with respect to all parameters.

A potential user of these sensitivity subroutines should be aware that the penalty function coded in SUMT-Version 4 does not have the factor "½" in the quadratic loss term (see equation (1) in Section 2). Therefore, the expressions for the several gradients have an additional factor of "2" in the computer program which does not appear in the supporting theory of Section 2.

4. New Computational Experience

In this Section, the four sample problems of Armacost and Fiacco (1974) are examined. Specifically, the convergence of the partial derivatives of the Lagrange multipliers is examined and the estimates of the gradient of the optimal value function obtained by the chain rule (equation (4)) and by the gradient of the Lagrangian (equation (3)) are compared. The problems are designated by the same letters as in the original paper.

Consider first a simple convex program. The problem is

$$\begin{aligned} &\text{minimize} && f(x,\epsilon) = x_1 + \epsilon_2 x_2 \\ &\text{subject to} && g_1(x,\epsilon) = \epsilon_1^2 - x_1^2 - x_2^2 \geq 0, \end{aligned} \qquad \text{B}$$

for $\epsilon_1 > 0$. The analytical solution point and its gradient are given in Armacost and Fiacco (1974) as

$$x(\epsilon) = \begin{bmatrix} x_1(\epsilon) \\ x_2(\epsilon) \end{bmatrix} = \begin{bmatrix} -\dfrac{\epsilon_1}{\sqrt{1+\epsilon_2^2}} \\ -\dfrac{\epsilon_1 \epsilon_2}{\sqrt{1+\epsilon_2^2}} \end{bmatrix} ,$$

$$f^*(\epsilon) = -\epsilon_1 \sqrt{1+\epsilon_2^2} .$$

$$\nabla_\epsilon x(\epsilon) = \frac{1}{\sqrt{1+\epsilon_2^2}} \begin{bmatrix} -1 & \dfrac{\epsilon_1\epsilon_2}{(1+\epsilon_2^2)} \\ -\epsilon_2 & \dfrac{-\epsilon_1}{(1+\epsilon_2^2)} \end{bmatrix},$$

and

$$\nabla_\epsilon f^*(\epsilon) = (\partial f^*(\epsilon)/\partial\epsilon_1,\ \partial f^*(\epsilon)/\partial\epsilon_2)$$

$$= \left(-\sqrt{1+\epsilon_2^2},\ -\epsilon_1\epsilon_2/\sqrt{1+\epsilon_2^2}\right).$$

The Lagrange multiplier and its gradient are analytically determined to be

$$u^*(\epsilon) = \sqrt{1+\epsilon_2^2}\,/\,2\epsilon_1,$$

and

$$\nabla_\epsilon u^*(\epsilon) = \left(-\sqrt{1+\epsilon_2^2}\,/2\epsilon_1^2,\ \epsilon_2/(2\epsilon_1\sqrt{1+\epsilon_2^2})\right).$$

The numerical example had $\epsilon_1 = 2$ and $\epsilon_2 = 1$ yielding the following numerical results:

$$f^* = -2\sqrt{2}, \qquad \nabla_\epsilon f^* = (-\sqrt{2}, -\sqrt{2}),$$

$$x^* = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{bmatrix}, \qquad \nabla_\epsilon x^* = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix},$$

$$u^* = \sqrt{2}/4 \approx 0.353, \qquad \nabla_\epsilon u^* = (-\sqrt{2}/8, \sqrt{2}/8) \approx (-.177, .177).$$

The numerical results obtained by the computer program are included in Table 2 for the optimal value function and Lagrange multiplier sensitivity. The values of the first order optimal value function sensitivity computed both by the chain rule (equation (4)) and by taking partial derivatives of the Lagrangian with respect to the parameters (equation (3))

TABLE 2

TRAJECTORY RESULTS FOR PROBLEM B

Subproblem	f	Lagrangian $\partial f/\partial \epsilon_1$	Lagrangian $\partial f/\partial \epsilon_2$	Chain rule $\partial f/\partial \epsilon_1$	Chain rule $\partial f/\partial \epsilon_2$	u	$\partial u/\partial \epsilon_1$	$\partial u/\partial \epsilon_2$
1	-1.9999	-1.9999	- .9999	-1.3333	-1.3333	.4999	-.3333	.1666
2	-2.5393	-1.5440	-1.2947	-1.4087	-1.4088	.3860	-.2100	.1761
3	-2.7765	-1.4439	-1.3833	-1.4139	-1.4139	.3609	-.1844	.1767
4	-2.8128	-1.4243	-1.4064	-1.4142	-1.4142	.3560	-.1790	.1768
5	-2.8245	-1.4127	-1.4123	-1.4142	-1.4142	.3531	-.1768	.1768
6	-2.8274	-1.4137	-1.4137	-1.4142	-1.4142	.3532	-.1767	.1768
7	-2.8282	-1.3899	-1.4141	-1.4142	-1.4142	.3475	-.1737	.1768
Analytical	-2.8282	-1.4142	-1.4142	-1.4142	-1.4142	.3537	-.1769	.1768

are presented in parallel. The results are also plotted in Figure 1 and portray the type of convergence experienced. While the previous results by Armacost and Fiacco (1974) clearly indicated a stability of the solution point and optimal value function and their gradients taken with respect to the parameters, Table 2 indicates that with the Lagrange multipliers, the same sort of stability is not found. It is well known that with barrier functions, the estimates of the Lagrange multipliers decrease in accuracy as the boundary is approached. The change in the value of u between subproblems 6 and 7 is an indication of this. It is no surprise, therefore, that the estimates of the gradient of the Lagrange multiplier behave in

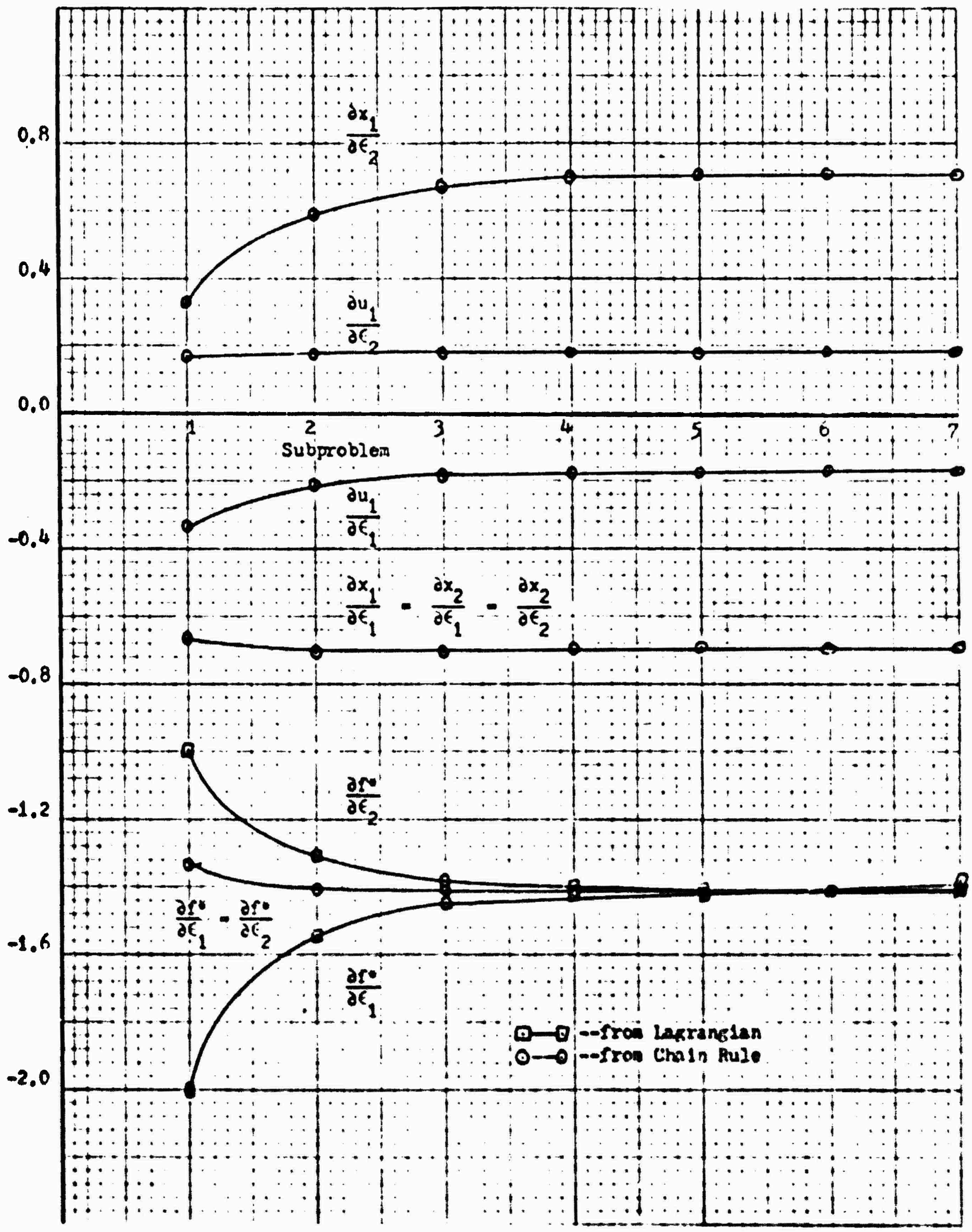

Fig. 1.--Partial derivatives for Problem 4.

a similar way. In addition, since the estimate of the gradient of the optimal value function obtained by evaluating the partial derivatives of the Lagrangian taken with respect to the parameters includes the estimates of the Lagrange multipliers, it too will not be as accurate an estimate as the boundary is approached.

The next problem considered is a nonconvex program with an equality constraint. The problem is to

$$
\begin{aligned}
\text{minimize} \quad & f(x,\epsilon) = x_1 + x_2 + \ln x_3 - x_4 \\
\text{subject to} \quad & g_1(x,\epsilon) = -x_1^2 + x_2 \geq 0, \\
& g_2(x,\epsilon) = x_1 \geq 0, \\
& g_3(x,\epsilon) = x_3 - \epsilon_1 \geq 0, \\
& h_1(x,\epsilon) = x_3^2 + x_4^2 - \epsilon_2^2 = 0,
\end{aligned}
\qquad \text{C}
$$

where $\epsilon_2 \geq \epsilon_1 \geq 0$ and $\epsilon_2 > 0$.

The analytical solution is:

$$f^*(\epsilon) = \ln \epsilon_1 - \sqrt{\epsilon_2^2 - \epsilon_1^2} \quad ,$$

$$x_1^*(\epsilon) = x_2^*(\epsilon) = 0, \quad x_3^*(\epsilon) = 1, \quad x_4^*(\epsilon) = \sqrt{\epsilon_2^2 - \epsilon_1^2},$$

$$u_1^*(\epsilon) = u_2^*(\epsilon) = 1,$$

$$u_3^*(\epsilon) = 1/\epsilon_1 + \epsilon_1 / \sqrt{\epsilon_2^2 - \epsilon_1^2} \, ,$$

and

$$w_1^*(\epsilon) = 1/(2\sqrt{\epsilon_2^2 - \epsilon_1^1} \, .$$

The numerical example used has $\epsilon_1 = 1$ and $\epsilon_2 = 2$. The numerical solution derived analytically for the solution point, Lagrange multipliers and their gradients are:

$$f^* \approx -1.732 \quad , \qquad \nabla_\epsilon f^* \approx (1.578, -1.157) \quad ,$$

$$x^* \approx \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1.732 \end{bmatrix} , \qquad \nabla_\epsilon x^* \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -.577 & 1.154 \end{bmatrix} ,$$

$$u^* \approx \begin{bmatrix} 1 \\ 1 \\ 1.732 \end{bmatrix}, \qquad \nabla_\epsilon u^* \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -.231 & -.385 \end{bmatrix},$$

$$w^* \approx .289, \qquad \nabla_\epsilon w^* \approx (.0962, -.1925).$$

The computed results for a trajectory sensitivity analysis are shown in Table 3.

The partial derivatives of the Lagrange multipliers shown in Table 3 are the only ones which are non-zero. Note that at the fifth and sixth subproblems, the estimates of the gradients of u_3 and w_1 are reasonably close to the true values determined analytically. It is at that point also that the estimates of u_3 and w_1 are the closest to their true values at the solution point. Notice also that the estimate of $\partial f^*/\partial\epsilon_2$ obtained from the partial derivative of the Lagrangian with respect to ϵ_2 is reasonably close to the true value. It is entirely dependent on w_1 and as the estimate for w_1 becomes less accurate, the error will be reflected in the first order sensitivity estimate of the optimal value function taken with respect to ϵ_2. In the following example, the need for careful attention to the differencing increment is illustrated when the parameters are the right-hand sides of the constraints. For Problem C, a sensitivity analysis was conducted at the final subproblem for a range of differencing increments. The results were that the sensitivity estimates remained fairly constant over the range of differencing increments from 10^{-7} to 10^{-12}. Thus, the source of error in this case is solely the lack of accuracy of the estimates of the Lagrange multipliers for the binding constraints.

Two related problems called the Shell Primal and the Shell Dual were presented by Armacost and Fiacco (1974). However, computational

TABLE 3

TRAJECTORY RESULTS FOR PROBLEM C

Subproblem	f	Lagrangian $\partial f/\partial \epsilon_1$	Lagrangian $\partial f/\partial \epsilon_2$	Chain rule $\partial f/\partial \epsilon_1$	Chain rule $\partial f/\partial \epsilon_2$	u_1	u_2	u_3	w_1	$\partial u_3/\partial \epsilon_1$	$\partial u_3/\partial \epsilon_2$	$\partial w_1/\partial \epsilon_1$	$\partial w_1/\partial \epsilon_2$
1	.857	1.808	-1.500	1.252	-.948	.999	1.999	1.808	.375	.931	-.891	.222	-.316
2	-1.027	1.568	-1.218	1.538	-1.128	1.000	1.366	1.568	.304	.164	-.512	.128	-.217
3	-1.550	1.571	-1.172	1.573	-1.150	.999	1.123	1.570	.292	-.123	-.416	.104	-.198
4	-1.686	1.575	-1.158	1.576	-1.154	1.001	1.030	1.575	.289	-.203	-.392	.098	-.193
5	-1.720	1.574	-1.161	1.577	-1.154	.996	1.007	1.574	.290	-.219	-.388	.097	-.193
6	-1.729	1.576	-1.168	1.577	-1.154	.996	1.000	1.576	.292	-.219	-.390	.097	-.194
7	-1.731	1.585	-1.109	1.577	-1.154	1.000	1.006	1.585	.277	-.258	-.369	.092	-.184
8	-1.732	1.581	-1.045	1.577	-1.154	1.001	1.002	1.581	.261	-.303	-.348	.087	-.174
Analytical	-1.732	1.578	-1.157	1.578	-1.157	1.000	1.000	1.578	.289	-.231	-.385	.096	-.192

results were presented only for the Shell Dual. (The Shell Dual was developed as a test problem by the Shell Development Company and used by Colville (1968) in his comparative analysis of nonlinear programming codes.) Computational results are presented below for both the Shell Primal and the Shell Dual. The first problem considered is the Shell Primal.

$$\text{minimize} \quad f(x,\epsilon) = \sum_{j=1}^{n} e_j x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_{ij} x_j + \sum_{j=1}^{n} d_j x_j^3$$

$$\text{subject to} \quad g_i(x,\epsilon) = \sum_{j=1}^{n} a_{ij} x_j - \epsilon_i \geq 0, \; i=1,\ldots,m, \qquad \text{D}$$

with $x_j \geq 0$, $j=1,\ldots,n$. The dual problem is much more difficult to solve and is the one most often used in computational comparisons. The Shell Dual is to

$$\text{maximize} \quad f(x,\epsilon) = \sum_{j=1}^{m} \epsilon_j y_j - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_{ij} x_j - 2 \sum_{i=1}^{n} d_i x_i^3$$

$$\text{subject to} \quad g_i(x,\epsilon) = e_i + 2 \sum_{j=1}^{n} c_{ji} x_j + 3 d_i x_i^2 - \sum_{j=1}^{m} a_{ji} y_j \geq 0, \; i=1,\ldots,n, \qquad \text{E}$$

with $x_i \geq 0$, $i=1,\ldots,n$, and $y_j \geq 0$, $j=1,\ldots,m$. In the numerical example used here, $n = 5$ and $m = 10$. The problem data is given in Table 3 of Armacost and Fiacco (1974) and in Appendix C. The parameters of the sensitivity analysis are the variables ϵ_i, $i=1,\ldots,10$, the components of the right-hand sides of the primal constraints.

As a brief aside, the computational solution of the Shell Dual provided the motivation for some of the recent work in parametric sensitivity analysis by Armacost and Fiacco. Specifically, Armacost and Fiacco (1974) noted that in solving the dual problem, the partial derivatives of the dual variables with respect to the right-hand sides of the primal constraints were obtained. With the correspondence between the dual variables and Lagrange multipliers and their interpretation as the partial derivatives of the optimal value function with respect to the right-hand sides of the primal constraints, it appeared that the second order partial derivatives of the optimal value function had been obtained. The calculations supported this conjecture since the matrix of partial derivatives of the dual variables with respect to the parameters was symmetric. Armacost and Fiacco (1976) have shown that when the parameters are the components of the right-hand side only, then the gradient of the Lagrange multiplier vector taken with respect to the parameters is the Hessian of the optimal value function. This matrix will be computed using the Shell Primal and then compared with the Hessian obtained by solving the Shell Dual.

In solving all of the sample problems, the option to screen the sensitivity estimates was used resulting in the partial derivatives being computed only for those parameters which affected the optimal value function by more than 0.1 percent of its current value. Annotated computer output for the final subproblem with sensitivity analysis data for the Shell Primal is shown in Figure 2. (The annotation applies to the computer output in Figures 3, 4, 5 and 6 as well.) The parameters are represented by the letter "A" vice "[illegible]" in the computer output. Similar output for the Shell Dual is shown in Figure [illegible]. (Compare the sensitivity analysis portion with Figure 4 of Armacost and Fiacco (1974).) The sensitivity estimates for both problems were obtained by conducting a trajectory sensitivity analysis, i.e., a sensitivity analysis performed at each subproblem along the minimizing trajectory. Since the Shell Dual is a maximization problem and SUMT is coded to solve a minimization problem, Problem [illegible] is solved

```
(1) ****************************************
        POINT=  43         DOTT=  0.11935440D-07      RHO=  0.61035160D-04      MAGNITUDE=  0.32240600 00      PHASE= 2
      F= -0.32348440D 02     P= -0.32340270D 02     G= -0.32349350 02     RSIGMA=  0.21635910D-02     H=  0.0
    THE CURRENT VALUE OF X IS
(2)  0.29999420 00       0.33344250 00       0.39999580 00       0.42835650 00       0.22405810 00
    THE CONSTRAINT VALUES
                         NOT INCLUDING THE NON-NEGATIVITIES
(3)  0.36295330 02       0.19525740 01       0.11017230-04       0.13956310 01       0.19975330-04       0.51641090-08
     0.38314150 02       0.56752360 02       0.58324370-03       0.68584710 00

   2ND  ORDER ESTIMATES
      F= -0.32348680 02      P= -0.32348320 02     G= -0.32348680D 02     RSIGMA=  0.0                   H=  0.0
    THE CURRENT VALUE OF X IS
(4)  0.30000000 00       0.33346770 00       0.40000000 00       0.42831030 00       0.22396480 00
    THE CONSTRAINT VALUES
                         NOT INCLUDING THE NON-NEGATIVITIES
     0.36295250 02       0.19523180 01       0.56718740-07       0.13958590 01       0.13176520-06       0.23446580-07
     0.38314260 02       0.56752480 02       0.73021380-06       0.68574280 00

   1ST  ORDER ESTIMATES
      F= -0.32348680D 02     P= -0.32348230 02     G= -0.32348680D 02     RSIGMA=  0.0                   H=  0.0
    THE CURRENT VALUE OF X IS
(5)  0.30000000 00       0.33346730 00       0.40000000 00       0.42831090 00       0.22396610 00
    THE CONSTRAINT VALUES
                         NOT INCLUDING THE NON-NEGATIVITIES
     0.36295250 02       0.19523220 01       0.48237230-07       0.13958560 01       0.80058370-07       0.20849710-07
     0.38314260 02       0.56752480 02       0.85756450-05       0.68574430 00

  LAGRANGE MULTIPLIERS
      F= -0.32348680D 02     P= -0.32348230 02    ,G= -0.32348680D 02     RSIGMA=  0.0                   H=  0.0
    THE CURRENT VALUE OF X IS
     0.20345440-03       0.18304550-03       0.15258950-03       0.14240680-03       0.27240770-03
    THE CONSTRAINT VALUES
                         NOT INCLUDING THE NON-NEGATIVITIES
(6)  0.16416250-05       0.31259820-04       0.51649300 01       0.43733020-04       0.30855280 01       0.11019110 02
     0.15930190-05       0.10754650-05       0.18464780 00       0.08992340-04

                         SENSITIVITY ANALYSIS

    THE VALUE OF R(RHO) IS 0.610350-04

     THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILLBE MADE IS
  X( 1)= 0.2999942       X( 2)= 0.3334425       X( 3)= 0.3999950       X( 4)= 0.4283565       X( 5)= 0.2240581       X(

     PARAMETER VALUE    DIFFERENCING INTERVAL
  1     -40.0000            0.100000-10
  2     -2.60000            0.100000-10
  3    -0.250000            0.100000-10
  4     -4.00000            0.100000-10
  5     -4.00000            0.100000-10       (8)
(7) 6   -1.00000            0.100000-10
  7     -40.0000            0.100000-10
  8     -60.0000            0.100000-10
  9      5.00000            0.100000-10
 10      1.00000            0.100000-10
```

Fig. 2.—Shell Primal subproblem and sensitivity output.

```
                         OPTIMAL VALUE FUNCTION SENSITIVITY

(9)  DF/DA( 1)= 0.1681470D-05 DF/DA( 2)= 0.3125846D-04 DF/DA( 3)=  5.164927     DF/DA( 4)= 0.4373254D-04 DF/DA( 5)=  3.055494
     DF/DA( 6)=  11.01909     DF/DA( 7)= 0.1592877D-05 DF/DA( 8)= 0.1075369D-05 DF/DA( 9)= 0.1043478     DF/DA(10)= 0.6899231D-04

     DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS  ——(10)
      3 , 5 , 6 , 9 ,

(11)  X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 3
     DX( 1)=-0.3999950     DX( 2)= 0.9733741D-01 DX( 3)=-0.1999975     DX( 4)= 0.2459346     DX( 5)=-0.6789674D-01 DX(

      U-DERIVATIVES WITH RESPECT TO PARAMETER 3
     DU( 1)=-0.3157874D-05 DU( 2)= 0.3459464D-05 DU( 3)=  4.071089     DU( 4)= 0.3981247D-04 DU( 5)=-0.5663126     DU( 6)=  3.471514
     DU( 7)=-0.1183379D-07 DU( 8)=-0.5719452D-08 DU( 9)= 0.1899914     DU(10)= 0.3693072D-04 DU(

           DF(X(R))/DA=    5.17404
     *********

      X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 5
     DX( 1)=-0.5723791D-06 DX( 2)=-0.1471101     DX( 3)=-0.3533089D-06 DX( 4)=-0.4917181D-01 DX( 5)= 0.9815517D-01 DX(

      U-DERIVATIVES WITH RESPECT TO PARAMETER 5
(12) DU( 1)= 0.1539952D-07 DU( 2)=-0.7536343D-05 DU( 3)=-0.5662908     DU( 4)=-0.1230715D-04 DU( 5)= 0.5051511     DU( 6)=-0.6158383
     DU( 7)=-0.4675935D-08 DU( 8)=-0.5579114D-08 DU( 9)= 0.2392121D-01 DU(10)= 0.1273260D-04 DU(

           DF(X(R))/DA=    3.06108
     *********

      X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 6
     DX( 1)=-0.1999961     DX( 2)= 0.9443110D-01 DX( 3)=-0.3499971     DX( 4)= 0.2281358     DX( 5)= 0.2794506D-01 DX(

      U-DERIVATIVES WITH RESPECT TO PARAMETER 6
     DU( 1)=-0.1675789D-06 DU( 2)= 0.6678611D-06 DU( 3)=  3.471577     DU( 4)= 0.3538894D-04 DU( 5)=-0.6158663     DU( 6)=  7.192956
     DU( 7)=-0.8293993D-08 DU( 8)=-0.1093249D-07 DU( 9)=  1.589096     DU(10)= 0.2588376D-04 DU(

(13)——     DF(X(R))/DA=    11.0395
     *********

      X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 9
     DX( 1)= 0.3127272D-[illegible] DX( 2)=-0.3957413D-01 DX( 3)= 0.3299407D-06 DX( 4)= 0.7578708D-01 DX( 5)= 0.1542692     DX(

      U-DERIVATIVES WITH RESPECT TO PARAMETER 9
     DU( 1)= 0.1559612D-09 DU( 2)=-0.6691631D-05 DU( 3)= 0.1099777     DU( 4)= 0.1185332D-04 DU( 5)= 0.2391431D-01 DU( 6)=  1.589064
     DU( 7)= 0.7919859D-09 DU( 8)= 0.4295927D-08 DU( 9)= 0.8347795     DU(10)=-0.2471617D-04 DU(

           DF(X(R))/DA=    0.103099
     *********
```

Fig. 2.--Continued.

Annotation

Identifier	Meaning
1	F = the value of the objective function at the current solution point.
2	The value of the components of the solution point of the current subproblem, here, $x_1,\ldots,x_5$.
3	The value of the constraints evaluated at the current solution point, i.e., $g_1,\ldots,g_{10}$.
4	The data corresponding to 1 - 3 above when the solution point is a second order estimate based on the values in 2.
5	The same as 4, but the solution point is a first order estimate based on the values in 2. These values and those in 4 are extrapolations from the current solution estimate.
6	The estimates of the Lagrange multipliers based on the current value of r (RHO) and the current estimates of the solution point, in this case as $u_i = r/g_i(x,\epsilon)$, i=1,...,10.
7	The value of the parameters, ϵ_i, i=1,...,10.
8	The value of the differencing increment used in the central differencing formula for each of the parameters.
9	The estimates of the gradient of the optimal value function calculated by equation (3).
10	The parameters whose associated partial derivatives affect the optimal value function by more than 0.001 of its current value.
11	The first order sensitivity of the solution point calculated by equation (2).
12	The first order sensitivity of the Lagrange multipliers calculated using the method described following equation (2).
13	The partial derivative of the optimal value function taken with respect to the indicated parameter and calculated by equation (4).

Fig. 2.--Continued.

```
******************************************
      POINT=  65       DOTT=  0.4411257D-07      RHO=  0.6103516D-04      MAGNITUDE=  0.5443617D 00      PHASE= 2
   F=  0.3234915D 02     P=  0.3235641D 02     G=  0.3234013D 02     RSIGMA=  0.7058842D-02     M=  0.0
  THE CURRENT VALUE OF X IS
   0.1684626D-05        0.3131440D-04        0.5174273D 01        0.4381135D-04        0.3061097D 01        0.1184054D 02
   0.1595861D-05        0.1077383D-05        0.1043453D 00        0.8915051D-04        0.3000026D 00        0.3334458D 00
   0.4300385D 00        0.4283569D 00        0.2240631D 00
  THE CONSTRAINT VALUES
                        NOT INCLUDING THE NON-NEGATIVITIES
   0.2238172D-03        0.1833846D-03        0.1528649D-03        0.1427815D-03        0.2729410D-03


 2ND  ORDER ESTIMATES
   F=  0.3234969D 02     P=  0.3234967D 02     G=  0.3234868D 02     RSIGMA=  0.0                      M=  0.0
  THE CURRENT VALUE OF X IS
   0.3367415D-08        0.7153088D-07        0.5174036D 01        0.1073438D-06        0.3061108D 01        0.1183950D 02
   0.3764172D-08        0.2540675D-08        0.1038710D 00        0.2040008D-06        0.3000000D 00        0.3334686D 00
   0.3999956D 00        0.4283077D 00        0.2239602D 00
  THE CONSTRAINT VALUES
                        NOT INCLUDING THE NON-NEGATIVITIES
   0.4845348D-06        0.3767802D-06        0.3349879D-06        0.1085657D-06        0.3294861D-06

 LAGRANGE MULTIPLIERS
   F=  0.3234868D 02     P=  0.3234992D 02     G=  0.3234868D 02     RSIGMA=  0.0                      M=  0.0
  THE CURRENT VALUE OF X IS
   0.3623270D 02        0.1949109D 01        0.1179589D-04        0.1393136D 01        0.1993898D-04        0.5154760D-05
   0.3824559D 02        0.5665128D 02        0.5849344D-03        0.6846305D 00        0.2034488D-03        0.1830437D-03
   0.1525845D-03        0.1424867D-03        0.2724017D-03
  THE CONSTRAINT VALUES
                        NOT INCLUDING THE NON-NEGATIVITIES
   0.2994602D 00        0.3328260D 00        0.3992752D 00        0.4274726D 00        0.2236204D 00


                        SENSITIVITY ANALYSIS

 THE VALUE OF R(RHO) IS 0.61035D-04

  THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILLBE MADE IS
X( 1)= 0.1684626D-05  X( 2)= 0.3131440D-04  X( 3)=  5.174273     X( 4)= 0.4381135D-04  X( 5)=  3.061097     X( 6)=  11.84054
X( 7)= 0.1595861D-05  X( 8)= 0.1077383D-05  X( 9)= 0.1043453     X(10)= 0.8915051D-04  X(11)= 0.3000026     X(12)= 0.3334458
X(13)= 0.4300389      X(14)= 0.4283569      X(15)= 0.2240631     X(

     PARAMETER VALUE    DIFFERENCING INTERVAL
  1      -40.0000           0.100000D-10
  2      -2.00000           0.100000D-10
  3      -0.250000          0.100000D-10
  4      -4.00000           0.100000D-10
  5      -4.00000           0.100000D-10
  6      -1.00000           0.100000D-10
  7      -40.0000           0.100000D-10
  8      -60.0000           0.100000D-10
  9       5.00000           0.100000D-10
 10       1.00000           0.100000D-10
```

Fig. 3.--Shell Dual subproblem and sensitivity output.

```
                    OPTIMAL VALUE FUNCTION SENSITIVITY

DF/DA( 1)= 0.0           DF/DA( 2)= 0.0           DF/DA( 3)= -5.174173     DF/DA( 4)= 0.0           DF/DA( 5)= -3.061010
DF/DA( 6)= -11.84066     DF/DA( 7)= 0.0           DF/DA( 8)= 0.0           DF/DA( 9)=-0.1044490     DF/DA(10)=-0.1776357D-03

DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS
 3 , 5 , 6 , 9 ,

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 3
DX( 1)=-0.3199323D-06 DX( 2)= 0.3474654D-05 DX( 3)=  4.064262     DX( 4)= 0.3996435D-04 DX( 5)=-0.5653876     DX( 6)=  3.465049
DX( 7)=-0.1187468D-07 DX( 8)=-0.5728193D-08 DX( 9)= 0.1894743     DX(10)= 0.3705778D-04 DX(11)=-0.3993361     DX(12)= 0.9718479D-01
DX(13)=-0.1996140     DX(14)= 0.2854635     DX(15)=-0.6779118D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 3
DU( 1)=-0.3999959     DU( 2)= 0.9736861D-01 DU( 3)=-0.1999975     DU( 4)= 0.2860099     DU( 5)=-0.6797010D-01 DU(

          DF(X(R))/DA=   -5.17412
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 5
DX( 1)= 0.1596979D-07 DX( 2)=-0.7565926D-05 DX( 3)=-0.5653817     DX( 4)=-0.1235465D-04 DX( 5)= 0.5043776     DX( 6)=-0.6151088
DX( 7)=-0.4395025D-08 DX( 8)=-0.5603023D-08 DX( 9)= 0.2377103D-01 DX(10)= 0.1277951D-04 DX(11)=-0.9774861D-05 DX(12)=-0.1468753
DX(13)=-0.1058967D-05 DX(14)=-0.4909590D-01 DX(15)= 0.9798144D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 5
DU( 1)=-0.5691225D-06 DU( 2)=-0.1471216     DU( 3)=-0.3519048D-06 DU( 4)=-0.4919960D-01 DU( 5)= 0.9818198D-01 DU(

          DF(X(R))/DA=   -3.06121
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 6
DX( 1)=-0.1681823D-06 DX( 2)= 0.6727627D-06 DX( 3)=  3.465049     DX( 4)= 0.3552231D-04 DX( 5)=-0.6151072     DX( 6)=  7.105060
DX( 7)=-0.9322921D-08 DX( 8)=-0.1096909D-07 DX( 9)=  1.588582     DX(10)= 0.2597367D-04 DX(11)=-0.1994387     DX(12)= 0.9430382D-01
DX(13)=-0.3493977     DX(14)= 0.2278306     DX(15)= 0.2817522D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 6
DU( 1)=-0.1999961     DU( 2)= 0.9445695D-01 DU( 3)=-0.3499971     DU( 4)= 0.2281903     DU( 5)= 0.2708145D-01 DU(

          DF(X(R))/DA=   -11.8397
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 9
DX( 1)= 0.1559149D-09 DX( 2)=-0.6715148D-05 DX( 3)= 0.1894744     DX( 4)= 0.1189638D-04 DX( 5)= 0.2377129D-01 DX( 6)=  1.588583
DX( 7)= 0.7948081D-08 DX( 8)= 0.4311413D-08 DX( 9)= 0.0343746     DX(10)=-0.2480389D-04 DX(11)= 0.9305713D-04 DX(12)=-0.3949690D-01
DX(13)=-0.2693861D-04 DX(14)= 0.7568968D-01 DX(15)= 0.1541232     DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 9
DU( 1)= 6.3110905D-06 DU( 2)=-0.3957025D-01 DU( 3)= 0.3284493D-06 DU( 4)= 0.7579224D-01 DU( 5)= 0.1542586     DU(

          DF(X(R))/DA=  -0.104003
*********
```

Fig. 3.--Continued.

by minimizing the negative of the objective function and therefore, the results in Fig 3 for the objective function value and the values of its partial derivatives must be multiplied by -1 to obtain the correct results. The variables X(1) - X(10) in the Shell Dual correspond to the Lagrange multipliers in the Shell Primal and the variables X(11) - X(15) in the Dual correspond to X(1) - X(5) in the Primal. The components of the Hessian of the optimal value function then are the partial derivatives of X(1) - X(10) in the Shell Dual and the u-derivatives in the Shell Primal. The dual variables, Lagrange multipliers, the partial derivatives of the optimal value function obtained by means of the Lagrangian, and those obtained using the chain rule are compared in Table 4.

TABLE 4

FIRST ORDER SENSITIVITY COMPARISON

	Shell Primal			Shell Dual		
i	u_i	$\partial f/\partial \epsilon_i$(Lag.)	$\partial f/\partial \epsilon_i$(C.R.)	x_i	$\partial f/\partial \epsilon_i$(Lag.)	$\partial f/\partial \epsilon_i$(C.R.)
1	$.168\text{x}10^{-5}$	$.168\text{x}10^{-5}$	-	$.168\text{x}10^{-5}$	0.0	-
2	$.312\text{x}10^{-4}$	$.312\text{x}10^{-4}$	-	$.313\text{x}10^{-4}$	0.0	-
3	5.1649	5.1649	5.1740	5.1742	5.1741	5.1741
4	$.437\text{x}10^{-4}$	$.437\text{x}10^{-4}$	-	$.438\text{x}10^{-4}$	0.0	-
5	3.0555	3.0554	3.0610	3.0610	3.0610	3.0612
6	11.8191	11.8190	11.8395	11.8405	11.8406	11.8397
7	$.159\text{x}10^{-5}$	$.159\text{x}10^{-5}$	-	$.159\text{x}10^{-5}$	0.0	-
8	$.107\text{x}10^{-5}$	$.107\text{x}10^{-5}$	-	$.107\text{x}10^{-5}$	0.0	-
9	.1046	.1046	.1038	.1043	.1044	.1040
10	$.889\text{x}10^{-4}$	$.889\text{x}10^{-4}$	-	$.891\text{x}10^{-4}$	$.177\text{x}10^{-3}$	-

Since the preliminary screening option was used, detailed sensitivity estimates for the parameters which correspond to the non-binding inequality constraints were not computed in Figures 2 and 3. In the Shell Primal, the Lagrange multipliers and the partial derivatives obtained by means of the Lagrangian correspond exactly. As noted in the discussions following Problems B and C, the estimates of the Lagrange multipliers for the binding constraints are very sensitive near the boundary of the constraint set. This explains the slight variation between these estimates and the other sensitivity estimates shown in Table 4 for the binding Primal constraints.

Now compare the second order sensitivity estimates with respect to the parameters which are the right-hand sides of the binding Primal constraints. Let H_D and H_P denote the submatrices of the Hessian of the optimal value function for the Dual and Primal respectively, obtained by deleting the rows and columns corresponding to the non-binding Primal inequality constraints. Thus, the components of H_D are $\partial x_i/\partial\epsilon_j$, $i,j=3,5,6,9$, and the components of H_P are $\partial u_i/\partial\epsilon_j$, $i,j=3,5,6,9$. From the computer output, these matrices are:

$$H_D = \begin{bmatrix} 4.0642 & -.5653 & 3.4650 & .1894 \\ -.5653 & .5043 & -.6151 & .0237 \\ 3.4650 & -.6151 & 7.1850 & 1.5885 \\ .1894 & .0237 & 1.5885 & .8343 \end{bmatrix},$$

and

$$H_P = \begin{bmatrix} 4.0710 & -.5663 & 3.4715 & .1899 \\ -.5662 & .5051 & -.6158 & .0239 \\ 3.4715 & -.6158 & 7.1929 & 1.5890 \\ .1899 & .0239 & 1.5890 & .8347 \end{bmatrix}.$$

Both H_D and H_P are symmetric, and while the agreement is not exact, it is very close as anticipated.

Armacost and Fiacco (1974) indicate that the differencing interval used in the sensitivity analysis can affect the accuracy of the results. For this reason, an option was provided by Armacost and Mylander (1973) to conduct the sensitivity analysis at the final subproblem for a range of differencing intervals. It is even more important to be cautious when dealing with right-hand side perturbations as the following discussion indicates. The results shown above for the Shell Primal and the Shell Dual were obtained with a trajectory sensitivity analysis and at the final subproblem, the differencing increment used in the central differencing formulas was 10^{-11}. The Hessian submatrices H_P and H_D were found to be very close. When the problems were solved using a sensitivity analysis at the final subproblem only with a differencing increment of 10^{-9}, the diagonal elements of H_P were considerably different from those of H_D. The problems were solved again with a sensitivity analysis performed at the final subproblem for a range of values of the differencing increment, ranging from 10^{-6} to 10^{-11}. The components of H_D remained constant as did the non-diagonal elements of H_P which were equal to the non-diagonal elements of H_D. The diagonal elements of H_P did not remain constant and their variation is depicted in Table 5.

The final example considered in this section is called the cattle feed problem. It was formulated and originally presented by van de Panne and Popp (1963). Armacost and Fiacco (1974) presented the cattle feed problem to illustrate an application of the sensitivity analysis. The additional sensitivity results are presented here for completeness. The problem is a chance-constrained program to determine the mix of inputs to cattle feed that will satisfy nutritive constraints and minimize the cost of the cattle feed. The protein content of the components is a random

TABLE 5

VARIATION IN THE COMPONENTS OF H_p

Δ	$\partial^2 f/\partial \epsilon_1^{\,2}$			
	i=3	i=5	i=6	i=9
10^{-6}	-3148.3	-383.81	-89157.9	.8342
10^{-7}	-27.228	-3.328	-851.345	.8347
10^{-8}	3.758	.466	-1.389	.8347
10^{-9}	4.067	.504	7.107	.8347
10^{-10}	4.071	.505	7.191	.8347
10^{-11}	4.071	.505	7.192	.8347

variable, normally distributed with a mean and variance determined experimentally. The application of the sensitivity analysis included the standard deviations as parameters with the interpretation that if the solution were sensitive to a standard deviation, more sampling would be indicated in order to obtain a sharper estimate. The statement of the problem is

$$\begin{aligned}
\text{minimize} \quad & f(x,\epsilon) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{subject to} \quad & g_1(x,\epsilon) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 - \epsilon_7 \geq 0, \\
& g_2(x,\epsilon) = \mu_1x_1 + \mu_2x_2 + \mu_3x_3 + \mu_4x_4 \\
& \qquad + \epsilon_5\sqrt{\epsilon_1^2x_1^2+\epsilon_2^2x_2^2+\epsilon_3^2x_3^2+\epsilon_4^2x_4^2} - \epsilon_6 \geq 0, \\
& h_1(x,\epsilon) = x_1 + x_2 + x_3 + x_4 - 1 = 0,
\end{aligned}$$

with $x_i \geq 0$, i=1,2,3,4. The notation is slightly different from Armacost and Fiacco (1974). Here, the parameters in the sensitivity analysis are denoted ϵ_i. The correspondence with the previous work is $\epsilon_i = \sigma_i$, i=1,2,3,4,

$\epsilon_5 = \phi$, $\epsilon_6 = p_m$ and $\epsilon_7 = o_m$. The problem data from Table 6 of Armacost and Fiacco (1974) are included in Appendix C. The sensitivity results for Problem F are shown in Figure 4. Again, the preliminary screening option was used to avoid calculating sensitivity estimates which did not affect the optimal value function significantly. Here, the components of the gradient of the optimal value function obtained directly from the gradient of the Lagrangian and the estimate of the gradient obtained by application of the chain rule are very close. The values of the Lagrange multipliers estimated in the SUMT program are $u_1 = 0.58037$, $u_2 = -.41005$ and $w_1 = -18.3738$. The multipliers u_1 and u_2 correspond to $\partial f^*/\partial \epsilon_7$ and $\partial f^*/\partial \epsilon_6$ respectively. Also note that second order sensitivity information for ϵ_6 and ϵ_7 is available since they are the right-hand sides of the two inequality constraints. Namely,

$$\partial^2 f^*/\partial \epsilon_7^2 = \partial u_1/\partial \epsilon_7 = .0112929,$$

$$\partial^2 f^*/\partial \epsilon_6 \partial \epsilon_7 = \partial u_1/\partial \epsilon_6 = .45706\text{x}10^{-6},$$

$$\partial^2 f^*/\partial \epsilon_7 \partial \epsilon_6 = \partial u_2/\partial \epsilon_7 = .45671\text{x}10^{-6},$$

and

$$\partial^2 f^*/\partial \epsilon_6^2 = \partial u_2/\partial \epsilon_6 = .46212\text{x}10^{-3}.$$

The results shown in Figure 4 were obtained using a sensitivity analysis at the final subproblem for a range of differencing increments. For other values of the differencing increments not shown here, the optimal value function sensitivity estimates with respect to ϵ_6 and ϵ_7 obtained directly from the gradient of the Lagrangian and the cross-partials of u_1 and u_2 with respect to ϵ_6 and ϵ_7 do vary somewhat while the other sensitivity estimates remain relatively constant. In a practical sense, however, the cross-partials of u_1 and u_2 with respect to ϵ_6 and ϵ_7 are constant since they are of the order of 10^{-6}.

```
                        SENSITIVITY ANALYSIS

   THE VALUE OF EPSMD( IS 0.976960-01

    THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILL BE MADE IS
X( 1)= 0.5197573      X( 2)= 0.3917960D-02  X( 3)= 0.3139958      X( 4)= 0.9396074D-01  X(

    PARAMETER VALUE    DIFFERENCING INTERVAL
 1     3.937000             0.100000-06
 2     7.962000             0.100000-06
 3      4.95000             0.100000-06
 4     0.780000             0.100000-06
 5      1.06500             0.100000-06
 6      21.5700             0.100000-06
 7      9.03000             0.100000-06


                  OPTIMAL VALUE FUNCTION SENSITIVITY

DF/DA( 1)= 0.3672851D-01 DF/DA( 2)= 0.3132176D-05 DF/DA( 3)= 0.2C01806     DF/DA( 4)= 0.1047445D-02 DF/DA( 5)= 0.5950205
DF/DA( 6)= 0.4107967     DF/DA( 7)= 0.5803659     DF/DA(

DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS
 1 , 3 , 5 , 6 , 7 ,


 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 1
DX( 1)=-0.9731279D-02 DX( 2)= 0.2347974D-02 DX( 3)=-0.1300178D-03 DX( 4)= 0.5075195D-02 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 1
DU( 1)=-0.4418810-01 DU( 2)=-0.9077652D-02 DU(

 W-DERIVATIVES WITH RESPECT TO PARAMETER 1
DW( 1)=-0.637410[illegible]     DW(

          DF(X(E))/DA=   0.9995420-01
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 3
DX( 1)=-0.1[illegible]D-01 DX( 2)= 0.3560012D-02 DX( 3)= 0.2227720D-04 DX( 4)= 0.1203000D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 3
DU( 1)= 0.7072077D-01 DU( 2)= 0.2444827D-02 DU(

 W-DERIVATIVES WITH RESPECT TO PARAMETER 3
DW( 1)= 0.2109153     DW(

          DF(X(E))/DA=   0.209173
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 5
DX( 1)=-0.4[illegible]D-01 DX( 2)= 0.1044535D-01 DX( 3)= 0.4449727D-04 DX( 4)= 0.3904742D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 5
DU( 1)= 0.1944951     DU( 2)= 0.0419763D-02 DU(

 W-DERIVATIVES WITH RESPECT TO PARAMETER 5
DW( 1)= 0.0437724     DW(

          DF(X(E))/DA=   0.4C2033
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 6
DX( 1)=-0.2509[illegible] DX( 2)=-0.0179127D-04 DX( 3)= 0.2072327D-02 DX( 4)= 0.8332479D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 6
DU( 1)= [illegible] DU( 2)= 0.4921191D-03 DU(

 W-DERIVATIVES WITH RESPECT TO PARAMETER 6
DW( 1)= 0.0627[illegible]D-02 DW(

          DF(X(E))/DA=   0.419620
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 7
DX( 1)=-0.00[illegible]D-01 DX( 2)= 0.4047[illegible]D-04 DX( 3)= 0.1007200     DX( 4)=-0.0042030D-01 DX(

 U-DERIVATIVES WITH RESPECT TO PARAMETER 7
DU( 1)= 0.1129251D-01 DU( 2)= 0.4567100D-04 DU(

 W-DERIVATIVES WITH RESPECT TO PARAMETER 7
DW( 1)= 0.9704294D-01 D .

          DF(X(E))/DA=   0.970910
*********
```

Fig. 4.--Sensitivity results for Problem F.

5. Large-scale, Multi-item Inventory Model

Traditionally, inventory models have been formulated to minimize some function of the ordering, holding and shortage (or backorder) costs subject to various constraints. Schrady and Choe (1971) have formulated an inventory model which appears to have much greater relevance for an inventory system such as the U. S. Naval supply system. For the Navy, the costs used in the traditional models may be quite artificial while the real objective of the system is to maximize the service to the Fleet, an objective equivalent to minimization of stockouts. In addition, the stock points of the Naval supply system have investment and reorder workload constraints that are real and binding. Schrady and Choe consider a multi-item inventory system with the specific objective function of minimizing the total time-weighted shortages. The decision variables are the "reorder quantities" and the "reorder points," the decisions of how much to order and when to order, for each item in the inventory. Clearly, it is of interest to know "if" and "by how much" these variables and the value of the objective function will change if certain parameters change. Schrady and Choe solved a small example problem using the SUMT computer code. McCormick (1972) has shown how the special structure of this inventory model can be used to facilitate the use of the SUMT code to solve very large inventory problems. Here, the sensitivity analysis is applied to this model, a large-scale inventory system, and it is illustrated by means of the example used by Schrady and Choe.

The model presented here is due to Schrady and Choe. An extension of the model by McCormick (1972) includes constraints on storage volume and probability of depletion. The sensitivity results are easily applied to the extended model. Several assumptions specify the nature of the model. The first is that all demand which occurs when the on-hand stock is zero, is back-ordered. The model is probabilistic in that the lead time demand is a random variable. Specifically, for the i^{th} variable, it is assumed that the demand which occurs during the time between the placement of an order and its receipt by the stock point is normally distributed with mean μ_i and variance σ_i^2. For the i^{th} item, let

c_i - item unit cost (in dollars),

λ_i - mean demand per unit time (in units),

r_i - reorder point,

Q_i - reorder quantity,

$\phi(x)$ - the Normal(0,1) density function, and

$\Phi(z)$ - the Normal(0,1) complementary cumulative distribution function $= \int_z^{\infty} \phi(x)\, dx$.

In addition, let K_1 be the investment limit in dollars, let K_2 be the reorder workload limit and let N be the total number of items in the inventory. The detailed development of the model is omitted here and the reader is referred to Schrady and Choe (1971) or McCormick (1972). A function needed from that development for the final model is

$$\beta_i(r_i) = \tfrac{1}{2}(\sigma_i^2 + (r_i - \mu_i)^2)\Phi((r_i - \mu_i)/\sigma_i) - \tfrac{1}{2}\sigma_i(r_i - \mu_i)\phi((r_i - \mu_i)/\sigma_i).$$

Then the general multi-item model of Schrady and Choe is

$$\text{minimize} \quad Z(Q,r) = \sum_{i=1}^{N} \beta_i(r_i)/Q_i \qquad \text{S-C}$$

$$\text{subject to} \quad g_1(Q,r) = K_1 - \sum_{i=1}^{N} c_i(r_i + Q_i/2 - \mu_i) \geq 0,$$

$$g_2(Q,r) = K_2 - \sum_{i=1}^{N} \lambda_i/Q_i \geq 0 ,$$

with r_i unrestricted, $Q_i \geq 0$, $i=1,\ldots,N$, $Q = (Q_1,\ldots,Q_N)^T$ and $r = (r_1,\ldots,r_N)^T$.

To put the problem in (x,ϵ) notation, make the following identifications for $i=1,\ldots,N$:

$$x_{2i-1} = Q_i,$$
$$x_{2i} = r_i,$$
$$\epsilon_{4i-1} = \mu_i,$$
$$\epsilon_{4i} = \sigma_i,$$
$$\epsilon_{4i+1} = c_i,$$
$$\epsilon_{4i+2} = \lambda_i,$$

and $\epsilon_1 = K_1$, $\epsilon_2 = K_2$. Rewrite Problem S-C as

$$\text{minimize} \quad f(x,\epsilon) = \sum_{i=1}^{N} \beta_i(x_{2i},\epsilon)/x_{2i-1} \qquad \text{S-C}(\epsilon)$$

$$\text{subject to} \quad g_1(x,\epsilon) = \epsilon_1 - \sum_{i=1}^{N} \epsilon_{4i+1}(x_{2i} + x_{2i-1}/2 - \epsilon_{4i-1}) \geq 0,$$

$$g_2(x,\epsilon) = \epsilon_2 - \sum_{i=1}^{N} \epsilon_{4i+2}/x_{2i-1} \geq 0,$$

$x_{2i-1} \geq 0$, x_{2i} unrestricted, $i=1,\ldots,N$, and

$$\beta_i(x_{2i},\epsilon) = \tfrac{1}{2}((\epsilon_{4i}^{\,2} + (x_{2i} - \epsilon_{4i-1})^2)\Phi((x_{2i} - \epsilon_{4i-1})/\epsilon_{4i})$$
$$- \epsilon_{4i}(x_{2i} - \epsilon_{4i-1})\phi((x_{2i} - \epsilon_{4i-1})/\epsilon_{4i})).$$

Schrady and Choe consider a three item example. The problem data and the initial starting point for the SUMT program are shown in Table 6.

TABLE 6

MULTI-ITEM INVENTORY PROBLEM DATA

Item	i=1	i=2	i=3
Data:			
$\mu_i = \epsilon_{4i-1}$	100	200	300
$\sigma_i = \epsilon_{4i}$	100	100	200
$c_i = \epsilon_{4i+1}$	1	10	20
$\lambda_i = \epsilon_{4i+2}$	1,000	1,500	2,000
Starting point:			
$Q_i = x_{2i-1}$	600	270	300
$r_i = x_{2i}$	200	260	400

In addition, $K_1 = \epsilon_1 = \$8,000$ and $K_2 = \epsilon_2 = 15$.

Figure 5 contains the computer output for the final subproblem of a trajectory sensitivity analysis for Problem S-C(ϵ) using the data of Table 6. The results indicate that the optimal value function is sensitive to parameters 2, 5, 8, 9, 12, and 13, i.e., K_2, c_1, σ_2, c_2, σ_3 and c_3 respectively. The fact that the solution is sensitive to the values of the standard deviations of the lead time demand of two items lets the decision maker know that since, these parameters were obtained by sampling, a possible action may be to conduct additional sampling in order to sharpen the estimate of the standard deviation.

The solution value is also very sensitive to all of the item costs. If the structure of Problem S-C is examined, this result is most surprising since the c_i appear only in the investment constraint and the optimal value function is not very sensitive to the investment limit K_1. Recall, however,

```
*****************************************
      POINT=  35        DOTT=  0.2999920D-07      RHO=  0.6103516D-04      MAGNITUDE=  0.2261735D-02      PHASE= 2
   F=  0.1299696D 02     P=  0.1299506D 02     G=  0.1298648D 02     RSIGMA= -0.1281691D-02     H=  0.0
 THE CURRENT VALUE OF X IS
  0.5331717D 03       0.2527781D 03       0.2455910D 03       0.2770077D 03       0.2850376D 03       0.4366108D 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.1178464D-01       0.9796656D-04


 2ND  ORDER ESTIMATES
   F=  0.1298684D 02     P=  0.1298702D 02     G=  0.1298684D 02     RSIGMA=  0.0              H=  0.0
 THE CURRENT VALUE OF X IS
  0.5331619D 03       0.2527777D 03       0.2455856D 03       0.2770083D 03       0.2850366D 03       0.4366122D 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
 -0.1322612D-03       0.2633644D-06


 1ST  ORDER ESTIMATES
   F=  0.1298684D 02     P=  0.1298707D 02     G=  0.1298684D 02     RSIGMA=  0.0              H=  0.0
 THE CURRENT VALUE OF X IS
  0.5331619D 03       0.2527774D 03       0.2455856D 03       0.2770083D 03       0.2850366D 03       0.4366123D 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
 -0.1141546D-03       0.3236619D-06


LAGRANGE MULTIPLIERS
   F=  0.1298684D 02     P=  0.1298707D 02     G=  0.1298684D 02     RSIGMA=  0.0              H=  0.0
 THE CURRENT VALUE OF X IS
  0.1144756D-06       0.2414574D-06       0.2485236D-06       0.2203374D-06       0.2141302D-06       0.1397930D-06
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.5177457D-02       0.6230203D 00


                       SENSITIVITY ANALYSIS

 THE VALUE OF R(RHO) IS 0.61035D-04

  THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILLBE MADE IS
X( 1)=  533.1717     X( 2)=  252.7761     X( 3)=  245.5910     X( 4)=  277.0077     X( 5)=  285.0376     X( 6)=  436.6106

   PARAMETER VALUE   DIFFERENCING INTERVAL
 1     8000.00          0.100000D-10
 2     15.0000          0.100000D-10
 3     100.000          0.100000D-10
 4     100.000          0.100000D-10
 5     1.00000          0.100000D-10
 6     100.000          0.100000D-10
 7     200.000          0.100000D-10
 8     100.000          0.100000D-10
 9     10.0000          0.100000D-10
10     1500.00          0.100000D-10
11     300.000          0.100000D-10
12     200.000          0.100000D-10
13     20.0000          0.100000D-10
14     2000.00          0.100000D-10
```

Fig. 5.--Computer output for Schrady-Choe inventory problem.

```
                 OPTIMAL VALUE FUNCTION SENSITIVITY

DF/DA( 1)=-0.5179757D-02 DF/DA( 2)=-0.6232204      DF/DA( 3)=-0.2832216D-04 DF/DA( 4)= 0.1185718D-01 DF/DA( 5)=  2.171260
DF/DA( 6)= 0.1169959D-02 DF/DA( 7)=-0.2832216D-03 DF/DA( 8)= 0.8970603D-01 DF/DA( 9)=  1.334539     DF/DA(10)= 0.2538508D-02
DF/DA(11)=-0.7662834D-03 DF/DA(12)= 0.1723395      DF/DA(13)=  1.445152      DF/DA(14)= 0.2185745D-02 DF/DA(

DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS
 2 , 5 , 8 , 9 ,12 ,13 ,

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 2
DX( 1)= -47.31873     DX( 2)=  5.226528     DX( 3)= -18.76101     DX( 4)=  6.196130     DX( 5)= -14.90654     DX( 6)=  9.967083

 U-DERIVATIVES WITH RESPECT TO PARAMETER 2
DU( 1)=-0.1623141D-03 DU( 2)=-0.1381545     DU(

          DF(X(R))/DA=  -0.625665
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 5
DX( 1)= -208.8685     DX( 2)= -31.79176     DX( 3)=  15.31396     DX( 4)= -10.34249     DX( 5)=  14.37551     DX( 6)= -20.00203

 U-DERIVATIVES WITH RESPECT TO PARAMETER 5
DU( 1)= 0.6063471D-03 DU( 2)= 0.1635012     DU(

          DF(X(R))/DA=    2.15929
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 8
DX( 1)=-0.5664421     DX( 2)=-0.1272665     DX( 3)= 0.2271028     DX( 4)=  1.033705     DX( 5)=-0.1084107     DX( 6)=-0.4918837

 U-DERIVATIVES WITH RESPECT TO PARAMETER 8
DU( 1)= 0.2325778D-04 DU( 2)= 0.6387492D-03 DU(

          DF(X(R))/DA=   0.8953820-01
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 9
DX( 1)=  0.190797     DX( 2)= -2.678276     DX( 3)= -4.971893     DX( 4)= -7.267635     DX( 5)=  3.652195     DX( 6)= -7.108662

 U-DERIVATIVES WITH RESPECT TO PARAMETER 9
DU( 1)= 0.2382230D-03 DU( 2)= 0.4891429D-01 DU(

          DF(X(R))/DA=    1.02862
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,12
DX( 1)= -1.237438     DX( 2)=-0.2610658     DX( 3)=-0.4033268     DX( 4)=-0.3839144     DX( 5)= 0.5842921     DX( 6)= 0.4463787D-01

 U-DERIVATIVES WITH RESPECT TO PARAMETER12
DU( 1)= 0.4288446D-34 DU( 2)= 0.1993932D-02 DU(

          DF(X(R))/DA=   0.172938
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,13
DX( 1)=-0.1669738     DX( 2)= -2.307243     DX( 3)= 0.4442831     DX( 4)= -3.170248     DX( 5)=-0.4251138     DX( 6)= -12.15228

 U-DERIVATIVES WITH RESPECT TO PARAMETER13
DU( 1)= 0.2749919D-03 DU( 2)= 0.3229495D-01 DU(

          DF(X(R))/DA=    1.43768
*********
```

Fig. 5.--Continued.

that when the partial derivatives are used as sensitivity estimates, one is effectively saying "how much will the objective function (or solution point) change if the parameter is increased by one unit?" Suppose c_1 (parameter 5) is increased by one unit from one to two. Using a linear estimate, the objective function is expected to increase by 2.16 to 15.15 and Q_1 (variable x_1) is expected to decrease by 208 units to 325. Figure 6 contains the computer output for the perturbed problem with $c_1 = 2$. Note that the value of the optimal value function increased to 14.99 and Q_1 decreased to 411. The changes were quite large and in the directions expected but of course not as large as the linear estimate. Notice, however, that this change in c_1 represented a 100 % increase in the value of the parameter. For purposes of comparison, consider a similar change in the investment constraint K_1 (parameter 1) to which the solution is apparently insensitive. Using the optimal value function sensitivity estimate (-.00517), a 100% increase in the value of the parameter is 8,000 and consequently, a linear estimate of the expected change in the optimal value function would be (8,000 x -.00517 =) -41.36. The above example does not imply that equivalent percentage changes in parameter values is relevant but rather is meant to emphasize that the sensitivity estimates are valid for a small neighborhood of the given parameter values and as such represent instantaneous changes.

The solution values in Figure 5 are slightly different from those presented in Schrady and Choe (1971). This difference is due to the use of

```
*********************************************
      POINT=  36        DOTT=  0.18494560-07      EHO=  0.61035160-04      MAGNITUDE=  0.16662010-02      PHASE= 2
   F=  0.14695770 02      F=  0.14994530 02      G=  0.14995280 02      PSIGMA= -0.12424690-02      H=  0.0
 THE CURRENT VALUE OF X IS
  0.41162500 03      0.22056830 03      0.25747930 03      0.26840210 03      0.29652410 03      0.41972050 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.10763580-01      0.79565440-04

 2ND  ORDER ESTIMATES
   F=  0.14995650 02      P=  0.14995830 02      G=  0.14995650 02      PSIGMA=  0.0                     H=  0.0
 THE CURRENT VALUE OF X IS
  0.41162310 03      0.22056840 03      0.25747810 03      0.26840260 03      0.29652310 03      0.41972970 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.18912080-04      0.14002520-06

 1ST  ORDER ESTIMATES
   F=  0.14995650 02      P=  0.14995870 02      G=  0.14995650 02      PSIGMA=  0.0                     H=  0.0
 THE CURRENT VALUE OF X IS
  0.41162010 03      0.22856840 03      0.25747810 03      0.26840260 03      0.29652310 03      0.41972980 03
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.25525020-04      0.19432540-06

LAGRANGE MULTIPLIERS
   F=  0.14975650 02      P=  0.14995870 02      G=  0.14995650 02      PSIGMA=  0.0                     H=  0.0
 THE CURRENT VALUE OF X IS
  0.14827850-06      0.26703240-06      0.23704080-06      0.22740200-06      0.20583540-06      0.14541580-06
 THE CONSTRAINT VALUES
                      NOT INCLUDING THE NON-NEGATIVITIES
  0.56735250-02      0.76710640 00

                      SENSITIVITY ANALYSIS

 THE VALUE OF R(RHO) IS 0.61035D-04

  THE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILLBE MADE IS
X( 1)=  411.6250      X( 2)=  228.5583      X( 3)=  257.4793      X( 4)=  268.4021      X( 5)=  296.5241      X( 6)=  419.7285

  PARAMETER VALUE   DIFFERENCING INTERVAL
 1      9000.00          0.100000D-10
 2      15.0000          0.100000D-10
 3      100.000          0.100000D-10
 4      100.000          0.100000D-10
 5      2.00000          0.100000D-10
 6      1000.00          0.100000D-10
 7      200.000          0.100000D-10
 8      100.000          0.100000D-10
 9      10.0000          0.100000D-10
10      1500.00          0.100000D-10
11      300.000          0.100000D-10
12      200.000          0.100000D-10
13      20.0000          0.100000D-10
14      2000.00          0.100000D-10
```

Fig. 6.—Computer output for perturbed Schrady-Choe problem.

```
                    OPTIMAL VALUE FUNCTION SENSITIVITY

DF/DA( 1)=-0.5073C44D-02 DF/DA( 2)=-0.767106S     DF/DA( 3)=-0.3791485D-04 DF/DA( 4)= 0.2464743D-01 DF/DA( 5)=  1.89680
DF/DA( 6)= 0.1065134D-02 DF/DA( 7)= 0.2906173D-03 DF/DA( 8)= 0.9584556D-01 DF/DA( 9)=  1.117848     DF/DA(10)= 0.2990437D-02
DF/DA(11)= 0.4035669D-01 DF/DA(12)= 0.184146      DF/DA(13)=  1.81946      DF/DA(14)= 0.2589044D-02 DF/DA(

DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS
 2 , 4 , 5 , 8 , 9 ,12 ,13 ,

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 2
DX( 1)= -34.6466      DX( 2)=  5.587090     DX( 3)= -10.43114     DX( 4)=  6.477016     DX( 5)= -15.63402     DX( 6)=  10.61381

 U-DERIVATIVES WITH RESPECT TO PARAMETER 2
DU( 1)=-0.1367365D-03 DU( 2)=-0.1602751     DU(

          OF(X(8))/DA=  -0.769859
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 4
DX( 1)= 0.2776041     DX( 2)=  1.674671     DX( 3)=-0.5746852D-01 DX( 4)=-0.5377614D-01 DX( 5)=-0.1666424D-01 DX( 6)=-0.132604

 U-DERIVATIVES WITH RESPECT TO PARAMETER 4
DU( 1)= 0.6419312D-03 DU( 2)= 0.3331831D-03 DU(

          OF(X(8))/DA=   0.2403680-01
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 5
DX( 1)= -73.49361     DX( 2)= -10.23380     DX( 3)=  9.602042     DX( 4)= -7.360548     DX( 5)=  9.516717     DX( 6)= -14.60248

 U-DERIVATIVES WITH RESPECT TO PARAMETER 5
DU( 1)= 0.4924825D-03 DU( 2)= 0.1305468     DU(

          OF(X(8))/DA=    1.09207
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 8
DX( 1)=-0.5650255     DX( 2)=-0.1277315     DX( 3)= 0.2540897     DX( 4)= 0.9765064     DX( 5)=-0.1059342     DX( 6)=-0.4577419

 U-DERIVATIVES WITH RESPECT TO PARAMETER 8
DU( 1)= 0.2319894D-04 DU( 2)= 0.6890143D-03 DU(

          OF(X(8))/DA=   0.9589540-01
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER , 9
DX( 1)=  5.97759      DX( 2)= -2.691771     DX( 3)= -9.790974     DX( 4)= -7.214006     DX( 5)=  3.03962      DX( 6)= -6.64[illegible]

 U-DERIVATIVES WITH RESPECT TO PARAMETER 9
DU( 1)= 0.2929199D-03 DU( 2)= 0.5596549D-01 DU(

          OF(X(8))/DA=    1.17105
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,12
DX( 1)=-0.9320127     DX( 2)=-0.2579912     DX( 3)=-0.3938431     DX( 4)=-0.3536755     DX( 5)= 0.6078397     DX( 6)= 0.3982221D-01

 U-DERIVATIVES WITH RESPECT TO PARAMETER12
DU( 1)= 0.4257077D-04 DU( 2)= 0.2113781D-02 DU(

          OF(X(8))/DA=   0.105405
*********

 X-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,13
DX( 1)= 0.2743289     DX( 2)= -2.213925     DX( 3)= 0.5191916     DX( 4)= -2.873637     DX( 5)=-0.5696231     DX( 6)= -11.55637

 U-DERIVATIVES WITH RESPECT TO PARAMETER13
DU( 1)= 0.2636435D-03 DU( 2)= 0.3528594D-01 DU(

          OF(X(8))/DA=    1.52531
*********
```

Fig. 6.—Continued.

two different approximations to the complementary cumulative normal distribution. While the solution values vary only slightly with these different estimates of the normal distribution, the variations in the Lagrange multipliers and some sensitivity estimates are greater. For the system library subroutine using the normal distribution error function (erf) to estimate the complementary cumulative distribution, the minimizing trajectory of subproblems showed that the estimate of the Lagrange multipliers deviated from what appeared to be a relatively constant value as the final subproblem was approached. As expected, the optimal value function sensitivity estimates obtained directly from the Lagrangian varied in almost direct proportion with the estimates of the Lagrange multipliers. In Figures 5 and 6, the estimates of the optimal value function sensitivity obtained by the two different methods (chain rule and Lagrangian) are in close agreement. This was not the case using the erf-related subroutine used by Schrady and Choe. This is the same effect experienced in other problems as discussed in Section 4. In this case, however, the x-derivatives are affected to a slight degree. The major source of this error appears to be the lack of necessary precision associated with the erf-related subroutine for the complementary cumulative normal distribution. The conclusion can only be that one must proceed with caution. From this and other examples, however, it appears that convergence of the Lagrange multiplier estimates along the minimizing trajectory is a good indication that the sensitivity estimates will be accurate.

This large-scale inventory example illustrates a potential "real world" application of sensitivity analysis. It also highlights the need for a careful interpretation of the sensitivity information as well as the recurrent call for caution in the use of a numerical algorithm.

6. Conclusions

The purpose of this paper was to present examples of computational implementation of sensitivity analysis with respect to the optimal value function and the Lagrange multipliers using the theoretical results of Fiacco (1973) and Armacost and Fiacco (1975, 1976). All of the problems presented led to new insights into the computational aspects of this type of sensitivity analysis.

The major conclusion is that the sensitivity estimates of the optimal value function obtained directly from the gradient of the Lagrangian, the partial derivatives of the Lagrange multipliers associated with the binding constraints, and to a lesser extent, the partial derivatives of the solution point, are dependent on the accuracy of the estimate of the Lagrange multipliers calculated in the penalty function algorithm and subsequently used in the gradient of the Lagrangian. If the estimates of the Lagrange multipliers along the minimizing trajectory converge to a common value, then it appears that the sensitivity estimates will converge to their true values provided the differencing increment is satisfactory. If, however, the Lagrange multipliers converge and then vary, the sensitivity estimates should be viewed with caution. (Note that the Lagrange multiplier estimates are available in the standard SUMT output and a trajectory analysis is not required.)

Armacost and Fiacco (1974) noted that the differencing increment used in the central differencing formulas was a potential source of error depending on the scaling of the problem. The same caution applies when dealing with Lagrange multiplier and optimal value function sensitivity.

The example problems which included right-hand side parameters indicated that the optimal value function second order sensitivity estimates are themselves very sensitive to the Lagrange multiplier estimates, the differencing increment and the scaling of the problem. It appears that further analysis is needed before this particular program can be used to take advantage of second order sensitivity estimates to improve algorithm performance. This is particularly true for the second partial derivatives of the optimal value function taken with respect to the right-hand side of a binding constraint.

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REFERENCES

Armacost, Robert L., and Fiacco, Anthony V. 1974. Computational experience in sensitivity analysis for nonlinear programming. Mathematical Programming. 6:301-326.

_______, and _______. 1975. Second-order parametric sensitivity analysis in NLP and estimates by penalty function methods. Technical Paper, Serial T-324. The Institute for Management Science and Engineering, The George Washington University.

_______, and _______. 1976. NLP sensitivity for R.H.S perturbations: A brief survey and recent second-order extensions. Technical Paper, Serial T-334. The Institute for Management Science and Engineering, The George Washington University.

_______, and Mylander, W. Charles. 1973. A guide to a SUMT-Version 4 computer subroutine for implementing sensitivity analysis in nonlinear programming. Technical Paper, Serial T-287. The Institute for Management Science and Engineering, The George Washington University.

Colville, A. R. 1968. A comparative study of nonlinear programming codes. IBM New York Scientific Center Technical Report 320-2947.

Fiacco, Anthony V. 1973. Sensitivity analysis for nonlinear programming using penalty methods. Technical Paper, Serial T-275. The Institute for Management Science and Engineering, The George Washington University.

McCormick, Garth P. 1972. Computational aspects of nonlinear programming solutions to large scale inventory problems. Technical Memorandum, Serial TM-63488. The Institute for Management Science and Engineering, The George Washington University.

Mylander, W. Charles, Holmes, Raymond L., and McCormick, Garth P. 1971. A guide to SUMT-Version 4: The computer program implementing the sequential unconstrained minimization technique for nonlinear programming. RAC-P-63, Research Analysis Corporation.

Schrady, D. A., and Choe, U. C. 1971. Models for multi-item continuous review inventory policies subject to constraints. Naval Research Logistics Quarterly. 18:541-463.

van de Panne, C., and Popp, W. 1963. Minimum-cost cattle feed under probabilistic protein constraints. Management Science. 9:405-430.

APPENDIX A

SUBROUTINES SENS, LMULT AND PRESEN

```
      SUBROUTINE SENS
C
C                    15 MARCH 1972
C
C  THIS VERSION OF THE SENSITIVITY ANALYSIS SUBROUTINE IS USED TO
C  COMPUTE THE DIRECTIONAL DERIVATIVES OF X AND F WITH RESPECT TO
C  CERTAIN PARAMETERS CODED IN THE ARRAY PAR(20). THE DIRECTIONAL
C  DERIVATIVES ARE ESTIMATED FOR ONE PARAMETER AT A TIME WITH NPAR BEING
C  THE NUMBER OF PARAMETERS INVOLVED IN THE SENSITIVITY ANALYSIS. THE
C  USE OF THE PARAMETERS PAR(20) MUST BE CONSISTENT THROUGHOUT THE
C  USERS SUBROUTINES.
C  THE SUBROUTINE IS USED FOR A SENSITIVITY ANALYSIS AT THE FINAL SUB-
C  PROBLEM OR FOR A SENSITIVITY ANALYSIS AT EACH SUBPROBLEM ALONG THE
C  MINIMIZING TRAJECTORY.  DPAR(20) IS THE ARRAY OF DIFFERENCING
C  INTERVALS CORRESPONDING TO THE PARAMETERS PAR(20).  DPAR(20) IS
C  ASSIGNED VALUES IN SUBROUTINE PARDIF.
C  THIS APPROACH TO SENSITIVITY ANALYSIS IS DUE TO A. V. FIACCO. THE
C  FIRST VERSION WAS CODED BY H. CAUSEY.  THE SECOND VERSION WAS CODED
C  BY W. C. MYLANDER.  THIS IS THE THIRD VERSION WHICH IS AN EXTENSION
C  OF THE SECOND VERSION TO PERMIT SENSITIVITY ANALYSIS ALONG THE
C  MINIMIZING TRAJECTORY, AND WAS CODED BY R. L. ARMACOST.
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*4 RMOIN,RATIO,EPSI,THETAC,XFP1,XEP2
      COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
      COMMON/EQAL/M,M1,M2
      COMMON/CNTNS/NT1,NT2,NT3,NT4,NT5,NT6,NT7,NT8,NT9,NT10
      COMMON/VALUE/F,G,DF,RSIGMA,RJ(40),RHO
      COMMON/TEST/DELX(20),DELAD(20),RMOIN,RATIO,EPSI,THETA0,
     1NTCTR,NUMIN,X1(20),X2(20),X3(20),XM2(20),XM1(20),PR1,
     2PR2,P1,F1,RJ1(40),DOTT,PGRAD(20),DIAG(20),
     3RLEV3,ACTLX,ESIGI,GI,NPHASE,NSATIS
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
      COMMON/EXOPT/NEXOP1,NEXOP2,NEXOP3,NEXOP4,NEXOP5,XEP1,XEP2
      DIMENSION DELT(20),DELTA(20),X2M(20),X3M(20)
      DIMENSION DELMU(40),DU(40),KTEST(20)
      NEXOP4 = 1
      NEXOP5 = 0
      DO 5 I=1,N
      DELTA(I) = DELX(I)
    5 DELT(I) = DELAD(I)
      CALL STORE
      AVAL = 1.0E-10
      MPM2 = M + M2
      ISENS = ISENS + 1
      CALL PARCIP
C  WRITE OUT X, RHO AND PAR.
      WRITE(6,10) RHO
   10 FORMAT(1H //33X,20HSENSITIVITY ANALYSIS    //
     115X,22HTHE VALUE OF RESMOI IS ,E12.5)
      WRITE(6,20)
   20 FORMAT(/5X, 40HTHE POINT AT WHICH THE ESTIMATE OF SENSITIVITY WILL
     1BE MADE IS       )
      DO 40 I=1,N,6
      II=MIN0(I+5,N)
      WRITE(6,30) ((J,X(J)), J=I,II)
   30 FORMAT( 6(2X,2HX(,I2,2H)=,G14.7) )
   40 CONTINUE
      WRITE(6,50) ((I,PAR(I),DPAR(I)), I=1,NPAR)
```

```
0033       50 FORMAT(45H0    PARAMETER VALUE   DIFFERENCING INTERVAL /
            1 (I4,2X,G13.5,4X,G14.5) )
0034          WRITE(6,55)
0035       55 FORMAT(//)
        C  EVALUATE F, G AND H.
0036          CALL RESINT(0,F)
0037          IF(MPMZ.EQ.0) GO TO 85
0038          DO 80 J=1,MPMZ
0039          CALL RESINT(J,RJ(J))
0040       80 CONTINUE
0041          IF(NCADIS.EQ.3) GO TO 85
0042          CALL PRESEN(DU,KTEST)
        C  COMPUTE DEL F.
0043       85 CALL GRAD1(0)
0044          DO 90 I=1,N
0045          X3H(I) = DEL(I)
0046       90 CONTINUE
        C  COMPUTE (DEL)**2 P - STORED IN A.
0047          CALL SECOND(2)
        C  PERFORM THE L-U DECOMPOSITION OF A.
0048          DO 100 I=1,N
0049          DELX(I)=0.0
0050      100 CONTINUE
0051          NPANT1
0052          NT3=1
0053          CALL INVERS(1)
        C  CHECK TO MAKE SURE AN ORTHOGONAL MOVE IS NOT ATTEMPTED.
0054          DO 110 I=1,N
0055          IF(DELX(I).EQ.0.0) GO TO 110
0056          WRITE(6,105)
0057      105 FORMAT(64H0 THE MATRIX OF SECOND PARTIALS IS NOT POSITIVE DEFINITE
            1, SENSITIVITY ANALYSIS IS TERMINATED     )
0058          GO TO 202
0059      110 CONTINUE
0060          DO 120 I=1,NPAR
0061          X2H(I) = PAR(I)
0062      120 CONTINUE
0063          DO 200 J=1,NPAR
0064          IF(NSELCT.NE.0) GO TO 115
0065          IF(KTEST(J).EQ.0) GO TO 200
0066    115   IF(NCADIS.EQ.3) GO TO 121
0067          CALL LMULT(0,DELMU,DEM,DU)
        C  COMPUTE D(DEL P)/DA(J) AND D(F)/DA(J) USING CENTRAL DIFFERENCING.
0068      121 PAR(J) =X2H(J) + DPAR(J)
0069          CALL RESINT(0,DF)
0070          IF(M.EQ.0) GO TO 126
0071          DO 125 I=1,M
0072          CALL RESINT(I,RJ(I))
0073          IF(RJ(I).GT.AVAL) GO TO 125
0074      123 DPAR(J)=0.1*DPAR(J)
0075          PAR(J) = X2H(J)
0076          WRITE(6,124) J,DPAR(J)
0077      124 FORMAT(16H RESETTING DPAR(,I2,3H)= ,G14.5)
0078          IF(DPAR(J).EQ.0.) GO TO 201
0079          GO TO 121
0080      125 CONTINUE
0081      126 IF(MZ.EQ.0) GO TO 128
0082          DO 127 I=1,MZ
0083          MZPI=M+I
0084          CALL RESINT(MZPI,RJ(MZPI))
0085      127 CONTINUE
0086    128   IF(NCADIS.EQ.3) GO TO 129
0087          CALL LMULT(1,DELMU,DEM,DU)
0088    129   CALL GRAD1(2)
0089          DO 130 I=1,N
0090          DELX(I)=DEL(I)
0091      130 CONTINUE
0092          DEM=2.0*DPAR(J)
0093          PAR(J)=PAR(J) - DEM
0094          CALL RESINT(0,VAL)
0095          IF(M.EQ.0) GO TO 136
0096          DO 135 I=1,M
0097          CALL RESINT(I,RJ(I))
0098          IF(RJ(I).GT.AVAL) GO TO 135
0099          GO TO 123
0100      135 CONTINUE
0101      136 IF(MZ.EQ.0) GO TO 138
0102          DO 137 I=1,MZ
0103          MZPI=M+I
0104          CALL RESINT(MZPI,RJ(MZPI))
0105      137 CONTINUE
```

```
130    IF(NCRC(4.EQ.0)) GO TO 139
       CALL LMULT(2,DELMU,DEM,DU)
139    CALL GRAD(2)
       UF=(DF-VAL)/DEM
       DO 140 I=1,N
       DELX(I)=(DELX(I) - DELXO(I))/DEM
  140  CONTINUE
       DA(J) = A2H(J)
C  HAVING ALREADY FACTORED A, SOLVE A*X=B FOR X,
C  WHERE H=UFLX AND B=DX/DA(J).
       CALL INVERS(2)
C  PRINT OUT DX/DA(J)
       WRITE(6,150) J
  150  FORMAT(47H  X-DERIVATIVES ARE WITH RESPECT TO PARAMETER ,I2)
       DO 170 I=1,N,6
       II=MIN0(I+5,N)
       WRITE(6,160) (JJ,DELX(JJ), JJ=I,II)
  160  FORMAT( 6(4H DX(,I2,2H)=,G14.7) )
  170  CONTINUE
       IF(NCRC(4.EQ.0)) GO TO 375
       IF(M.EQ.0) GO TO 301
       CALL LMULT(1,DELMU,DEM,DU)
       WRITE(6,350) J
350    FORMAT(/ 41H  U-DERIVATIVES WITH RESPECT TO PARAMETER    ,I2)
       DO 351 I=1,M,6
       II = MIN0(I+5,M)
       WRITE(6,352) ((JJ,DU(JJ)),JJ=I,II)
352    FORMAT(6(4H DU(,I2,2H)=,G14.7))
351    CONTINUE
301    IF(M2.EQ.0) GO TO 375
       CALL LMULT(4,DELMU,DEM,DU)
       WRITE(6,360) J
360    FORMAT(/ 41H  W-DERIVATIVES WITH RESPECT TO PARAMETER    ,I2)
       DO 361 I=1,M2,6
       II = MIN0(I+5,M2)

       WRITE(6,362) ((JJ,DU(JJ+M)), JJ=I,II)
362    FORMAT(6(4H DW(,I2,2H)=,G14.7))
361    CONTINUE
375    CONTINUE
C  COMPUTE DF/DA(J).
       DO 185 I=1,N
       DF = DF + X1H(I)*DELX(I)
  185  CONTINUE
C  PRINT DF/DA(J).
  *    WRITE(6,190) DF
  190  FORMAT(/ 13X,13HDF(X(K))/DA= ,G14.6/10H *********/)
  200  CONTINUE
       GO TO 202
  201  WRITE(6,106) J
  106  FORMAT(1H ,21HTERMINATING PARAMETER,I3,10H DUE TO DRAD = 0   /)
       GO TO 200
  202  NT=NH
       CALL OBJECT
       DO 205 I=1,N
       DELX(I) = DELTX(I)
  205  DELX1(I) = DELT(I)
       RETURN
       END
```

```
              SUBROUTINE LMULT(IND,DELMU,DEM,DU)
              IMPLICIT REAL*8(A-H,O-Z)
              REAL*4 RHOIN,RATIO,EPS1,THETAO,XFP1,XFP2
              COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
              COMMON/VALUE/F,G,DF,FSIGMA,RJ(40),RHO
              COMMON/EQAL/ ME, M1, M2
              COMMON/CRST/DELX(20),DELX(20),RHOIN,RATIO,EPS1,THETAO,
             1NTCTH,NUMIN(,X1(20),X2(20),X3(20),XF2(20),XP1(20),PR1,
             2PR2,F1,F1,RJ1(40),DOTT,PGRAD(20),DIAG(20),
             3DELV,ADELX,NSIG1,G1,NPHASE,NSATIS
              DIMENSION DELMU(40),DU(40)
              MMZ = M + M2
              GO TO (1,2,3,3), IND
        C IND = 0
              DO 100 I=1,MMZ
        100   DELMU(I) = 0.
              RETURN
        C IND = 1
        1     DO 50 I=1,MMZ
        50    DELMU(I) = RJ(I)
              RETURN
        C IND = 2
        2     DO 60 I=1,MMZ
        60    DELMU(I) = (DELMU(I) - RJ(I))/DEM
              RETURN
        3     DO 70 I=1,MMZ
              CALL GRAD(I)
              CALL FESTNT(I,VAL)
              SUM = 0.
              DO 71 JJ=1,N
        71    SUM = SUM + DEL(JJ)*DELX(JJ)
              IF(IND.EQ.4) GO TO 80
              DU(I) =-(SUM + DELMU(I))*RHO/(VAL**2)
              GO TO 70
        80    DU(I) = 2.*(SUM + DELMU(I))/RHO
        70    CONTINUE
              RETURN
              END
```

```
      SUBROUTINE DPRSEN(DU,KTEST)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
      COMMON/VALUE/F,G,FO,FSIGMA,FJ(40),RHO
      COMMON/EQUAL/ N, M1,MZ
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
      DIMENSION GX(40),DU(40),KTEST(20),KLIST(20)
      MPMZ = M + MZ
      FTEST = 0.001 * DABS(F)
      DO 100 J=1,NPAR
      KTEST(J) = 0
      PAR(J) = PAR(J) + DPAR(J)
      CALL FESTNT(0,DF)
      IF(MPMZ.EQ.0) GO TO 20
      DO 10 I=1,MPMZ
      CALL FESTNT(I,DU(I))
   10 CONTINUE
   20 DFM = 2. * DPAR(J)
      PAR(J) = PAR(J) - DFM
      CALL FESTNT(0,XF)
      IF(MPMZ.EQ.0) GO TO 40
      DO 30 I=1,MPMZ
      CALL FESTNT(I,GX(I))
   30 CONTINUE
   40 DFEPS = (DF - XF)/DFM
      IF(MPMZ.EQ.0) GO TO 60
      DO 50 I=1,MPMZ
   50 DU(I) = (DU(I) - GX(I))/DFM
   60 SUM = DFEPS
      IF(M.EQ.0) GO TO 80
      DO 70 I=1,M
   70 SUM = SUM - RHO/FJ(I)*DU(I)
   80 IF(MZ.EQ.0) GO TO 95
      TSUM = 0.
      DO 90 I=1,MZ
      IM = I+M
   90 TSUM = TSUM + FJ(IM)*DU(IM)
      SUM = SUM + TSUM * 2./RHO
   95 DEL(J) = SUM
      PAR(J) = PAR(J) + DPAR(J)
      DTEST = DABS(DEL(J))
      IF(DTEST.GE.FTEST) KTEST(J) = 1
  100 CONTINUE
      WRITE(6,600)
  600 FORMAT(22X,34HOPTIMAL VALUE FUNCTION SENSITIVITY     //)
      DO 200 I=1,NPAR,5
      II=MIN0(I+4,NPAR)
      WRITE(6,601) ((JJ,DEL(JJ)), JJ=I,II)
  601 FORMAT( 5(7H DF/DA(,I2,2H)=,G14.7))
  200 CONTINUE
      JJ = 0
      DO 250 J=1,NPAR
      IF(KTEST(J).EQ.0) GO TO 250
      JJ = JJ + 1
      KLIST(JJ) = J
  250 CONTINUE
      IF(JJ.EQ.0) GO TO 300
      WRITE(6,602)
  602 FORMAT( /51H DETAILED SENSITIVITY RESULTS FOLLOW FOR PARAMETERS  )
      WRITE(6,603) (KLIST(I), I=1,JJ)
  603 FORMAT(1H 40(I2,2H ,))
      WRITE(6,604)
  604 FORMAT(/)
      RETURN
  300 WRITE(6,605)
  605 FORMAT( 43H THERE ARE NO DETAILED SENSITIVITY RESULTS      //)
      RETURN
      END
```

APPENDIX B

USER SUBROUTINES FOR SCHRADY-CHOE PROBLEM

```
      SUBROUTINE READIN
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
901   FORMAT(I5,4F12.0)
      READ(5,901) NI,PAR(1),PAR(2)
      WRITE(6,901) NI,PAR(1),PAR(2)
      DO 100 I=1,NI
      READ(5,901)    IDENT(I),(PAR(4*I-2+J),J=1,4)
      WRITE(6,901)  IDENT(I),(PAR(4*I-2+J),J=1,4)
100   CONTINUE
      NPAR = 4*NI+2
      RETURN
      END
```

```
      SUBROUTINE RESTNT(IN,VAL)
      IMPLICIT REAL*8(A-H,O-Z)

      COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
      COMMON/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
      VAL = 0.
      IF(IN.EQ.0) GO TO 300
      IF(IN.EQ.1) GO TO 100
200   DO 250 I=1,NI
      IJ = 2*I-1
      QQ = X(IJ)
      IF(QQ.LE.0.) GO TO 180
250   VAL = VAL + PAR(4*I+2)/QQ
      VAL = PAR(2) - VAL
      RETURN
180      VAL = -1.0
      RETURN
100   DO 150 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      RR = X(IJ1)
      IF(RR.LT.0.) GO TO 180
      QQ = X(IJ)
      IF(QQ.LT.0.) GO TO 180
150   VAL = VAL + PAR(4*I+1)*(RR+QQ/2.-PAR(4*I-1))
      VAL = PAR(1) - VAL
      RETURN
300   DO 350 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      QQ = X(IJ)
      RR = X(IJ1)

      UU = PAR(4*I-1)

      SS = PAR(4*I)

      DELTA = RR - UU
      ZN = DELTA / SS
      CALL ANORML(ZN,FI,DEN)
      PHI(I) = FI
      DENSE(I) = DEN
      BETA(I) =0.5*((SS*SS+DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I))
350   VAL = VAL + BETA(I)/QQ
      RETURN
      END
```

```
      SUBROUTINE GRADI(IN)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
      COMMON/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
      IF(IN.EQ.0) GO TO 300
      IF(IN.EQ.1) GO TO 100
      DO 250 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      OO = X(IJ)
      DEL(IJ1) = 0.
  250 DEL(IJ) = PAR(4*I+2)/OO/OO
      RETURN
  100 DO 150 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      DEL(IJ1) = -PAR(4*I+1)
  150 DEL(IJ) = DEL(IJ1)/2.
      RETURN
  300 DO 350 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      OO = X(IJ)
      RR = X(IJ1)

      UU = PAR(4*I-1)

      SS = PAR(4*I)

      DELTA = RR - UU
      ZN = DELTA / SS
      CALL ANDTR(ZN,FI,DEN)
      PHI(I) = FI
      DENSE(I) = DEN
      BETA(I) =0.5*((SS*SS+DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I))
      DEL(IJ1)=(DELTA*PHI(I)-SS*DENSE(I))/OO
  350 DEL(IJ) = -BETA(I)/OO/OO
      RETURN
      END

      SUBROUTINE MATRIX(IN,IKK)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SHARE/X(20),DEL(20),A(20,20),N,M,MN,NP1,NM1
      COMMON/INV/BETA(20),PHI(20),DENSE(20),IDENT(20),NI
      COMMON/SEN/PAR(20),DPAR(20),NPAR,ISENS
      IF(IN.EQ.0) GO TO 300
      IF(IN.EQ.1) GO TO 100
      DO 250 I=1,NI
      IJ = 2*I-1
      OO = X(IJ)
  250 A(IJ,IJ) = -2.*PAR(4*I+2)/OO**3
      RETURN
  100 IKK = 1
      RETURN
  300 DO 350 I=1,NI
      IJ1 = 2*I
      IJ = IJ1 - 1
      OO = X(IJ)
      RR = X(IJ1)

      UU = PAR(4*I-1)

      SS = PAR(4*I)

      DELTA = RR - UU
      ZN = DELTA / SS
      CALL ANDTR(ZN,FI,DEN)
      PHI(I) = FI
      DENSE(I) = DEN
      BETA(I) =0.5*((SS*SS+DELTA*DELTA)*PHI(I)-SS*DELTA*DENSE(I))
      A(IJ,IJ) =  2.*BETA(I)/OO**3
      A(IJ,IJ1)=-(DELTA*PHI(I)-SS*DENSE(I))/OO/OO
  350 A(IJ1,IJ1) = PHI(I)/OO
      RETURN
      END

      SUBROUTINE ANDTR(XX,PHI,DENSE)
      IMPLICIT REAL*8(A-H,O-Z)
      AX = DABS(XX)
      T = 1.0/(1.0+0.2316419*AX)
      DENSE = 0.3989422*DEXP(-XX*XX/2.)
      PHI = DENSE*T*((((1.330274*T -1.821256)*T+1.781478)*T
     1 - 0.3565638)*T + 0.3193815)
      IF(XX) 1,2,2
    1 PHI = 1.0 - PHI
    2 RETURN
      END
```

APPENDIX C

SHELL PROBLEM DATA

i \ j		1	2	3	4	5	
c_j		-15	-27	-36	-18	-12	
c_{ij}	1	30	-20	-10	32	-10	
	2	-20	39	-6	-31	32	
	3	-10	-6	10	-6	-10	
	4	32	-31	-6	39	-20	
	5	-10	32	-10	-20	30	
d_j		4	8	10	6	2	b_i
a_{ij}	1	-16	2	0	1	0	-40
	2	0	-2	0	0.4	2	-2
	3	-3.5	0	[illegible]	0	0	-0.25
	4	0	-2	0	-4	-1	-4
	5	0	-9	-2	1	-2.8	-4
	6	2	0	-4	0	0	-1
	7	-1	-1	-1	-1	-1	-40
	8	-1	-2	-3	-2	-1	-60
	9	1	2	3	4	5	5
	10	1	1	1	1	1	1

CATTLE FEED PROBLEM DATA

j		c_j	a_j	μ_j	σ_j
1	Barley	24.55	2.3	12.0	0.53 (parameter 1)
2	Oats	26.75	5.6	11.9	0.44 (parameter 2)
3	Sesame Flakes	39.00	11.1	41.8	4.50 (parameter 3)
4	Groundnut Meal	40.00	1.3	52.1	0.79 (parameter 4)

ϕ = -1.645 (parameter 5) corresponds to probability of 0.95

p_m = 21 (parameter 6)

o_m = 5 (parameter 7)

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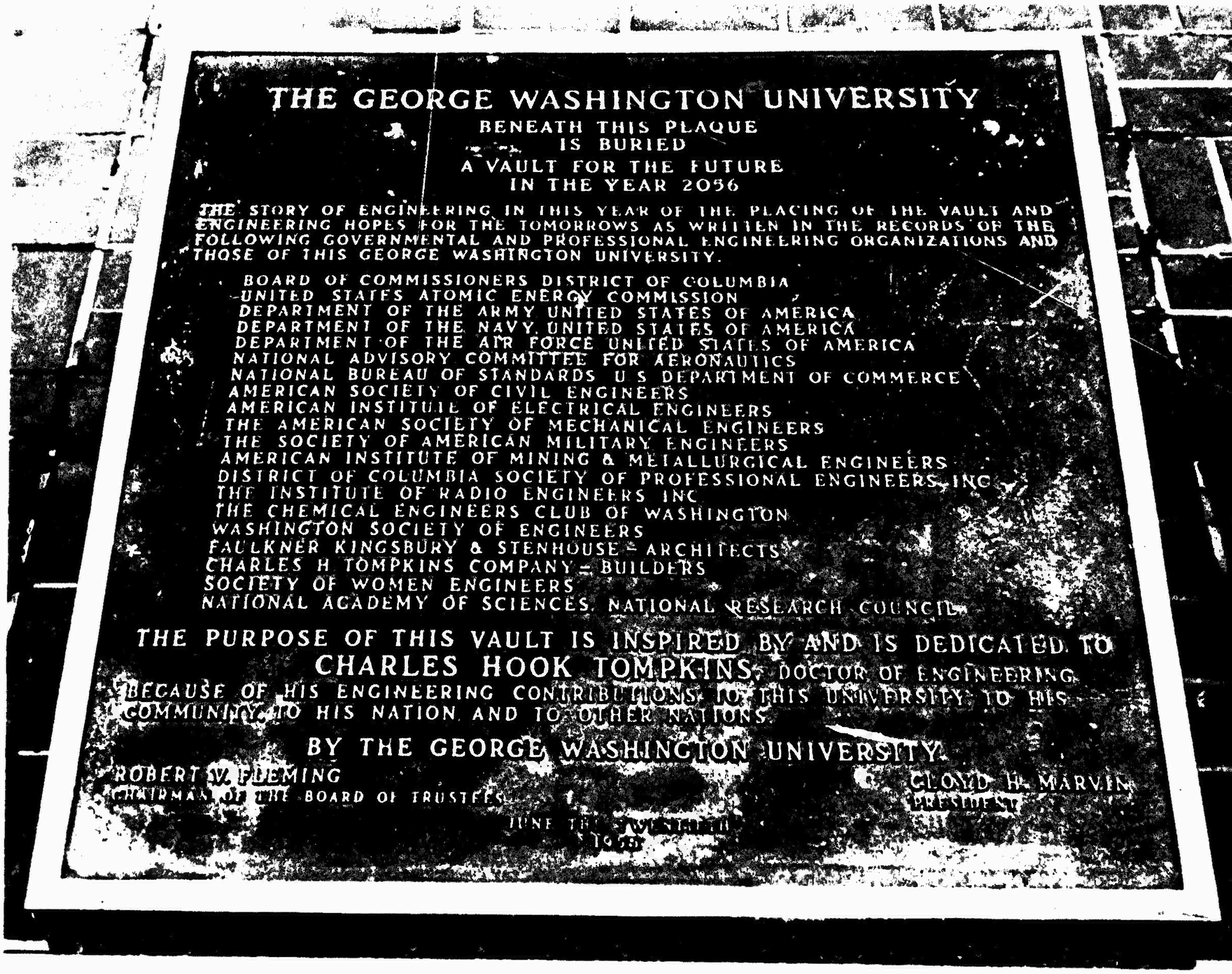

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