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On the Feasibility of Detecting Low Level Acoustic Signals in the Ocean by Use of a Laser Heterodyne Detector

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This report provides a basis for assessing the feasibility of using a laser doppler velocimeter for the measurement of low amplitude acoustic particle velocity in the sea at very low frequencies.

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The writing of this report was supported by NAVSEA Code 06H1, Mr. Carey Smith, under the program management of Mr. Charles Walker.

Purpose of the Report

The purpose of this report is to provide a basis for assessing the merits of a proposal by General Electric Corp. to detect acoustic signals in the water by a laser interferometer in which the laser beam itself is totally submerged. The basis we will use for making this assessment will be to compare the proposed device with state-of-the-art Navy hydrophone capability. For ready reference we take our model of (one of) the best Navy hydrophones available to date to be the USRD H56, whose relevant performance we summarize as follows:

Henriquez (J. Acous. So. Am. 52, p 1450 (1972)) discusses the design of the H56 hydrophone, which he constructed to measure ambient noise in 10 Hz to 60 kHz band. The sensitive element is a tangentially poled PZT-5 cylinder, of which two are incorporated in a single unit. The equivalent noise pressure level of the H56 is 38 dB re $1\,\mu Pa$ in a 1 Hz band at 100 Hz. The free-field voltage sensitivity is -185 dB re $1V/\mu Pa$. Thus the internal noise of the H56 (in a 1 Hz-band) appears as $10^{-7.35}$ volt at the terminals. Knudsen seastate zero is 62 dB re $1\,\mu Pa$ in a 1-Hz band at 100 Hz. This noise in the water appears as $10^{-6.15}$ volt at the terminals. Thus seastate zero is greater than the internal noise of the H56 by a factor of (approximately) 10 in the frequency region less than 1000 Hz, making the H56 ideally suited to the study of ambient sea noise.

The H56 active PZT element has a capture area (i.e., the area which "sees" the wave front) of approximately 200 π mm^2. In a typical single beam gas laser experiment of the doppler-velocimeter type the capture area is about $15\pi \times 10^{-3}\,mm^2$. The ratio of capture areas is thus about 13,000 in favor of the H56. The H56 is a pressure-sensitive device. The proposed virtual acoustic sensor is a particle velocity-sensitive device. Thus the H56 measures the scalar aspect of the acoustic field while the proposed device measures (one component of) the vector aspect of the acoustic field.

The chief problem of the virtual acoustic sensor is minuteness of the threshold signal in comparison to the dominating noise of the medium and of the detection circuit itself. The following report is devoted to an analysis of this signal-to-noise ratio problem.

Summary and Conclusions

We list below the chief conclusions of this investigation.

I. The method of detection of an acoustic signal reviewed in this report is that of a laser heterodyne device designed to measure either of two different quantities (1) a sinusoidal displacement of particles in a fluid (2) a fluid velocity (turbulent or laminar). In the first case the success in detection rests on the capability of measuring magnitude of the power spectrum of a photodetector current in the presence of noise, rather than in the capability of measuring a frequency shift. The important physical quantity in the signal processing is the modulation index,* not a Doppler shift. Hence the acoustic sensor in question is a true displacement device, rather than a velocity device. In the second case the acoustic sensor is designed to measure fluid velocity. Success in detection rests on the capability of measuring Doppler shift in the presence of noise. It is a true Laser Doppler Velocimeter.

Conclusions on the two applications are presented below.

II. As an acoustic sensor of displacement we estimate the capabilities of the laser heterodyne as follows:

a. The theoretical magnitude of modulation index which meets the Navy threshold requirements for detection of a submarine in sea state 1 at 100 Hz is of the order of 8×10^{-6} radian. The possibility of detecting an index of this small magnitude is the core of the feasibility study in the accompanying report.

b. Under the assumption that shot noise is the only noise in the circuitry of an acoustic displacement sensor Massey (1968, Proc. IEEE 56, 2157) calculated that displacements of the order of 10^{-12} M† could be measured in the laboratory for a laser wavelength of 6330×10^{-10} M, equivalent to a modulation index of approximately 2×10^{-5} radian. Thus the Navy threshold to be achieved is (somewhat smaller than) an order of magnitude less than the displacement laser heterodyne capability calculated by Massey to be available in the laboratory in the presence of shot noise.

*For definition see page 71.

†M = meter.

c. Power spectral broadening can be due to causes other than shot noise. When non-shot noise is the dominant feature, the detectable modulation index is shown by mathematical modeling to be a function of the ratio of the acoustic frequency ω_s to the total noise bandwidth ΔB (in units of radians/sec). Orders of magnitude calculations show the following: when the bandwidth of noise in a power spectrum is larger than the acoustic frequency*, the modulation index is of the order of unity. The smallest displacement that can be detected is then about 4×10^{-8} meter. This is five orders of magnitude greater than the Navy threshold. In contrast when the noise bandwidth in a power spectrum is a fraction of the acoustic frequency the detection capability is much greater. Calculation shows that a ratio $\omega_s / \Delta B \sim 7$ corresponds to a capability of detecting a modulation index of 10^{-5}, which is nearly the Navy threshold. This means that at an acoustic frequency of 100 Hz the noise bandwidth must not exceed 14 Hz in order to measure acoustic displacements of the order of 10^{-12} to 10^{-13} meter. The possibilty of reducing noise bandwidth to the limit $\Delta B \sim \omega_s / 7$ is open to investigation.

d. In all calculations of c. we have assumed that the received signal from the scattering volume is large enough to overcome the inherent noise in the photodetector circuit. Under certain simplifying assumptions this noise is

$$i_N = e \sqrt{2\left(\frac{\lambda_\ell \eta}{h c_\ell}\right) P_s B}$$

(e = electron charge, λ_ℓ = laser wavelength, η = detector quantum efficiency, P_s is the received power, B = detector circuit bandwidth, h = Planck's constant, c_ℓ = speed of light).

The received power collected over an area Δa in direction θ is given by

$$\Delta P_s(\theta) = \beta(\theta) P_0 \exp(-\alpha(R_1+R_2)) \frac{(\Delta a)\ell}{R_2^2}$$

($\beta(\theta)$ = scattering function, P_0 = laser power, α = attenuation coefficient, R_1, R_2 distance to / from the scattering volume), ℓ = depth of scattering volume).

Combining the two equations shows that the minimum detectable displacement (to an order of magnitude) is

*This is approximately the case of a *single* laser beam disturbed by noise (at 100 Hz) due to Brownian motion (see Part IV). Multiple beams (or "diversity") may remove this limit (see Part VI.).

$$x_{s\,MIN} \approx 10^{-16}\sqrt{\frac{B}{\Delta P_s}} \quad (meter)^*$$

At a distance of 30 meters in sea-water the backscattered (i.e. θ = 180°) laser power into a lens of diameter 0.1 meter is of the order of $10^{-8} P_o$, where P_o is the laser output power. Hence the minimum detectable signal (order of magnitude) is

$$x_{s\,MIN} \approx 10^{-12}\sqrt{\frac{B}{P_o}} \quad (meter)$$

Hence we conclude that to achieve the Navy threshold at a distance of 30 meters one requires a laser power whose magnitude in watts is about equal to the detector circuit bandwidth magnitude in herz, provided the receiving aperture has a diameter of 0.1 meter. This required laser power can be reduced by increasing the receiving aperture and/or the depth of the scattering volume. The possibility of increasing receiving aperture is very important to the success of the laser heteodyne detector, and is open to investigation.

e. The effect of Brownian motion is serious in that it sets an irreducible magnitude to the noise bandwidth Δ B noted in c. above. However this is true only if the capture area (i.e. the scattering volume) is very small. It is well-known that increasing capture area reduces the effects of random inputs into a detection system by affording an opportunity for increasing the S/N ratio (see "diversity," Part VI.).

g. The effect of "platform motion" is serious only if this motion is correlated to, or has a Fourier component in the same pass band, as the acoustic signal to be processed.

h. The effect of medium inhomogeneities including gas bubbles is serious in that it degrades the laser beam coherence over long path length. However such degradation can be overcome by increasing the receiver capture area.(see Hodara (1966) Proc. IEEE 54, 368)

i. The acoustic particle displacement (hence particle velocity)measured, is one component of a three-component vector, namely the component in-line between receiver and scattering volume. Two component LDV's have been constructed (Greated 1971, J. Phys. E. 4, 585; Blake 1972, J. Phys. E. 5, 623; Grant and Orioff (1973), Appl. Optics 12, 2913). The vector nature of particle displacement does not appear to be a major problem.

j. A space array of laser beams (= virtual array) is essentially an assembly of electro-optic hydrophones of dipole type. The signal processing of the returns from such an array

*For derivation see page 69.

is conventional. The virtual array has the same size as a real array at the same frequency.

k. Since a virtual array of laser beams measures the local particle displacement the effects of reflecting, diffracting or scattering bodies must be taken into account. The virtual array must be located far enough away from such bodies in order to give a true statement of the incoming acoustic particle motion.

III. As a <u>sensor of fluid velocity</u> we estimate the capabilities of the laser heterodyne as follows:

a. Yeh and Cummins (1964 Appl. Phys. Letters <u>4,</u> 176) concluded that they could detect (in the laboratory) constant (i.e. laminar) velocities as low as 4×10^{-5} meter/sec at a scattering angle of 30°.

b. Edwards et. al (1971, J. Appl. Phys. <u>42,</u> 837) estimated that under conditions (in the laboratory) where thermodynamic diffusion of molecules was the limiting factor they could detect constant velocities as low as 10^{-5} meter/sec.

Both of these values are five orders of magnitude greater than the Navy threshold.

c. The basic limit in the use of the laser Doppler velocimeter for measurement of <u>turbulent</u> velocity fluctuations is the <u>Doppler ambiguity</u> (or uncertainty in measuring a frequency shift) due to extraneous time-varying modulation of the laser beam. These modulations are introduced by finite transit time of particles through the scattering volume, turbulent fluctuations across the scattering volume, mean velocity gradients, and circuit noise. Doppler ambiguity limits spatial and temporal resolution.

d. The only measurable velocity inturbulent flow is the Eulerian random velocity $u_o(t)$ averaged over the scattering volume. This is the sum of a mean velocity $\bar{u}_o(t)$ and a fluctuating velocity $u_o'(t)$. The power spectrum of turbulence consists of a mean (Doppler) frequency shift broadened by the spectrum of the fluctuating components. As noted in c. the resolution of the power spectrum of turbulence (that is, its separation out of the noise) is limited by the Doppler ambiguity (DA). If the frequency broadening of the turbulent velocity fluctuations (namely the quantity we wish to measure) is of the same order as the broadening due to Doppler ambiguity (which is the noise we wish to avoid) then there is no way of telling them apart. If ω_o is the mean Doppler shift due to $u_o'(t)$ then the condition of resolution is

$$\frac{u}{\bar{u}} > \frac{DA}{\omega_0}$$

(This Doppler ambiguity poses a severe limit in the determination of the turbulence spectrum.) It is fundamental to recognize that the measurement of laminar flow $\bar{u}$ which is non-random differs from the measurement of turbulent flow $u_0(t)$ which is random. In the latter case there is a largest wavenumber (or highest cut-off frequency) that is measurable for a fixed Reynolds number and fixed scattering angle. Thus the entire power spectrum of velocity turbulence is unattainable. A simple estimate of the largest measurable wavenumber is $k_{MAX} = 2\pi/L$ in which L is the largest dimension of the scattering volume. Thus if L is a number fixed by the LDV the largest turbulence wavenumber measurable is $2\pi/L$, and the rest of the spectrum is unresolvable. Hence if the presence of submarine turbulence is to be determined by examining turbulence scale sizes less than L meters, the LDV method fails. It can be revived by reducing the scattering volume. However, such reduction is accompanied by increase in Doppler ambiguity since space is sampled over a shorter time interval. Mathematical modeling shows that there is an optimum size of scattering volume, L_{OPT}, given by

$$L_{OPT} = \ell_0 \frac{1}{\sqrt{2}}\left(\frac{1}{1.27}\right)\left(R_e \sin\frac{\theta}{2}\right)^{\frac{1}{2}}, \qquad \ell_0 = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

(ϵ = rate of dissipation of turbulent energy per unit mass, ν = kinematic viscosity, R_e = Reynolds number based on the mean velocity = $\bar{u}\lambda_\epsilon / 2\nu \sin\frac{\theta}{2}$), θ = angle of scattering). Wavenumbers greater than $2\pi/L_{OPT}$ are not resolvable because of Doppler ambiguity. The symbol ℓ_0 is the inner scale (meters) of turbulence. When θ = 180°,

$$R_e = \frac{\bar{u}_0 \lambda_\epsilon}{2\nu \sin\frac{\theta}{2}} = \frac{\bar{u}_0\,(5\times10^{-7})}{2\times10^{-6}} = 0.25\,\bar{u}_0$$

so that

$$L_{opt} = \ell_0\,(0.57)(0.5)\,\bar{u}_0^{1/2} \approx \frac{\ell_0}{4}\,\bar{u}_0^{1/2}$$

If the turbulent velocity is 1 meter/sec, the optimum scattering volume is 1/4 of the inner scale of turbulence. This is a very severe restriction. Any attempt to decrease the scattering volume only increases the Doppler ambiguity.

e. George and Lumley (1973, J. Fluid Mech. 60, p. 321) state that " . . . estimates show that the possibility of measuring dissipation spectra in high-speed or in geophysical flows using Doppler velocimeters is quite remote."

Of course the detection of submarine-induced velocity turbulence may not depend on determination of dissipation spectra. It will however depend on determining some portion of the velocity spectrum (versus wavenumber). The portion that can possibly be used is a subject of investigation.

f. The basic analysis (presented in Part II of this report) emphasizes temporal correlation of the velocity spectrum at a single point in space. By use of two velocimeters to sample different scattering volumes one may make two-point velocity correlations. In this way additional statistical moments of the velocity turbulence can be obtained. These may be essential in detecting the presence of a wake of a submarine.

Two-point velocity correlations in a turbulent fluid have been successfully measured by Clark(1970 Ph. D. Thesis, U. of Virginia, Charlottesville).

It is to be noted that single-point statistics are insufficient to characterize turbulent flow.

g. C. J. Bates (July 1974, DISA INFO. No. 16, p. 5-10, 779 Susquehanna Ave., Franklin Lakes, New Jersey 07417) has studied Doppler ambiguity bandwidths in (laboratory) pipe flow of water using an LDV. In a 10 inch pipe at flow rate of 1.4 ft/sec (Reynolds number 2.056 x 10^5) he found the following ratios of spectral broadening:

		Core	Wall
(1)	$\frac{\text{transit time spectral broadening}}{\text{mean Doppler frequency shift}}$	0.636	0.636
(2)	$\frac{\text{turbulence spectral broadening}}{\text{transit time spectral broadening}}$	1.074	38
(3)	$\frac{\text{mean velocity gradient spectral broadening}}{\text{turbulence spectral broadening}}$	0.000719	0.75

Thus, in accordance with IIc. above the smallest unambiguous displacement that can be measured in Bates' experiment is estimated to be 10^{-8} meter.

IV. The key problem of the laser heterodyne detector of low level acoustic signals is its very small S/N ratio. To improve this ratio General Electric Co. proposes to use the diversity technique. Diversity is discussed in Part VI of this report. Successful application of diversity is based on these assumptions: (1) a multiplicity of "channels" (or copies) of the transmitted signal, (2) Rayleigh fading in each channel, (3) the received signal plus noise are statistically independent.

In the laser heterodyne detector the incoming signal plus noise is generally unknown, although pattern recognition leads to positive identification. The noise of the system is primarily due to the medium itself, and is correlated over significant distances in the medium for many frequencies of interest. There is also some question as to whether the amplitudes of the received signals are Rayleigh distributed.

The application of diversity thus will require more insight into the fading properties of the transmission channel from laser to field points and back. If the spatial sampling is at least a half-wavelength of the acoustic frequency to be detected, and if the fading is statistically independent between channels it is anticipated that diversity transmission will increase the S/N ratio significantly. The nominal increase is 10 $\log_{10}N$, where N is the number of channels.

V. Rough estimates of the detection limits of laser heterodyne systems may be made with knowledge of times, distances, absorptions, etc., which are of significance to the generation and transmission of laser light in water. These estimates appear below.

Significant Distances and Times

RMS acoustic particle displacement, planewave equivalent to seastate 1 at 100 Hz	3.4×10^{-13}M
Radius of inner most electron orbit of the hydrogen atom	0.5×10^{-10}M
Laser frequency	5.8×10^{14}Hz
Laser wavelength	5.145×10^{-7}M
Sound Speed in water	1.5×10^{3}M
One period of acoustic wave at 100 Hz	1×10^{-2}s
Speed of light in water	2.26×10^{8}Ms^{-1}
One period of laser light	1.72×10^{-15}s
2-way travel time over 30 M range in water	2.65×10^{-7}s
Laser wavelength in water	3.86×10^{-7}M
Time to traverse one quarter laser wavelength in water	0.426×10^{-15}s
Depth of Volume Interrogation(=half laser wavelength) in water(this depth can be made longer).	1.93×10^{-7}M

Ratio of laser power backscattered ΔP to laser power transmitted P_0 in an "average ocean" at range R for a scattering length of 7.5 meter (at 100 Hz) and a circular receiving aperture of diameter 0.1 meter:

R (meter)	$\Delta P / P_0$
10	1.30×10^{-7}
20	1.20×10^{-7}
30	1.96×10^{-9}
40	4.05×10^{-10}
50	9.53×10^{-11}
60	2.43×10^{-11}
70	6.58×10^{-12}
80	1.85×10^{-12}
90	5.38×10^{-13}
100	1.61×10^{-13}

Brownian Motion and Brillouin Effect

a. The mean distance travelled in time t by a colloidal particle in water at temperature (T) = 300°K, dynamic viscosity $\eta = 10^{-3}M^{-2}s$, diameter a of particle = $3 \times 10^{-6}M$, is

$$\sqrt{\bar{r}^2} = \sqrt{\frac{kTt}{\pi \eta a}} = \sqrt{\frac{1.38 \times 10^{-23} \times 3 \times 10^{2}}{\pi \times 10^{-3} \times 3 \times 10^{-6}}}\sqrt{t}$$

$$= 0.662 \times 10^{-6}\sqrt{t} \text{ meter}$$

Thus the mean distances travelled in significant times due to Brownian motion are:

(a) in one period of acoustic wave at 100 Hz	$0.7 \times 10^{-7}M$
(b) in one period of laser light	$2.75 \times 10^{-14}M$
(c) in time to interrogate two quarter waves of laser wavelength (2-way travel over distance equal to a quarter wavelength)	$1.9 \times 10^{-14}M$
(d) in time for acoustic amplitude to go from zero to maximum (i. e. one quarter period)	$0.33 \times 10^{-7}M$
(e) the time to cover the two-way travel between laser and particle (2 x 30M = 60M)	$0.341 \times 10^{-9}M$

b. When observed with visible light the Brillouin effect in water is equivalent to an acoustic wave at approximately 10^{10} Hz modulating the laser beam. The laser wavelength is then associated with two satellites at wavelength separation of ~ 0.05A (see Appendix E).

Currently Available Laser Systems

Many types of lasers useful for heterodyne experiments are available. The following set of specifications can be considered representative of more recent design achievement.

SPECIFICATIONS*

Brightness:	Model K 1500 is normally furnished with selected ruby laser crystals to provide a brightness up to 7.5 x 10^{13} watts/cm^2/steradian. Through special selection techniques, the K 1500 can be supplied with brightness of 3 x 10^{14} watts/cm^2/steradian.
Peak Power:	1.1 Gigawatt (Max.)
Beam Divergence:	Available from 2.4 mr FAHE down to 1.2 milliradians
Line Width:	Less than 0.06 Å per single line achieved spectral component with optional accessory (KLMS).
Pulse Energy:	10-15 Joules
Ruby Life:	Depends on power levels; typically 300-400 shots at 1.1 gigawatt; much longer at lower power levels.
Beam Size:	Approximately 1.9 x 1.76 cm (elliptical). Can be corrected to round shape with optional sapphire prism.
Repeatability:	±10% for 10 shot series
Wave Length:	6943 Å (Ruby)
Q-switch:	Pockels Cell
Pulse Width:	10-15 Nanoseconds
Jitter:	±10 Nanoseconds
Repetition Rate:	2 PPM
Power Supply:	10 KV
Laser Head:	4" x 9/16 Ruby Oscillator and 9" x 3/4" Ruby Amplifier. Water cooled; helical flash lamp.

*KORAD K1500 (2520 Colorado Ave., Santa Monica, Calif. 90404).

Comment: The Summary and Conclusions noted above, together with auxiliary data on Brownian motion, Brillouin effect, available laser systems, etc. were presented to afford the reader a condensed background of pertinent facts for judging the merit of the proposed use of laser heterodyne systems to measure acoustic signals. A deeper appreciation of the optical-acoustical interaction requires a more detailed mathematical model. This modeling is presented in the remainder of the report. The reader should bear in mind however that while the results of this investigation are based on an intensive effort to reach the key issues involved the points of view presented do not exhaust the range of possibilities in regard to both analysis and experiment.

Introduction

Present day techniques for the detection of underwater sound are based on the local interaction of acoustic pressure with a piezoactive device (vz. hydrophone) which converts alternating pressure into alternating electric current. The limit of detectability is circuit noise (Johnson noise). Sound is also made visible by shadow photography, or schlieren, in which the local change in illumination depends on the change in the gradient of density of the fluid as the pressure wave passes through. Since acoustic pressure (or density) waves are scalars the determination of the direction of the wave is accomplished by arrays of hydrophones in which phase differences between elements are used to maximize sensitivity in the direction of the approaching wave. These arrays must be several wavelengths long to be effective, but length can be reduced by use of acoustic multipoles.

Hydrophone arrays in current use suffer the following deficiencies. (1) at low frequencies an array several wavelengths in size becomes very long, making for costly structures to hold them, or making for errors due to lack of stability in arrays which are trailed behind ships (towed arrays). (2) all sound detection is local. Hence if the array is placed on the ship hull it is disturbed very strongly by presence of the interfering hull, by flow noise, by bubbles, by hull motion, etc. If towed, the long array suffers from local flow noise, from bearing ambiguities, from catenary curvature in the steady drag condition etc. Other deficiencies in hydrophone arrays are sensitivity to depth of operation, lack of precision calibration, and phase and amplitude errors due to non-uniform elements, and/or non-uniform spacing.

It is clearly advantageous to devise a system of sound detection which is not local. A promising technique is to use an optical heterodyne to detect the phase difference between a reference laser beam and a laser beam scattered from suspended particles in motion in afluid. The feasibility of this technique is investigated in the body of this report.

Magnitude of Threshold Underwater Acoustic Signal to Be Detected

In anti-submarine warfare the threshold of detection of a submarine by passive listening is a somewhat arbitrary number. It will be taken here to be the magnitude of a signal of the level of noise in the ocean at sea state 1 at specified frequency in a one herz band. For purposes of this analyses we choose a frequency of 100 Hz. The average deep-water ambient spectrum level of plane wave equivalent of rms noise in a 1 Hz band is found in several references (Horton, Fundamentals of Sonar, p. 61, Urick, Principles of Underwater Sound for Engineers, p. 168). Two values of noise are quoted, one without shipping noise, and the second with shipping noise. We choose here the smallest magnitude of the threshold by excluding shipping noise. This we state the threshold of detection in terms of the following selected plane wave equivalents:

100 HZ	Plane Wave Equiv. in 1 Hz Band
Spectrum pressure level	-70 dB re N/M^2
(or) Particle velocity	$2.13 \times 10^{-10}\ Ms^{-1}$
(or) Power spectrum level	-132 dB re 1 watt M^{-2}
(or) Particle displacement	3.4×10^{-13} M (= 3.4×10^{-3} Angstrom)

The addition of shipping noise increases the threshold amplitude by approximately 20 dB. As the frequency rises the level of ambient-noise spectra first increases (i.e. the ocean becomes more noisy) to about - 65 at 500 Hz, then declines at a rate of about 6 dB per octave until it reaches the thermal noise limit.

Proposed Method of Detection

In the publication "Virtual Operture Sonar" R.M. Ameigh et al, (General Electric Co. 973-SH-347-973-SO-142-01, March 1970) it is proposed to use an optical Doppler radar to operate underwater for the measurement of the motion of natural colloidal-type particles suspended in sea water. Devices of this type have the generic name of Laser Doppler Velocimeter (LDV). Since LDV's come in different arrangements it will be useful here to describe three varieties that are in common use. The first type is the local-oscillator heterodyne arrangement shown in Fig. 1.

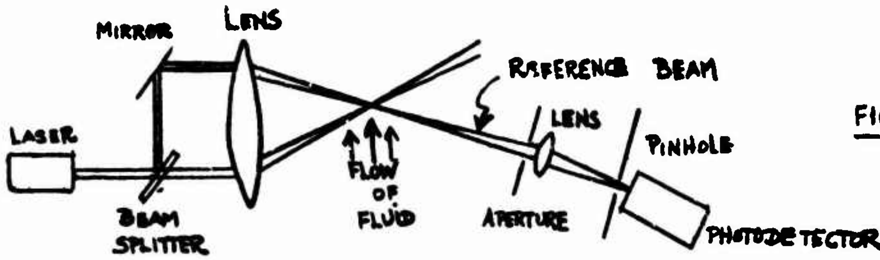


FIG. 1

In this system a local oscillator beam (or "reference beam") couples with the scattered light from a second beam at the focal point of the principal lens and is then viewed directly by the photodetector which detects the Doppler (Goldstein and Kried, (1967), J. Appl. Mech 34, 813). The second type or differential heterodyne arrangement is shown in Fig. 2.

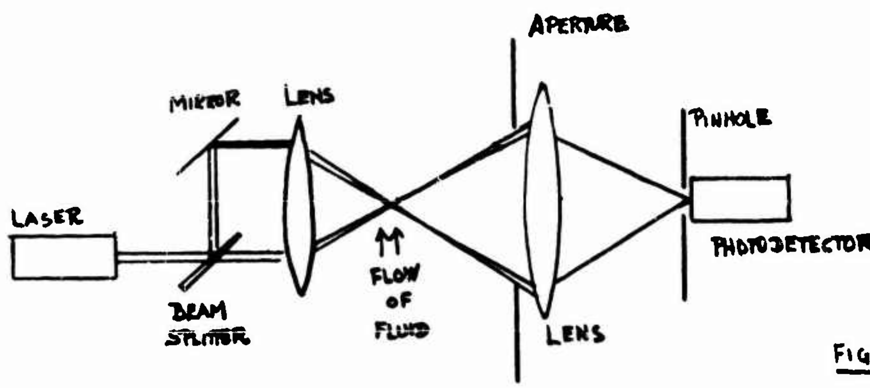


FIG. 2

In this arrangement the light scattered from the common focal region is focused by a lens on to the photodetector which detects the Doppler shift due to motion in the focal volume (Rudd, 1969 J. Phys. E. 2, 55). The third type is the symmetric heterodyne arrangement, Fig. 3

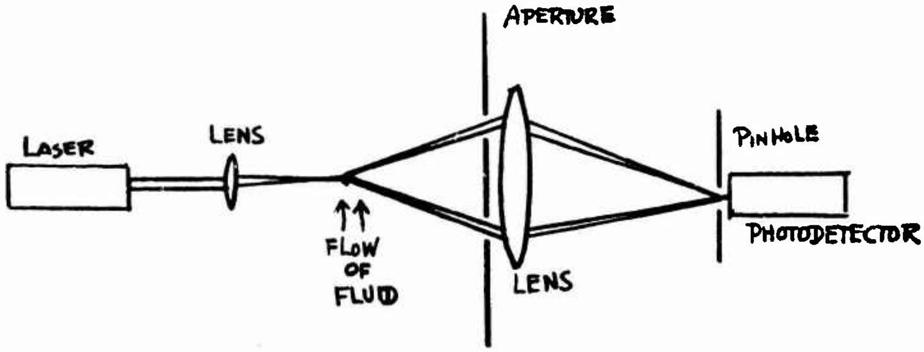


FIG. 3

In this arrangement the scattered light bundle is stopped by an aperture with two slit openings, which are combined by a lens and focused on the photodetector.

It is noted that in all three arrangements of LDV it is necessary to use the heterodyne technique to retain phase information. This means a reference signal (of coherent light) must be mixed with the scattered light. The method of providing the reference signal distinguishes the varieties of LDV.

Unified Mathematical Modeling

Two beam interferometry is briefly reviewed in Appendix A. More complicated devices have been reviewed by Wang (1972, J. Phys. E. 5, 763). He has proposed unified mathematical models for the varieties of LDV shown above. These are listed here for easy reference.

Type I. Local oscillator heterodyne

A reference beam, amplitude a_o, phase ϕ_o, frequency ω_I is assumed to have the form,

$$E_{Lo}(t) = a_o \exp(-i\omega_I t + i\phi_o)$$

The scattered beam is a collection of light scattered by N partices, velocity $\underline{V}_p$ phase ϕ_p, frequency ω_I, amplitude a_p, i.e.

$$E_s(t) = \sum_{p=1}^{N} a_p \exp\left[-i\omega_I t + i\underline{K}\cdot\underline{v}_p t + i\phi_p\right]$$

in which $\underline{K} = \underline{k}_s - \underline{k}_I$ being the scattered wave vector. In brief: there is one incident and one (generally different) scattered direction. The power spectral density of the photodetector current is,

$$S(\omega) = \frac{e^2}{2\pi}\alpha\left(a_0^2 + \sum_1^N a_p^2\right) + e^2\alpha^2\left(a_0^2 + \sum_1^N a_p^2\right)\delta(\omega)$$

$$+ e^2\alpha^2 a_0^2 \sum_1^N a_p^2\, \delta(\omega \pm \underline{K}\cdot\underline{v}_p)$$

$$+ e^2\alpha^2 \sum_{p=1}^{N}\sum_{q\neq p}^{N} a_p^2 a_q^2\, \delta\left[\omega + \underline{K}\cdot(\underline{v}_p - \underline{v}_q)\right]$$

in which e, α are photodetector parameters.

Type II Differential heterodyne

Fig. 2 shows that there are two incident beams with wave vectors $\underline{k}_{I1}$ and $\underline{k}_{I2}$. The scattered light has the mathematical form,

$$E_s(t) = \sum_1^N a_p \exp(-i\omega_I t)\left\{\exp\left[i(\underline{k}_s - \underline{k}_{I1})\cdot\underline{v}_p t + i\phi_{p_1}\right]\right.$$

$$\left. + \exp\left[i(\underline{k}_s - \underline{k}_{I2})\cdot\underline{v}_p t + i\phi_{p_2}\right]\right\}$$

In brief: there are two incident directions and one (generally different) scattered direction. The power spectral density of the photodetector current is,

$$S(\omega) = \frac{e^2\alpha^2}{\pi}\sum_1^N a_p^2 + 4e^2\alpha^2\left(\sum_1^N a_p^2\right)\delta(\omega) + e^2\alpha^2\sum_1^N a_p^4\,\delta(\omega \pm (\underline{k}_{I2} - \underline{k}_{I1})\cdot\underline{v}_p)$$

$$+ e^2\alpha^2 \sum_{p=1}^{N} \sum_{q \neq p}^{N} a_p^2 a_q^2 \Big\{ \delta\left[\omega \pm (\mathbf{k}_s - \mathbf{k}_{I1}) \cdot (\mathbf{v}_p - \mathbf{v}_q)\right]$$

$$+ \delta\left[\omega \pm (\mathbf{k}_s - \mathbf{k}_{I2}) \cdot (\mathbf{v}_p - \mathbf{v}_q)\right] + \delta\left[\omega \pm (\mathbf{k}_{I2} - \mathbf{k}_{I1}) \cdot \mathbf{v}_p \pm (\mathbf{k}_s - \mathbf{k}_{I1}) \cdot (\mathbf{v}_p - \mathbf{v}_q)\right]$$

$$+ \delta\left[\omega \pm (\mathbf{k}_{I2} - \mathbf{k}_{I1}) \cdot \mathbf{v}_p \pm (\mathbf{k}_s - \mathbf{k}_{I2}) \cdot (\mathbf{v}_p - \mathbf{v}_q)\right]$$

<u>Type III</u> <u>Symmetric Heterodyne</u>

In the symmetric heterodyne the scattered light field is given by

$$E_s(t) = \sum_{1}^{N} a_p \exp(-i\omega_I t) \Big\{ \exp\left[i\,(\mathbf{k}_{s_1} - \mathbf{k}_I) \cdot \mathbf{v}_p t + i\phi_{p_1}\right]$$

$$+ \exp\left[i\,(\mathbf{k}_{s_2} - \mathbf{k}_I) \cdot \mathbf{v}_p t + i\phi_{p_2}\right]\Big\}$$

In brief: there is <u>one</u> incident direction and <u>two</u> scattered directions. The power spectral density is quite complex, but it can be retrieved in the case of common particle velocity by replacing $(\mathbf{k}_{I2} - \mathbf{k}_{I1})$ with $(\mathbf{k}_{s_2} - \mathbf{k}_{s_1})$ in the above formula for Type II.

<u>Concluding Remarks</u>: The power spectra formulas derived by Wang (loc. cit) are spectral <u>lines</u>, the linewidths not being considered. These spectra serve to identify the prominent features of the Doppler – induced frequency shifts. However, by failing to present linewidths the spectra do not assist us in estimating the detectability of these lines in the presence of noise originating both in the fluid and in the LDV itself. Since the inherent noises in the systems will be our chief concern we will be compelled to find and develop

more detailed models. A list of the important noises in LDV systems is given here.

Noises in LDV Systems

The linewidths of the Doppler-shifted spectrum originate from the following causes:

1. laser light non-chromaticity.
2. Brownian motion of the colloidal particles.
3. velocity gradients across the scattering volume.
4. fluctuations of velocity in the scattering volume (i.e. turbulence).
5. angular uncertainties due to the divergence of the incident beam and detector angular aperture.
6. finite passage of scattering particles through the laser beam.
7. temporal jitter of the electronic LDV apparatus.
8. temporal and spatial changes in the index of refraction of the scattering fluid.

In order to account for such a profusion of quantities which broaden the Doppler-spectrum we must find their order of appearance in the more detailed model cited above. Since the model to be developed is very complex it will be useful to first present a physical picture of a generic LDV, following which we will develop the math model with greater understanding.

Part I

Simplified Model of the Laser Heterodyne Detector of Sound

Simple Model of a LDV

The optical heterodyne (or Laser Doppler Velocimeter (LDV)) can be modeled in a simple way (Rudd, J. Sci. Inst. Vol. 2 Series 2 1969, p.55), Fig. 4

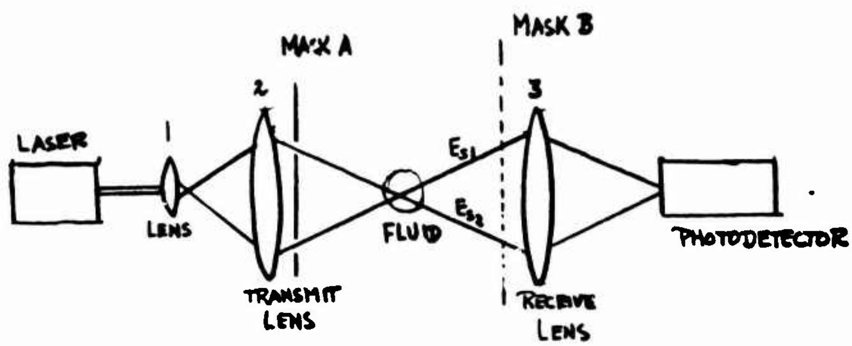


FIG. 4

Here, a laser beam is first expanded (Station 1) then focused by a transmission lens in the fluid (2) then collected by a receiving lens (3) on to a photodetector. In order to produce "beats" (i.e. a fringe system) the beam from (2) is masked (MASKA) by a screen with two splits before it is brought to a focus in the fluid. Alternatively the beam is masked at position 3 by MASK B, after it leaves the focus in the fluid. Mask A generates a real set of fringes. Mask B generates a virtual set of fringes.

The two positions of the masks correspond to two systems of LDV in current use. A system with Mask A describes the Goldstein-Kreid experiment (J. Appl. Mech. 1967 34 813-8). In this experiment a real set of fringes is generated at the focal volume on the fluid by beam splitting the laser light, then bringing the two beams to a focus at an arrival angle 2ϕ between them. Scattering then occurs, which is then detected (as in Fig. 1).

A system with Mask B describes the Yeh-Cummins experiment (Appl. Phys. Letters 4 176-8 1964). Here scattering occurs first. The scattered light is then brought together with the unscattered (or reference) beam (at angle 2ϕ) to form a virtual fringe system which is then detected by the photodetector.

Simple Description of Photocurrents

When MASK A is the position of the effective screen, and a set of real fringes is available, a scattering particle entering the focal region and crossing the fringes sampies the succession of light and dark bands. The scattered light is alternately bright and dark, with maximum brightness occurring in the center of the fringe system. The two scattered laser beams E_{s_1}, E_{s_2} are modulated by the fluid flow (at velocity V_f), and are multiplied by the photomultiplier to give an electric current proportional to their product, i.e.

$$I \propto \frac{E_{s_1}^2}{2} + \frac{E_{s_2}^2}{2} + E_{s_1} E_{s_2} \cos(2\pi f_D t)$$

in which f_D is the Doppler shift in the laser frequency due to a (constant) velocity of the scattering particle. A sketch of this current is shown in Fig. 5

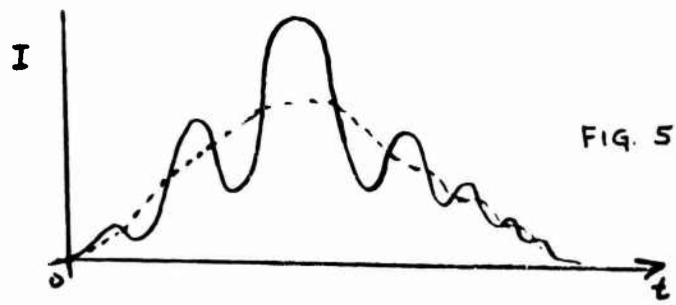


FIG. 5

When MASK B is in position a set of virtual fringes is found in the focal volume. A particle crossing this volume at constant velocity V alternately scatters and does not scatter light as it is illuminated with the light and dark bands of the fringes. The scattered light E_s is combined with a reference beam E_R (i.e. the second beam generated by MASK B) which is detected as a current I where

$$I \propto \frac{E_R^2}{2} + \frac{E_s^2}{2} + E_R E_s \cos(2\pi f_D t)$$

A sketch of this current is shown here,

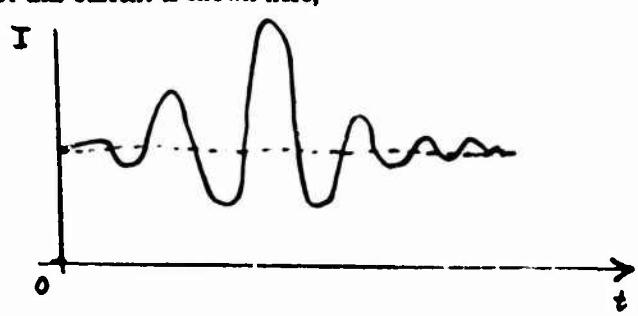


FIG. 6

The Doppler Shift

To understand the Doppler shift (which is the same in both experiments) we first consider the focal volume to be infinite, i.e. we first consider an infinite set of fringes to be available, each of equal intensity. A scattering parti e crossing there fringes at right angles (to the fringe lines) samples the alternating light and darkness (as noted earlier). This has the same effect as making the particle stationary and allowing the set of fringes to slide over the particle. Let the speed of the fringe motion be V_f. If the wavenumber of the fringes is $K_E (= 2\pi/\Lambda_E)$ then the radian frequency of appearance of (say) the brightest scattered light is $K_E V_f$ (radians per sec), that is, the time required to ride over one wavelength is $\Lambda_E / V_f = T$ (in units of seconds/cycle), so that the frequency of alternating darkness and light is $T^{-1} = f_D$ which is a "Doppler Shift." Since light frequency Ω_E is given by $K_E C_E$ we also see that $\omega_D = \Omega_E (V_f / C_E)$, or $f_D = f_E (V_f / C_E)$

We next take the focal volume to be finite, that is, we consider the set of fringes to be bounded in space. The time history of the scattering of light is then as follows: zero before the (finite) set of fringes rides over the particle; abruptly oscillatory when the fringes cross the particle; abruptly zero again at the termination of the fringe set. The frequency of appearance of the light and dark scattered light must be measured from the <u>entire</u> time record of $-\infty$ to $+\infty$. This record can be visualized as a infinite sinusoid screened by a black screen with a single rectangular window. The time history of scattered light is thus a product of two functions (sinusoid times window). The Fourier spectrum of this time history is the convolution of the spectrum of each function separately, i.e. the convolution of a delta function $\delta(\Omega - \Omega_E)$ and a $\sin x/x$. The resultant spectrum (in the range $\omega = 0$ to $\omega \to \infty$) is a $\sin x/x$ centered at Ω_E. Thus the spectral representation of the time history of the crossing of a particle through a finite set of fringes has a spectral width of the first lobe of the $\sin x/x$ spectrum. If Δt is the (finite) time of transit of the particle then the bandwidth of the first lobe of $\sin x/x$ is $\Delta f = (\Delta t)^{-1}$ (Bracewell "The Fourier Transform" p. 128, and p. 368). Thus the spread in the Doppler, Δf_D, due to finite transit time Δt_D is $\Delta f_D = (\Delta t_D)^{-1}$ This spread is essentially a representation of <u>uncertainty</u> in the Doppler (see Figs. 5, 6).

A second approach to the calculation of Doppler spread due to finite transit time is to use the model given by Fig. 1, and consider the masks to be two slits of the equivalent interferometer. The aperture of the equivalent slits is the spatial function

$$E\left(\frac{x}{\lambda}\right) = \Pi\left(\frac{x+W}{w}\right) + \Pi\left(\frac{x-W}{w}\right) \tag{I.3}$$

in which Π is a rectangle function of unit height, centered at $x = \pm W$, and w units wide. The Fourier transform of this aperture is,

$$P(f) = \frac{2w}{\lambda}\,\mathrm{sinc}\left(\frac{wf}{\lambda}\right)\cos\left(2\pi\frac{W}{\lambda}f\right) \tag{I.4}$$

(Bracewell, p. 283). The power spectrum is (approximately) the square of this, or

$$S(f) = 4\left(\frac{w}{\lambda}\right)^2\left(\frac{\sin(\pi w f/\lambda)}{\pi w f/\lambda}\right)^2\cos^2\left(2\pi\frac{W}{\lambda}f\right) \tag{I.5}$$

If we set the variable $x/\lambda = t$, ($\frac{x}{\lambda}$ is interpreted as time) then the "width of the interferometer" is calculated in seconds, $w/\lambda = \Delta t$ sec. and the first zero of $S(f)$ occurs at $\Delta t\, f = 1$, or $f = (\Delta t)^{-1}$ (as before). We thus model the spectrum of the photodetector current by $S(f)$.

Basic Coherence Requirements

In the model of Fig. 4 the basic optical system is that of a two beam interferometer. A review of this system will help define coherence requirements. Let $E_1 (= A_1 e^{iks_1})$ be the complex amplitude of the first beam at distance S_1 from the two-slit screen, and let $E_2 (= A_2 e^{iks_2})$ be the second complex amplitude observed at the same observation point but traveling distance S_2. The total intensity of light at P is

$$I = EE^* = A_1^2 + A_2^2 + 2A_1A_2\cos(k\ell) \tag{I.6}$$

in which $\ell = s_1 - s_2$ is the optical path difference. Since $\cos(k\ell)$ is periodic we define an order number (K.D. Mielenz, p. 89 "Electron Beam and Laser Technology" ed. L. Marton, 1968), $m = \frac{k\ell}{2\pi} = N + \varepsilon$, $0 \le \varepsilon < 1$, N integral.

Now, to quantities of small $\Delta\ell$, Δk

$$\Delta m = \left[\left(\frac{\partial m}{\partial \ell}\right)^2(\Delta\ell)^2 + \left(\frac{\partial m}{\partial k}\right)^2(\Delta k)^2\right]^{1/2} \tag{I.7}$$

This means that the phase kl has an uncertainty in order number $(= \Delta m)$ due to an uncertainty of path difference $\Delta\ell$ caused by finite extension of the source (i.e. of the slit) and to an uncertainty in wavenumber Δk due to lack of monochromaticity. Assume next that there is an incremental uncertainty in kl i.e.

$$kl = 2\pi m + 2\pi\,|\Delta m| \tag{I.8}$$

If $|\Delta m| = 1/2$, it is seen that the change in intensity ΔI due to uncertainty in m is

$$\Delta I = A_1^2 + A_2^2 - 2A_1A_2\cos(kl)$$

Thus the uncertainty in intensity cancels the cosine term of the intensity itself leading to the conclusion that the condition $|\Delta m| = 1/2$ results is incoherent superposition of the two light beams, with the disappearance of observable interference effects. Thus the limit of coherence is given by

$$(k\Delta l)^2 + (l\Delta k)^2 = \pm\pi \tag{I.9}$$

One can arbitrarily define a coherence condition if $|\Delta m| \ll 1/4$, or if,

$$(k\Delta l)^2 + (l\Delta k)^2 \ll \pi/2 \tag{I.10}$$

The two parameters, Δl, Δk, require separate investigation.

Case I. Assume source (i.e. slit) is infinitesimaly small so that l is exactly specifiable, (i.e. $\Delta l = 0$) and assume that the source emits a spectrum of finite width $(\omega_2-\omega_1) = 2(\Delta k)C_E$.

The uncertainty in the order number is Δm where

$$\Delta m = \pm\frac{l}{2\pi}\frac{(\omega_2-\omega_1)}{2C_E} = \frac{\pm l\,(f_2-f_1)}{2C_E} \tag{I.11}$$

The temporal coherence length of the source is thus found from the condition $l_{coh}\Delta k = \pm\pi$ to be

$$l_{coh}\frac{(\omega_2-\omega_1)}{2C_E} = \pi, \quad \left(\text{or} \quad l_{coh} = 2\pi\left[\frac{1}{(\omega_2-\omega_1)/C_E}\right]\right) \tag{I.12}$$

Thus the greatest path difference over which fringes can be observed is controlled by the frequency spread of the source. When the source is white light, $\omega_2-\omega_1$ is very large, and the temporal coherence length is small, i.e. fringes are observed only over very small path distances. In contrast laser light has a value $(\omega_2-\omega_1)/C_E \approx 2\times10^{-12}$. Hence $l_{coh} \sim 2\pi/2\times10^{-12} \approx 3\times10^{12}$ cm or 30 million kilometers.

In a multimode laser the Doppler width is much greater, e.g. a neon line is $(\omega_2-\omega_1)/C_E \sim$ 0.3 rad/cm. Since the mode separation in a laser of length L is $\Delta k_c = \pi/L$, it is seen that the number of possible modes is

$$N = \frac{(\omega_2-\omega_1)}{C_E}\cdot\frac{1}{\Delta k_c} \tag{I.13}$$

For a laser length $L = 100\,\mathrm{cm}$, $\Delta k_c \sim 0.03$ rad/cm. Thus 10 modes may be in operation. The coherence length with 10 modes is thus reduced to

$$l_{coh} = \frac{2\pi}{0.3} \sim 20\ \mathrm{cm}$$

Case II. Spatial Coherence

Assume one (slit) source has a finite (width) dimension (say QQ'), and emits perfect monochromatic light ($\omega_2 = \omega_0 = \omega_1$). Let α be the angle at observation point P which encloses QQ' (Fig. 7*). From Q there is emitted a wave which arrives at P simultaneously with a wave from the corresponding Q of the second slit, with a path difference l. The geometric picture of two wavefronts at P can be duplicated at Q by assuming P to issue two wavefronts which intersect at Q (of one slit) with the same path difference l. The wavefronts W_1, W_2 also intersect in the plane of the figure along a line. Assume point A is on this line of intersection. A is chosen such that one wavefront passes through Q' of the slit. The second wavefront is at a path difference (to Q') given by $Q'S$. If the slit dimension QQ' is projected on the line of intersection to give dimension g, then the path difference $Q'S$ is (approximately) $g\alpha$. Thus the effect of finite slit size is to introduce a path length uncertainty, from zero at Q then along all intermediate points to $\pm g\alpha$ at Q'. Using the center of the slit as reference the uncertainty is zero to $\pm g\alpha/2$ which is Δl of the above formula Eq. I.7. The source is spatially coherent if

$$\frac{2\pi}{\lambda_E}\left(\frac{g\alpha}{2}\right) \ll \frac{\pi}{2} \qquad \text{or} \qquad g\alpha \ll \frac{\lambda_E}{2} \tag{I.14}$$

The largest source (width) with which interference fringes can be observed is called g_{coh} which is twice the above number, or

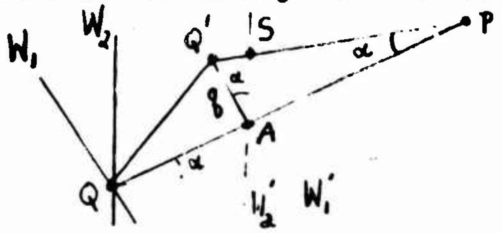
$$g_{coh} = \lambda_E/\alpha \tag{I.15}$$

Spatial coherence is thus controlled by slit width and wavelength of the laser light.

Statistical Nature of Coherence

Interference by a single pair of wavetrains is too rapid for any detector to follow. Under actual experiment the set of fringes generated by the equivalent mask is the statistical average of a large number of wavetrains. The coherence time of a typical gas laser is of the order of 10^{-2} sec. Although this allows transient tracking by a detector of

* FIG 7



single pair interference fringes, the time required for the remainder of the experiment and signal processing only allows statistical averages.

For ideal two beam interference the intensity at the observation point is (as noted above)

$$I = A_1^2 + A_2^2 + 2 A_1 A_2 \cos(kl) \tag{I.16}$$

and the fringe visibility is

$$V = \frac{2 A_1 A_2}{A_1^2 + A_2^2} \tag{I.17}$$

For a general two beam interference experiment this is replaced by a statistical statement,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, |\gamma_{12}(\tau)| \cos[\alpha_{12}(\tau)] \tag{I.18}$$

in which $\gamma_{12}(\tau)$ is the complex degree of mutual coherence of two points S_1, S_2 in the wavefield, $|\gamma_{12}|$, α_{12}. are its magnitude and angle, and τ is a time shift between the signals from S_1, and S_2. The angle $\alpha_{12}(\tau)$ is obtained from the phase condition for the appearance of bright fringes,

$$\alpha_{12}(\tau) = 2\pi N, \qquad N = \text{integer} \tag{I.19}$$

The magnitude $|\alpha_{12}|$ is obtained from observation of the two beam interferometer measurement of visibility at P, called V_{meas},

$$|\gamma_{12}(\tau)| = V_{meas} \times \frac{I_1 + I_2}{2\sqrt{I_1 I_2}} \tag{I.20}$$

$$V_{meas.} = \frac{I_{MAX} - I_{MIN}}{I_{MAX} + I_{MIN}} \tag{I.21}$$

Degradation of coherence is caused by several factors: (1) increase in path length (2) change in particle numbers and particle density in the fluid (3) change in index of refraction of the fluid. A typical experiment to test changes in coherence is to generate a set of interference fringes in space, and observe the distortion of these fringes due to the above factors. The degradation of coherence in a two-beam interferometer experiment using an argon laser, 2 mm beam, 5145A° has been studied by R. E. Lee et al (Opto-electronics) 5 (1973) 41-51. These authors used distilled water, coastal water and fabricated salt water as transmitting media. All water samples showed approximately the same value of normalized degree of coherence (about 0.97), independent of their nature and independent of path length (for paths up to 3 meters). Agitating the water mechanically by moving a stick back and forth along the water trough showed no change in fringe pattern, except in the fabricated salt water where a transient reaction (i.e. spotty unsteady motion of the fringes) was observed due to a transient non-homogeneous condition of the water-salt mixture. This transient disappeared after a few moments. The interference patterns generated in space were recorded by a traversing photomultiplier. These records show (a) relatively smooth oscillations of light intensity due to fringes for tap water and distilled water (b) ragged or spiky oscillation of light intensity for coastal water and fabricated salt water, both containing suspended particles. The spikes appear to correspond to the observed low frequency local wandering of the fringes. It is concluded that any inhomogeneity of the medium which disturbs the phase uniformity of the laser beams will decrease the visibility of the fringes (i.e. decrease coherence). As noted earlier, these disturbers are (1) suspended particles which introduce (moving) delta function-type density changes (2) temperature changes which also introduce mass density changes in the medium. When these inhomogenities are functions of time they cause the fringe system to undergo wriggly, sloshing motion, with transient brightening and fading. The frequency spectrum of the photomultiplier current shows considerable broadening ("noise", "Doppler noise").

<u>Coherence Requirements for LDV</u>

The temporal coherence requirements for a two-beam interferometer noted above are somewhat modified when fluid motion is specifically to be measured. If the spread of wavelengths in the source is $\lambda_2^E - \lambda_1^E$ and the spread of the Doppler is $f_2 - f_1$ then one requires that

$$\frac{\lambda_2^E - \lambda_1^E}{\lambda_0^E} \leq \frac{f_2 - f_1}{f_0}, \qquad \lambda_0^E, f_0 = \text{nominal laser wavelength, freq.} \tag{I.22}$$

(Rudd, loc. at eq. (13)). Similarly, the spatial coherence requirement for an LDV is given by specifying the width d of the equivalent slit in the two-beam interferometer,

$$d < \frac{\lambda_0}{2\sin\alpha}\left(\frac{f_0}{f_2 - f_1}\right) \tag{I.23}$$

(Rudd, eq. (16)). The definition of α is the same as in section on Basic Coherence Requirements.

A Phasor Description of the Received Signal in an LDV

A single particle crossing the (finite) fringe pattern at constant velocity samples the pattern and generates the (scattered) wave packet shown in Fig. (6). If its motion is random it will generate pieces of wavepackets in a random manner, thus introducing random noise whose spectral bandwidth is inversely proportional to the time duration of the packet. A time record of random particle motion in an LDV will be a series of spikes.

If many particles are simultaneously crossing the fringe pattern, each scattered wave-packet will have different phase. The net amplitude of the wavefront scattered by all the particles will be proportional to the square root of their number.

If the particle samples the fringe pattern periodically (frequency ω_s) it will generate a frequency modulation of the original carrier wave (frequency Ω_E) that generated the fringe pattern. The net time function of the scattered wave (for particle velocity $\underline{v}$, acoustic modulation μ_{FM}) is, $\underline{E}_s = \underline{A}_o \, \mathcal{R}e \, \exp i\left\{\Omega_E t + \mu_{FM}\sin\omega_s t + \underline{K}\cdot\underline{v}t\right\}$

or

$$\underline{E}_s = A_o \cos\left\{\Omega_E t + \mu_{FM}\sin\omega_s t + \underline{K}\cdot\underline{v}t\right\} \tag{I.24}$$

in which μ_{FM} is the modulation index ($= \underline{K}\cdot\underline{h}$, $\underline{h}$ defined below), and $|K| = \Omega_E / C_s$.

If the brightness pattern is subject to fading then amplitude A will be a function of time ($= A(t)$) so that the scattered wave will be amplitude modulated.

If N particles participate in the scattering process during an observation time the time history of the scattered wave will be

$$\underline{E}_s(t) = \sum_m A_n(t)\,\mathcal{R}e\, e^{i\underline{K}\cdot\underline{v}t + i\Omega_E t + i\mu_{FM}\cos\omega_s t} + \sum_{N-P}^{N} A_n(t)\,\mathcal{R}e\, e^{i\phi_n(t) + i\Omega_E t_o + i\underline{K}\cdot\underline{v}t_o}\,\delta(t-t_o) \tag{I.25}$$

in which P particles move with a signal (frequency ω_s) and $N-P$ move randomly. If particles all move with same signal, but have random components of motion as well, then

$$\underline{E}_s(t) = \sum_m \mathcal{R}e\left[A_m(t) + \sum_k B_k(m,t)\, e^{i\phi_k(m,t)}\,\delta(t-t_k)\right] e^{i\Omega_E t + i\mu_{FM}\sin\omega_s t + i\underline{K}\cdot\underline{v}t} \tag{I.26}$$

This model says that the n'th particle will undergo k random steps (the exact number depending on m), scattering a spike at each step, while simultaneously scattering a modulated carrier. If the signal is greater than the spikes the scattered light will appear as the time function, Fig. 8(a), 8(b)

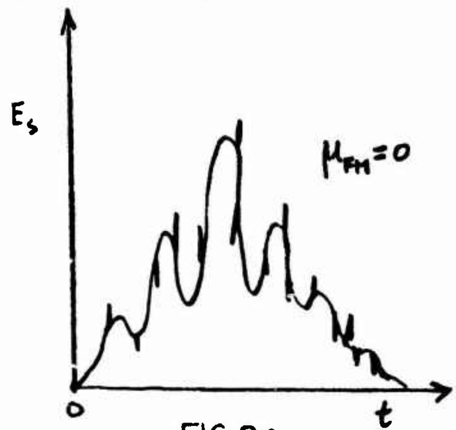


FIG 8a.

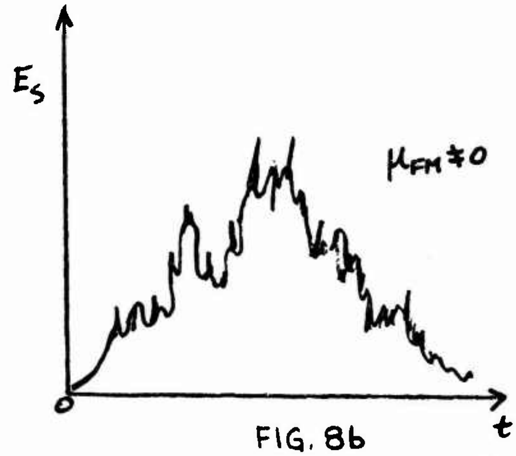


FIG. 8b

If the signal is very small relative to the spikes, then this time function appears as Fig. 8C,

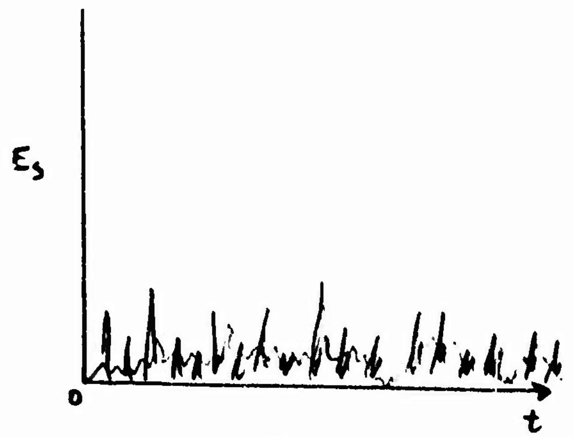


FIG. 8c

Finally let us suppose that all the particles simultaneously undergo a (nonsignal) platform motion which results in a (possibly random) phase and amplitude change, $Ce^{i\eta(t)}$

Then,

$$E_s(t) = \sum_m \mathcal{R}_e \left[A_m(t) + \sum_k B_k(m,t_k) e^{i\phi_m(m,t_k)} \delta(t-t_k) \right] C(t) e^{i\eta(t)} \tag{I.27}$$

$$\cdot\, e^{i\Omega_s t + i\mu_{FM}\cos\omega_s t + i\underline{K}\cdot\underline{v}t}$$

Scattered Field In terms of the Simplified Model (Edwards et al J. Appl. Phys. 42, 837 (1971)).

Let the finite scattering volume ΔV have N particles at time $t=0$. The distance moved by the m^{th} particle is $\underline{r}_{om}(t) = \underline{r}_{om}(0) + \Delta \underline{r}_{om}(t) + \underline{v}t + \underline{h} \sin \omega_s t$

The scattered laser field may be modeled as follows:

$$E_s(\underline{r},t) = \mathrm{Re}\, \underline{A} \sum_m \frac{\Delta \rho}{\rho_o}\left(\underline{r}_{om}(0) + \Delta r_{om}(t) + \underline{v}t + \underline{h} \sin \omega_s t \right)$$

$$\times \exp i \left[(\underline{K} - \underline{K}') \cdot \left(r_{om}(0) + \Delta \underline{r}_{om}(t) + \underline{v}t + h\left[\underline{r}_{om}(0)\right] \sin \omega_s t \right) \right]$$

$$\times \left(\exp - i \Omega_E t\right) \Delta V(\underline{r}_{om})$$

(The significance of the terms appearing in this math model are detailed in Appendix H) Here we desire to describe the scattering process in terms of the simple model of particles sampling fringes. The following points are to be noted:

(1) the initial phase of E_s is $\exp i\, (\underline{K} - \underline{K}') \cdot \underline{r}_{om}(0)$.

(2) each random walk step $\Delta \underline{r}_{om}(t)$ samples a fraction of the fringes and introduces spikes, plus wavering noise in the scattered light associated with phase changes $\exp i\, (\underline{K} - \underline{K}') \cdot \Delta \underline{r}_{om}(t)$.

(3) the steady velocity $\underline{V}$ samples the entire fringe set and generates a complete wave packet of the sinusoid $\cos(2\pi f_D t)$ multiplied by the window function defined by the scattering volume.

(4) the sinusoidal particle displacement $\underline{h} \sin \omega_s t$ samples the fringes periodically. The number of fringes sampled depends on the amplitude $\underline{h}$ and number of cycles $\omega_s t$. If $\omega_s t$ is small then the sinusoidal sampling appears as a (nearly constant) velocity $\underline{h}\omega_s$, additive to $\underline{V}$, *i.e.* the apparent Doppler is $(\underline{K} - \underline{K}') \cdot (\underline{V} + \underline{h}\omega_s)$. If many cycles $\omega_s t$ occur

during the observation time then the number of fringes sampled depend on $\underline{h}$. To clarify the physical picture we assume the steady velocity V is zero. At time t and position $\underline{r}_{on}(0)$ inside the fringe set, the amplitude of particle displacement is $\underline{h}\,[r_{on}(0)]$. Let $2h = \ell$ fringes (ℓ not generally integer). If ℓ is less than one fringe (namely, less than a half wavelength of light at frequency Ω_E) the scattered electric field exhibits little phase modulation (it is black, gray or white, almost independent of time). If ℓ is exactly two neighboring fringes, the scattered field is nearly sinusoidal, with equal plus and minus amplitudes. If ℓ is more than two fringes, plus a fraction, the scattered electric field is periodic, but not sinusoidal, see Fig. (9).

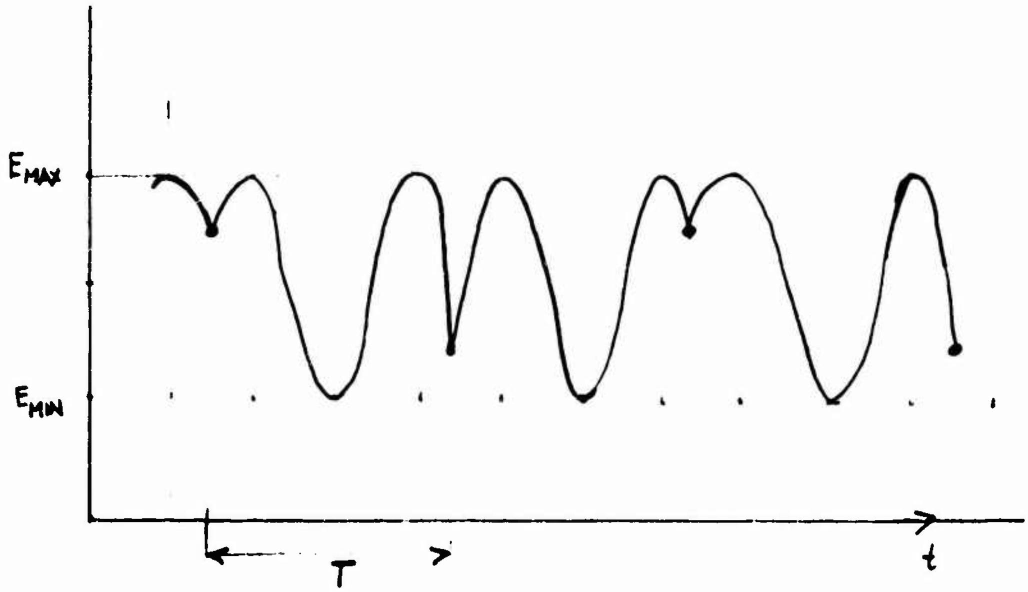


FIG. 9

in which the period of sampling is $T = 2\pi/\omega_s$. The modulation index M is $(\underline{K} - \underline{K}') \cdot \underline{h}[\underline{r}(0)]$ in which $|\underline{K}| = \Omega_E/c_E$ The Jacobi expansion of $\exp(iM\sin\theta)$ is found in Watson "Bessel Functions." page 22),

$$e^{iM\sin\theta} = J_0(M) + 2\sum_{m=1}^{\infty} J_{2m}(M)\cos 2m\theta \pm 2i\sum_{m=0}^{\infty} J_{2m+1}(M)\sin(2m+1)\theta$$

in which $J_m(M)$ is the m^{th} order Bessel function (of the first kind), and $\theta = \omega_s t$. For very small M

$$e^{iM\sin\omega_s t} \approx J_0(M) + 2i\,J_1(M)\sin\omega_s t \approx 1 + iM\sin\omega_s t$$

Thus for very small modulation index the phase modulation appears as a (sinusoidal) amplitude modulation. Considering only a partial product we have

$$\mathrm{Re}\left[e^{-i\Omega_E t}\, e^{i(\underline{K}-\underline{K}')\cdot \underline{h}[r_{an}(0)]\sin\omega_s t}\right]$$

$$\approx \mathrm{Re}\,(\cos(\Omega_E t) - i\sin\Omega_E t)(1 + iM\sin\omega_s t)$$

$$= \cos\Omega_E t + M\sin\Omega_E t\sin\omega_s t$$

The spectrum of the modulation is given by

$$S(\omega) = \frac{M}{4}\left\{\delta(\omega + \Omega_E + \omega_s) + \delta(\omega + \Omega_E - \omega_s) + \delta(\omega - \Omega_E + \omega_s) + \delta(\omega - \Omega_E - \omega_s)\right\}$$

In the positive half of the frequency spectrum the modulation appears as two side bands, at a distance $\pm\omega_s$ of Ω_E (Bracewell "Fourier Transforms" p. 184). Thus the spectrum of scattered the light amplitude is typically that of a sinusoidal FM (or AM) modulated carrier. However, these spectral lines have no finite width, and thus do not account for noise.

Conclusion to Part I

In Part I we have presented a physically appealing model of the laser Doppler velocimeter in terms of a set of laser interference fringes crossed by occulting particles. This model is useful in providing insight into the scattering process, thus allowing experimentalists to master the details of the phenomena in their laboratory experiments. It is however very important to comprehend the exact nature of what the experiments are measuring, and to construct a quantative model of the complex nature of the "noises" in the experiment which set lower limits in detectability of the fluid motion. A very detailed math model has been devised by George and Lumley (J. Fluid Mech. 60, p. 321 (1973)). We adopt this model in all succeeding discussions.

Part II

Mathematical Modeling

Summary of Math Modeling

The scattering volume is assumed to be three-dimensional (effective dimensions $\sigma_1, \sigma_2, \sigma_3$) and to contain more than one scattering particle in motion. A first step in math modeling is to project the volume on to the xy plane, Fig. (10). Next the random velocities of the light scattering particles are averaged over particle position on the scattering volume to give the instantaneous equivalent of an ensemble averaged (random) velocity $U_0(t) = (u_0(t), V_0(t))$ (which is the Eulerian velocity reported by the detection system as producing the Doppler shift), and a (random) deviation from this average, $\Delta U(\underline{x},t) = (\Delta u(\underline{x},t), \Delta V(\underline{x},t))$ which the particle at $\underline{x}$ possesses. The displacement of a particular particle (at x) in time t is thus assumed to be the sum of a displacement $\underline{D}_0(t) = (\underline{X}(t), Y(t))$ (the ensemble average for all particles) due to $U_0(t)$ and $\Delta \underline{D}(\underline{x},t) = (\Delta X(\underline{x},t), \Delta Y(\underline{x},t))$ due to $\Delta U(\underline{x},t)$ unique to the particle at $\underline{x}$.

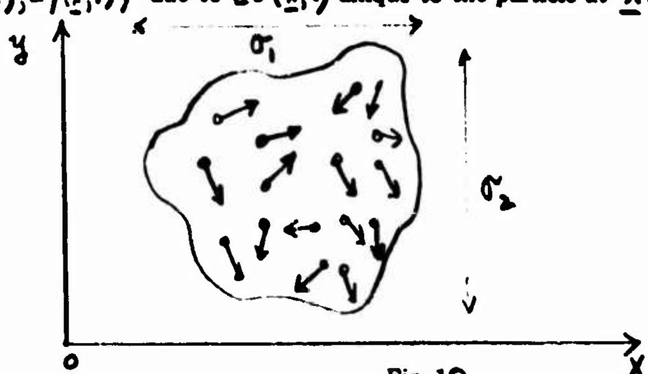


Fig. 10

The analysis proceeds by restricting attention to a single dimension, (say x-dimension). The random Eulerian velocity $u_0(t)$ is separated into a random mean velocity $\bar{u}_0(t)$ and a fluctuation about the random mean $u_0'(t) = u_0(t) - \bar{u}_0(t)$. A time average of $\bar{u}_0(t)$ is labeled $\bar{u}_0$. The square of the random deviation $\Delta u[\underline{x}(a,t), t]$ is ensemble averaged over particle position $\underline{a}$ and time averaged over t to give the average mean - square velocity deviation $\overline{(\Delta u)^2}$ When the motion is laminar and deterministic,

$$\bar{u}_0 = \bar{u}, \qquad \overline{u'^2} = 0 = \overline{(\Delta u)^2}$$

The time-averaged random mean velocity $\bar{u}_0$ (in general turbulent flow) is assumed to contribute the Doppler spectral line in the photodetector (electric) current. The Doppler shift of this line is $\bar{\omega}_0 = K\bar{u}_0$ where K is the x-component of the "vector wavenumber". The fluctuating quantities u_0', Δu are assumed to contribute spectral noise (called Doppler broadening) centered on the spectral line at $\bar{\omega}_0$

Assuming the fluctuations u_0', Δu to be Gaussian distributed one can show by math modeling that the frequency bandwidth ΔB which measures the Doppler broadening (on $\bar{\omega}_0$) is,

$$(\Delta B)^2 = K^2\,\overline{u_0'^2} + K^2\,\overline{(\Delta u)^2} + \frac{\bar{u}^2}{2\sigma_1^2}$$

Here the term $\bar{u}^2/2\sigma_1^2$ is the spectral broadening due to finite transit time of the 'average' particle traveling at averaged speed $\bar{u}$ through the effective (projected) volume σ_1. In the absence of fluctuations the broadening of the spectrum due to finite travel time is called the Doppler ambiguity of RADAR.

When other sources of time fluctuation of the scattering particles are present they can be assumed to be Gaussian distributed. The most important of these contribute bandwidths as follows: $\Delta\omega_G^2$ due to gradients in the (laminar) mean velocity $\bar{u}$ across the volume, $\Delta\omega_\beta^2$ due to Brownian motion, and $\Delta\omega_s^2$ due to non-monochromaticity of the laser light beam. Assuming these to be important one writes the frequency bandwidth of spectrum to be

$$(\Delta B)^2 = K^2\,\overline{u_0'^2} + (DA)^2$$

$$(DA)^2 = K^2\,\overline{(\Delta u)^2} + \frac{\bar{u}^2}{2\sigma_1^2} + \Delta\omega_G^2 + \Delta\omega_\beta^2 + \Delta\omega_s^2$$

Here DA is called the Doppler ambiguity of the model.

Two important ratios can be formed from these assumptions and definitions. These are: (1) $DA/\bar{\omega}_0$ the ratio of the Doppler ambiguity to the Doppler shift, (2) $K\sqrt{\overline{u_0'^2}}/K\bar{u}$ or $u'/\bar{u}$ the ratio of fluctuation in u' to the laminar $\bar{u}$ which is equivalent to the ratio of spectral broadening due to u' to the line spectrum due to $\bar{u}$. These ratios are useful in setting limits to resolution in applications of the LDV to turbulence measurements.

The basic model of the LDV is the equivalent two-beam interferometer using waves of laser light. The Doppler (electric) current $i(t)$ reported by the photodetector of the interference light intensity is

$$i(t) = F(t)\cos KX + G(t)\sin KX = (F^2 + G^2)\cos\left[KX(t) - \phi(t)\right]$$

in which $F(t)$, $G(t)$ account for fluctuations in the scattering process and finite beam widths of the laser beams, and $\phi(t)$ is a phase (of light intensity) given by $\tan^{-1}(G/F)$. Thus the photodetector current is represented by a phasor whose amplitude and phase are random functions of time. The probability distribution of F and G are assumed Gaussian. The fluctuation $d\phi/dt$ of the phase is important in the determination of the Doppler spectrum since the equivalent Doppler frequency is $\omega_1 = d/dt\left[KX(t) - \phi(t)\right]$. The correlation properties of $\dot{\phi}(t)$ are modeled as filtered noise in an FM receiver. Its power spectrum is labeled $N(\alpha)$. It has been found (by experiment) to be white with a magnitude proportional to the Doppler ambiguity DA (i.e. $N(\alpha) \approx N(0) = 0.368\,DA$).

Under the assumption that the scattering volume is larger than the wavelength of the laser light $(i.e. K^2\sigma_1^2 \gg 1)$ it can be shown by math modeling that the correlation of $F(t)$ is approximately the same as the correlation for $G(t)$, that the cross-correlation of F and G is approximately zero, and that the time average over fluctuations (but not the mean) is,

$$\overline{\cos K\left[X(t) - X(t')\right]} \approx \exp\left\{-\frac{K^2\sigma_x^2}{2}\right\}\cos K\left[\hat{X}(t) - \hat{X}(t')\right]$$

Here,

$$\sigma_x^2 \approx \overline{u_0'^2}\,(t-t')^2$$

and $\hat{X}(t)$ is the time average (or mean) of the random displacement $X(t)$. Similarly, under the same assumption of volume size, the math model shows that

$$\frac{\overline{F(t)\,F(t')}}{\overline{F^2(t)}} \approx \exp\left\{-\frac{K^2\sigma_\Delta^2}{2}\right\} + \frac{\hat{X}(t)-\hat{X}(t')}{4\sigma_1^2}\Big\}$$

where

$$\sigma_\Delta^2 \approx \overline{(\Delta u)^2}\,(t-t')^2$$

Thus the normalized correlation of the random photodetector current is modeled as the correlation of a phasor with random amplitude and phase,

$$\frac{\overline{i(t)\,i(t')}}{\overline{i^2(t)}} = \exp\left\{-\frac{K^2\sigma_\Delta^2}{2} + \frac{\hat{X}(t)-\hat{X}(t')}{4\sigma_1^2}\right\}\exp\left\{-\frac{K^2\sigma_2^2}{2}\right\}$$

$$\cdot \cos K\left[\hat{X}(t)-\hat{X}(t')\right]$$

in which the exponentials account for the fluctuations in both amplitude and phase.

The basic math model is first used to calculate the power spectrum of the photodetector current. In this modeling we require an explicit form for the average (time-varying) displacement $\hat{X}(t)$ to be attributed to <u>all</u> the particles. We assume that all the particles have a volume averaged velocity $\bar{u}_0$ and an acoustic displacement $X_s \sin \omega_s t$. With this choice of $\hat{X}(t)$ it is found by Fourier transformation of the autocorrelation of $i(t)$ that the Doppler spectrum consists of two Gaussian peaks at $\omega = \pm K\bar{u}_0$ with amplitude proportional to $J_0(KX_s)$ each peak in turn associated with two satellite Gaussian peaks $\Delta\omega = \pm\omega_s$ with amplitude proportional to $J_1(KX_s)$ making a total of two groups of peaks with three peaks in each group. The "spectral lines" in one group are thus: $K\bar{u}_0$, $K\bar{u}_0+\omega_s$, $K\bar{u}_0-\omega_s$. The math model also predicts that <u>each</u> spectral line will be broadened by fluctuations and finite transit time. The Doppler broadening will be ΔB as defined above, i.e.

$$\Delta B = \left[K^2\overline{u_0'^2} + (DA)^2\right]^{1/2}$$

This model will be now be used to determine limits of detection of the acoustic signal in the presence of noise, and to estimate the limits of detection of turbulence.

In the case of particle displacements due to a mean velocity $\bar{u}_0$ and an acoustic displacement $X_s \sin \omega_s t$ the spectrum of the photodetector current centered at $K\bar{u}_0$ together with one satellite is shown in Fig. 10a. Here the acoustic signal is so weak that the acoustic satellite is everywhere less than the noise accompanying the Doppler shift due to $K\bar{u}_0$. Increasing the amplitude of the acoustic signal raises the spectrum of the satellite to position B_2, B_3 (Fig. 10b). At the same time the Gaussian spectrum at $K\bar{u}_0$ diminishes to A_2, A_3. At the point marked by the circle the satellite just emerges from the noise. The amplitude of acoustic signal required to reach this condition is the theoretical minimum detectable acoustic signal for given frequency ω_s and Doppler broadening ΔB. The irreducible minimum spectral noise is that of the photodetector.

In the case where the particle displacements are due to turbulence one must measure fluctuations of velocity in the fluid flow by observing the fluctuations in the Doppler frequency ω_0. If the frequency broadening of these velocity fluctuations are of the same order as the broadening due to the Doppler ambiguity (=DA, defined above), then one cannot distinguish between fluctuations u' and fluctuations giving rise to the Doppler ambiguity. The limit of resolution of u' relative to $\bar{u}$ (i.e. of turbulent change velocity relative to laminar flow), is then given by the condition

$$\frac{u'}{\bar{u}} > \frac{DA}{\bar{\omega}_0}$$

Thus the measurement of turbulent u' is limited by the Doppler ambiguity.

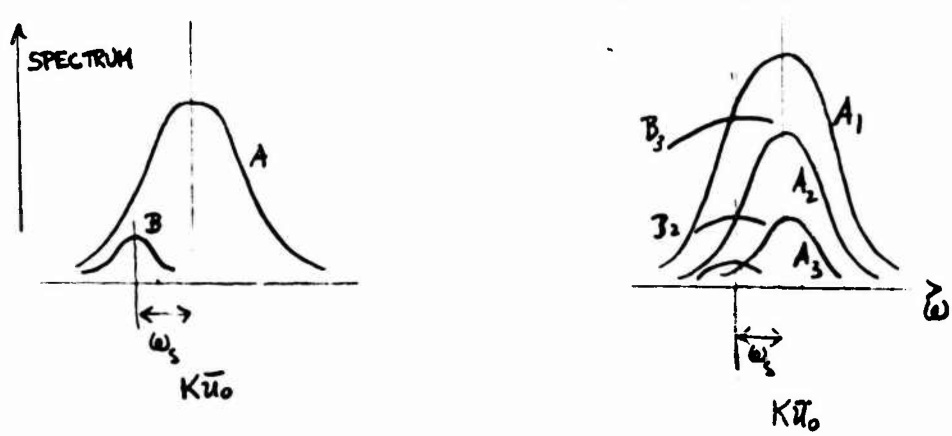


FIG 10a. FIG. 10b

Conclusion

This summary has served to note the following points: the only hydrodynamic velocity that is measurable in a laser Doppler velocimeter (LDV) is the Eulerian velocity $u_o(t)$, which is an average overall the scattering particles. The Lagrangian velocity of individual particles (in the case of flow of many particles) is not measurable. The principal sources of spectral noise associated with the scattering volume are finite transit time of particles crossing the scattering volume, the temporal fluctuations in $u_o(t)$ itself, and the random motion of individual particles about the mean Eulerian velocity $\overline{u_o}(t)$ as functions of particle position in the scattering volume. The basic model of the LDV is a two beam interferometer. The basic model of the electric current in the photodetector is that of phasor with random amplitude and random phase. The calculation of minimum detectable signal is based on comparison of the size of the acoustic spectrum to the spectral noise in the absence of acoustic signals. The irreducible noise in the system is that of the photodetection process. The possibility of reducing all other noises remains open to research.

Model of George and Lumley

It is known (see Appendix C) that the light intensity $\underline{S}$ falling on the detector is $\underline{S} = \frac{c}{4\pi}\sqrt{\frac{\epsilon}{\mu}}\, E_r^* E_s \hat{a}_p$, where $\hat{a}_p$ is a unit vector normal to the photosensitive area, and E_r is the reference beam. Assume the coordinates of the photosensitive area are x'', y'', z'' and that an element of area is $dx''dz''$ so that $\hat{a}_p = \hat{j}''$, where $\hat{j}''$ is the unit vector in the direction of y''. Let M be a constant which converts light intensity into electric current (dimensions: volt^{-1}). The total electric current detected by the LDV is,

$$i_{TOT} = \frac{Mc}{4\pi}\sqrt{\frac{\epsilon}{\mu}} \iint \left| E_r^*(x'',y'',z'') \sum_m E_{s_m}(x'',y'',z'') \right| dx''dz'' \qquad \text{(II.1)}$$

in which there are m scattering particles. If the scattering particle is small relative to a wavelength of laser light the scatterer is a simple monopole so that in the coordinates of the reference beam (double prime coordinates) the scattered field is

$$E_s(\underline{R}) = \frac{c_p}{j\lambda_\epsilon}\, E_i(\underline{x}_p')\, \frac{\exp\{j k |\underline{R} - \underline{x}_p'|\}}{|\underline{R} - \underline{x}_p''|}, \qquad j = \sqrt{-1} \qquad \text{(II.2)}$$

in which $\underset{\sim}{x}_p''$ is the position of the particle in the coordinates of the reference beam, $\underset{\sim}{x}_p'$ is the position of the same particle in the coordinates of the scattered beam, C_p is a scattering coefficient constant, and

$$|\underline{R} - \underset{\sim}{x}_p''|^2 = (x'' - x_p'')^2 + (y'' - y_p'')^2 + (z'' - z_p'')^2 \tag{II.3}$$

(Note again that the double primed coordinates are on the photodetector). Since

$$\iint_S E_r^*(\underline{R}) \frac{\exp\{jk|\underline{R} - \underset{\sim}{x}_p''|\}}{j\lambda|\underline{R} - \underset{\sim}{x}_p''|} dx'' dz'' = E_r^*(x_p'', y_p'', z_p'') \tag{II.4}$$

it is seen that the electric current due to one particle is

$$i = \frac{Me}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\, C_p\, E_i(x_p', y_p', z_p')\, E_r^*(x_p'', y_p'', z_p'') \tag{II.5}$$

The reference beam and scattered beam cross each other at angle θ. The photodetector is set to detect the reference beam only.

As noted earlier, LDV's are essentially based on two-beam interference. The "slit-sources" are two laser beams which are characterized by finite widths. It is customary to assume that the distribution of light amplitude in each beam width is Gaussian. Thus the light beams are modeled as

$$E_r(\underline{x}'') = E_{or} \exp\left\{-\frac{(x''^2 + z''^2)}{2\sigma^2}\right\} \exp\{iK_E y''\} e^{-i\Omega_E t}$$

$$E_i(\underset{\sim}{x}'') = E_{oi} \exp\left\{-\frac{(x'^2 + z'^2)}{2\sigma^2}\right\} \exp\{iK_E y'\} e^{-i\Omega_E t} \tag{II.6}$$

$$K_E = \Omega_E / c_E \tag{II.7}$$

in which E_{0k}, E_{0i} are amplitudes at the center of the beam (as measured at the focus) and σ is given by

$$\sigma = \frac{\sqrt{2}\, f_0 \lambda_E}{\pi d} \tag{II.8}$$

where f_0 is the focal length of the lens, λ_E is the wavelength of light, and d is the distance (in the cross-section of the light beam) between the e^{-1} intensity points of the beam.

It is convenient next to introduce a local coordinate system $\underset{\sim}{X}$ fixed at the intersection point of the two crossed beams. A particle in $\underset{\sim}{X}$ is located at $\underline{X}_p = (x_p, y_p, z_p)$, where one <u>defines</u> X_p in terms of x', x'', y', y'' by the relations,

$$\begin{aligned} x' &= \left(x_p \cos\tfrac{\theta}{2} + y_p \sin\tfrac{\theta}{2}\right), \quad y' = \left(-x_p \sin\tfrac{\theta}{2} + y_p \cos\tfrac{\theta}{2}\right) \\ x'' &= \left(x_p \cos\tfrac{\theta}{2} - y_p \sin\tfrac{\theta}{2}\right), \quad y'' = \left(x_p \sin\tfrac{\theta}{2} + y_p \cos\tfrac{\theta}{2}\right) \end{aligned} \tag{II.9}$$

In terms of X_p the net current due to a particle p is

$$i_{p\,NET} = \frac{Mc}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\; C_p \exp - \left\{ \frac{x_p^2}{2\sigma_1^2} + \frac{y_p^2}{2\sigma_2^2} + \frac{z_p^2}{2\sigma_3^2} \right\} E_{0k} E_{0i} \cos K x_p \tag{II.10}$$

$$K = 2 K_E \sin\frac{\theta}{2} \tag{II.11}$$

This is the analog of the classic formula for a two-beam interferometer (see Eq. A5), in which

$$\sigma_1 = \frac{\sigma}{\sqrt{2}\cos\frac{\theta}{2}}\,; \qquad \sigma_2 = \frac{\sigma}{\sqrt{2}\sin\frac{\theta}{2}}\,; \qquad \sigma_3 = \frac{\sigma}{\sqrt{2}} \tag{II.12}$$

The exponential term in Fig. II-10 accounts for the finite cross section of the laser beams.

Note that the plane x_p, y_p is normal to the plane of the two laser beams, and effectively describes the observing screen in the conventional 2-beam Young interference experiment. The coordinate y_p is normal to the effective observing screen.

Now x_p, y_p, z_p are Eulerian coordinates of particle position, and so are functions of Lagrangian initial position $\underline{a}$, i.e. $\underset{\sim}{x}_p = \underset{\sim}{x}_p(a,t)$ (see Eringen, Mech. of Continuies p. 7). Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be the velocity components (in the local coordinate system) of identified particles, i.e. $\mathcal{U} = \mathcal{U}(\underline{a},t)$ etc. Then

$$x_p(\underset{\sim}{a},t) = \underset{\sim}{a}_p + \int_0^t \mathcal{U}(\underset{\sim}{a}_p, t_1)\, dt_1 \,, \text{ etc.}$$

(II.13)

However, $\mathcal{U}(\underset{\sim}{a}_p, t)$ is not observable in the case under consideration where there are many particles in the scattering volume. The observable velocity is $u_0(t)$ which is an ensemble average over all scattering particles. To perform the averaging process one introduces a probability function $g(\underset{\sim}{a})$ that a particle of specific size is present in the scattering volume. The scattering volume itself is specified by $w[x_p(\underline{a},t)]d\underset{\sim}{a}$ where $w(\underset{\sim}{x})$ is the exponential in Eq. II.10. Thus the observable velocity is

$$u_0(t) = \frac{1}{N_\varepsilon} \int \mathcal{U}(\underline{a},t)\, g(\underline{a})\, w[x_p(\underline{a},t)]\, d\underset{\sim}{a}$$

(II.14)

and N_ε is the expected number of particles in the volume. At any instant a particle velocity deviates from the observable average by amount $\mathcal{U}(\underline{a},t) - u_0(t)$. The time-averaged deviation is

$$\Delta(\underline{a},t) = \int_0^t \left[\mathcal{U}(\underline{a},t_1) - u_0(t_1)\right] dt_1$$

(II.15)

If the effective displacement over all particles is $X(t)$ then by definition,

$$X(t) = \int_0^t u_0(t_1)\,dt_1$$

(II.16)

Hence the (Eulerian) coordinate of the particle at $\underset{\sim}{a}$ is,

$$x_p(\underset{\sim}{a},t) = a_p + X(t) + \Delta(\underset{\sim}{a}_p,t)$$

(II.17)

that is, a particle which is initially at $\underset{\sim}{a}_p$ moves an average distance $X(t)$ (in the $\underset{\sim}{x}_p$ system) plus a random distance (along the x_p coordinate) of amount $\Delta(\underset{\sim}{a}_p,t)$. Hence the total current due to m particles is

$$i(\underset{\sim}{a},t) = \sum_m \frac{Mc}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\, C_p\, e^{x_p - \left\{\frac{x_m^2}{2\sigma_1^2}+\frac{y_m^2}{2\sigma_2^2}+\frac{z_m^2}{2\sigma_3^2}\right\}} E_{on} E_{oi} \cos K\left[a_m + X + \Delta_m\right]$$

(II.18)

This current is again a function of particle positions $(\underset{\sim}{a})$ and is unobservable. The current $i(t)$ averaged over all $\underset{\sim}{a}$ will be observable. Hence one again introduces the probability function $g(\underset{\sim}{a})$ and writes

$$i(t) = \int i(\underset{\sim}{a},t)\, g(\underset{\sim}{a})\, d\underset{\sim}{a}$$

(II.19)

Two random variables are in $i(\underset{\sim}{a},t)$ namely random particle position $(=\underset{\sim}{a})$ and random particle displacement $(=\Delta(\underset{\sim}{a},t))$. Noting that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, one can write

$$i(t) = F(t)\cos KX + G(t)\sin KX = \left(F^2+G^2\right)^{1/2}\cos\left[KX(t) - \phi(t)\right]$$

(II.20)

in which

$$F(t) = \int A \cos K \left[\underset{\sim}{a} + \Delta(\underset{\sim}{a}, t) \right] g(\underset{\sim}{a}) d\underset{\sim}{a}$$

(II.21)

$$G(t) = -\int A \sin K \left[\underset{\sim}{a} + \Delta(\underset{\sim}{a}, t) \right] g(\underset{\sim}{a}) d\underset{\sim}{a}$$

(II.22)

$$A(\underset{\sim}{x}) = \exp - \left\{ \frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} + \frac{z^2}{2\sigma_3^2} \right\}$$

(II.23)

Since the photodetector gives a signal proportional to the time derivative of the phase the output of the velocimeter is a frequency shift

$$\omega_1 = K \frac{dX}{dt} - \frac{d\phi}{dt} = K u_0(t) - \dot{\phi}$$

(II.24)

The observed (averaged) velocity $u_0(t)$ is a random function of time, with mean value $\overline{u_0(t)}$, fluctuation $u_0(t) - \overline{u}_0(t) = u_0'(t)$ and mean-square fluctuation $\overline{u_0'(t)^2} = \overline{[u_0(t) - \overline{u}_0(t)]^2}$. The local (non-averaged) velocity $u(\underset{\sim}{x}, t)$ is also a random function, with mean value $\overline{u}(\underset{\sim}{x})$ and fluctuation $u(\underset{\sim}{x}, t) - \overline{u}(\underset{\sim}{x})$. The covariance $R_{11}(x, x')$ is the x-component time averaged product of $u(\underset{\sim}{x}, t)$ at $\underset{\sim}{x}$ and $\underset{\sim}{x}'$,

$$R_{11}(x, x') = \overline{[u(\underset{\sim}{x}, t) - \overline{u}(\underset{\sim}{x})][u(\underset{\sim}{x}', t) - \overline{u}(\underset{\sim}{x})]}$$

(II.25)

Now the time averaged mean-square fluctuation $\overline{u_0'^2}$ can be written in terms of R_{11} provided the space average product $\overline{g(\underset{\sim}{a}) g(\underset{\sim}{a}')}$ is known. It can be shown (George & Lumley, p. 360) that if the particles are distributed as a Poisson process, then

$$\overline{g(\underset{\sim}{a})\, g(\underset{\sim}{a}')} = N_\varepsilon^2 + N_\varepsilon\, \delta(\underset{\sim}{a} - \underset{\sim}{a}')$$

(II.26)

where N_ε is the expected number of particles in the scattering volume. The formula for $\overline{u_0'^2}$ is then

$$\overline{u_0'^2} = \int R_{11}(\underset{\sim}{x}, \underset{\sim}{x}')\, w(\underset{\sim}{x})\, w(\underset{\sim}{x}')\, d\underset{\sim}{x}\, d\underset{\sim}{x}' + \frac{1}{N_\varepsilon} \int \overline{u^2(\underset{\sim}{x})}\, w^2(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$w(\underset{\sim}{x}) = \frac{A(\underset{\sim}{x})}{(2\pi)^{\frac{3}{2}}\, \sigma_1 \sigma_2 \sigma_3}$$

(II.27)

Noting that

$$R_{11}(\underset{\sim}{x}, \underset{\sim}{x}') = \int \Phi_{11}(\underset{\sim}{k})\, e^{-i \underset{\sim}{k} \cdot (\underset{\sim}{x} - \underset{\sim}{x}')}\, d\underset{\sim}{k}$$

(II.28)

and defining

$$\mathcal{W}(\underset{\sim}{k}) = \hat{w}(\underset{\sim}{k})\, \hat{w}^*(\underset{\sim}{k})$$

(II.29)

where $\hat{w}(\underset{\sim}{k})$ is the Fourier transform of $w(\underset{\sim}{x})$, it is seen that

$$\overline{u_0'^2} = \int \Phi_{11}(\underset{\sim}{k})\, \mathcal{W}(\underset{\sim}{k})\, d\underset{\sim}{k} + \frac{1}{N_\varepsilon}\left(\overline{u'^2} + \overline{u}^2\right)$$

(II.30)

under the assumptions that the fluctuations of $u(x)$ conform to the picture "homogeneous turbulence." Note that $\overline{u'^2}$ is the particle-averaged variance of $u(\underset{\sim}{x})$ (over the scattering volume) and $\overline{u}^2$ is the averaged square of $u(\underset{\sim}{x})$ at $\underset{\sim}{x}$. Under the further assumption that

$N_\varepsilon \gg 1$ and $|\mathcal{W}(\underset{\sim}{k})| > N_\varepsilon^{-1}$, one reduces this last formula to

$$\overline{u'^2} \approx \int \Phi_{11}(\underset{\sim}{k})\, \mathcal{W}(\underset{\sim}{k})\, d\underset{\sim}{k} \tag{II.31}$$

In words: the time and particle averaged mean-square fluctuation of the observable velocity (in the X direction) is obtained by integrating the 11-component of the power spectrum of fluctuating flow at wave number k times the weighted volume of $\underset{\sim}{k}$ space $\mathcal{W}(\underset{\sim}{k})\, d\underset{\sim}{k}$ (which tells what weight to assign to $\Phi_{11}(\underset{\sim}{k})$ at $\underset{\sim}{k}$). The weighted volume is essentially a statement of the finite boundaries of intersection volume in real space. From the definition of $w(\underline{x})$, it is seen that

$$\mathcal{W}(\underset{\sim}{k}) = \exp - \left\{ \frac{k_1^2}{2K_1^2} + \frac{k_2^2}{2K_2^2} + \frac{k_3^2}{2K_3^2} \right\} \tag{II.32}$$

in which $\underline{k} = (k_1, k_2, k_3)$ and

$$K_1 = \frac{1}{\sqrt{2}\,\sigma_1} \quad ; \quad K_2 = \frac{1}{\sqrt{2}\,\sigma_2} \quad ; \quad K_3 = \frac{1}{\sqrt{2}\,\sigma_3} \tag{II.33}$$

This form of $\mathcal{W}(\underset{\sim}{k})$ has an important meaning. Noting that $\sigma_1, \sigma_2, \sigma_3$ essentially define the x, y, z boundaries of the scattering volume one sees that K_1, K_2 and K_3 are the "cut-off wave numbers arising from the spatial Fourier decomposition of the scattering volume. Similarly k_1, k_2, k_3 are the wave numbers associated with the spatial Fourier decomposition of the fluid velocity field. Now if $k_1 > K_1$, that is, if the spatial variation of fluid velocity is smaller than the scattering volume in the x-direction then $\exp - k_1^2/2B_1^2$ is small. This means that the k_1 component (or its corresponding velocity change) is severely attenuated by the measurement volume. This conclusion is of prime importance in the question of velocity resolution of the velocimeter.

The subscripts of Φ_{11} have special significance. Let $V_i(\underline{x})$ and $V_j(\underline{x})$ be two velocity vector fields (in Cartesian index notation). Then the velocity correlation tensor is the spatial average

$$R_{ij}(\underline{r}) = \overline{V_i(\underline{x})\, V_j(\underline{x}+\underline{r})} \tag{II.34}$$

In particular $R_{11}(\underline{r})$ is the correlation of a velocity vector field at $\underline{x}$ and at $\underline{x}+\underline{r}$. The spectral density (tensor) $\Phi_{ij}(\underline{k})$ is the Fourier transform of $R_{ij}(\underline{r})$, and $\Phi_{11}(\underline{k})$ is the spectral density component of the average product at $\underline{x}$ and $\underline{x}+\underline{r}$ of the same velocity field. Since only one-dimensional velocity spectrums can be measured one defines

$$F_{11}^{1}(k_1) = \iint_{-\infty}^{\infty} \Phi_{11}(k_1, k_2, k_3)\, dk_2\, dk_3 \tag{II.35}$$

The notation $F_{11}^{1}(k_1)$ means the following: the 11 subscript signifies the same velocity field, and superscript 1 indicates the first (say x) component. Note that

$$\overline{u'^2} = \int_{-\infty}^{\infty} F_{11}^{1}(k_1)\, dk_1 \tag{II.36}$$

i.e. the mean-square fluctuation of the x-component of velocity spatially averaged is given by the integral shown. The measured one-dimensional spectrum $F_0(k_1)$ is quite different since gives the temporally averaged mean-square value of the average (effective) velocity in the scattering volume,

$$\overline{u_0'(t)^2} = \int_{-\infty}^{\infty} F_0(k_1)\, dk_1 \tag{II.37}$$

and

$$F_0(k_1) = \exp\left\{-\frac{k_1^2}{2\bar{K}_1^2}\right\} \iint_{-\infty}^{\infty} \Phi_{11}(k_1, k_2, k_3) \times \exp\left\{-\frac{k_2^2}{2\bar{K}_2^2} - \frac{k_3^2}{2\bar{K}_3^2}\right\} dk_2\, dk_3 \tag{II.38}$$

Note again that $F_0(k_1)$ is the measured spectrum and F_{11}' is the theoretical spectrum. The ratio $F_0(k_1)/F_{11}'(k_1)$ is the <u>velocimeter transfer function</u>. Typical values based on the assumption of isotropic turbulence, and a special form of the energy spectrum is shown in Fig. 4, p. 322, George and Lumley .

<u>Particle Displacement</u>

The observed x-component instantaneous velocity, averaged over all particles, is $u_0(t)$. This is a random variable since the number of particles varies from moment to moment in the scattering volume. If we assign a probability distribution to $u_0(t)$ we can find its mean value $\overline{u_0(t)}$ (see Fig. 11). The effective displacement due to $\overline{u_0(t)}$ is $\hat{X}(t)$, where

$$\hat{X}(t) = \int_0^t \overline{u_0(t_1)}\, dt_1 = \bar{u}\, t \tag{II.39}$$

Here $\bar{u}$ is a fictitious average <u>constant</u> velocity such that the effective displacement $\hat{X}$ is the product $\bar{u}\, t$. Now $\hat{X}(t)$ itself is a random function of time. Thus at time t the fluctuation in <u>observable</u> (x-component of) displacement is

$$X'(t) = X(t) - \hat{X}(t) \tag{II.40}$$

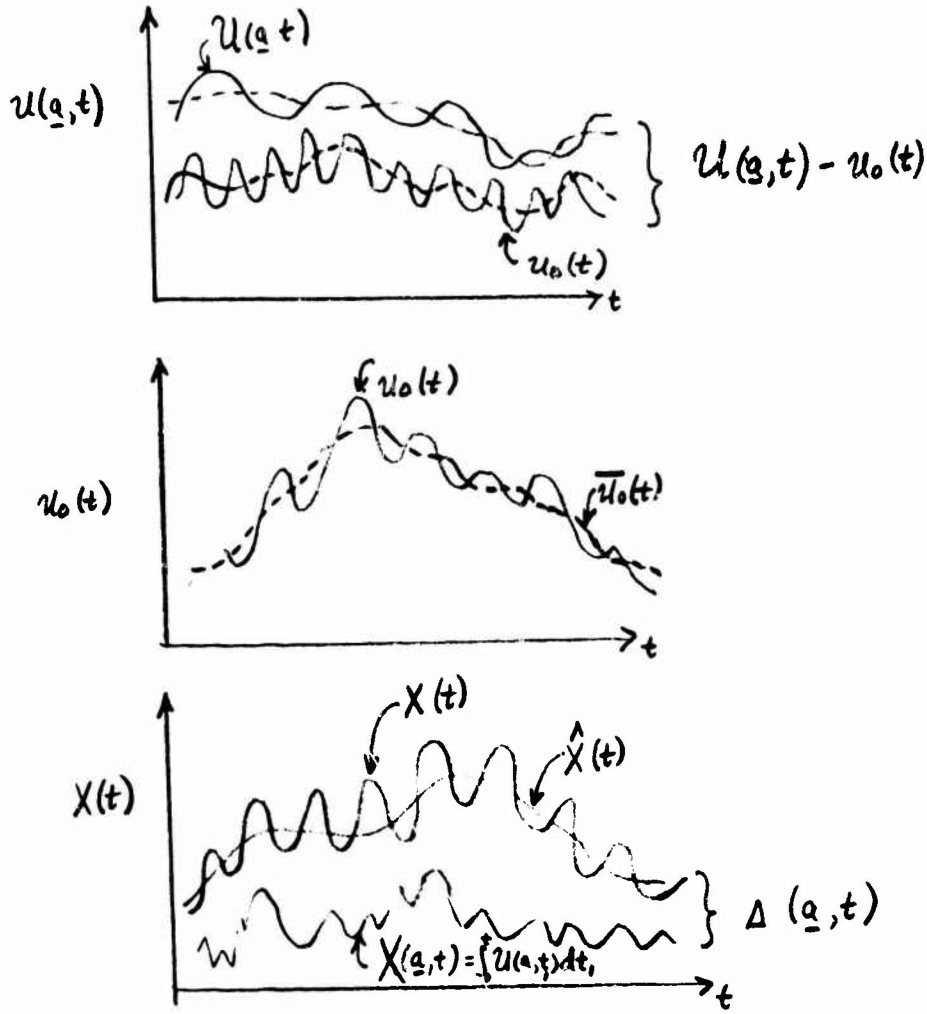


Fig. 11a, 11b, 11c

In distinction to observable $u_0(t)$ there is the unobservable x-component of velocity of a particular particle , viz. $\mathcal{U}(\underline{a},t)$. The difference between $\mathcal{U}(\underline{a},t)$ and $u_0(t)$, averaged over-time, results in a random x-displacement $\Delta(\underline{a},t)$. The total (random) instantaneous displacement is therefore

$$x(t) = a + \int_0^t \mathcal{U}(\underline{a},t_1)\,dt_1 - \int_0^t u_0(t_1)\,dt_1 + \int_0^t u_0(t_1)\,dt_1 \qquad \text{(II.11)}$$

or

$$X(t) = a + \Delta(\underline{a},t) + X(t)$$

In this form $X(t)$ is unobservable because of $\Delta(\underline{a},t)$. A method of averaging over all particle position is discussed in the next section.

Measurable Correlation of the Photodetector Current

We desire to form (1) the total current $i(t)$ and (2) the normalized correlation of current,

$$(1)\quad i(t) = \int i(\underline{a},t)\, g(\underline{a})\, d\underline{a}$$

$$(2)\quad \frac{\overline{i(t)\, i(t')}}{i^2(t)} \tag{II.42}$$

(II,43)

in which the random variables are $\underline{a}$, $X(t)$ and $\Delta(\underline{a},t)$. The function $g(\underline{a})$ is already assigned (see Eq. II.26). To $X(t)$ we assign the probability density

$$P\left[X(t)-\hat{X}(t),\ X(t')-\hat{X}(t')\right] = \frac{1}{(2\pi)^{1/2}\sigma_X} \exp\left\{-\frac{\left[(X(t)-\hat{X}(t))-(X(t')-\hat{X}(t'))\right]^2}{2\sigma_X^2}\right\}$$

(II.44)

and to $\Delta(\underline{a},t)$, considered a variable, we assign

$$P\left[\Delta(\underline{a},t),\ \Delta(\underline{a},t')\right] = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_\Delta} \exp\left\{-\frac{\left[\Delta(\underline{a},t)-\Delta(\underline{a},t')\right]^2}{2\sigma_\Delta^2}\right\}$$

(George and Lumley, Eqs. 4.1.2, 4.1.3) (II.45)

in which σ_X and σ_Δ are standard deviations. A full calculation leads to the (measureable) result that

$$\frac{\overline{i(t)\, i(t')}}{\overline{i^2(t)}} = \exp - \left\{ \frac{\frac{K^2\sigma_\Delta^2}{2} + \frac{(\hat{X}-\hat{X}')^2}{4\sigma_1^2}}{1+\frac{\sigma_x^2}{2\sigma_1^2}+\frac{\sigma_\Delta^2}{2\sigma_1^2}} \right\} \exp - \left\{ \frac{K^2\sigma_x^2}{2} \, \frac{1}{1+\frac{\sigma_x^2}{2\sigma_1^2}+\frac{\sigma_\Delta^2}{2\sigma_1^2}} \right\}$$

$$\times \cos \left\{ \frac{K(\hat{X}-\hat{X}')}{1+\frac{\sigma_x^2}{2\sigma_1^2}+\frac{\sigma_\Delta^2}{2\sigma_1^2}} \right\}$$

(II.46)

(George & Lumley, Eq. 4.1.6)

This is the correlation coefficient for the Doppler current between times t and t'.

We return now to the quantity $\mathcal{U}[\underline{X}(\underline{a},t)]$ which is the unobservable particle velocity at point $\underline{X}$, (where $\underline{X}$ is the Eulerian, or fixed, coordinate), and the averaged (over particles) particle velocity $\mathcal{U}_0(t)$ (independent of position in the volume) and form the mean-square fluctuation.

$$\overline{\left[\mathcal{U}\left[\underline{X}(\underline{a},t)\right] - \mathcal{U}_0(t)\right]^2}$$

(II.47)

This quantity differs for each $\underline{X}$. If we average over volume the result is an average mean-square velocity deviation, labelled $\overline{(\Delta\mathcal{U})^2}$. It is the mean-square difference between the averaged mean square fluctuation at the center of the scattering volume, and the mean-square velocity fluctuation averaged over the entire volume. Thus it is a measure of velocity deviations across the scattering volume. If the fluid medium is incompressible, and statistically homogeneous, stationary and isotropic, it is directly seen that

$$\overline{(\Delta u)^2} = \int \Phi_{11}(\underline{k}) \left\{ 1 - \mathcal{I}r(\underline{k}) \right\} d\underset{\sim}{k}$$

or (II.48)

$$\overline{(\Delta u)^2} = \int_{-\infty}^{\infty} \left[F_{11}'(k_1) - F_0(k_1)\right] dk_1 \tag{II.49}$$

Hence $\overline{(\Delta u)^2}$ accounts for the difference between the volume-averaged $F_0(k_1)$ spectrum and the true spectrum at the coordinate point. We next assume the "optical size" of the scattering volume in the x-direction (namely $K\sigma_1$) is of such magnitude that $K^2\sigma_1^2 \gg 1$. Taking $|K| = 4\pi/\lambda_E$, λ_E being the optic wavelength, this assumption means that the scattering volume is many wavelengths of light in the x-direction. Now the autocorrelation of photodetector current depends on the quantities $\sigma_x^2/4\sigma_1^2$, $\sigma_\Delta^2/2\sigma_1^2$, where σ_x is the standard deviation of the assumed Gaussian form of the joint probability of the efffective random displacement of the averaged particle at the center of the scattering volume at times t, t', and σ_Δ is the standard deviation of the assumed Gaussian form of the joint-probability of the random displacement difference $\Delta(\underline{a}, t)$ between a particle at $\underline{a}$ and the "averaged" central particle. For small time difference $(= t - t')$

$$\sigma_x^2 \approx \overline{u'^2}\,(t-t')^2; \quad \sigma_\Delta^2 \approx \overline{(\Delta u)^2}\,(t-t')^2 \tag{II.50}$$

(again: $\overline{u'^2}$ is the mean square fluctuations of the center; $\overline{(\Delta u)^2}$ is the mean square fluctuations relative to the center). Using $\exp\{-K^2\sigma_1^2\}$ as a characteristic number, it can be shown that when $K^2\sigma_1^2 \gg 1$, $\overline{F(t)\,F(t')} \approx \overline{G(t)\,G(t')}$, $\overline{F(t)\,G(t')} \approx 0$, and $\overline{\cos K(X - X')} = \exp\{-\frac{1}{2}K^2\sigma_x^2\}\cos K(\hat{X} - \hat{X}')$, valid to the order of the characteristic number. Thus, in this approximation,

$$\frac{\overline{i(t)\,i(t')}}{\overline{i^2(t)}} = \exp\left\{-\frac{K^2\sigma_\Delta^2(t,t')}{2} + \frac{(\hat{X}(t) - \hat{X}(t'))}{4\sigma_1^2}\right\}\exp\left\{-\frac{1}{2}K^2\sigma_x^2(t,t')\right\}$$

$$\times \cos K\left(\hat{X}(t) - \hat{X}(t')\right). \tag{II.51}$$

A more convenient form is

$$\frac{\overline{i(t)\,i(t+\tau)}}{\overline{i^2(t)}} = \frac{\overline{F(t)\,F(t+\tau)}^{\,t}}{\overline{F^2(t)}} \; \overline{\cos K\left[X(t)-X(t+\tau)\right]}^{\,t}$$

(II.52)

in which the superscript t means time average, and

$$\frac{\overline{F(t)\;F(t+\tau)}^{\,t}}{\overline{F^2(t)}^{\,t}} = \exp\left\{-\frac{K^2\sigma_\Delta^2(\tau)}{2} + \frac{\hat{X}(t)-\hat{X}(t+\tau)}{4\sigma_1^2}\right\}$$

(II.53)

The availability of the normalized correlation of the photodetector current leads directly to a calculation of the power spectrum. It will be convenient to discuss each factor of the correlation separately.

The Fourier transform of $Q(\tau) = \overline{\cos K[X(t)-X(t+\tau)]}^{\,t}$ (namely of $\exp\left\{-\frac{K\sigma_x^2(\tau)}{2}\right\}$ $\cos K[\hat{X}(t)-\hat{X}(t+\tau)]$) is

$$Q(\omega) = \frac{1}{2}\exp\left\{-\frac{(\omega-K\bar{u}_0)^2}{2K^2\overline{u_0'^2}}\right\} + \frac{1}{2}\exp\left\{-\frac{(\omega+K\bar{u}_0)^2}{2K^2\overline{u_0'^2}}\right\}$$

(II.54)

The Fourier transform of $P(\tau) = \frac{\overline{F(t)F(t+\tau)}^{\,t}}{\overline{F^2(t)}^{\,t}}$ is

$$P(\omega) = \exp\left\{-\frac{\omega^2}{2\left(\frac{\bar{u}^2}{2\sigma_1^2} + K^2\overline{(\Delta u)^2}\right)}\right\}$$

Here (as before) it is desirable to define all relevant terms again; thus, $\overline{u}_0$ is the time averaged value of $u_0(t)$, $\overline{u_0'^2}$ is the time averaged value of $[u_0(t)-\overline{u}_0]^2$; $\overline{u}$ is the time averaged value of the non-random $X(t)/t$, $\overline{(\Delta u)^2}$ is the time averaged mean-square deviation of the random velocity off center from the velocity on center. The Fourier spectrum $I(\omega)$ of the photodetector current is then

$$2I(\omega) = \int_{-\infty}^{\infty} P(\omega)\, Q(\omega-\omega_1)\, d\omega_1$$

$$" = \exp\left\{-\frac{(\omega-K\overline{u}_0)^2}{2\left[K^2\overline{u_0'^2}+\frac{\overline{u}^2}{2\sigma_1^2}+K^2\overline{(\Delta u)^2}\right]}\right\}$$

$$+\exp\left\{-\frac{(\omega+K\overline{u}_0)^2}{2\left[K^2\overline{u_0'^2}+\frac{\overline{u}^2}{2\sigma_1^2}+K^2\overline{(\Delta u)^2}\right]}\right\} \qquad \text{(II.55)}$$

In words: the normalized power spectrum of the photodetector current arising from the motion of particles which scatter light in a heterodyne laser-doppler-velocimeter consists of two Gaussian peaks centered at $\pm K\overline{u}_0$ where $\overline{u}_0$ is the time average of the velocity $u_0(t)$, which itself is the (observable) average of the velocities of all the individual particles at some instant t present in the scattering volume. The symbol K is $\frac{4\pi}{\lambda_E}\sin\frac{\theta}{2}$, where λ_E is the wavelength of the laser light, and θ is the included angle between the two beams of the interferometer. The Gaussian peaks are broadened by the effects of several time-varying quantities. The first of these is $\overline{u}^2/2\sigma_1^2$. To see its full meaning we discuss it as if it were the only quantity affecting the spectrum. The entity σ_1 is recognized as the effective size of the scattering volume (in the x-direction), and $\overline{u}$ is the time averaged particle velocity of all the particles in the absence of turbulence. Essentially $\bar{u}$ is the velocity of all particles through the volume, acting in laminar flow. Hence $\sigma_1/\bar{u}$ is the effective time of transit, and $\bar{u}/\sigma_1$ is the bandwidth of frequencies associated with this time of transit. In effect the laser light is modulated by the on-off switch of the finite scattering volume, resulting in an uncertainty in the Doppler shift that is ascribed to $\bar{u}_0$.

When the flow becomes turbulent one sees two additional effects which serve to broaden the spectrum. First there is the effect of the fluctuation of the (particle-averaged) central velocity $u_0(t)$ about its mean $\bar{u}_0$. The time scale of this occurrence is

$$t = \left(K^2 \overline{u_0'^2}\right)^{-\frac{1}{2}} \quad , \quad u_0'(t) = u_0(t) - \bar{u}_0 \tag{II.56a}$$

and the associated bandwidth is $\left(K^2 \overline{u_0'^2}\right)^{1/2}$. The fact that $u_0(t) - \bar{u}_0(t)$ exists introduces an uncertainty in the Doppler ascribed to $\bar{u}_0$. Second, the velocities of the particle are no longer laminar. At any time t there is a difference between the velocity of an individual particle $u(\underline{a}, t)$ and the velocity averaged over all particles $u_0(t)$. The time average of the square of this (random) difference is $\overline{(\Delta u)^2}$. The time scale of this turbulence-induced velocity variation across scattering volume is

$$t = \left(K^2 \overline{(\Delta u)^2}\right)^{-\frac{1}{2}} \tag{II.56b}$$

and the associated bandwidth is $\left(K^2 \overline{(\Delta u)^2}\right)^{\frac{1}{2}}$.

Power Spectrum of the Photodetector Current

In the presence of a sinusoidal phase modulation due to an acoustic signal we represent the photodetector current by the form

$$i(t) = \mathrm{Re}\left[F^2(t) + G^2(t)\right]^{1/2} \exp i\left[KX - Kx_s \cos\omega_s t - \phi\right] \tag{II.57}$$

Since this function is periodic one can separate the modulation index KX_s from the time variation $\cos\omega_s t$, viz

$$i(t) = \mathcal{R}_e \left[F^2(t) + G^2(t) \right]^{\frac{1}{2}} e^{i[KX-\phi]} \sum_{m=-\infty}^{\infty} (-i)^m J_m(KX_s) e^{im\omega_s t}$$

(II.58)

We next form the covariance of the random variable $i(t)$. To do this we make the basic assumption that the acoustic signal is uncorrelated with the turbulence of the medium. This assumption allows us to treat the acoustic term separately while obtaining the covariance. Thus the covariance of the acoustic term is

$$R_{\omega_s} = \sum_{m=0}^{\infty} \epsilon_m J_m^2(KX_s) \cos m\omega_s t, \qquad \epsilon_m = \text{Neuman number}$$

(II.59)

The normalized covariance of $\left[F^2(t) + G^2(t)\right]^{\frac{1}{2}} \exp i[KX-\phi]$ is given by Eq. II.46. In the approximation that

$$\frac{\sigma_x^2}{2\sigma_1^2} + \frac{\sigma_\Delta^2}{2\sigma_1^2} \ll 1$$

it is seen that the normalized covariance of $i(t)$ is

$$\frac{\overline{i(t)\, i(t')}}{\overline{i^2(t)}} = \frac{\overline{F(t)\, F(t')}}{\overline{F^2(t)}} \exp\left(-\tfrac{1}{2} K^2 \sigma_x^2\right) \cos K(\hat{X} - \hat{X}')$$
$$\times \sum_{0}^{\infty} \epsilon_m^2 J_m^2 (KX_s) \cos m\omega_s t$$

in which

$$\frac{\overline{F(t)\,F(t')}}{\overline{F^2(t)}} = \exp\left\{-\frac{K^2\sigma_\Delta^2}{2} + \frac{(\hat{X}-\hat{X}')^2}{4\sigma_1^2}\right\} \tag{II.60}$$

The power spectrum of $i(t)$ can be obtained by taking the Fourier transform of this function. We note first that the FT given by Eq. II.55 accounts for all terms in the power spectrum except the acoustic term. By the modulation theorem (p. 108 Bracewell) if $F(\omega)$ is the transform of $f(t)$ then the transform of $f(t)\cos\omega_0 t$ is $\frac{1}{2}F(\omega-\omega_0)+\frac{1}{2}F(\omega+\omega_0)$. Thus the power spectrum $I(\omega)$ of the photodetector current in the presence of an acoustic signal is given by

$$I(\omega) = \frac{1}{4}\sum_{m=0}^{\infty}\frac{\epsilon_m}{2}J_m^2(Kx_s)$$

$$\times\left\{\exp\left[-\frac{(\omega-K\bar{u}_0+m\omega_s)^2}{2(\Delta B)^2}\right] + \exp\left[-\frac{(\omega-K\bar{u}_0-m\omega_s)^2}{2(\Delta B)^2}\right] + \exp\left[-\frac{(\omega+K\bar{u}_0+m\omega_s)^2}{2(\Delta B)^2}\right] + \exp\left[-\frac{(\omega+K\bar{u}_0-m\omega_s)^2}{2(\Delta B)^2}\right]\right\} \tag{II.61}$$

$$(\Delta B)^2 = K^2\overline{u_0'^2} + \frac{\bar{u}^2}{2\sigma_1^2} + K^2\overline{(\Delta u)^2}$$

<u>In words:</u> In the presence of an acoustic signal which gives an average displacement of $x_s \sin\omega_s t$ to all the particles of the scattering volume, and in the presence of a steady (constant) average velocity $\bar{u}_0$ of <u>all</u> the particles, the power spectrum of the photodetector current exhibits the following character:

(1) there are two Gaussian peaks at frequencies $\pm K\bar{u}_0$ (= case of m=0).

(2) the standard deviation of the distributions in each case is the Doppler ambiguity ΔB.

(3) there are four additional satellite Gaussian peaks: two centered at $K\bar{u}_0 \pm m\omega_s$, and two centered at $-K\bar{u}_0 \pm m\omega_s$, (= case of $m \neq 0$).

(4) the relative amplitude of the satellites $m=1$ to the main peak is proportional to the ratio of the Bessel functions $J_0^2(Kx_s)/J_1^2(Kx_s)$.

Instantaneous Signal

The photodetector current i(t) has been modeled as

$$i(t) = F(t)\cos\left[KX(t)\right] + G(t)\sin\left[KX(t)\right]$$

in which $F(t)$, $G(t)$ are functions obtained by integrating over random initial position and random deviation from initial position for all particles in the illuminated volume and $X(t)$ is the time averaged displacement of the random "center velocity" $u_0(t)$. This formula can be altered to give a different (physical) insight into the detection process. The basic process (as noted earlier) is a two-beam interferometer experiment; which calls for an intensity of light (note: $i(t)$ is proportional to light intensity) with a sinusoidal component Kx_p (see Eq. II.20). In the above model $X(t)$ is the analogy of x_p and the illuminating (i.e. scattering volume) source has a random character described by $F(t)$ and $G(t)$. One can say that the equivalent two-beam interferometer exhibits a fluctuating amplitude versus time. Thus one can write $i(t)$ as a phasor with fluctuating amplitude, and fluctuating phase,

$$i(t) = A(t)\cos\left[KX(t) - \phi\right], \quad A(t) = \left[F^2(t) + G^2(t)\right]^{1/2}, \quad \phi = \tan^{-1}\frac{G(t)}{F(t)} \tag{II.63}$$

At time t_1 there are four random quantities, $F(t_1), \dot{F}(t_1), G(t_1), \dot{G}(t_1)$. Similarly at time t_2 there are four additional random quantities. The autocorrelation $R_N = \overline{\dot{\phi}(t)\,\dot{\phi}(t+\tau)}$ is that of an eight-dimensional Gaussian distribution. It is known that

$$R_N(\tau) = \frac{1}{2}\left(\frac{\ddot{\rho}}{\rho} - \frac{\dot{\rho}^2}{\rho^2}\right)\log\left(1-\rho^2\right)$$

$$\rho(\tau) = \frac{\overline{F(t)F(t+\tau)}}{\overline{F^2}} = \frac{\overline{G(t)G(t+\tau)}}{\overline{G^2}} \tag{II.64}$$

Now according to Part II. Sect. 1 (above) the functions F, G have Gaussian correlations. Thus

$$\rho(\tau) = \exp\left\{-\frac{1}{2}(DA)^2\tau^2\right\} \tag{II.65}$$

From this one obtains

$$R_N(\tau) = -\frac{1}{2}(DA)^2 \log(1-\rho^2) \tag{II.66}$$

The Fourier power spectrum $N(\Omega)$ of $\dot{\phi}(t)$ is therefore given by

$$N(\Omega) = F\left\{\dot{\phi}(t)\right\} = \int_{-\infty}^{\infty} R_n(\tau)\, e^{i\Omega\tau}\, d\tau \tag{II.67}$$

or

$$N(\Omega) = \frac{1}{4\pi^{\frac{1}{2}}}(DA)\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \exp\left\{-\frac{\Omega^2}{4n(DA)^2}\right\} \tag{II.68}$$

This is approximated by the two values depending on the relative magnitude of Ω and DA,

(1) $N(\Omega) \simeq N(0) \qquad \Omega < DA$

(2) $N(\Omega) \simeq \dfrac{(DA)^2}{2\Omega} \qquad \Omega \gg DA$

where,

$$N(0) = 0.368\ DA \qquad (\Omega < DA). \tag{II.69}$$

We define the Doppler ambiguity spectrum as the power spectrum due to temporal character of the center x- component of velocity $u_0(t)$ (i.e. $\bar{u}_0 + u_0'$). This is given by spectrum of finite transit and spectrum of u_0'. We define the turbulence power spectrum as that associated with the fluctuations in phase $\dot{\phi}$ as defined above. Now the rate of change the phase $KX - \phi$ is the important contributor to the Doppler. One labels this ω_1 where

$$\omega_1 = K u_0(t) - \dot{\phi} \tag{II.70}$$

Both u_0 and $\dot{\phi}$ can be written as a Fourier expansion

$$u_0(t) = \int_{-\infty}^{\infty} e^{i\Omega t} dZ(\Omega)$$

$$\dot{\phi}(t) = \int_{-\infty}^{\infty} e^{i\Omega t} dN_1(\Omega) \tag{II.71}$$

The autocorrelations of u_0 and $\dot{\phi}$ are

$$\overline{u_0(t)\, u_0(t+\tau)} = \int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\tau}\, \overline{dZ(\Omega)\, dZ^*(\Omega')}$$

$$\overline{\dot{\phi}(t)\, \dot{\phi}(t+\tau)} = \int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\tau}\, \overline{dN(\Omega)\, dN(\Omega')} \tag{II.72}$$

Since the amplitudes $dZ(\Omega)$ and $dN(\Omega)$ are orthogonal (in frequency), we write

$$\begin{cases} \overline{dZ(\Omega)\, dZ^*(\Omega')} = 0 & \Omega \neq \Omega' \\ \quad = \dfrac{F_0\left(\frac{\Omega}{\bar{u}}\right)}{\bar{u}} d\Omega, & \Omega = \Omega' \end{cases}$$

$$\begin{cases} \overline{dN_1(\Omega)\, dN_1(\Omega')} = 0 & \Omega \neq \Omega' \\ \quad = N(\Omega)\, d\Omega, & \Omega = \Omega' \end{cases} \tag{II.73}$$

Now the limit of resolution of turbulence spectrum is set at a frequency Ω_0 such that the Doppler ambiguity spectrum is equal to the turbulence spectrum, i.e.

(Doppler Ambiguity)=(turbulence)

$$\frac{K^2 F_{11}'\left(\frac{\Omega_0}{\bar{u}}\right)}{\bar{u}} = N(\Omega_0)$$

(II.74)

(dimensions: (1) $K: M^{-1}$; (2) $F_0: M^3 s^{-2}$; (3) $\bar{u}: Ms^{-1}$; (4) $\Omega_0: s^{-1}$; (5) $N: s^{-1}$).
It is convenient to define $\Omega_0/\bar{u} = k_0$; $k_0\eta = \tilde{k}_0$; $[\epsilon^{\frac{1}{4}}\nu^{5/4}]^{-1}F(k) = \tilde{F}(\tilde{k})$ in which ϵ is the Kolmogorov rate of dissipation of turbulent energy per unit mass, ν is the kinematic viscosity, and $\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$ is the Kolmogorov microscale. Adopting the approximation that

$$N(\Omega_0) \simeq N(0) = 0.368\, DA$$

we are required to find appropriate values for $DA = \left[K_1^2\bar{u}^2 + K^2\overline{(\Delta u)^2}\right]^{1/2}$, $K_1^2 = \frac{1}{2\sigma_1^2}$.
For small scattering angle one can write

$$\overline{(\Delta u)^2} \simeq \frac{\epsilon}{15\nu}\left[\frac{1}{K_2^2}\right], \qquad K_2^2 = \frac{K_1^2 \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

(II.75)

(George and Lumley, Eq. 3.2.8).

Hence

$$(\epsilon^{\frac{1}{4}}\nu^{5/4})F_{11}'(\tilde{k}_0) = 9.33\times10^{-3} K_1\eta R_e\left\{1 + \frac{(2\pi)^2}{15 R_e^2 K_1^4\eta^4 \sin^2\frac{\theta}{2}}\right\}^{1/2}$$

(II.76)

(George and Lumley, Eq. 5.5.2). Here R_e is the Reynolds number based on the smallest length L_λ that can be resolved in the mean flow direction, i.e.

$$R_e = \frac{\bar{u} L_\lambda}{\nu}, \qquad L_\lambda = \frac{\lambda}{2 \sin\frac{\theta}{2}} \tag{II.77}$$

The above equation for $F_{11}'(\tilde{k}_0)$ is a key result. <u>In words</u>: the one dimensional velocity spectrum of turbulence $F_{11}'(\tilde{k}_0)$ can be determined for given Reynolds number R_e of flow, given angle θ between the two beams (scattered and reference) of the equivalent 2-beam interferometer, and given size σ_1 of scattering volume in the x direction as it appears in the parameter $K_1 = \frac{1}{\sqrt{2}\,\sigma_1}$. In the square-root brackets the first term (namely unity) represents the (relative) contribution of finite transit time. It contributes the <u>Doppler ambiguity spectrum</u>. The second term gives the (relative) contribution of turbulence to the broadening of the spectrum of velocity. For fixed R_e and θ the spectral broadening in $F_{11}'(\tilde{k}_0)$ is a function of the size parameter $K_1\eta$. Two regimes of variation of $F_{11}'(\tilde{k}_0)$ with $K_1\eta$ can be found. In the first, $K_1\eta$ is made very small. This is equivalent to making the scattering volume very large (relatively speaking). The spectrum broadening then is due to turbulence, and <u>increases as the size of the volume of scattering increases.</u> In the second regime $K_1\eta$ is made very large, i.e. the scattering volume is made very small. The spectrum broadening is then due to finite transit time (Doppler ambiguity broadening), its magnitude increasing with reduction in size of the scattering volume. Between the two regimes there is evidently a minimum broadening of the spectrum. This occurs at the value of $K_1\eta$ which makes $F_{11}'(\tilde{k}_0)$ a minimum. By inspection the condition for minimum $K_1\eta$ is given by making the contributions of finite transit time and turbulence of equal magnitude, i.e.

$$\frac{(2\pi)^2}{15 R_e K_1^4 \eta^4 \sin^2\frac{\theta}{2}} = 1, \quad \text{or} \quad (K_1\eta)_{opt} = \frac{1.27}{\sqrt{R_e \sin\frac{\theta}{2}}} \tag{II.78}$$

The smallest value of the turbulence spectrum that than be measured with minimum ambiguity due to finite transit time is thus

$$\left[F_{11}'(\tilde{k}_0)\right]_{OPT} = \frac{1.68 \times 10^{-2}\, Re^{\frac{3}{2}}}{\left(\sin\frac{\theta}{2}\right)^{1/2}} \tag{II.79}$$

In words: when it is desired to use the laser doppler velocimeter to measure turbulence it is found that the power spectrum of velocity turbulence is broadened by "large" scattering volume, which renders the determination of the "spectral line" associated with a given wavenumber $\tilde{k}_0$ uncertain. The broadening of the spectrum is reduced by decreasing the size of the scattering volume. Hence one expects that the smallest available volume of scattering would generate the narrowest spectral lines. However as the volume of scattering is reduced the broadening due to finite transit time is increased, making the spectral line of velocity again uncertain. Evidently the narrowest spectral line is achieved by using a compromise size of scattering volume. This compromise size is fixed by $(K_1\eta)_{opt}$. The smallest spectral level $F_{11}'(\tilde{k}_0)$ that is optimized by scattering volume is $\left[F_{11}'(\tilde{k}_0)\right]$ opt. Hence the highest value of $\tilde{k}_0$ that can be measured with minimized ambiguity due to finite transit time is found by expressing $F_{11}'(\tilde{k}_0)$ as a function of some turbulence model, and solving for $\tilde{k}_0$ implicitly.

The implicit solution for $\tilde{k}_0$ can be illustrated by use of the model of isotropic turbulence. Here,

$$F_{11}'(\tilde{k}_0) = \frac{1}{2}\int_{\tilde{k}_0}^{\infty} \frac{E(k)}{k}\left[1 - \frac{\tilde{k}_0^2}{k^2}\right]^2 dk \tag{II.80}$$

The form of $E(k)$ is particularized by choice. Let $E(k)$ be that of Pao (Phys. Fluids 8, 1063, (1965)),

$$E(k) = \alpha k^{-5/2} \exp\left\{-\frac{3}{2}\alpha(k\eta)^{4/3}\right\}$$

(II.81)

Substituting this form into the above integral, and performing the integration leads to a model of $F_{11}'(\tilde{k}_0)$ which can be plotted versus $\tilde{k}_0$. On the same graph one can plot $[F_{11}'(\tilde{k}_0)]_{opt}$ for various choices of R_e and θ. The intersection of the turbulence model (Pao's model) with the ambiguity model $[F_{11}'(\tilde{k}_0)]$ opt. gives the value of $\tilde{k}_0$, which is the wavenumber for which the ratio of the turbulence spectrum to ambiguity spectrum is unity. This is the largest value of wavenumber of the turbulence power spectrum that can be measured with the minimum ambiguity (that is, under conditions of the minimum spectral broadening due to finite transit time), for fixed R_e and θ.

Seeing that every choice of R_e and θ is equivalent to a choice of $\tilde{k}$, one can plot R_1 ($= R_e / \sin\frac{\theta}{2}$) versus $\tilde{k}_0$. Suppose we desire the maximum $\tilde{k}_0$ $(= \Omega_0 \eta / \bar{u})$ to be unity. Then the graph shows $R_1 = 0.10$. For an experiment in which θ is fixed such that $2 \sin\frac{\theta}{2} = 0.2$ and the wavelength $\lambda = 6.3 \times 10^{-5}$ cm it is found from the Reynolds number that the smallest measurable velocity (the optimal velocity in the above sense) is $\bar{u} \simeq 1.5$ cm/sec.

In actual experiments the optimum scattering volume can be determined by a simple formula once the Reynolds number and angle of scattering are known. (see Appendix B).

Concluding Remarks to Part II

In Part II the laser Doppler velocimeter is mathematically analyzed using the model of George and Lumley. The photodetector (electrical) current $i(t)$ in this model is a phasor which is a random function (in both amplitude and phase) of the particle displacement across the laser fringes (see Eq. II.20). By assigning probability densities to the fluctuating component of the volume averaged displacement, and to the deviation displacement of an individual particle from the volume averaged displacement, one can find an ensemble average of i(t), and then form the normalized autocorrelation of i(t) at times t and t (see Eq. II.46). Under the assumption that the size of scattering volume is several wavelengths of laser light large, and that $t-t'$ is not too great, one can calculate the power spectrum of the random current $i(t)$ (see Eq. II.55). The particle displacement is then considered to contain a sinusoidal component due to the presence of an acoustic signal in the scattering volume. The calculation of the power spectrum of photodetector current is then repeated leading to the result shown in Eq. II.63. In applications to measurement of turbulent velocities one is required to find the true one-dimension power spectrum $F_{11}'(k)$ of these random velocities for different scale sizes (i.e. wavenumbers k). Because of finite size of the laser beam, and the existence of random motion from particle to particle within the scattering volume the measured spectrum $F_0(k)$ differs from the true spectrum $F_{11}'(k)$. The measured spectrum $F_0(k)$ can be summed for all values of k to give the time-averaged mean-square fluctuation of the observable Eulerian velocity

The true spectrum $F_{11}'(k)$ can be summed for all values of k to give the time-averaged mean-square fluctuation of the unobservable velocity $u(t)$ which is $u_0(t)+\Delta u(t)$ where Δu is the deviation of $u(t)$ across the scattering volume. Under certain assumptions $F_{11}'(k)$ can be calculated (see Pao's spectrum Eqs. II.80, II.81). A theoretical limit of spectral resolution (of the turbulent velocity spectrum) can be calculated using a good model for the noise due to the phase fluctuations of the equivalent phasor $i(t)$. The model is that of filtered noise in an FM receiver, called $N(\Omega)$. By approximation $N(\Omega)$ is taken as a white spectrum with a fixed value $N(0) = 0.368\left[K^2\overline{(\Delta u)^2} + \overline{u}^2/2\sigma_1^2\right]^{1/2}$ (see Eq. II.69). The limit of resolution of the turbulent velocity spectrum $F_{11}'(k)$ can then be obtained by setting $N(0)$ as the threshold (see Eq. II.74) of $F_{11}'(k)$. The limit thus found is Eq. (II.76) where $F_{11}'(\tilde{k}_0)$ is found to be a function of the Reynolds number (based on the laminar velocity $\overline{u}$) and the wavenumber of the laser light $K=(4\pi/\lambda_E)$ $\sin\frac{\theta}{2}$, where θ is the angle between the two beams of the laser interferometer. When the magnitude of $F_{11}'(k)$ is found for given Reynolds number and θ, the value of $\tilde{k}_o$ is then (implicitly) determinable. This is the "cut-off wavenumber" (see Eq. II.79). At this wavenumber the ambiguity due to finite transit time and turbulence is minimal. It is essentially the largest wavenumber (hence the smallest velocity) that can be measured.

The model of George and Lumley reviewed and extended above is the most complete. Additional models are briefly discussed in Appendices F, G, H and I. These serve to display somewhat different points of view.

Part III

Communications Theory Model

In the model of George and Lumley (Part II) the photodetector current $i(t)$ is a phasor with random amplitude and random phase,

$$i(t) = \left[F^2(t) + G^2(t)\right]^{1/2} \cos\left[KX(t) - \phi(t)\right]$$

we restrict the generality of this model and consider only the cosine function, replacing the random amplitude by a constant. Furthermore we take the displacement $X(t)$ to be deterministic, and write

$$X(t) = \bar{u}t + x_s \cos\omega_s t$$

in which $\bar{u}$, X_s and ω_s are defined in Part II. The only random quantity is a phase $\phi(t)$ This $\phi(t)$ represents noise in the laser Doppler velocimeter and is modeled as if shaped by an RC filter, bandwidth $\Delta\omega_N$, power spectrum $\mathcal{W}_N(f)$ and covariance K_V given by

$$\mathcal{W}_N(f) = \frac{4\psi}{\Delta\omega_N\left[1 + \frac{\omega^2}{\Delta\omega_N^2}\right]}, \quad K_V = \psi e^{-\Delta\omega_N|t|}, \quad K_N(0) \equiv \psi \tag{III.1}$$

It is convenient to regard the frequency $K\bar{u} = \omega_0$ as a carrier frequency, the quantity $Kx_s \cos\omega_s t$ as a phase modulation of this carrier, and $\phi(t)$ as a noise modulation. In complex notation the photodetector current then is assumed to have the form

$$i(t) = A \,\mathrm{Re}\, \exp i\left\{\omega_0 t - Kx_s \cos\omega_s t - \phi(t)\right\}$$

Since this is a periodic function it can be represented by a Fourier expansion,

$$i(t) = A \,\mathrm{Re}\left[\sum_{-\infty}^{\infty} (-i)^m J_m(Kx_s)\, e^{im\omega_s t + i\omega_0 t - i\phi(t)}\right] \tag{III.2}$$

The covariance function of this modulated carrier is then derived to be

$$K_V(\zeta) = \frac{A^2}{2} \sum \epsilon_m J_m^2(Kx_s)\, e^{-(Kx_s)^2(1-e^{-|\zeta|})} \cos\left(\frac{\omega_0\zeta}{\Delta\omega_N}\right) \cos\left(\frac{m\omega_s\zeta}{\Delta\omega_N}\right) \tag{III.3}$$

(Middleton, "Intro. Statis. Comm. Theory." p. 612).

Here $\zeta = \Delta\omega_N t$, J_m is the Bessel function of first kind of integer order, and $\epsilon = 1$ when $m = 0$, $\epsilon_m = 2$ when $m \neq 0$. The spectral distribution corresponding to this covariance function is $\mathcal{W}_V$ where

$$\mathcal{W}_V(\omega) = \frac{A^2 e^{-(Kx_s)^2}}{\Delta\omega_N} \sum_{m=0}^{\infty} \frac{\epsilon_m}{2} J_m^2(Kx_s) \sum_{n=0}^{\infty} \frac{(Kx_s)^n}{n!} \left(\frac{n}{n^2+\beta_{+m}^2} + \frac{n}{n^2+\beta_{-m}^2} \right) \quad \text{(III.4)}$$

$$\beta_{\pm m} = \frac{\omega - \omega_0 \mp m\omega_s}{\Delta\omega_N}$$

(Middleton, p. 613). (Note, only positive frequencies appear here.)

This power spectrum has a dual character. It is noted in the formula for $K_V(\zeta)$ that

$$e^{(Kx_s)^2 e^{-|\zeta|}} = e^{(Kx_s)^2\left[1 - |\zeta| + \frac{|\zeta|^2}{2!} - \cdots\right]}$$

$$'' \quad = e^{(Kx_s)^2}\left[1 - (Kx_s)^2|\zeta| + 2\frac{(Kx_s)^4}{2!}|\zeta|^2 - \cdots\right] \quad \text{(III.5)}$$

The Fourier transform (FT) of unity is a delta function at the origin of frequency. The FT of $\cos \omega_0 t$ is,

$$FT\{\cos\omega_0 t\} = \cos\left(\frac{\omega_0\zeta}{\Delta\omega_N}\right)\} = \frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0)$$

If we multiply this time function by $\cos m\omega_s t$ then

$$FT\left\{\cos\omega_0 t \cos m\omega_s t\right\} = \frac{1}{4}\delta(\omega - \omega_0 + m\omega_s) + \frac{1}{4}\delta(\omega - \omega_0 - m\omega_s)$$
$$+ \frac{1}{4}\delta(\omega + \omega_0 + m\omega_s) + \frac{1}{4}\delta(\omega + \omega_0 + m\omega_s)$$

Thus centered around $\omega = \omega_0$ there are two delta functions (or discrete spectra) for every value of m. The magnitudes of these spectral lines are proportional to $J_m^2(Kx_s)$. We return now and examine III.4 at the special value $n = 0$ and $\omega = \omega_0 + m\omega_s$. In this case $\beta_{+m} = 0$ and $\beta_{-m} = \frac{2m\omega_s}{\Delta\omega_N}$. The value of the first term is known, viz.

$$\lim_{\substack{n\to 0\\ \beta\to 0}} \frac{n}{n^2 + \beta_{+m}^2} = \pi\,\delta(\beta_{+m}), \quad \pi\int \frac{n}{n^2+\beta_{+m}^2}\, d\beta_{+m} = 1 \quad \text{(III.6)}$$

(see Morse & Feshbach, Meth. Theo. Phys. 813). Thus the term $n \to 0, \beta \to 0$ yields a discrete spectrum and the term $n \neq 0, \beta \neq 0$ yields a continuous spectrum, i. e.

$$\mathcal{W}_v(\omega)_{m=0} = \frac{A^2 e^{-(Kx_s)^2}}{\Delta\omega_N} J_0^2(Kx_s)\left\{\delta(\omega-\omega_0)\pi + \sum_{m=1}^{\infty}\frac{(K^2x_s^2)^m}{m!}\left(\frac{m}{m^2+\frac{(\omega-\omega_0)^2}{\Delta\omega_N^2}}\right)\right\} \tag{III.7}$$

Now let $m=1$, and take $\beta_{+1} = \omega-\omega_0-\omega_s$, $\beta_{-1} = \omega-\omega_0+\omega_s$. We then choose a frequency $\omega \to \omega_0+\omega_s$. Then $\beta_{+1} \to 0$ and $\beta_{-1} \to 2\omega_s$. Hence for the choice $m = 1$ the spectrum again has both discrete and continuous values,

$$\mathcal{W}_v(\omega)_{m=1} = \frac{A^2 e^{-(Kx_s)^2}}{\Delta\omega_N} J_1^2(Kx_s)\Bigg\{\delta(\omega-\omega_0-\omega_s)\pi + \delta(\omega-\omega_0+\omega_s)\pi$$
$$+\sum_{m=1}^{\infty}\frac{(K^2x_s^2)^m}{m!}\left(\frac{m}{m^2+\left(\frac{\omega-\omega_0+\omega_s}{\Delta\omega_N}\right)^2} + \frac{m}{m^2+\left(\frac{\omega-\omega_0-\omega_s}{\Delta\omega_N}\right)^2}\right)\Bigg\} \tag{III.8}$$

A sketch of the total spectrum is shown below:

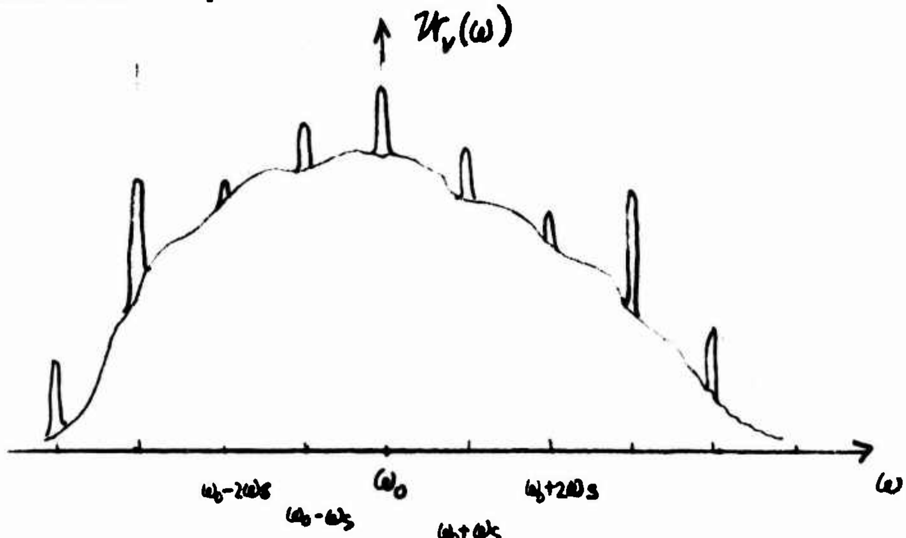


FIG. 12

We next consider the continuous spectrum of $m=0$ at the value of $\omega = \omega_0+\omega_s$ and take Kx_s to be small. Then the noise due to the peak at ω_0 is

$$\mathcal{W}_v(\omega_0+\omega_s)_{m=0} \approx A^2\frac{e^{-K^2x_s^2}}{\Delta\omega_N} J_0^2(Kx_s)\left(\frac{K^2x_s^2}{1+\left(\frac{\omega_s}{\Delta\omega_N}\right)^2}\right) \tag{III.9}$$

The continuous spectrum of $m = 1$ at $\omega = \omega_0 + \omega_s$ is

$$\mathcal{H}_v(\omega_0 + \omega_s) = \frac{A^2}{\Delta\omega_N} e^{-(Kx_s)^2} J_1^2(Kx_s) K^2 x_s^2 \left\{ 1 + \frac{1}{1 + \left(\frac{2\omega}{\Delta\omega_N}\right)^2} \right\} \quad \text{(III.10)}$$

There are two signals in this special case: (1) noise due to the main peak at the Doppler frequency $K\bar{u}_0$ (2) noise at the frequency of the acoustic satellite. We now assume that the criterion for detection of an acoustic signal is that it should equal the satellite noise at the satellite frequency. Hence we state that an acoustic signal of magnitude equal to the continuous noise at the satellite frequency will be detectable if it rises above the noise contributed by the Doppler ambiguity of the central peak (i.e. $m=0$) in the absence of an acoustic signal. This means that we must find a value of Kx_s such that,

$$J_1^2(Kx_s)\left[1 + \frac{1}{1 + \left(\frac{2\omega}{\Delta\omega_N}\right)^2} \right] = \frac{J_0^2(Kx_s)}{1 + \left(\frac{\omega_s}{\Delta\omega_N}\right)^2}$$

or

$$\frac{J_1^2(Kx_s)}{J_0^2(Kx_s)} = \frac{1 + 4\left(\frac{\omega_s}{\Delta\omega_N}\right)^2}{2\left[1 + 3\left(\frac{\omega_s}{\Delta\omega_N}\right)^2 + 2\left(\frac{\omega_s}{\Delta\omega_N}\right)^4\right]} \quad \text{(III.11)}$$

These formulas will be used in the following sections to make numerical estimates of the minimum detectable modulation index.

Part IV

Calculations

Summary of Measured Displacements

Massey (1968, "An Optical Heterodyne Ultrasonic Image Converter," Proc. IEEE 56, 2157) used a local oscillator heterodyne apparatus to sense the vibration amplitude distribution on a reflective resonant diaphragm placed in a liquid acoustic medium. The phase modulation of the laser beam was $\phi(t) = 4\pi \frac{Z(x,y)}{\lambda_E} \sin \omega_1 t$ in which λ_E is the laser wavelength, ω_1 is the acoustic signal frequency and $Z(x,y)$ is the amplitude of vibration of the diaphragm. The dominant noise process was shot noise in the photodetector. For a circuit bandwidth B, and electronic charge e the rms noise current is

$$i_N = \sqrt{2e(I_{L0}+I_S)B}$$

in which I_{L0}, I_S are currents in the detector due to the local oscillator beam, or signal beam acting alone. Assuming a total laser power P_0, quantum efficiency η, and setting $I_S = I_{L0}$, one finds that

$$I_S = I_{L0} = \frac{\lambda_E \eta e P_0}{2hc_E}$$

in which h is Planck's constant, c_E is the speed of light. Now the minimum detectable amplitude is found by equating the noise i_N to the first side band current. It is found that

$$Z_{MIN} = \left(\frac{hc\lambda_E B}{\pi^2 \eta P_0}\right)^{1/2}$$

Thus the minimum detectable amplitude depends on the square-root of the circuit bandwidth and inversely as the square-root of the laser power. In a typical case

$$\lambda_E = 6330 \times 10^{-10} M \qquad f_1 = 1 MHz$$
$$B = 20 kHz$$
$$\eta = 5 \times 10^{-2}$$
$$P_0 = 10^{-3} W$$

it is found that

$$Z_{MIN} = 2.2 \times 10^{-12} M \qquad (peak)$$

Yeh and Cummins (1964, Appl. Phys. Letters. 4, 176) also used a LDV to measure fluid flow based on the local oscillator heterodyne principle. They considered that they could detect constant velocites as low $4 \times 10^{-5} M s^{-1}$ at a scattering angle of $30°$.

Edwards et al, (1971, J. Appl. Phys. 42 837) analyzed and measured steady flow in a LDV based on the local oscillator heterodyne technique. In order to resolve the Doppler signal they estimate that the (constant) velocity $\underline{v}$ must be such that

$$\underline{K} \cdot \underline{v} > D K^2, \qquad K^2 = \underline{K} \cdot \underline{K}$$

in which D is the translational diffusion coefficient, and $|K| = \frac{4\pi}{\lambda_E} \sin^2 \frac{\theta}{2}$. In the case of $\lambda_E = 6328 \times 10^{-10} M, \theta = 45°, D = 10^{-12} M^2 s^{-1}$, they found that approximately

$$|v| > 10^{-5} M s^{-1}$$

Wang (1972; J. Phys. E. 763) found the following S/N dependencies;

local oscillator heterodyne: $S/N \sim n_p \sigma_p \left(\frac{\lambda_E^2}{l_s^2} \right)$

differential or symmetrical heterodyne: $S/N \sim \left(\frac{1}{n_p^{5/3}} \right) \sigma_p \left(\frac{\lambda_E^2}{l_p^2} \right)$

in which n_p is the partical number density, σ_p is the Mie scattering cross section of the partical, λ_E is the laser wavelength, ls is the linear dimension (say depth) of the scattering volume, and l_p is the average partical size. Thus for the case of the local oscillator heterodyne a high particle concentration and small focal volume are needed. It is insensitive to particle size and has broad frequency spreading. In contrast the differential (or symmetrical) heterodyne system requires low particle concentration, is sensitive to particle size and has low frequency spreading.

Criterion for Signal Detection

We assume Kx_s to be small, and consider only two terms (i.e. $m=0$, $m=1$) in the equation for the power spectrum of the photodetector (Eq. II.61). We then state the following criterion for signal detection: an acoustic signal is considered detectable if it has a magnitude given by Eq. II.61 at $m=1$ at a frequency $\omega = K\bar{u}_0 + \omega_s$, for that magnitude of Kx_s which makes it equal to the noise of $m=0$ at the same frequency: in symbols,

$$J_1^2(Kx_s)\left\{1 + e^{-\frac{(2\omega_s)^2}{2(\Delta B)^2}}\right\} = J_0^2(Kx_s)\, e^{-\frac{\omega_s^2}{2(\Delta B)^2}}$$

or

$$\frac{J_1^2(Kx_s)}{J_0^2(Kx_s)} = \mathcal{C}, \qquad \mathcal{C} = \frac{e^{-\frac{\omega_s^2}{2(\Delta B)^2}}}{1 + e^{-\frac{(2\omega_s)^2}{(\Delta B)^2}}}$$

$$(\Delta B)^2 = K\overline{u_0'^2} + \frac{\bar{u}^2}{2\sigma_1^2} + K^2\overline{(\Delta u)^2}$$

Discussion: When the noise band width exceeds the frequency of the acoustic signal to the extent that $\omega_s/(\Delta B) \ll 1$, then $\mathcal{C} \simeq 0.5$ and $\sqrt{\mathcal{C}} \approx 0.7$ For this case

$$J_1(Kx_s) \approx 0.7\, J_0(Kx_s)$$

with a solution,

$$Kx_s = \frac{4\pi x_s}{\lambda_E} \approx 1.1$$

or

$$x_s = \frac{1.1}{4\pi}\lambda_E$$

We choose $\lambda_E \approx 5\times10^{-7}$ M. Hence the threshold signal is

$$x_s = \frac{1.1}{4\pi} \times 5\times10^{-7} = 4.4\times10^{-8}\ \mathrm{M}$$

In contrast when $\omega_s/\Delta B >> 1.1$, $C \approx e^{-\frac{\omega_s^2}{2(\Delta B)^2}}$. If we choose $\omega_s = 3(\Delta B)$, then $C = e^{-4.5} = 0.011$, $\sqrt{C} = 0.105$. Thus

$$J_1(Kx_s) = 0.105 \, J_0(Kx_s)$$

with the solution,

$$Kx_s \approx 0.2, \quad \text{or} \quad x_s \approx \frac{0.2}{4\pi} \times 5\times10^{-7} \approx 8\times10^{-9} M$$

A table of minimum detectable signals is presented here, based on the approximation that

$$J_1(Kx_s) \approx \frac{Kx_s}{2}; \quad i.e. \; x_s = \frac{\sqrt{C}}{4\pi} 2\lambda_E \approx \frac{10}{4\pi}\sqrt{C}\times10^{-7}$$

Table I

ω_s / (ΔB)	X_s (meter)
5	1.5×10^{-10}
6	9.8×10^{-12}
7	3.8×10^{-13}

Thus, in order to achieve detection of the particle displacement $3.4\times10^{-13} M$ associated with sea state 1 the noise band must be less than $1/7$ of the radian frequency of the signal. At $f = 100 HZ$ we therefore require $\Delta B \leq (1/7) 200\pi \approx 90 s^{-1}$, i.e. $\Delta f \approx 14$ Hz.

A different criterion of signal detectability is to assume that the discrete spectrum of Eq. III. 8 at $\omega = \omega_0 + \omega_s$ is equal to the continuous spectrum at the same frequency. Thus the condition becomes

$$K^2 x_s^2 \left\{ 1 + \frac{1}{1 + \left(\frac{2\omega_s}{\Delta\omega_K}\right)^2} \right\} = \pi$$

As an example let $\omega_S = \omega_N$ Then

$$K^2 X_S^2 = \pi (0.83)$$

$$X_S = \frac{1.62}{4\pi} \times 5\times10^{-7} = 6.4\times10^{-8} M$$

Shot Noise

We assume the optical signal is strong enough so that the dominant noise process in the electronic system of the photodetector is shot noise (see Oliver, 1961 "Signal-to-Noise ratios in photolectric mixing", Proc. I RE (Correspondence) 49, p. 1960). The rms noise current in the local oscillator heterodyne system is then

$$i_N = \sqrt{2e(I_{Lo} + I_S)B}$$

(see discussion in "Summary of Measured Displacements"). Under the assumption that $I_{Lo} = I_S$ one has

$$I_{Lo} = I_S = \frac{\lambda_E \eta e P_o}{2 h c_E} .$$

Hence

$$i_N = e\sqrt{2\left(\frac{\lambda_E \eta}{h c_E}\right) P_o B} .$$

Now choosing $\lambda_E = 5.145\times10^{-7} M$, $\eta = 0.05$, $h = 6.6\times10^{-34}$ joule sec, $c_E = 3\times10^8$ m/sec, one has

$$\frac{2\lambda_E \eta}{h c_E} = \frac{(2)(5.145\times10^{-7})\times5\times10^{-2}}{6.6\times10^{-34}\times3\times10^{8}} = 2.6\times10^{17} .$$

Also, the electronic charge is

$$e = 1.6\times10^{-19} \text{ Coulomb}$$

Hence

$$i_N = (1.6\times10^{-19})(5.1)10^{8}\sqrt{P_o B} = (8.16)10^{-11}\sqrt{P_o B} \text{ amperes.}$$

If the system of detection is shot-noise limited then this electric current is the threshold of detection. In the general case of an unselected laser this noise threshold is

$$\dot{i}_N = 5.09 \times 10^{-7} \sqrt{\lambda_E \eta P_0 B}$$

in which all quantities are in MKS units.

Minimum Detectable Signal In the Presence of Shot Noise (Massey 1968, loc. cit)

We assume here that the only noise in the system of a local oscillator heterodyne is shot noise. The photodetector electric current in the presence of an acoustic signal is then

$$i(t) = I_{Lo} + I_s + 2\sqrt{I_{Lo} I_s}\cos(\omega_0 t + K x_s \cos\omega_s t)$$

in which $I_{Lo} = \frac{E_R^2}{2}$, $I_s = \frac{E_s^2}{2}$. The side band structure has the form

$$\dot{i}_{SB} = \sqrt{2 I_{Lo} I_s}\Big\{ J_0(K x_s)\cos\omega_0 t + \sum_{m=1}^{\infty} J_m(K x_s)\cos[(\omega_0 + m\,\omega_s)t] - \sum_{m=1}^{\infty} J_n(K x_s)\cos[(\omega - m\,\omega_s)t] \Big\}$$

Assuming $K x_s \ll 1$, then $J_0(K x_s) \approx 1$, $J_1(K x_s) \simeq K x_s/2$, $J_n(K x_s) \approx 0$, for $m > 1$.

Then the amplitude of current in <u>one</u> side band is

$$\dot{i}_{SB} \approx \sqrt{2 I_{Lo} I_s}\,\frac{K x_s}{2}\,.$$

If this is equated to the shot noise current the minimum detetestable acoustic amplitude is

$$(x_s)_{MIN} = \left(\frac{h c_E \lambda_E B}{\pi^2 \eta P_0}\right)^{\frac{1}{2}}.$$

Choosing $\lambda_E = 5.145 \times 10^{-7}$, $\eta = 0.05$, $h = 6.6 \times 10^{-34}$, $c_E = 3 \times 10^{8}$, it is seen that

$$(x_s)_{MIN} \approx 4.6 \times 10^{-16}\sqrt{\frac{B}{P_0}}\ \text{(meters)}$$

Scattering Equation

The scattering function is definded by the relation

$$\beta(\theta) = \frac{dI(\theta)}{E\,dV} \qquad (M^{-1}\,str^{-1})$$

in which $I(\theta)$ is the light intensity (watts) in the θ-direction, E is the irradiance (watts/M^2) dV is the volume of scattering (M^3). The total scattering coefficient b for non-polarized light) is

$$b = 2\pi \int_0^{\pi} \beta(\theta) \sin\theta \, d\theta \qquad (M^{-1})$$

Now let the differential scattering volume be the simple geometric figure of a cross section A normal to the beam and a depth l. The product AE is the power of the incident beam P_i. Hence the power scattered into the solid angle $d\Omega$ is

$$\Delta P_s(\theta) = \Delta I(\theta)\,\Delta\Omega(\theta) = \beta(\theta) P_i \,\Delta\Omega\, l$$

The incident power is related to the laser output power P_o

$$P_i = P_o e^{-\alpha R_1}$$

in which α is the attenuation coefficient and R_1 is the distance travelled. The solid angle Ω is that intercepted by a lens of aperture Δa at distance R_2, i.e.

$$\Delta\Omega = \frac{\Delta a}{R_2^2}$$

Thus the power received over Δa is

$$\Delta P_s(\theta) = \beta(\theta) P_o e^{-\alpha(R_1+R_2)} \frac{(\Delta a) l}{R_2^2}$$

Calculations

From Duntley (1963), J.O.S.A. 53, (214-233) the "average ocean water" can be described as having an attenuation $\alpha = 0.05/M$, and a scattering coefficient $\beta(180°) = 6\times10^{-4}/$ meter steredian.

Assuming $R_1 = R_2 = R$ it is seen that

$$\frac{\Delta P_s(180°)}{P_0} = 6\times10^{-4} e^{-0.10R} \frac{(\Delta a)\ell}{R^2}$$

If we take a receiving lens of diameter 0.1 meter, a depth of scattering volume at least a half-wavelength of acoustic signal at 100HZ (= 7.5 meters), and a working range of 30 meters, it is seen that

$$\frac{\Delta P_s(180°)}{P_0} = (6\times10^{-4}) e^{-3.0} \frac{\pi}{4}\times10^{-2}\times\frac{7.5}{30^2}$$

$$'' \qquad = 1.96\times10^{-9}$$

If "clear water" is considered then $\beta(\theta) \approx 2\times10^{-4}$, and the backscattered light is

$$\frac{\Delta P_s(180°)}{P_0} \approx \frac{2}{3}\times10^{-9}$$

Modulation Index and Doppler Shift

The modulation index appears in Eq. II. 57 as the symbol KXs in which $K = \left(\frac{4\pi}{\lambda_s}\right)\sin^2\frac{\theta}{2}$.

The threshold particle displacement at 100HZ is 3.4×10^{-13} M. Choosing the angle of observation of the scattered light to be 180° we see that the threshold modulation index is calculated to be

$$(KX_s)_{MIN} = \frac{4\pi\times3.4\times10^{-13}}{5.145\times10^{-7}} = 8.3\ (10^{-6})$$

The Doppler shift in the presence of an acoustic signal is given by Eq. II.61 . Here it is seen that a constant velocity $\overline{u}_0$ shifts the frequency of the carrier by an amount $K\overline{u}_0 = (4\pi/\lambda_s)\overline{u}_0$. The acoustic signal shifts the frequency be the amount mω_s, where m is the number of the side-band. We see then that an acoustic modulation of the laser carrier does not appear as a true Doppler, namely as a frequency shift dependent on velocity, but rather as a frequency shift due to phase modulation, whose magnitude (of shift) is independent of the associated modulation index, but whose amplitude of power spectrum at the frequency of shift does depends on this index. The detection method is not that of an laser Doppler velocimeter measuring a constant velocity, but rather of an optical heterodyne measuring sinusoidal displacement. Ultimately the detection process rests on the ability to measure magnitude of spectrum rather than the ability to measure frequency shift.

Particle Size, Particle Density, Volume Scattering Function, Attenuation in the Ocean

Suspended material in the ocean which is retained by a 45 micron filter is called particulate matter. The amount of particulate matter (in milligrams per liter) ranges from 0.04 in the surface water of the North Atlantic to 18 in the region of coastal waters. An average oceanic total is 0.8 to 2.5 (See Table II).

Particle size distribution in various ocean bases is shown in Fig. 12(a) in which is a graph of the number of particles per cubic centimeter versus the diameter of the particle. A statistical analysis leads to a statement that the mean squared particle radius is about 14 x 10^{-12} M^2, so that the "mean particle diameter" is about 7 microns. According to Mie theory the scattering cross-section of a single sphere (β_s) which is much larger than an optical wavelength is twice the geometric cross section.

Hence $\beta_s = 2\pi \times 14 \times 10^{-12} = 8.8 \times 10^{-11}\ M^2$

Table III shows the volume scattering function of pure water and sea water as a function of wavelength. Fig. 12(b) shows the attenuation of sea water as a function of wavelength.

Motion of the Scattering Particle

The mathematical model is based on the assumption that the motion of the "colloidal" particle suspended in the water faithfully records the acoustic particle velocity. To check under what conditions this can be true, we take the colloidal particle to be a sphere, radius a, density ρ_s and assume it is oscillating in a fluid medium of density ρ_f, kinematic viscosity ν_f at frequency ω_s and net amplitude U. Then the forces $\mathcal{F}$ exerted by the fluid on the sphere are twofold: (1) the accelerative (2) the viscous force. According to Lamb (Hydrodynamics, p. 644, Eq. (26)), the expression for $\mathcal{F}$ is

TABLE II. Amount of suspended particulate matter

Area	Depth (m)	Suspension (mg/l)	Reference
(a) total			
Oceanic	deep water	0·05 (average)	Jacobs and Ewing, 1969
North Atlantic	surface water	0·04–0·15	Folger and Heezen, 1967
Oceanic	—	0·8–2·5 (average)	Lisitsyn, 1959
Pacific, coast	—	1·6	Goldberg *et al.*, 1952
Coastal	—	6·0–18·0	Postma, 1954
(b) organic fraction			
Atlantic	—	0·04–0·17	Riley *et al.*, 1965
North Atlantic	—	0·05–0·2	Gordon, D. C., 1970a
North Atlantic	deep water	0·01–0·02	Gordon, D. C., 1970a
Central Pacific	surface water	0·02	Gordon, D. C., 1971
Central Pacific	deep water	0·005	Gordon, D. C., 1971
(c) inorganic fraction			
Atlantic, offshore	—	0·05–1·0	Armstrong, 1958, 1965
Coastal	—	0·16–1·20	Armstrong, 1958, 1965

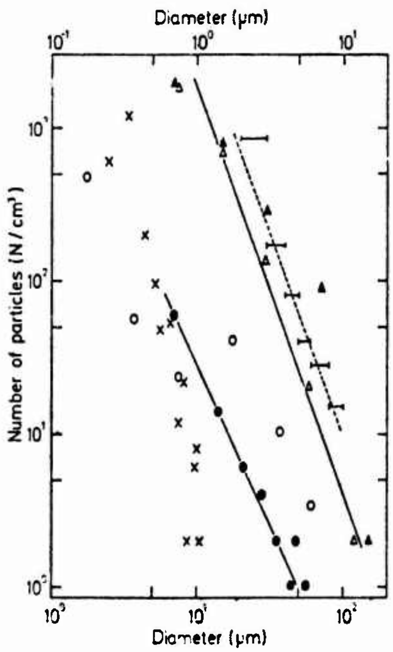


Fig. 12a Examples of particles size distributions:

Upper scale:

△ Kullenberg, Pacific deep, 1953;
|—| Brun-Cottan, Coulter counter, 500 m depth, Mediterranean, 1971.

Lower scale:

● Gordon D.C., microscope, organic matter, surface Atlantic, 1970;
× Carder *et al.*, Coulter counter, Pacific surface, 1971;
▲ Jerlov, microscope, fiord, 1955;
○ Ochakovsky, microscope, Mediterranean, 1966a.

TABLE III Volume scattering function at 90° and total scattering coefficient for pure water and sea water as a function of the wavelength.

λ(nm)	350	375	400	425	450	475	500	525	550	575	600	
$\beta_{90}(10^{-4}\ m^{-1})$	6·47	4·80	3·63	2·80	2·18	1·73	1·38	1·12	0·93	0·78	0·68	Pure water
$b^{*}(10^{-4}\ m^{-1})$	103·5	76·8	58·1	44·7	34·9	27·6	22·2	17·9	14·9	12·5	10·9	
$\beta_{90}(10^{-4}\ m^{-1})$	8·41	6·24	4·72	3·63	2·84	2·25	1·80	1·46	1·21	1·01	0·88	Pure sea water
$b^{*}(10^{-4}\ m^{-1})$	134·5	99·8	75·5	58·1	45·4	35·9	28·8	23·3	19·3	16·2	14·1	(S = 35–39‰)

* Computed according to eq. (11) with $\delta = 0{\cdot}09$ which leads to $b = 16{\cdot}0 \times \beta(90)$.

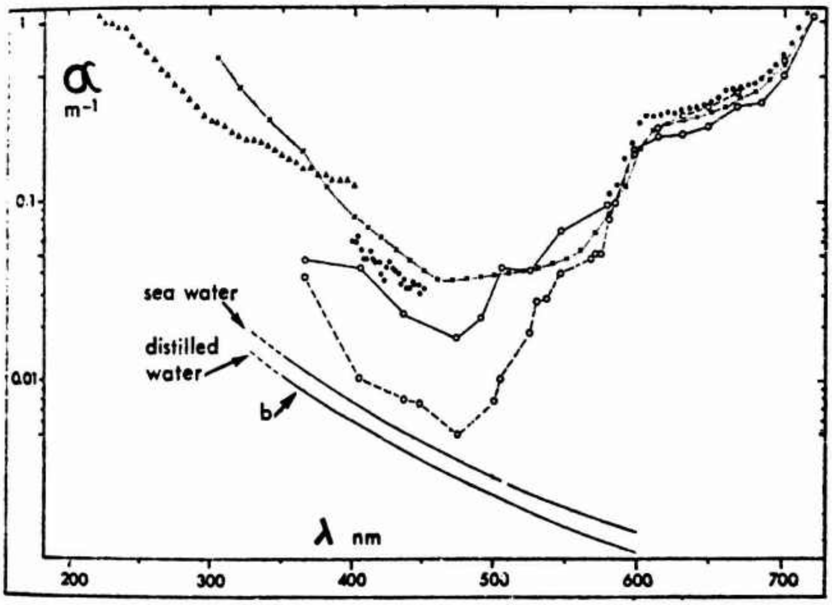


Fig. 12b Attenuation curves in the near ultraviolet and in the visible part of the spectrum.

▲ Lenoble-Saint Guily (1955), path length: 400 cm;
× · · Hulburt (1934) (1945), path length: 364 cm;
● Sullivan (1963), path length: 132 cm;
○— Clarke-James (1939), path length: 97 cm (Ceresin lined tube);
○ · · · James-Birge (1938), path length: 97 cm (Silver lined tube).

Total scattering coefficient for pure water and pure sea water as a function of wavelength, according to Table III.

$$\mathfrak{F} = -\frac{4}{3}\pi\rho_f a^3\left(\frac{1}{2}+\frac{9}{4\beta a}\right)\frac{dU}{dt} - 3\pi\rho_f a^3\omega\left(\frac{1}{\beta a}+\frac{1}{\beta^2 a^2}\right)U$$

in which $\beta^2 = \omega/2\nu$. If the period of oscillation is made infinitely long, then $\beta^2 a^2 \ll \beta a \ll 1$ and the term in dU/dt (i.e., the inertial increment due to the gross motion of the fluid) becomes negligible. Under these conditions the magnitude of the force exerted by the fluid on the sphere is $\mathfrak{F} = 6\pi\rho_f \nu a U$. This is the stokesian force resisting the slow descent of a particle in a viscous fluid under the action of gravity. The equation of forced motion arising from this resistive force is therefore

$$\frac{4}{3}\pi a^3 \rho_s \frac{dU_s}{dt} = 6\pi\mu_f a U$$

in which $\mu_f = \rho_f \nu$ is the fluid dynamic viscosity. Thus, for particles of diameter D,

$$\frac{dU_s}{dt} = \mathcal{K}U, \qquad \mathcal{K} = \frac{18\mu_f}{D^2\rho_s}$$

Since U is the differential motion of the water, i.e., $U = U_f - U_s$, we can assume U_f to be the acoustic excitation and write $U_f = U_0 \exp j\omega t$ At the steady state frequency ω the velocity U_s reduces to

$$U_s = \frac{U_0 \exp j\omega t}{\frac{j\omega}{\mathcal{K}} + 1}$$

This equation defines the properties of a low-pass filter with a cut-off frequency $\omega_{c.o.} = \mathcal{K}$ As long as $\left|\frac{\omega}{\mathcal{K}}\right| \ll 1$ the motion of the colloidal particle will faithfully follow the motion of the acoustic wave. To calculate the magnitude of the cut-off frequency, we note that the dynamic viscosity of water at $20^\circ C$ (in centipose) $= 0.01$ dyne sec/cm^2 $= 10^{-3}$ NsM^{-2}. Assuming particle sizes of order 7 x 10^{-6} meter, one calculates $\mathcal{K} = \omega_0 = \left(\frac{18}{49}\right)10^9/\rho_s$. If the density of the colloidal particle is the same as that of water, the cut-off frequency is

$$f_{c.o.} = \frac{18 \times 10^6}{49 \times 2\pi} = 58.5 \text{ kHz}$$

If $\left|\frac{\omega}{\mathcal{K}}\right| \ll 1$, then $|U_s - U_0| \simeq \frac{\omega}{\omega_0} U_0$. By choosing $\omega \ll \omega_0$, one sees that the difference between the motion of the colloidal particie and that of the acoustic wave is negligible.

It appears from the above argument that for (low-frequency) periodic motion, which is long relative to the optical process duration, the forces involved are purely resistive, and the particle motion is not sensibly different from the wave motion. However, when the wave motion is complex due to presence of soft reflective walls, diffraction, etc., the forces acting on a colloidal particle are along more than one coordinate. The motion of the particle is then oval, or elliptical, and the equations written above no longer hold. It could then be said that in complex sound fields, it can hardly be expected that a colloidal particle will faithfully follow the motion of the acousitic wave since it would then be required to be nearly indistinguishable from the medium itself, assumed free.

Brownian Motion

According to the theory of Brownian motion, the mean of the square of the distance travelled by a particle in a fluid during a time t is given by the equation

$$\overline{r^2} = 6Dt$$

(See Lundau, Lifshitz "Fluid Mechanics", p. 227ff). Here D is the diffusion coefficient (dimensions M^2s^{-1}). For spherical particles of radius a diffusing slowly in a medium of dynamic viscosity μ , it is known that $D = \frac{kT}{6\pi\mu a}$ Thus, the mean distance travelled in time t is

$$\sqrt{\overline{r^2}} = \sqrt{\frac{kTt}{\pi\mu a}} \; ; \qquad t = \frac{\overline{r^2}\,\pi\mu a}{kT}$$

It is important to es mate the time required for the particle to move a "decorrelation distance". Arbitrarily (but reasonably), this distance is taken to be a quarter wavelength of the laser light. Let $\lambda_l = 3860\,\mathring{A}$, $\mu = 10^{-3}\,NM^{-2}s$, $a = 3\times10^{-6}\,M$, Boltzman's constant = 1.38×10^{-23} $NMK^{\circ -1}$, $T = 300^\circ K$ Then the decorrelation time is

$$t_d = \left(\frac{3.86\times10^{-7}}{4}\right)^2 \frac{\pi\times10^{-3}\times3\times10^{-6}}{1.38\times10^{-23}\times3\times10^{2}} = 2.1\times10^{-2}\,s$$

The "decorrelation speed" is $\sqrt{\overline{r^2}}/t_d$ or

$$C_d = \frac{(3.86 \times 10^{-7}/4)}{2.1 \times 10^{-2}} = 4.6 \times 10^{-6}\ \mathrm{Ms}^{-1}$$

Thus, C_d is of the order of 10^{-3} mm/sec.

The possibility of decorrelation of the motion due to the acoustic wave by the Brownian motion of the suspended particles must be seriously considered, if the duration of the process required to sample the wave exceeds 0.02 sec. At 100 Hz this allows about 2 complete cycles to be sampled. However, at 50 Hz only 1 cycle can be sampled. Thus, there is a threshold frequency for doppler detection of particle velocity, estimated here at 50 Hz.*

Spectral Broadening Due to Brownian Motion

The probability of finding a Brownian particle in the distance interval r and $r+dr$ is proportional to $\exp(-r^2/4Dt)$ (See Landau, Liftshitz, p. 227). For two-way travel the phase change $\Delta\phi$ due to Brownian motion is $\Delta\phi = 2k\Delta r = \frac{4\pi\Delta r}{\lambda}$. Hence, the probability of finding the motion of the particle in phase $\Delta\phi$ is proportional to $\exp\left\{-(\Delta\phi)^2/(64\pi^2 \frac{D}{\lambda^2} t)\right\} = \exp\left\{-\frac{(\Delta\phi)^2}{B_B t}\right\}$ where $B_B = \frac{64\pi^2 D}{\lambda^2}$. In accordance with the mathematical model sketched earlier, the spectrum of the first sideband due to acoustic modulation is 1/2 of this quantity. Thus, the spectral broadening due to Brownian Motion is

$$\frac{B_B}{2} = \frac{32\pi^2 D}{\lambda_E^2} = \frac{16}{3}\frac{\pi k T}{\lambda_E^2 \mu a}$$

For a laser wavelength in water of 3.86×10^{-7} M, $\mu = 10^{-3}$ NsM^{-2}, $a = 3 \times 10^{-6}$ M, $k = 1.38 \times 10^{-23}$ NM$^{\circ}$K^{-1}, $T = 300^{\circ}$K the spectral broadening is 155 Hz.

Multiple Scattering

The transport equation can be used to model multiple scattering. Let $\Psi(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3\underset{\sim}{r}\, d^3\underset{\sim}{v}$ be the number of photons in volume $d^3\underset{\sim}{r}$ at $\underset{\sim}{r}$ with velocities in $d^3 v$ at $\underset{\sim}{v}$ at time t. Let $\sigma(\underset{\sim}{r}', \underset{\sim}{v}', t' \mid \underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3 r'\, d^3 v'\, dt$ be the number of photons scattered from $d^3\underset{\sim}{r}'$ with velocities in $d^3\underset{\sim}{v}'$ at $\underset{\sim}{v}'$ at time t' into volume $d^3 r$ with velocities $d^3\underset{\sim}{v}$ at $\underset{\sim}{v}$ and time t. Let $S(\underset{\sim}{r}, \underset{\sim}{v}, t)\, d^3 r\, d^3 v$ represent the rate at which photons are introduced into $(\underset{\sim}{r}, \underset{\sim}{v})$ by the laser. Then the most general time dependent transport equation is

*This conclusion is restricted to the case of a single scattering particle whose Brownian motion disturbs the acoustic particle velocity over short times, and over an assumed characteristic length of $\lambda_E/4$. In the multiple particle case the Brownian motion is random. Over long (enough) times it only adds noise to the detection process but does not set threshold frequencies. The characteristic length is then not significant.

$$\frac{\partial \Psi}{\partial t} + \underline{v} \cdot \underline{\nabla} \Psi + v\sigma\,(\underline{r}, \underline{v}, t) = \int v'\sigma\, \Psi' d^3r'\, d^3v' dt' + S$$

This is a differential-integral equation in the unknown function Ψ and known function σ.

It is too intractable to solve. Simplification is usually accomplished by considering only the axisymmetric case in which r is replaced by z, μ (z being the direction of propagation, μ the polar angle to this direction), and considering only uniform velocity $\underset{\sim}{V}$ so that Ψ is independent of velocity. Then, the transport equation reduces to

$$\left(\sigma_s + \sigma_a - \frac{\mu}{L} + \frac{\partial}{\partial t}\right) \Psi(\mu) = \int dt' \int_{-1}^{+1} \sigma(\mu, \mu', t')\, \Psi(\mu', t')\, d\mu'$$

where σ_s, σ_a are the scattering and absorption cross-sections, respectively, and L = diffusion length. The solution is assumed to have the form $\Psi(\mu) e^{-z/L}$. If multiple scattering is neglected, then, approximately

$$(\sigma_s + \sigma_a)\, \Psi_1(\mu) + \frac{\partial \Psi_1(\mu)}{\partial t} = \frac{\mu}{L} \Psi_0(\mu)$$

in which Ψ_1, is number of photons simply scattered from the initial set Ψ_0. The first term on the $l.h.s.$ is the number of singly scattered photons that are lost by absorption and scattering. By changing subscript 0 to 1 and subscript 1 to 2, this same formula can be used to estimate the number of photon that are doubly scattered. A repetition of this bookkeeping process leads to the GE conclusion that 27% of the received laser light will be multiply scattered and 73% simply scattered (See GE Report 973-SH-347, March 1970, R. M. Ameigh, et al, p. 28).

This conclusion on the partition of the incident photons into single and multiple scattered photons is based on neglect of the integral in the transport equation which accounts for the transfer of photons from direction μ' to direction μ ; i.e., the volume effect is neglected leaving only the linear effect. A true account of multiple scattering must rely on solving the integral equation per se.

While single scattering accounts for observed scattering effects over short path lengths, it is not valid over long path lengths. In the latter region multiple scattering is the dominant factor.

Part V

The Effect of Medium Distortions On the S/N Ratio of an Optical Heterodyne System

Fried (Proc. IEEE 55, p. 57, 1967) has developed a theoretical formula for the limit of improvement of the S/N ratio in the presence of a distorting medium. We review here the conclusions most relevant to our study.

Let the photodetector be modeled as a circular aperture of diameter D. By use of an appropriate weighting function $W(\underline{x})$ for the contribution of each elementary area $d\underline{x}$ of this circle the total photocurrent is derived to be

$$i = \frac{1}{2}\int_D d\underline{x}\, W(\underline{x})\,\eta\,(E_o + E_s)^*(E_s + E_o) = \frac{\pi}{8} D^2 \eta\,(E_s + E_o)^*(E_s + E_o)$$

(V.1)

in which η is the quantum efficiency of the detector, E_s is the complex scattered signal field and E_o the complex local oscillator field of the optical heterodyne. On the assumption that $E_o > E_s$ it is seen that the information-carrying part of the photocurrent is

$$i_s = \eta \frac{\pi D^2}{4} |E_o||E_s| \cos(2\pi \Delta f t + \Delta\phi)$$

(V.2)

in which $\Delta f = f_s - f_o$, $\Delta\phi = \phi_s - \phi_o$, and $|E_o|, |E_s|$ are the amplitudes of the local oscillator beam, and the scattered beam respectively.

Let G be the gain associated with the current amplifier, and R = resistance of the load in the detection circuit. Then the time average of the signal in the photodetector is

$$S = \overline{(i_s G)^2 R} = \frac{1}{2}\left[\eta \frac{\pi D^2}{4} |E_o||E_s|\right]^2 G^2 R \qquad \text{(V.3)}$$

For circuit noise fluctuation current i_N (= shot noise), the amplified noise is $i_N G$ and the noise power per unit bandwidth is

$$N = e G^2 \eta \frac{\pi}{4} D^2 |E_s|^2 R$$

(V.4)

in which e is the electronic charge. Hence the S/N ratio per unit bandwidth is

$$\frac{S}{N} = \frac{\eta}{e} \frac{\pi D^2}{8} |E_s|^2$$

(V.5)

Thus S/N is proportional to the total signal power collected (i.e. $\frac{\pi}{8} D^2 |E_s|^2$) and proportional to the quantum efficiency η/e of the detector.

When the wave field scattered from the medium is random one replaces the non-random phase ϕ_s by the random phase $\phi_s(\underline{x})$, and the non-random amplitude $|E_s|$ by the random amplitude $|\overline{E_s}| \exp [\ell(\underline{x})]$, in which $|\overline{E_s}|$ is the rms value of $E_s(\underline{x})$, and $\ell(\underline{x})$ is the log amplitude i.e. $\ell(\underline{x}) = \ln E_s(\underline{x})/|\overline{E}|$ The signal current i_s is then derived to be

$$i_s = \eta |E_o| |\overline{E_s}| \int d\underline{x}\, W(\underline{x}) \exp[\ell(\underline{x})] \cos[2\pi \Delta f t + \phi_s(\underline{x}) - \phi_o]$$

(V.6)

The information bearing signal is

$$S = \overline{(i_s G)^2} R = (\eta |E_o| |\overline{E_s}| G)^2 R \; \overline{\left\{ \int d\underline{x}\, W(\underline{x}) \exp[\ell(\underline{x})] \cos[2\pi \Delta f t + \phi_s(\underline{x}) - \phi_o] \right\}^2}$$

(V.7)

or

$$S = (\eta |E_o| |\overline{E_s}| G)^2 R \; \frac{1}{2} \iint d\underline{x}\, d\underline{x}'\, W(\underline{x}) W(\underline{x}') \exp[\ell(\underline{x}) + \ell(\underline{x}')] \cos[\phi_s(\underline{x}) - \phi_s(\underline{x}')]$$

In the last step the time average has been dropped because: (1) the average over ϕ_o vanishes; (2) the spectra of $\phi_s(\underline{x})$ and $\ell(\underline{x})$ are within the information bandwidth of the detector,

$\phi_s(\underline{x})$ and $\ell(\underline{x})$ therefore being virtually constant over the averaging period. Since S is random one can form the ensemble average $\langle S\rangle$. In terms of a change of variable, $\underline{r} = \underline{x} - \underline{x}'$, $\underline{r}' = \frac{1}{2}(\underline{x} + \underline{x}')$, this average is

$$\langle S\rangle = \frac{1}{2}\left(\eta |E_o| \overline{|E_s|} G\right)^2 R \iint d\underline{r}\, d\underline{r}'\, W(|\underline{r}' + \tfrac{1}{2}\underline{r}|)\, W(|\underline{r} - \tfrac{1}{2}\underline{r}|) \exp\left[-\tfrac{1}{2}\mathcal{D}(r)\right]$$

(V.8)

where $\mathcal{D}(r)$ is the wave-structure function defined as the sum of the structure function for phase $\mathcal{D}_\phi(r)$, and amplitude $\mathcal{D}_\ell(r)$, i.e.

$$\mathcal{D}(r) = \mathcal{D}_\phi(r) + \mathcal{D}_\ell(r)$$

$$\mathcal{D}_\phi(r) = \langle [\phi(\underline{x}) - \phi(\underline{x}')]^2 \rangle$$

$$\mathcal{D}_\ell(r) = \langle [\ell(\underline{x}) - \ell(x')]^2 \rangle$$

$$r = |\underline{r}| \qquad \text{(V.9)}$$

By choosing $W(\underline{x})$ properly it can be shown that

$$\langle S\rangle = \pi \left(\eta |E_o| \overline{|E_s|} G\right)^2 R \int_0^D r\, dr\, K_o(r) \exp\left[-\tfrac{1}{2}\mathcal{D}(r)\right]$$

$$K_o(r) = \frac{1}{2}\left[D^2 \cos^{-1}\left(\frac{r}{D}\right) - r\left(D^2 - r^2\right)^{1/2}\right]$$

(V.10)

The shot noise in the detector system is the same as Eq. V.4. Thus the S/N ratio is

$$\langle S\rangle/N = 4\,\frac{\eta}{e}\,\frac{\overline{|E_s|}^2}{D^2} \int_0^D r\, dr\, K_o(r) \exp\left[-\tfrac{1}{2}\mathcal{D}(r)\right]$$

$$\text{"} \quad = \frac{\pi r_o^2}{8}\left(\frac{\eta}{e}\right) \overline{|E_s|}^2\, \psi\left(D/r_o\right) \qquad \text{(V.11)}$$

$$\psi\left(\frac{D}{r_o}\right) = \frac{32}{\pi r_o^2 D^2} \int_0^D r\, dr\, K_o(r) \exp\left[-\tfrac{1}{2}\mathcal{D}(r)\right]$$

(V.12)

The symbol r_0 is introduced for the following reason. A graph of $\psi(\mathcal{D}/r_0)$ vs. $\mathcal{D}/r_0$ in log-log coordinates has the following form,

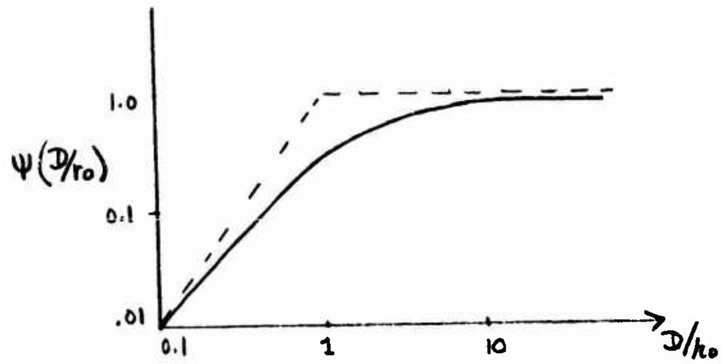


FIG. 13 (V.13)

This graph is based on several assumptions: (1) both $\phi(\underline{x})$ and $\ell(\underline{x})$ obey Gaussian statistics (2) the turbulence in the fluid is isotropic and homogeneous (3) the Kolmogoroff theory of turbulence applies (i.e. $\mathfrak{D} = \mathcal{A} r^{5/3}$, $r_0 \simeq \left(\frac{6.88}{\mathcal{A}}\right)^{3/5}$, $\mathfrak{D}(r) = 6.88\left(\frac{r}{r_0}\right)^{5/3}$, $\mathcal{A} = C_N^2 l_0^{2/3}$, C_N^2 = structure constant defined by Tatarski, "Propagation in Turbulent Media" 1961 Eq. 3.50). From the graph it is surmised that $\langle S \rangle / N$ reaches an asymptotic limit as the aperture $\mathcal{D}$ is increased, and that r_0 is the smallest $\mathcal{D}$ for which this limit is available. Thus increasing the aperture beyond r_0 does not increase the $\langle S \rangle / N$ ratio, given the existence of wave front distortion. (See next section for critical comment.)

Fried has calculated r_0 for many cases of distortion in atmospheric turbulence (see his Fig. 8). We note here that for the case: (1) electromagnetic wavelength $\lambda = 0.5 \mu M = 5000 \mathring{A}$ (2) zenith angle $\theta = 0$ (3) altitude of receiver $H = 7$ kilometers (4) daytime; the magnitude of r_0 is approximately 0.12 meters (say 5 inches). However in the case of underwater turbulence the calculation of r_0 is a subject of further investigation.

Recently, considerable progress has been made in the mathematical theory of the propagation of laser beams in strongly turbulent media. A summary of developments is provided by Prokhorov et al. Proc. IEEE 63, p. 790 (1975), where the subject of saturation of intensity of scintillations is treated in great depth.

Limits on VAS Imposed By The Inhomogeneities of the Medium

The VAS is equivalent to an array of remote sensors located in an inhomogeneous medium. The medium imposes limits on its performance. Fried (see previous section) has discussed the effect of the medium on the photodetector performance for a single sensor. He concludes that the S/N ratio cannot be improved by increasing the receiver aperture, once a critical aperture size is reached. This conclusion must be reviewed in light of known statistical antenna theory. Since this theory is very extensive we focus here attention only on one key theoretical point, namely on the maximum attainable directivity of a statistically perturbed array.

Shifrin ("Statistical Antenna Theory," Golem Press, 1971) provides a convenient summary. Let $D_0 (= 2L/\lambda)$ be the directivity of a continuous line of length L located on the x-coordinate axis receiving a (monochromatic) wavelength λ. Also let $\phi(x)$ be a random phase of (normalized) position $x = 2z/L$. It is first assumed that $\phi(x)$ is normally distributed with zero mean, variance α (independent of coordinate x), and correlation function $\rho_\phi(|x - x_1|)$. The mathematical form of ρ_ϕ is assumed to be of the form

$$\rho_\phi(|x - x_1|) = \exp\left\{-\frac{(x - x_1)^2}{L_\phi^2}\right\}, \qquad L_\phi = \frac{\ell_\phi}{(L/2)} \tag{.1}$$

in which L_ϕ is the normalized correlation length for phase, and ℓ_ϕ is the non-normalized correlation length for phase. The mean directive gain $\bar{D}$ is given by

$$\bar{D} = \frac{D_0}{4} \iint_{-1}^{+1} e^{-\alpha[1 - \rho_\phi]} \, dx \, dx_1 \tag{.2}$$

Let θ be the far field pattern angle, measured from the normal to the center of the line array, and let the generalized angle be defined by $\psi = (kL/2)\sin\theta = (\pi L/\lambda)\sin\theta$. When discussing directivity of an array that sees a wavefront with (Gaussian) phase correlation ρ_ϕ it is convenient to define a function $I(L_\phi, \psi, \pm\psi_1)$ in which

$$I(L_\phi, \psi, \pm\psi_1) = I\left(L_\phi, \frac{\pi L}{\lambda}\sin\theta, \pm\frac{\pi L}{\lambda}\sin\theta_1\right)$$

$$= \iint_{-1}^{+1} \exp\left\{-\frac{(x-x_1)^2}{L_\phi^2} + j\frac{\pi L x}{\lambda}\sin\theta \mp j\frac{\pi L x_1}{\lambda}\sin\theta_1\right\} dx\,dx_1$$

$$j = \sqrt{-1} \tag{.3}$$

In terms of these quantities, it is seen by expanding the exponential that the mean directive gain is

$$\bar{D} = D_0 e^{-\alpha}\left[1 + \frac{1}{4}\sum_1^\infty \frac{\alpha^m}{m!} I\left(\frac{L_\phi}{\sqrt{m}}, 0, 0\right)\right] \tag{.4}$$

A plot of $\bar{D}$ vs $\lg L/\lambda$ for various values of phase variance α and correlation distance l_ϕ is shown below:

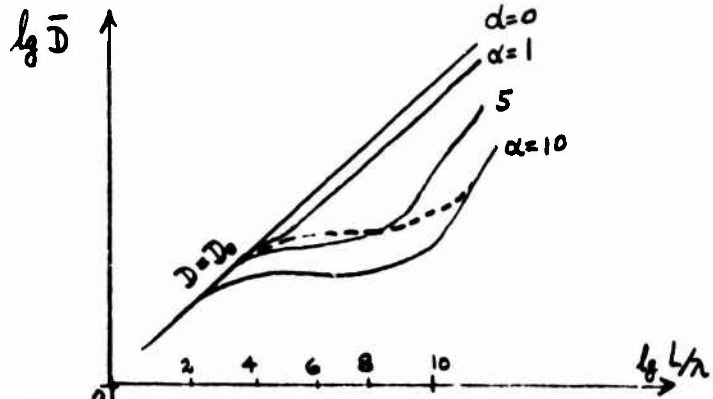


FIG. 14

(Shifrin, Fig. 12.3)

Here, $l_\phi = 50\,M$ (full curves), $l_\phi = 100\,M$ (broken curve). These curves can be discussed in terms of the parameter $L_\alpha = L_\phi/\sqrt{\alpha}$, i.e. the ratio of normalized correlation length of phase to the standard deviation of phase. Two cases are of interest.

Case I. Let $L_\phi \gg 1$, that is, the antenna dimension is much smaller than the correlation length l_ϕ. Then choose $L_\alpha \gg 1$. It is seen that $D \sim D_0$. This means the fluctuations of the medium have no effect on antenna gain. Now choose $L_\alpha \ll 1$, meaning that the standard deviation for phase is very large. Then

$$\bar{D} = \sqrt{\frac{2}{\pi\bar{\gamma}^2}}$$

in which $\overline{\gamma^2}$ is the mean value of the square of the (random) angle of arrival (see Fig. 15).

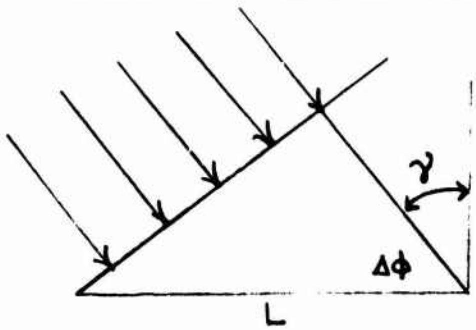


FIG. 15

It is seen that antenna gain $\bar{D}$ does not depend on the length of the antenna. This is the region of directive gain saturation. However as the length of the antenna increases the gain saturation disappears and the mean directive gain increases with antenna length without limit.

Case II. Let $L\phi << 1$ (that is, the antenna dimensions are much larger than the correlation length ℓ_ϕ). Once again, for sufficiently large phase fluctuations there is a region where $\bar{D}$ is independent of length of the antenna. This is the region of directive gain saturation. When the antenna length is increased beyond this region the gain $\bar{D}$ increases with length of the array beyond all limit.

Summarizing: An inhomogeneous medium limits the performance of a linear array only over a limited region (region of gain saturation) of array lengths. Beyond this region the directive gain of the antenna increases with length beyond all limit. The limits of S/N discussed by Fried in the previous section are seen to apply only over limited detector apertures. If the aperture is made sufficiently large (according to Shifrin), there is always a directivity gain for a fixed standard deviation of phase fluctuation. Thus S/N can be increased by increasing the aperture provided the aperture is large enough.

Part VI

Signal Processing

We take a large two-dimensional seismic array to be roughly analogous to the projected VAS array, and discuss seismic signal processing in a brief review.

Seismic signals are always mixed with seismic noise. Several methods of processing to reject noise are known. In the first method (Burg, Geophys. 29, p. 693-713, 1964) both signal and noise are assumed to be stationary multidimensional random processes with known cross-correlation functions. A linear (Wiener) filter is designed which produces an output that is a minimum mean-square-error version of the signal. In a second method (Claerbout, Geophys. 29 p. 197, 1964) the known noise correlation function is used to design a linear processor for the seismometer outputs which provides a minimum-mean-squared-error prediction of the noise over a short interval ahead. This prediction is subtracted from the actual seismometer output, thus greatly reducing the noise level. However the signal waveform is distorted. In the third method (Capon et al. Proc. IEEE 55, p. 192, 1967) the seismic noise is assumed to be a zero-mean, time stationary multidimensional random process. The signal is assumed to be a single plane wave propagating in a homogeneous, linear, nondispersive medium, but is an otherwise unknown time function. The signal processing therefore takes on the character of generating an estimate of an unknown time function and uses the theory of maximum likelihood (or minimum-variance) unbiased estimator approach. The essential feature of this approach is to design a noise-rejection filter based on a matrix of filter weighting coefficients which are obtained by use of the calculus of variations on a system of equations. Briefly, let there be a total of P sensors in the seismic array, and let the noise in each sensor constitute one component of a multidimensional random process, $\underline{N} = \hat{\underline{\iota}}_1 N_1(t) + \hat{\underline{\iota}}_2 N_2(t) + \cdots + \hat{\underline{\iota}}_p N_p(t)$

Assume further that each N_i is Gaussian, so that the total $\underline{N}$ has a multidimensional Gaussian distribution with zero means. The covariance matrix of $\underline{N}$ is defined as

$$K(t_1,t_2) = \begin{pmatrix} K_{11}(t_1,t_2) & K_{12}(t_1,t_2) & \cdots\cdots & K_{1P}(t_1,t_2) \\ \cdots & \cdots & & \cdots \\ K_{P1}(t_1,t_2) & \cdots & \cdots & K_{PP}(t_1,t_2) \end{pmatrix}$$

with

$$K_{ij}(t_1,t_2) = \mathcal{E}\left[N_i(t_1)\, N_j(t_2)\right]$$

where $\mathcal{E}$ is the expected value.

Now let the time function of the incoming wave be sampled $2\nu+1$ times from $-\nu$ to ν, and designate two distinct samples by integers m, n. Then the minimum-variance unbiased estimator has weighting coefficients θ_j, $j=1, 2 \ldots P$, determined by solving the matrix of equations

$$\sum_{j=1}^{P} \sum_{m'=-\nu}^{\nu} \theta_j(m'|n)\, K_{jk}(m,m') + \lambda_{\nu mn} = 0$$

in which $\lambda_{\nu mn}$ are $2\nu+1$ Lagrangian multipliers chosen to satisy the constraints that

$$\sum_{j=1}^{P} \theta_j(m|n) = \delta_{mn}, \qquad \textit{for all } m,n$$

Thus knowing K_{ij}, and the constraints, one finds a set of P coefficients θ_j, $j = 1,2 \ldots P$. These coefficients form the basis of constructing a <u>digital filter</u> which (by its construction) gives the optimum estimate of the <u>unknown time function</u> constituting the signal, simultaneously rejecting noise. The synthesis of the digital (two sided) filter is carried out in the frequency domain. It is noted that the estimator thus found is also the maximum-likelihood estimator.

Diversity

Diversity is a signal processing technique aimed at improving the reliability of reception of signals that are subject to fading in the presence of random noise. To discuss diversity one requires a brief summary of the nature of a fluctuating channel.

We assume that there exists in a communications channel many "copies" of a signal, travelling different paths. Each copy has an identifiable amplitude and phase which vary slowly with time. At a particular receiving point the instantaneous time signature of arrival of signals is the sum of two or more sinusoids with vary amplitudes and phase. This sum shows fluctuations in signal strength called multipath interference fading. By choosing many different channels one can obtain many time signatures which differ noticeably from each other, some fading, some flaring up, etc. If these signatures (consisting of signals plus noise) are widely enough spaced in location (or time, or frequency) as to be statistically independent, one can combine them in an appropriate manner so as to obtain better or more reliable reception of the "message". Such combinations form the basis of diversity methods of signal enhancement. It is to be emphasized that enhancement is possible only when there is a random character to the received signal itself, and when diverse copies of the signal can be obtained, or generated, from a single original.

A convenient and much used model of a fluctuating channel is the "Rayleigh-fading" model. In this model a transmitter projects a signal at frequency ω_c which arrives at the receiver in the form

$$e(t) = A(t) \cos \left[\omega_c t + \phi(t)\right]$$

The random amplitude $A(t)$ is Rayleigh distributed, with probability density

$$p(A) = \frac{2A}{a^2} e^{-A^2/a^2} \qquad A > 0$$

in which a is the $\sqrt{\overline{A^2}}$ in an (appropriate) interval T. The random phase is usually assumed to be uniformly distributed between 0 and 2π with a value $\pi/2$. Thus $e(t)$ is a narrow band gaussian process with zero mean and variance $\sigma_e^2 = a^2/2$. If $e(t)$ is expressed in quadrature form

$$e(t) = e_c(t) \cos \omega_c t + e_s(t) \sin \omega_c t$$

then e_s, e_c are independent gaussian random variables.

According to the Rayleigh-fading law the probability that the received signal will fall below A' in any time interval fitting T is

$$P\left(A \leq A'\right) = 1 - \exp\left(-\frac{A}{a}\right)^2$$

Half the time the signal will be greater than the median $A_M \approx 0.693a^2$. A sketch of various models of amplitude fading channels is shown below:

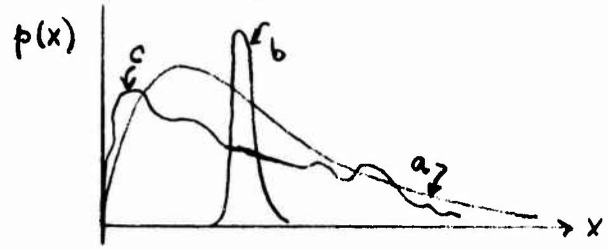


FIG. 16

In these (see D.G. Brennan, MIT Lincoln Lab.) probability densities, curve a is a Rayleigh channel, curve b shows mild fading, curve c shows deep, frequent and persistent fading.

We assume now that a number of copies of the signal (or "message") are available. These are to be sent from source to receiver. Upon arrival each copy is assumed to show fading in the presence of noise of the transmission path itself or of the receiver. To apply the technique of diversity signal processing the fading patterns of the original copy must be statistically independent. To assure this several methods of transmission are used: (1) copies of the signal are sent in time succession as a sequence of pulses. The signal is said then to be time diversified. (2) copies of the signal are transmitted on different carrier frequencies. The signal is said to be frequency diversified. (3) Copies of the signal are received from different spatial locations in the medium. The signal is then said to be spatially diversified. In all cases statistically independent noise is assumed to corrupt the signal.

When K copies of the received signal are available they can be signal processed in various ways. A simple technique is coherent processing. In this procedure the K received signals are added, and then squared to form the power. The signal power is proportional to K^2 and the noise power is proportional to K Hence the improvement in S/N ratio is equal to the number K of independent copies available. Such processing however assumes the quality (good or poor) of all signals is identical. A more general type of processing is to weight each available copy ($= f_k(t)$) by a weighting factor a_k and then add them together to form $f(t)$, where

$$f(t) = \sum_{k=1}^{K} a_k f_k(t)$$

Different choices of a_k lead to different processing techniques. If only one a_k is different from zero at any time, one has switching techniques. Trying each a_k one at a time is called scanning diversity. Trying a group of a_k simultaneously and selecting the best is called optimal selection diversity. If all the a_k are used, and given equal magnitudes the processing is called equal-gain combining. If the a_k's are adjusted according to the quality of $f_k(t)$ so as to yield the maximum S/N the result is called maximal-ratio combining.

These different procedures can be calculated to show the S/N gain of diversity over signal channel reception. A set of charts have been prepared by Brennan (1959, Proc. IRE 47 p. 1075-1102) for the cases of two-channel, four-channel diversity, and theoretical gain for a large number of channels. For example, a typical case shows that ten (fading) copies can be made to deliver a 10 dB improvement over a single copy when the technique of maximal-ratio combining is used.

Conclusion to the Report

We have presented above a comprehensive review of all factors pertinent to the detection of acoustic phenomena by use of a laser heterodyne detector. The principal problem in proposed Navy applications of this technique (as replacement of standard Navy hydrophones) is the very small signal to noise ratio in a single beam. This limits detection threshold in noisy environments to (a few) orders of magnitude greater than the thresholds currently available to the Navy. However, considerable improvement in S/N ratio is foreseen in the proposed use of multiple beams (i.e., "diversity"). The gain is anticipated to be proportional to the number of beams employed.

APPENDICES

Appendix A

Analysis of Two Beam Interference

Two point sources of light separated a distance d in a dark screen superimpose on an observing screen at distance $\mathcal{D}$. The total field at point $P(x)$ in the observing screen is

$$E_p = \frac{E_1}{|\underline{r}_1|} \exp j\left[\omega t - \underline{K}_E \cdot \underline{r}_1 + \phi(t')\right] + \frac{E_2}{|\underline{r}_2|} \exp j\left[\omega t - \underline{K}_E \cdot \underline{r}_2 + \phi(t'')\right] \tag{A.1}$$

If we assume:

(1) $E_1 = E_2 = E_i$

(2) $\phi(t') = \phi(t'') \sim 1$

(3) $|\underline{r}_2| - |\underline{r}_1| = \Delta r, \quad \Delta r \ll |\underline{r}_1|, \; \Delta r \ll |r_1|$

then

$$E_p = 2 E_i \cos\left(\frac{\pi x_p}{\lambda} \cdot \frac{d}{\mathcal{D}}\right) \tag{A.2}$$

Now let the angle of intersection of the two beam be θ. Then

$$\frac{d}{\mathcal{D}} = 2 \sin\frac{\theta}{2} \tag{A.3}$$

so that

$$E_p = 2 E_i \cos\left(2 \frac{\pi x_p}{\lambda} \cdot \sin\frac{\theta}{2}\right), \quad K_E = \frac{\Omega_E}{C_E} \tag{A.4}$$

The intensity at P is I_p given by

$$\begin{aligned} \underline{I}_p &= \tfrac{1}{2}\, \underline{c}\, K_0 E_i^2 \cos^2 \pi\left(\frac{d}{\mathcal{D}} \cdot \frac{x}{\lambda}\right) \\ \text{"} &= \underline{c}\, K_0 E_i^2 \left[1 + \cos 2\left(\frac{2\pi x_p}{\lambda} \sin\frac{\theta}{2}\right)\right] \\ \text{"} &= \underline{c}\, K_0 E_i^2 \left[1 + \cos \mathcal{K} x_p\right], \quad \mathcal{K} = \frac{4\pi}{\lambda} \sin\frac{\theta}{2} \end{aligned} \tag{A.5}$$

in which K_0 is the dielectric constant, and $\underline{c} = c\hat{a}$.

If the sources have finite width QQ' the projection of which on to a plane normal to OP is q and if α is the angle enclosed by the rays from Q and Q' then a path difference $\frac{q\alpha}{2}$ is introduced, as measured form origin O at the center of the width. The phase difference is $K_E \frac{q\alpha}{2}$. Eq. A.5 constitutes the basic formulation appearing in George and Lumley's model, Part II of this report.

Appendix B

Dimensionless Wavenumbers and Optimum Scattering Volume (George and Lumley loc. cit).

The symbol η is the Kolmogorov microscale ($\eta = (\nu^3/\epsilon)^{1/4}$) and has the dimensions of length. The symbol α is a frequency, appearing in the spectra for velocity, phase etc. The symbol $\bar{u}$ is the mean flow velocity. The symbol $\tilde{k}_o$ is the dimensionless wavenumber for which the ratio of turbulence to ambiguity spectrum is unity. By definition

$$\tilde{k}_o = \frac{\alpha_o \eta}{\bar{u}} \tag{B.1}$$

Thus if $\tilde{k}_o = 1.0$ the mean velocity $\bar{u}$ is equal to the velocity of the fluid of displacement equal to the Kolmogorov microscale multiplied by frequency α_o.

The symbol $k_x = (2^{1/2}\sigma_1)^{-1}$ where σ_1 is the size of the scattering volume in the x-direction. The symbol $k_x \eta = \tilde{k}_x$ is thus a ratio of the Kolmogorov microscale to the "x-component of scattering volume." The statement that $\tilde{k}_x$ is optimum when

$$\tilde{k}_{x\,opt} = \frac{1.27}{\left(R_e \sin\frac{\theta}{2}\right)^{1/2}} = \left(\frac{\eta}{\sqrt{2}\,\sigma_1}\right)_{opt} \tag{B.2}$$

means

$$(\sigma_1)_{opt} = \frac{\eta \left(R_e \sin\frac{\theta}{2}\right)^{1/2}}{1.27\sqrt{2}} \tag{B.3}$$

in which the Reynolds number is

$$R_e = \frac{2\pi\, \bar{u}^2}{\nu\, \bar{\omega}_o}, \qquad \bar{\omega}_o = K_\epsilon \bar{u}_o \tag{B.4}$$

The symbol $m_x = (2^{1/2}\sigma_2)$ where σ_2 is the size of the scattering volume perpendicular to the flow, but in the plane of σ_1 For small θ ,

$$m_x \simeq k_x \sin\theta/2 \,/\cos\theta/2 \tag{B.5}$$

Thus,

$$\tilde{m}_{x\,opt} = \tilde{k}_{x\,opt}\, \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \simeq 1.27 \left(\frac{\sin\frac{\theta}{2}}{R_e}\right)^{1/2} \tag{B.6}$$

The optimum volume of scattering should be determined by this formula.

Appendix C

Optic-Acoustic Interactions

Optic-acoustic interactions are reviewed in Appendices C, D and E.

We consider first a nonstationary medium in which the time-rate of change of the index of refraction is much slower than the frequency of the electric field. In symbols $(\partial m/\partial t)(\frac{1}{m}) << T^{-1}$ where m is the index of refraction and T is the period of oscillation of the electric field. Now let the change in index $(= m_1)$ be caused by the passage of a soundwave, i.e. caused by a change in mass density. Thus when the sound wave is present the instantaneous index of refraction $(= m)$ is the sum of the unperturbed value m_0 and the perturbation m_1. In the absence of true sources the propagation of the electric vector E is governed (to quantities of first order in m_1) by the linear wave equation

$$\nabla^2 \underline{E} - \frac{(m_0+m_1)^2}{C_E^2}\frac{\partial^2 \underline{E}}{\partial t^2} = 0 , \quad \text{or } \nabla^2 \underline{E} - \frac{1}{C_E^2}\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{(2m_1/m_0)}{C_E}\frac{\partial^2 \underline{E}}{\partial t^2} \tag{C.1}$$

in which terms of order m_1^2 have been omitted. The perturbation in the index of refraction m_1 is proportional to the perturbation in fluid density ρ_1, i.e

$$\frac{m_1}{m_0} = \frac{\rho_1}{\rho_0} \tag{C.2}$$

thus we write the wave equation for $\underline{E}$ in the form

$$\nabla^2 \underline{E} - \frac{1}{C_E^2}\frac{\partial^2 E}{\partial t^2} = \frac{2\rho_1}{\rho_0}\frac{1}{C_E^2}\frac{\partial^2 E}{\partial t^2} \tag{C.3}$$

The transit of the electric vector in the medium creates a mechanical body force $\underline{F}_E$ (dimensions: NM^{-3}) which acts as a source of acoustic pressure p the propagation of which is governed by the equation

$$\nabla^2 p - \frac{1}{C_s^2}\frac{\partial^2 p}{\partial t^2} = \underline{\nabla} \cdot \underline{F}_E$$

Now the acoustic density is proportional to the acoustic pressure, and the mechanical stress is proportional to the electric field. We write $p = K_s \rho_1$, $\underset{\sim}{F}_E = K_E \underset{\sim}{E}$.

Thus,

$$\nabla^2\rho_1 - \frac{1}{C_s^2}\frac{\partial^2\rho_1}{\partial t^2} = \frac{B_E}{B_s}\,\underline{\nabla}\cdot\underline{E} \tag{C.4}$$

Taken together we see that the waves of density and electric field form a coupled system given by Eqs. (C1) and (C2). Returning to Eq. (C1) we let $\rho_1 = \rho_1(\underline{x},t)$ and seek a solution $E(x,t)$ by Fourier transformation of the time coordinate with zero initial conditions,

$$(\nabla^2+k^2)E(\underline{x},\omega) = \left(\frac{2}{\rho_0}\right)\rho_1(\underline{x},\omega) * \left(-\frac{\omega^2}{C_E^2}\right)E(\underline{x},\omega) = Q(\underline{x},\omega) \tag{C.5}$$

in which $*$ signifies the convolution operation. The symbol $Q(\underline{x},\omega)$ describes a (fictitious) source. The electric field at any point is obtained by the use of the Green's function for the infinite domain

$$E(\underline{x},\omega) = \int_{\underline{x}_0} Q(\underline{x}_0,\omega)\,G_\omega(\underline{x},\underline{x}_0)\,d\underline{x}_0 \tag{C.6}$$

This formula is an integral equation in the scattered part of the electric field, the scattering itself being due to inhomogeneity in mass density of the medium. A few important applications of this formula to the interaction of acoustic and electric fields are discussed by Morse and Ingard (Chap. XIII). They are (a) the Debye-Sears effect (b) Bragg scattering (c) Brillouin scattering. These topics are briefly reviewed here.

A. Debye – Sears Effect

Let the perturbation of the mass density be due to a periodic progressive ultrasonic wave travelling in the positive x-direction. The harmonic frequencies are $\omega_s^{(n)}$, and the wave numbers are $k_s^{(n)}, n = 1, 2 \cdots$ .... It is assumed that the width of the sound beam is small.

Now let the incident light be a monochromatic plane wave, $\underline{E} = E_0 \cos(K_E\cdot\underline{x} - \Omega t)$ where $\underline{K_E} = (\hat{\imath}\cos\alpha + \hat{\jmath}\cos\beta + \hat{k}\cos\gamma)|\underline{K}|$, α, β, γ, being direction cosines, and $K_E = \Omega/C_E$. This light is scattered by the "grating effect" of the periodic ultrasonic wave. Choosing for example then n'th Fourier component of this wave, and writing the mass density as sinusoid;

$$\frac{2\rho_1}{\rho_0} = \eta^{(n)}\cos\left(k_s^{(n)}x - \omega_s^{(n)}t\right)$$

one then seeks a solution to Eqs. (3) and (4). In general the solution process is difficult to carry through. Approximate answers are obtainable by the following procedure. First it is assumed the perturbation is so small that the electric field in the source term Q $(\underset{\sim}{x},\omega)$ is effectively the incident light. This is the Born, or single scattering, approximation. Secondly it is assumed that the width of the sound beam ($= 2L$) is so small that multiple reflection (as from parallel layers) is negligible in establishing the scattered electric field. Thirdly, both electrical and mechanical fields are taken as 2-dimensional (i.e. coordinates x, z), so that $K_E = \hat{i} K_x + \hat{k} K_z$ and $|K_E|^2 = K_x^2 + K_z^2$. Thus using the 2-dimensional Green's function $(i/4)\, H_0^{(1)}\left(|K_E|\sqrt{(x-x_0)^2+(z-z_0)^2}\right)$, and taking E_0 to be parallel to the y-axis, one arrives at the following approximation to the scattered field due to the n'th harmonic components of the periodic ultrasonic wave,

$$\underset{\sim}{E} = E_+ \sin\left[\underset{\sim}{K}_+ \cdot \underset{\sim}{r} - (\Omega+\omega_s)t\right] + E_- \sin\left[\underset{\sim}{K}_- \cdot \underset{\sim}{r} - (\Omega-\omega_s)t\right] \quad \text{(C.7)}$$

where

$$\underset{\sim}{E}_\pm = \left[\frac{\Omega^2 L \eta^{(n)} E_0}{c(\Omega\pm\omega_s)\cos\theta_s^{(n)\pm}}\right] \frac{\sin\left[(\Omega L/c)\cos\theta\cos\theta_s^{(n)\pm}\right]}{(\Omega L/c)\cos\theta\cos\theta_s^{(n)\pm}} \quad \text{(C.8)}$$

$$\underset{\sim}{K}_\pm \cdot \underset{\sim}{r} = \frac{\Omega}{c} x \sin\theta_s^{(n)\pm} + \frac{(\Omega\pm\omega_s)}{c} z \cos\theta_s^{(n)\pm} \quad \text{(C.9)}$$

$$\frac{\Omega}{c} x \sin\theta_s^{(n)\pm} = |\underset{\sim}{K}_E| \sin\theta \pm k_s, \quad \cos\theta_s^{(n)\pm} = \sqrt{1-\sin^2\theta^{(n)\pm}} \quad \text{(C.10)}$$

The symbol $\theta_s^{(n)\pm}$ is the angle of scattered light.

In words: each Fourier component of the periodic mass density perturbation induced by the ultrasonic wave scatters the light (incident at angle θ) into two plane waves traveling in directions $\theta_s^{(n)+}$ and $\theta_s^{(n)-}$ respectively where

$$\sin\theta_s^{(n)\pm} = \frac{|K_E|\sin\theta}{\left(\frac{\Omega\pm\omega_s^{(n)}}{c}\right)} \pm \frac{k_s^{(n)}}{\left(\frac{\Omega\pm\omega_s^{(n)}}{c}\right)} \quad \text{(C.11)}$$

When $\omega_s^{(n)} \ll \Omega$ (as is usually the case) the scattering angles are given by

$$\sin\theta_s^{(n)} \approx \sin\theta \pm \frac{k_s^{(n)}}{|K_E|} \quad \text{(C.12)}$$

Let $\lambda_s^{(1)}$ be the fundamental component of the sound wave, and Λ_E the wavelength of the incident light wave, then

$$\sin\theta_s^{(n)\pm} = \sin\theta \pm n\frac{\Lambda_E}{\lambda_s} \quad \text{(C.13)}$$

Thus on each side of the direction of the incident beam will be found two diffracted plane waves for each harmonic of mass density appearing in the periodic ultrasonic beam. The frequencies of these waves are $\Omega + n\omega_s$ ("up Doppler") and $\Omega - n\omega_s$ ("down Doppler"). Their amplitudes are proportional to the amplitude $\eta^{(n)}$ of the n'th order harmonic of the ultrasonic wave (which diminishes with increasing n). For an arbitrary periodic distribution of mass density a family of plane waves will appear in space distributed in angle according to Eq. (6C). These diffracted waves make up the Debye-Sears effect. It is a notable feature of this phenomenon that the diffracted waves appear for <u>any</u> angle of incidence θ .

Bragg Reflection

A plane wave train of monochromatic light (wavelength Λ_E) crosses a plane wave train of monochromatic sound (wavelength λ_s) at an angle of incidence θ_i. The sound of train constitutes parallel layers of density from which the light is reflected. We consider three wavefronts of sound at phases 0, 2π and 4π radians (that is 2 λ_s long), and a single wavefronts of light at zero phase. The light is first reflected by the zero phase wavefront of sound at an angle θ_i. In order for a second reflection to occur the light must reach the second layer, travelling a distance d_E. It is seen that d_E must be equal to or less than $2\lambda_s$. Thus the angle θ_i is given by

$$\sin \theta_i = \frac{d_E}{2\lambda_s} \tag{C.14}$$

Now we choose d_E to be Λ_E (one wavelength). This sets the requirement that the second reflected wavetrain be in phase with the first reflected wave. This is equivalent to the geometrical requirement that the wavefronts 1 and 2 be reflected into wavefronts 3 and 4 shown in the Fig. C1. This is the Bragg angle requirement that

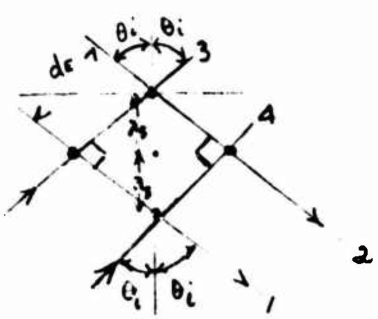


FIG. C1

$$\sin \theta_i = \frac{\Lambda_E}{2\lambda_s} \tag{C.15}$$

For given Λ_E and λ_s it is seen that only one angle will satisfy this equation. If we set $\theta_i' = 90^o$ the light is reflected exactly backward, provided the wavelength of light is twice the wavelength of sound. In going from wavefronts 1 and 2 into wavefronts 3 and 4 the light wave is scattered through a total angle $\theta_1 = 2\theta_i'$. However the wavelength λ_s is created by a wave moving at velocity C_s, and characterized by a frequency $f_s = C_s/\lambda_s$. The frequency of the light $f_E = C_E/\Lambda_E$ is thus Doppler shifted by an amount $\pm f_s$, that is, by an absolute amount,

$$f_s = 2 f_E \frac{C_s}{C_E} \sin \frac{\theta_1}{2} \tag{C.16}$$

This equation gives the spatial and temporal relations between all the variables. If, for example, we are to observe reflection of light from a sonic wavetrain at "scatter" angle θ_1 then for a selection of frequency of light f_E we must employ a frequency of sound given by the above equation. In a typical case (Morse and Ingard, p. 819) f_E is taken as 10^{15} Hz, C_E as $3 \times 10^8 \, MS^{-1}$ and C_s as $10^3 \, MS^{-1}$. Then the observation of back reflection (at Bragg angle 180°) requires an ultrasonic frequency of some 10^{10} Hz. If f_s is less than 10^8 Hz the reflection angle is very small, that is, the reflection is very nearly in the same direction as the incident wave.

Appendix D

Thermal Modes of Fluid Motion (Ref. "Theoretical Acoustics", Morse & Ingard, Chap. XIII.)

The generation of acoustic pressure p and the diffusion of acoustic temperature τ form a coupled system, described by the equations,

$$\nabla^2 p - \mathcal{L}_1 p = -\mathcal{L}\alpha\tau \tag{D.1}$$

$$\nabla^2 \tau - \mathcal{L}_2 \tau = \mathcal{L}_3 p \tag{D.2}$$

in which

$$\mathcal{L}_1 = \frac{\gamma}{c_s^2}\left(\frac{\partial^2}{\partial t^2} - l_v' c_s \frac{\partial}{\partial t}\nabla^2\right)$$

$$\mathcal{L}_2 = \frac{1}{l_h c_s}\frac{\partial}{\partial t}$$

$$\mathcal{L}_3 = \left(\frac{\gamma - 1}{\gamma\alpha}\right)\mathcal{L}_2$$

$$\gamma = c_p/c_v\,; \quad l_v' = \left(\frac{4}{3} + \frac{\eta}{\mu}\right) l_v\,; \quad l_v = \frac{\mu}{\rho c_s}\,; \quad \alpha = \frac{\beta}{K_T}\,; \quad l_h = \frac{K}{\rho c_p c_s}$$

Here, C_p, C_v are specific heats at constant pressure and constant temperature, η is the bulk viscosity modulus, μ is the shear viscosity modulus, β is the coefficient of thermal expansion, K is the thermal conductivity and K_T is the bulk (elastic) modulus at constant temperature. Eqs. (1) and (2) are acoustic equations. A more general equation of fluid dynamics is the Navier-Stokes equation,

$$\rho\frac{\partial u_l}{\partial t} = -\mathrm{grad}\, p + \left(\eta + \frac{4}{3}\mu\right)\nabla^2 u_l \tag{D.3}$$

$$\rho\frac{\partial u_t}{\partial t} = -\mu\, \mathrm{curl}\,(\mathrm{curl}\, \underline{u}_t) \tag{D.4}$$

in which U_l and U_t are the longitudinal and transverse parts of the fluid velocity, p is the acoustic pressure, ρ is the equilibrium fluid density.

Three species of fluid motion are predicted by Eqs. (D.1) thru (D.4). These are (1) longitudinal waves of acoustic pressure p (= Eq. D.1), (2) transverse shear waves (= Eq. D.4) (3) thermal diffusion of temperature (= Eq. D.2) which then gives rise to acoustic waves (= Eq. D.1). We will discuss thermal motion and the accompanying thermal modes of sound generation.

In the thermal mode $\alpha\tau$ is very much greater than p. The diffusion of temperature in this case is given by Eq. (2). To apply this equation we first assume the acoustic pressure on the r.h.s is negligible, and take the temperature pertubation to have the space and time dependence of a plane highly damped travelling wave, $\tau \propto \exp i(\underline{k}\cdot\underline{r}-\omega t)$. Then, to first order,

$$\nabla^2\tau = -k^2\tau \tag{D.5a}$$

$$\frac{\partial\tau}{\partial t} = -i\omega\tau \tag{D.5b}$$

From Eq. (2) it is then seen that the effective square of the wave number is complex,

$$k^2 = \frac{i\omega}{l_h c_s} \tag{D.5c}$$

The acoustic pressure corresponding to this diffusion of temperature is found from Eq. (1) in combination with Eq. (5a), (5b), and (5c), i.e.

$$p \sim \frac{i\gamma\alpha\omega}{c_s}\left(l_h - l_v'\right)\tau \tag{D.6a}$$

The effective perturbation of pressure is not this p alone but the combination $p-\alpha\tau$. The term $\alpha\tau$ describes the reduction in pressure due to heat conduction (i.e. describes the departure from the assumed adiabatic condition of wave propagation). Corresponding to waves of acoustic pressure are waves of acoustic density ρ_1 which to first order are related to each other by the formula $p-\alpha\tau = c_s^2\rho_1$ or

$$\rho_1 = \frac{p-\alpha\tau}{c_s^2} = \frac{i\gamma\alpha\omega\left(l_h-l_v'\right)}{c_s}\tau - \frac{\alpha\tau}{c_s^2} \tag{D.6b}$$

The perturbation in fluid velocity associated with thermal motion is found from Eq. (3) by neglecting the viscosity term on the r.h.s since this term will yield only a second-order effect of termperature on velocity. Thus one finds

$$\underline{u}_\ell = \frac{\gamma\alpha}{\rho C_s}\left(\ell_\ell - \ell_v'\right) \operatorname{grad} \tau \quad \text{(D.6c)}$$

The set of equations comprised in (5)-(6) form a convenient description of the thermal mode of fluid motion in terms of the parameters τ, p, k^2, ρ_1 and $\underline{u}_\ell$.

Decay Rates of Temperature and Pressure Fluctuations

Assume the modes of motion noted above arise from random fluctuations in pressure (Δp) and termperature (ΔT). Temperature fluctuations decay at the rate

$$K \nabla^2 (\Delta T) = \rho C_p \frac{\partial (\Delta T)}{\partial t} \quad \text{(D.8)}$$

Assuming the spatial dependence of ΔT has the form $\exp i\underline{q}\cdot\underline{r}$, we reduce this equation to the form

$$\frac{\partial (\Delta T)}{\partial t} = -\beta (\Delta T), \qquad \beta = \frac{K}{\rho C_p} q^2 \quad \text{(D.9)}$$

Hence the time decay of (spontaneous) fluctuations in temperature of a fluid follows the law $\Delta T \propto \exp(-\beta t)$, in which β is the temporal decay constant.

Now let there be a pressure fluctuation (spontaneous) of amount $\Delta p(\underline{r}, t)$ whose spatial dependence is $\cos \underline{q}\cdot\underline{r}$, $q = \frac{\omega_s}{C_s}$. The mode of fluid motion if found by solving

$$\nabla^2 p - \alpha_1 p = 0 \quad \text{(D.10)}$$

for the complementary solutions in time. The result is

$$\Delta p(\underline{r}, t) \propto e^{-\alpha t}\left[\cos(\underline{q}\cdot\underline{r} + \omega_s t) + \cos(\underline{q}\cdot\underline{r} - \omega_s t)\right] \quad \text{(D.11)}$$

which describes two waves traveling in opposite directions (at angle $\angle \underline{q}\cdot\underline{r}$) at frequency $\omega_s = q C_s$ where $q(\omega)$ satisfies the dispersion formula obtainable from (D.10). The temporal decay constant is

$$\alpha = \left(\frac{4\mu}{3\rho} + \frac{\eta}{\rho}\right)\frac{q^2}{2} = l_v' q^2 c_s/2 \qquad \text{(D.12)}$$

which is also obtained from the complementary solution of (D.10). The time $t = \alpha^{-1}$ will be called the effective life time of the wave.

Under forced drive Eq. (D.10) has a forcing function $F(\underline{k},t)$ on the r.h.s. By Fourier transformation of time (under zero initial conditions) the acoustic pressure is seen to be given by

$$p(\underline{k},\omega) = \frac{-\frac{c_s^2}{\gamma}\hat{F}(\underline{k},\omega)}{\frac{c_s^2 q^2}{\gamma} - \omega^2 + 2i\omega\alpha} \qquad \text{(D.13)}$$

at $\omega = \omega_s$ the lifetime of ω_s (i.e. its effective influence on the spectrum of $p(\underline{k},\omega)$) is of order α^{-1} (dimensions of seconds). The bandwidth of p under forced drive is therefore of order $\frac{1}{\alpha^{-1}}$, that is, of order α.

In sum: Spontaneous fluctuations of pressure are propagated as damped traveling waves, and spontaneous fluctuations of temperature are diffused as critically damped waves. A spontaneous spatial sinusoidal distribution of pressure at wave number q propagates away as two damped waves in opposite directions at frequency $\omega_s = q c_s$. Each propagating acoustic mode is associated with a propagating mass density. The mathematical form of these damped waves have the same form as the forced drive of a damped harmonic oscillator (Eq. (D.10)).

We next consider the interaction of light waves with mass density fluctuation associated with thermal motion of a fluid. Let $\Delta\rho(\underline{k},t)$ be a spontaneous fluctuation in mass density. Choosing (as before) the thermodynamic variables the pressure and temperature we write.

$$\Delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_T \Delta p + \left(\frac{\partial\rho}{\partial T}\right)_p \Delta T \qquad \text{(D.14)}$$

We consider the effects of temperature and pressure separately. We first take the spatial Fourier transform of the pressure and temperature fluctuations, and write.

$$\Delta p(\underset{\sim}{k},t) = \int \Delta p(\underset{\sim}{r},t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r} \tag{D.15}$$

$$\Delta T(\underset{\sim}{k},t) = \int \Delta T(\underset{\sim}{k},t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r} \tag{D.16}$$

where the volume of integration is finite (as it must be in any real case). Treating Δp as a random variable we next calculate the temporal autocorrelation $R_p(\underset{\sim}{k},\tau)$ of the pressure,

$$R_p(\underset{\sim}{k},\tau) = \langle \Delta p(k,t+\tau)\,\Delta p^*(k,t)\rangle \tag{D.17}$$

in which the symbols $\langle\rangle$ represent a temporal average (= integration) where the integration is over finite time. The intensity of the pressure fluctuation is obtained by setting $\tau = 0$. Similarly the autocorrelation of the temperature fluctuations is given by

$$R_T(\underset{\sim}{k},\tau) = \langle \Delta T(\underset{\sim}{k},t+\tau)\,\Delta T^*(\underset{\sim}{k},t)\rangle \tag{D.18}$$

Assuming further that the pressure and temperature fluctuations are uncorrelated we can find the autocorrelation of mass density $R_\rho(\underset{\sim}{k},t)$ by addition, i.e.

$$R_\rho(\underset{\sim}{k},\tau) = \left(\frac{\partial\rho}{\partial p}\right)_T^2 R_p(\underset{\sim}{k},\tau) + \left(\frac{\partial\rho}{\partial T}\right)_p^2 R_T(\underset{\sim}{k},\tau) \tag{D.19}$$

From previous analysis (see Eq. (D.11)) the temporal character of a pressure fluctuation has been involved as the response of a damped simple harmonic oscillates for each wave-number k_s , and associated frequency $\omega_s = k_s c_s$. Thus one can write

$$R_p(k,\tau) = R_p(\underset{\sim}{k},0)\, e^{-\alpha|\tau|} \cos\omega_s\tau \tag{D.20}$$

Similarly,

$$R_T(\underset{\sim}{k},\tau) = R_T(\underset{\sim}{k},0)\, e^{-\beta|\tau|} \tag{D.21}$$

These formulas will be used in the discussion of Brillouin scattering (see Eqs. E6, E7, etc).

Appendix E

Brillouin Scattering (Ref. "Theoretical Acoustics", Morse & Ingard, Chap. XIII.)

The interaction of light waves with inhomogeneities of mass density in a fluid has been discussed in Appendix C in the case of Debye-Sears effect. In the present application the perturbations of mass density are assumed to be due to changes in pressure and to be spontaneous (i.e. random). Let $\Delta p(r,t)$ be the random fluctuation in pressure. Its spatial Fourier transform $\Delta p(k,t)$ is given by

$$\Delta p\,(k,t) = \int \Delta p(r,t)\, e^{-ik\cdot r}\, d^3 r \tag{E.1}$$

where

$$\Delta p\,(r,t) = \int \Delta k\,(k,t)\, e^{ik\cdot r}\, d^3k/(2\pi)^3 \tag{E.2}$$

Assume the volume V of pressure inhomogeneities to be finite, and assume the incident light to be a plane wave $E_i = E_0 \exp i\,(K\cdot r_0 - \Omega t)$. Using the Green's function for unbounded space and approximating the local electric field by E_i (=Born approximation), one formulates Eq. (C.6) for the scattered wave by

$$E_s(r,t) = \frac{1}{\varrho_0}\left(\frac{\partial\varrho}{\partial p}\right)_T \int dt_0 \int d^3r_0 \int d^3k\, \Delta p(k,t)\, e^{ik\cdot r_0}\, \frac{E_0\Omega^2}{c_E^2}\; e^{iK\cdot r_0 - i\Omega t}\, G\,(r,t\,|\,r_0,t_0) \tag{E.3}$$

in which the dimensions of G are meter^{-1} sec^{-1}, and

$$G = \frac{1}{4\pi\,|r-r_0|}\,\delta\left(t_0 - t + \frac{|r-r_0|}{c}\right) \tag{E.4}$$

In the far field $|r-r_0| \simeq r$ and $|r-r_0|/c \simeq r - \frac{K'\cdot r_0'}{K'}$, where $K' = |K'|$. Performing all mathematical operations on (E.3) one arrives at the formula

$$E_s(r, K'-K, t) = \frac{1}{\varrho_0}\left(\frac{\partial\varrho}{\partial p}\right)_T \frac{E_0\Omega^2 V}{4\pi r\, c_E^2}\, \Delta p(K'-K,t)\, e^{iK'r - i\Omega t} \tag{E.5}$$

in which we have set $\underset{\sim}{k} = \underset{\sim}{K}' - \underset{\sim}{K}$. Now the intensity of light waves (I) is given by the Fourier transform of the time averaged autocorrelation of E_s, (brackets indicate time average, or division, by time interval),

$$I_{\Delta p}(r, \underline{K}'-\underline{K}, \Omega') = \int d\tau \; \langle E_s(r, \underline{K}'-\underline{K}, t+\tau)\, E_s^*(r, \underline{K}'-\underline{K}, t) \rangle e^{i\Omega'\tau}$$

$$'' = \frac{1}{\varrho_0^2}\left(\frac{\partial \varrho}{\partial p}\right)_T^2 \left(\frac{E_0 \Omega^2 V}{4\pi r C^2}\right)^2 \int_{-\infty}^{\infty} \frac{R_{\Delta p}(\underline{K}'-\underline{K}, \tau)}{\Theta} e^{i(\Omega'-\Omega)\tau} d\tau \qquad \text{(E.6)}$$

in which Θ is the effective duration of the pressure inhomogeneity. The autocorrelation of intensity of pressure has the same form as that of a damped harmonic oscillator, as noted earlier, (see Eq. D20),

$$R_{\Delta p}(\underline{K}'-\underline{K}, \tau) = R_{\Delta p}(\underline{K}-\underline{K}, 0)\, e^{-\alpha|\tau|} \cos \omega_s \tau$$

$$\omega_s = |\underline{K}' - \underset{\sim}{K}| c_s \qquad \text{(E.7)}$$

Thus, the intensity of the scattered light is given by

$$I_{\Delta p}(r, \underline{K}'-\underline{K}, \Omega') = \left(\frac{\partial \varrho}{\partial p}\right)_T^2 \frac{1}{\varrho_0^2} \left(\frac{E_0 \Omega^2 V}{4\pi r C_E^2}\right) \langle R_{\Delta p}(\underline{K}'-\underline{K}, 0) \rangle$$
$$\times \int_{-\infty}^{\infty} e^{-\alpha\tau} \cos \omega_s \tau \; e^{i(\Omega - \Omega')\tau} \qquad \text{(E.8)}$$

The integral is standard. The final result is,

$$I_{\Delta p}(r, \underline{K}-K', \Omega') = \left(\frac{\partial \varrho}{\partial p}\right)_T^2 \left(\frac{E_0 \Omega^2 V}{4\pi r C_E^2}\right)^2 \frac{\langle R_{\Delta p}(\underline{K}'-\underline{K}, 0) \rangle}{\varrho_0^2}$$
$$\times \left\{ \frac{\alpha}{\alpha^2 + [\Omega' - (\Omega - \omega_s)]^2} + \frac{\alpha}{\alpha^2 + [\Omega' - (\Omega + \omega_s)]^2} \right. \qquad \text{(E.9)}$$

In words: the intensity of scattered light in the direction $\underline{K}'-K$ at distance r due to spontaneous pressures in a fluid is the sum of two contributions, namely, the two <u>Brillouin lines</u>. The first line is at frequency $\Omega' = \Omega + \omega_s$, and the second line is at $\Omega' = \Omega - \omega_s$. The bandwidth of these lines is α, and the corresponding lifetime is α^{-1}.

When mass density fluctuations are due to fluctuations in temperature a similar mathematical treatment leads to the result that

$$I_{\Delta T}(r, \underline{K}'-\underline{K}, \Omega) = \frac{1}{\rho_0^2}\left(\frac{\partial \rho}{\partial T}\right)_p^2 \left(\frac{E_0 \Omega^2 V}{4\pi r c_s^2}\right)\left[\frac{R_{\Delta T}}{\Theta}(\underline{K}'-\underline{K}, 0)\right]\frac{\beta}{\beta^2+(\Omega'-\Omega)^2} \tag{E.10}$$

This is the scattered Rayleigh line which appears in the direction $\underline{K}'-\underline{K}$, centered at frequency Ω (= incident frequency) with bandwidth β (or effective lifetime of β^{-1}).

The total intensity of scattered light is the same of scattering due to pressure and temperature. However, by using an appropriate equation of state relating pressure perturbations in the fluid to temperature perturbations one can reduce all Brillouin scattering to thermal motions of the fluid.

The intensity of light derived earlier was expressed in Lorentizian form which is equivalent to the form of the response of a simple harmonic oscillator to a random excitation. An analog of Eq. (D13) is the RLC circuit with forced E_f (volts) at frequency f. For a random emf the random current is given by

$$\langle i_f^2 \rangle = \frac{\langle E_f^2 \rangle}{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}, \qquad \omega = 2\pi f \tag{E.11}$$

Now if the random emf consists of fluctuations in the resistor R (= thermal noise) it was shown by Nyquist that in the small frequency interval between f, $f+\Delta f$ one has

$$\langle E^2 \rangle = 4RkT\Delta f \tag{E.12}$$

in which k = Boltzman's constant, T = absolute temperature. The mean square current accompanying thermal noise is therefore

$$\langle i_f^2 \rangle = \left(\frac{2kT\Delta f}{L}\right)\left[\frac{\alpha}{\alpha^2 + \frac{1}{4}\left(\omega - \frac{\omega_0^2}{\omega}\right)^2}\right], \qquad \omega_0^2 = \frac{1}{LC} \tag{E.13}$$

Here $\alpha = R/2L$ and α^{-1} is the effective life of the fluctuation, kT is energy per degree of freedom and $kT\Delta f$ is the power in the fluctuation.

Instead of an electrical problem one can consider a single degree of freedom mechanical system of a mass M, spring K, and viscous resistance R. The equivalent Nyquist formulas for random (spontaneous) noise in the system are

$$\langle F^2(t)\rangle = 2RkTt \tag{E.14a}$$

in which F is the force impulse (Newton sec) which operates during time t sec, and

$$\langle x^2\rangle = \frac{2kTt}{R} \tag{E.14b}$$

in which x is the amplitude of free Brownian motion in time. Thus,

$$\langle F^2(\Delta f)\rangle = 4RkT\Delta f \tag{E.14c}$$

is the (time) average of forces with frequencies between f and $f + \Delta f$. In words: A system at temperature T exhibits random impulses during any time interval t, the mean square average of which is given by Eq. (E.14a). These impulses act on an extraneous particle causing it to undergo Brownian motion. The motion is damped by viscosity.

Experimental Observation of Brillouin Lines

Brillouin lines are observed by high resolution spectroscopy. A modern technique is the laser interferometer. According to Eq. C.16 the shift in laser beam frequency $\Delta f_E / f_E$ due to the velocity C_s of the acoustic wave generated randomly by the Brillouin effect is

$$\left|\frac{\Delta f_E}{f_E}\right| = 2\,\frac{C_s}{C_E}\,\sin\frac{\theta}{2} = \frac{\Delta\lambda_E}{\lambda_E}$$

in which C_E is the speed of light in the fluid, and λ_E is its wavelength. Assuming $\theta = 180^\circ$, $C_s = 1.5 \times 10^3\ Ms^{-1}$, $C_E = 2.26 \times 10^8\ Ms^{-1}$ (in water), it is seen that

$$\frac{\Delta\lambda_E}{\lambda_E} = 1.3 \times 10^{-5}$$

The wavelength separation $\Delta\lambda_E$ depends on the laser wavelength to be used. Let $\lambda_E = 3.86 \times 10^{-7}$ M (in water). Then

$$\Delta\lambda_E = 5.1 \times 10^{-12} \text{ M} \approx 0.05 \text{ Å}; \qquad \Delta f_E = f_s = 7.6 \times 10^{9} \text{ Hz}.$$

This is the magnitude of Brillouin effect that can be detected with visible light.

Appendix F

A Case of Scattering of Sound Waves by Inhomogeneities

Let there be a region R (in a fluid) in which the change in compressibility is described by the factor $\gamma_K(\underline{r},t)$

$$\gamma_K(\underline{r},t) = \begin{cases} \dfrac{\Delta K}{K_0} & \text{inside } R \\ 0 & \text{outside } R \end{cases} \tag{F.1}$$

The acoustic pressure satisfies the equation

$$\nabla^2 p - \frac{1}{C_s^2}\frac{\partial^2 p}{\partial t^2} = \frac{1}{C_s^2}\gamma_k(\underline{r},t)\frac{\partial^2 p}{\partial t^2} \tag{F.2}$$

Assuming $p(\underline{r},t) = p(r)e^{-i\omega t}$, and using the Green's function $g_\omega(\underline{r}|\underline{r}_0)$ for unbounded space, it is seen that the total field is

$$\begin{aligned} p(\underline{r}) &= p_i(\underline{r}) + p_s(\underline{r}) \\ \text{"} \quad &= p_i(\underline{r}) + \int k^2 \gamma_K(\underline{r},t)\, p(\underline{r})\, g(\underline{r}|\underline{r}_0)\, d^3 r_0 \end{aligned} \tag{F.3}$$

in which

$$g(\underline{r}|\underline{r}_0) = \frac{1}{4\pi|\underline{r}-\underline{r}_0|}e^{ik|\underline{r}-\underline{r}_0|}$$

In the far field

$$g(\underline{r}|\underline{r}_0) \to \frac{1}{4\pi r} e^{ikr - ik\hat{\underline{r}}\cdot\underline{r}_0} \quad , \quad \underset{\sim}{k}_s = k\hat{r} \tag{F.4}$$

The scattered field is

$$p_s(\underline{r}) = \frac{e^{ikr}}{4\pi r} \int k^2 \gamma_k(\underline{r}_0, t) p(\underline{r}_0) e^{-ik\hat{\underline{r}}\cdot\underline{r}_0} d^3\underline{r}_0 \tag{F.5}$$

Using the Born approximation we set the total pressure $p(r_0)$ to be the incident pressure $p_i(r_0)$. Let the incident beam be finite, and choose its form to be

$$p_i(\underset{\sim}{r}_0) = A(\underline{r}_0) \exp i(\underline{k}_i \cdot \underline{r}_0)$$

Then,

$$p_s(\underline{r}) = \frac{k^2}{4\pi r} e^{ikr} \int A(\underline{r}_0) \gamma_k(\underset{\sim}{r}_0, t) e^{i(\underset{\sim}{k}_i - \underset{\sim}{k}_s)\cdot\underset{\sim}{r}_0} d^3\underset{\sim}{r}_0 \tag{F.6}$$

If the inhomogeneities are in the form of particles the volume integration can be replaced by a sum,

$$p_s(\underset{\sim}{r}) = \frac{Ak^2 e^{ikr}}{4\pi r} \sum_m \gamma_k(\underline{r}_{om}, t) e^{i(\underset{\sim}{k}_i - \underset{\sim}{k}_s)\cdot r_{om}} \Delta V(r_{om}) \tag{F.7}$$

Finally we allow r_{om} to be a function of time, $r_{om}(t)$. We see then that the incident wave is phase and amplitude modulated.

When there are many scattering inhomogeneities we write

$$p_s(\underline{r}) = \frac{k^2 e^{ikr}}{4\pi r} \sum_n \int A(\underline{r}_{on}) \gamma_K(r_{on}) e^{i(\underline{k}_i - \underline{k}_s)\cdot \underline{r}_{on}} d^3\underline{r}_{on} \tag{F.8}$$

The value of the integral depends on the size of the scattering volume $\Delta V(\underline{r}_{on})$ relative to the wavelength of the incident wave, λ_s. If $\Delta V(\underline{r}_{om}) \ll \lambda_s$, then the phase function (given by the exponential) is constant during the integration process. In this case (Rayleigh scattering)

$$p_s(r) = \frac{k^2 e^{ikr}}{4\pi r} \sum_n e^{i(\underline{k}_i - \underline{k}_s)\cdot \underline{r}_{on}} P(\underline{r}_{on}) \tag{F.9}$$

where,

$$\begin{aligned} P(r_{om}) &= \int A(r_{on}) \gamma_K(\underline{r}_{on}) d^3\underline{r}_{on} \\ &= A(\underline{r}_{om}) \gamma_{Kon} \Delta V(r_{on}). \end{aligned} \tag{F.10}$$

Appendix G

Volume Scattering Due to Perturbations of Fluid Density

In volume scattering due to perturbations on mass density the scattered field obeys the wave equation

$$\nabla^2 \underline{E}_s - \frac{1}{C_E^2}\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{2\Delta\varrho}{\varrho_o}\frac{1}{C_E^2}\frac{\partial^2 \underline{E}}{\partial t^2} \tag{G.1}$$

Now

$$\frac{\Delta\varrho}{\varrho_o} = \frac{1}{\varrho_o}\left(\frac{\partial\varrho}{\partial p}\right)_T \Delta p + \left(\frac{\partial\varrho}{\partial T}\right)\frac{\Delta T}{\varrho_o} + \left(\frac{\partial\varrho}{\partial M}\right)\frac{\Delta M}{\varrho_o} \tag{G.2}$$

in which ΔM represents chemical (= mass) concentration.
Consider only mass changes, $\varrho = \varrho(M)$. Then

$$\nabla^2 \underline{E}_s - \frac{1}{C_E^2}\frac{\partial^2 E}{\partial t^2} = \frac{2}{\varrho}\left(\frac{\partial\varrho}{\partial M}\right)\Delta M \frac{\partial^2 \underline{E}_s}{C_E^2\,\partial t^2} = \underline{\gamma}(\underline{r}_o, t) \tag{G.3}$$

Then let

$$\Delta M(\underline{r}, t) = \int \Delta M(\underline{k}, t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d^3\underset{\sim}{k} \tag{G.4}$$

Assume next that the electric field E_s on the $r.h.s$ is the incident field $E_o \exp i[\underset{\sim}{K}\cdot\underset{\sim}{r} - \Omega t]$ and use the free-field Green's function $(1/4\pi)(|\underline{r}-\underline{r}_o|)^{-1}\delta(t_o - t + |\underline{r}-\underline{r}_o|/c)$ to solve the wave equation. Thus

$$\underline{E}_s(\underline{r}, t) = \iint \underline{\gamma}(\underline{r}_o, t)\, G(\underline{r}, \underline{r}_o | t, t_o)\, d^3\underline{r}_o\, dt_o \tag{G.5}$$

or

$$\underset{\sim}{E}_s(\underset{\sim}{r},t) = \frac{2}{\rho_0}\left(\frac{\partial\rho}{\partial M}\right)_{P,T}\frac{1}{C_E^2}(-\Omega^2)\iint \Delta M(\underset{\sim}{k},t)e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_0}d^3\underset{\sim}{k}\;\frac{1}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}_0|} \quad \text{(G.6)}$$

$$\times\,\delta\left(t_0-t+\frac{|\underset{\sim}{r}-\underset{\sim}{r}_0|}{c}\right)E_0 e^{i[\underset{\sim}{K}\cdot\underset{\sim}{r}_0-\Omega t_0]}d^3\underset{\sim}{r}_0\,dt_0$$

As before, assume $|\underset{\sim}{r}-\underset{\sim}{r}_0| \simeq r-(\underset{\sim}{K}'/K)\cdot\underset{\sim}{r}_0$, $|K'| = K = \Omega/c_E$. Integration over t_0 gives

$$\underset{\sim}{E}_s(\underset{\sim}{r},t) = \frac{E_0}{C_E^2}(-\Omega^2)\frac{2}{\rho_0}\left(\frac{\partial\rho}{\partial M}\right)_{P,T}\int \Delta M(\underset{\sim}{k}_0,t)e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_0}d^3\underset{\sim}{k} \quad \text{(G.7)}$$

$$\times\, e^{i\left[\underset{\sim}{K}\cdot\underset{\sim}{r}_0-\Omega\left(t-\frac{r}{c}+\frac{(\underset{\sim}{K}'/K)\cdot\underset{\sim}{r}_0}{c}\right)\right]}d^3\underset{\sim}{r}_0$$

Now of the volume $V(\underset{\sim}{r}_0)$ is large enough the integration over $\underset{\sim}{r}_0$ is an infinite integration over an imaginary exponential which yields a delta function

Thus

$$\underset{\sim}{E}_s'(\underset{\sim}{r},t) = \frac{-E_0\Omega^2V}{4\pi rC_E^2}\frac{2}{\rho}\left(\frac{\partial\rho}{\partial M}\right)_{P,T}\int \Delta M(\underset{\sim}{k},t)e^{iK'r-i\Omega t} \quad \text{(G.8)}$$

$$\times\,\delta(\underset{\sim}{k}+\underset{\sim}{K}-\underset{\sim}{K}')\,d^3\underset{\sim}{k}$$

$$\underset{\sim}{E}_s(r,t\,|\,\underset{\sim}{K}'-\underset{\sim}{K}) = -\frac{E_0\Omega^2V}{4\pi rC_E^2}\frac{2}{\rho_0}\left(\frac{\partial\rho}{\partial M}\right)_{P,T}\Delta M(\underset{\sim}{K}'-\underset{\sim}{K},t)e^{iK'r-i\Omega t} \quad \text{(G.9)}$$

The autocorrelation R_E is

$$R_E(\tau) = \langle E_s(t+\tau)E^*(t)\rangle \qquad \text{dimensions: field}^2$$

or

$$R_E(\tau) = \left(\frac{E_0\Omega^2V}{4\pi rC_E^2}\right)^2\left(\frac{2}{\rho_0}\left(\frac{\partial\rho}{\partial M}\right)_{P,T}\right)^2\langle\Delta M(\underset{\sim}{K}'-\underset{\sim}{K},t)\Delta M^*(\underset{\sim}{K}'-\underset{\sim}{K},t+\tau)\rangle \quad \text{(G.10)}$$

The value $R_E(0)$ gives the intensity of the scattered electric field. The power spectrum $|E(\omega)|^2$ of E_s (dimensions: field^2 x sec^2), is the Fourier transform of R_E over the finite interval T,

$$\frac{2\pi}{T}\left|E(\Omega')\right|^2 = \int_{-T/2}^{T/2} R_E(\tau)\, e^{i\Omega'\tau}\, d\tau$$

or

$$\frac{2\pi}{T}\left|E(\Omega')\right|^2 = \left(\frac{E_0\Omega^2 V}{4\pi\hbar c_E^2}\right)\left[\frac{2}{\varrho_0}\left(\frac{\partial\varrho}{\partial M}\right)_{P,T}\right]^2 \int_{-T/2}^{T/2} \langle \Delta M(\underline{K}'-\underline{K},t)\,\Delta M^*(\underline{K}'-\underline{K},t+\tau)\rangle\, dt \tag{G.11}$$

Now the spectral intensity density $\mathcal{H}_E(\Omega')$ of the electric field is the limit

$$\mathcal{H}_E(\Omega') = \lim_{T\to\infty} \frac{2\pi}{T}\left|E(\Omega')\right|^2 \tag{G.12}$$

Hence

$$\mathcal{H}_E(\underline{K}'-\underline{K},\Omega') = \left(\frac{E_0\Omega^2 V}{4\pi\hbar c_E^2}\right)^2\left[\frac{2}{\varrho_0}\left(\frac{\partial\varrho}{\partial M}\right)_{P,T}\right]^2 \times \int R_{\Delta M}(\underline{K}'-\underline{K},\tau)\, e^{i\Omega'\tau}\, d\tau \tag{G.13}$$

(dimensions: field^2 x sec)

In words: If we observe the scattered electric field in direction $\underline{K}'-\underline{K}$ as a function of frequency we can find the autocorrelation of the density field by Fourier transformation. Alternatively, if we know the autocorrelation of the mass density field we can calculate the electrical spectral intensity density (dimensions: electric field^2 x sec).

Appendix H

Scattering of Light from Particles Suspended in a Fluid

(Edwards et al. J. Appl. Phys. 42, 837, (1971)).

The scattered electric field E_s is given by

$$\underline{E}_s(\underline{r},t) = \underline{E}_0 \frac{\Omega_E^2}{4\pi r c_E^2} \frac{1}{\varrho_0}\left(\frac{\partial \varrho}{\partial M}\right)_{P,T} \int \Delta M(\underline{r}_0,t) e^{i[(\underline{K}-\underline{K}')\cdot \underline{r}_0 - \Omega t]} \times e^{iK'r} d^3\underline{r}_0 \tag{H.1}$$

Let us now consider individual scatters rather than a continuum. To do this we replace the volume integration over r_0 by a sum over individual scattering particles, i.e.

$$\underline{E}_s(\underline{r},t) = \underline{A} \sum_n \Delta M(r_{on},t) \exp i\left[(\underline{K}'-\underline{K})\cdot \underline{r}_{on} - \Omega t\right] \Delta V_n(r_{on}) \quad \left(\text{dimensions: } \frac{\text{electric field}}{\text{acoustic pressure}}\right)$$

$$\underline{A} = \frac{2\underline{E}_0}{\varrho_0}\left(\frac{\partial \varrho}{\partial M}\right)_{P,T} \frac{e^{ikr}}{r} \frac{\Omega_E^2}{4\pi c_E^2} \tag{H.2}$$

in which ΔV is the volume of a scatterer. Furthermore let us take the location vectors r_{on} to be functions of time (namely we take the particles to be in motion) and write

$$\underline{r}_{on}(t) = \underline{r}_{on}(0) + \Delta \underline{r}_{on}(t) + \underline{v}t$$

This reads: the n'th particle, initially at $\underline{r}_{on}(0)$ moves $\Delta r_{on}(t)$ in time t relative to the fluid, which is itself in motion with velocity V.

The density perturbation ΔM is a function of space and time. Let us take an arbitrary time to be the origin, and write

$$E_s(\underline{r},0) = A \sum_n \Delta M(\underline{r}_{on}(0),0) \exp i\left[(\underset{\sim}{K}-\underset{\sim}{K}')\cdot \underset{\sim}{r}_{on}(0)\right] \Delta V_n(\underset{\sim}{r}_{on}) \tag{H.3}$$

Also, at time t

$$\underline{E}_s(r,t) = \underset{\sim}{A} \sum_m \Delta M\left[\underline{r}_{om}(0) + \Delta \underline{r}_{om}(t) + \underline{v}t\right] \exp i\left[(\underline{K}-\underline{K}')\cdot (\underline{r}_{om}(0) + \Delta\underline{r}_{om}(t) + \underline{v}t)\right] \exp(-i\Omega_E t)\, \Delta V_m(\underline{r}_{om}) \quad \text{(H.4)}$$

In this analysis we require the intensity of the scattered electric field. We can obtain this by use of an autocorrelation function $R_E(\tau)$, defined as

$$R_E(\tau) = \langle\langle E_s(\underline{r},0)\, E_s^*(r,\tau) \rangle\rangle \quad \text{(H.5)}$$

in which the double angle brackets indicate ensemble averages over (1) initial position $r_{om}(0)$; (2) random displacements $\Delta r_{om}(\tau)$. To form an ensemble average over initial position we introduce (see Edwards et al) a probability $P_r(\underset{\sim}{r_o}(0))$ per unit volume that a scattering center will be found in volume V. To form an ensemble average over displacements $\Delta\underline{r}_{om}(\tau)$ we introduce a probability per volume $P_r(\Delta r_{om}(\tau),\tau)$ that the n'th particle will move $\Delta\underline{r}_{om}$ units in time τ. The autocovariance of the scattered field then becomes

$$R_E(\tau) = A A^* \mathrm{Re}\, e^{-i\Omega\tau} \int \sum_m P_r(\Delta\underline{r}_{om}(\tau),\tau) \exp -i\left[(\underline{K}-\underline{K}')\cdot\Delta\underset{\sim}{r}_{om}(\tau)\right] \times \int P_r(r_{om}(0)) \exp -i\left[(\underline{K}-\underline{K}')\cdot\underline{v}\tau\right] \Delta M(r_{om}(0),0)\, \Delta M(r_{om}(0)+\Delta r_{om}(\tau) + \underline{v}\tau)\, d^3 r_{om}(0)\, d^3\Delta\underline{r}_{om}(\tau) \left[\Delta V(r_{om}(\tau))\right]^2 \quad \text{(H.6)}$$

(dimensions: field^2)

We note that only one sum on m is used in the product $\sum_n\sum_m$ required by the definition of R_E since the contributions $m \neq n$ all vanish in view of the assumed statistical independence of the scattering centers. This equation can be written in another way by defining an autocorrelation function for the density,

$$R_{\Delta M}(\Delta r_{om}(\tau) + \underline{v}\tau) = \int P_r(r_{om}(0)) \left[\Delta M(r_{om}(0)) \cdot \Delta M(r_{om}(0) + \Delta r_{om}(\tau) + \underline{v}\tau)\right] d^3\underset{\sim}{r}_{om}(0) \left[\Delta V(\underline{r}_{om})\right]^2 \quad \text{(H.7)}$$

(dimensions: acoustic pressure^2)

Thus,

$$R_E(\tau) = AA^* \mathcal{R}e\, e^{-i\Omega\tau} \int \sum_m P_r(\Delta r_{om}(t),\tau) \exp -i\left[(\underline{K}-\underline{K}')\cdot \underline{\Delta r}_{om}(\tau)\right]$$

$$\cdot R_{\Delta M}(\Delta r_{om}(\tau)+\underline{V}\tau) \exp -i(\underline{K}-\underline{K}')\cdot\underline{V}\tau \, d^3\underline{\Delta r}_{om} \qquad \text{(H.8)}$$

The power spectrum $|E_s(\Omega')|^2$ of the scattered field is obtained by Fourier transformation of $R_E(\tau)$ over a finite interval T,

$$\frac{2\pi}{T} \left| E_s(\Omega') \right|^2 = \int_{-T/2}^{T/2} R_E(\tau) e^{i\Omega'\tau} d\tau \qquad \text{(H.9)}$$

The spectral intensity density $\mathcal{H}_E(\Omega')$ of the electric field is the limit

$$\mathcal{H}_E(\Omega') = \lim_{T\to\infty} \frac{2\pi}{T} \left| E(\Omega') \right|^2 = \int_{-\infty}^{\infty} R_E(\tau) e^{i\Omega'\tau} d\tau$$

(dimensions: electric field^2 x sec)

In words: a time record of the received signal observed in direction $\underline{K}-\underline{K}'$ is used to form an autocorrelation function $R_E(\tau)$ where τ is the time shift. After $R_E(\tau)$ is formed we then set $\tau=0$ to obtain the intensity of the received electrical signal. This intensity is a function of the autocorrelation of the density field. In the absence of an acoustic signal the autocorrelation $R_E(\tau)$ will correspond to the autocorrelation of noise. The ratio of signal power to noise power will have the form

$$\frac{S}{N} = \frac{R_E(0)_{\Delta M}}{R_E(0)_N} \qquad \text{(H.10)}$$

if we set a threshold $S/N = 1$, we establish the minimum detectable intensity of electric signal as,

$$R_E(0)_{\Delta M} = R_E(0)_N \qquad \text{(H.11)}$$

The function $P_\lambda(r)$ is an amplitude weighting of the laser light, which depends on the position of the scatterer relative to a characteristic size (or dimension). For example, if the characteristic dimension is L_c and the position $r > L_c$ then we would expect P to be very small. The vector position $\underline{r}$ is a function of time. If V is the velocity of the fluid then $\underline{v}t$ is a component of $\underline{r}$. Hence P(r) = P(r(t)).

Now the autocorrelation R_E of the scatter field is

$$R_E \;(\underline{k},\tau) = \mathrm{Re} \;\langle\langle E_s(\underset{\sim}{k},0)\, E_s^*(\underset{\sim}{k},\tau)\rangle\rangle \tag{H.12}$$

in which $\langle\langle\;\;\rangle\rangle$ means averaging over (1) initial position of the scatterers (2) over displacements relative to fluid (= random deviations). Let $Q(\Delta\underline{r}+\underline{v}t)$ be defined as

$$Q\,(\Delta\underset{\sim}{r}+\underline{v}\tau) = \text{const.} \int_{-\infty}^{\infty} P_r(r(0))\, P[r(0)+\Delta\underline{r}+\underline{v}\tau]\; \exp(-i\,\underset{\sim}{k}\cdot\underset{\sim}{v}\tau)\, d\underset{\sim}{r}(0) \tag{H.13}$$

If one selects a (spherical) finite volume as a model of the collection of scatterers and writes

$$P_r(\underline{r}) = \frac{I_0^{1/2}}{(2\pi\sigma^2)^{3/4}}\, e^{-\frac{r^2}{4\sigma^2}} \tag{H.14}$$

then

$$Q = I_0\, e^{-\frac{(\Delta\underline{r}+\underline{v}\tau)^2}{8\sigma^2}}\, e^{-i(\underline{K}\cdot\underline{v})\tau} \tag{H.15}$$

The ratio $\sqrt{2}\,\sigma/|v|$ gives the <u>residence time</u> of the scattering centers in the sample volume. The effect of random displacements is expressed in the Δr dependency, conjoined with a probability function $P_r(\Delta r,\tau)$ which shows the probability that a scattering particle has moved $\Delta\underset{\sim}{r}$ in time τ . Thus the generalized autocorrelation $\mathcal{Q}(\Delta\underset{\sim}{r}+\underline{v}\tau,\underline{k})$ is,

$$\mathcal{Q}\,(\Delta r+\underline{v}\tau,\underset{\sim}{k}) = P_r(\Delta\underset{\sim}{r},\tau)\; Q(\Delta\underset{\sim}{r}+\underline{v}\tau,k)\, e^{-i\underset{\sim}{k}\cdot\Delta\underset{\sim}{r}} \tag{H.16}$$

(the minus sign in the exponential is derived from the <u>conjugate</u> electric field). Thus, for a <u>continuum</u> of particles, the autocorrelation function is

$$R_E\;(\underset{\sim}{k},\tau) = \text{const.}\int \mathcal{Q}\,(\Delta r+\underline{v}\tau,\underset{\sim}{k})\, d^3\Delta\underset{\sim}{r} \tag{H.17}$$

The power spectrum $S(\underset{\sim}{k},\omega)$ is then obtained by Fourier transformation

$$S(\underset{\sim}{k},\omega) = \int_0^{\infty} R_E(\underset{\sim}{k},\tau)\, e^{-i\omega\tau}\, d\tau \qquad \text{(H.18)}$$

In words: The detection process for measuring the scattered electric field is modeled as a harmonic oscillator, frequency $\underline{K}\cdot\underline{V}$, damped by two terms, viz. the Δr effect due to random motion of the particle relative to the average motion of the fluid, and $\underline{V}\tau$ effect, arising from finite transit time of the particles in the scattering volume.

Specific Example

Assume the $\Delta\underline{r}$ effect can be modeled on the theory of the random walk, i.e.

$$P_r(\Delta r,\tau) = \frac{1}{(4\pi D\tau)^{3/2}}\, e^{-\frac{(\Delta r)^2}{4D\tau}} \qquad \text{(H.19)}$$

in which D is the diffusion coefficient of the particles in the fluid. Choosing the amplitude weighting function P_r to be given by Eq. (H.14) (in which a scale size σ is specified) it is seen that the power spectrum or the scattered field is

$$S(\underset{\sim}{k},\omega) = \text{const.}\ \mathrm{Re} \int_0^{\infty} e^{i(\omega_0-\omega-\underset{\sim}{k}\cdot\underset{\sim}{r})\tau} \int_{-\infty}^{\infty} \frac{1}{(4\pi D\tau)^{3/2}} \cdot e^{-\frac{\Delta r}{4D\tau} - \frac{(\Delta\underset{\sim}{r}+\underline{V}\tau)^2}{8\sigma^2}}\, e^{-i\underset{\sim}{k}\cdot\Delta\underset{\sim}{r}}\, d^3(\Delta\underset{\sim}{r})\, d\tau \qquad \text{(H.20)}$$

We first allow $\underline{V}$ to be zero. Then, using the convolution theorem for Fourier transforms one arrives at

$$S(\underset{\sim}{K},\omega) = \text{const.}\ \mathrm{Re} \int_0^{\infty} \exp\left[i(\omega_0-\omega)\tau\right] \int_{-\infty}^{\infty} \exp\left[-(\underline{K}-\underline{K}')D\tau - 2K'^2\sigma^2\right] d\underset{\sim}{k}'\, d\tau \qquad \text{(H.21)}$$

An approximate spectrum can be obtained by expanding the diffusion term in a Taylor Series about $\underset{\sim}{K}'=0$, and then integrating. The result is

$$S(\underline{K},\omega) - \text{const.}\left[\frac{1/K^2D}{1+\gamma^2} + \frac{3/K^2D}{4K^2\sigma^2}\,\frac{(\gamma^4-12\gamma^2+3)}{(1+\gamma^2)^3} + \cdots\right] \qquad \text{(H.22)}$$

$$\gamma = \frac{\omega_0 - \omega}{K^2 D}$$

We next allow $\underline{V}$ to be finite, and proceed again to expand the diffusion term in a Taylor series about $\underline{K}'=0$. For small volumes of interrogation we can use only the first term in this expansion, namely,

$$
\begin{aligned}
S(\underline{K},\omega) &= \text{const.}\; \mathrm{Re}\left[\frac{\sigma}{V}\exp(z^2)\,\mathrm{erfc}(z)\right] \\
z &= \frac{\sqrt{2}\,\sigma}{V}\left[K^2 D + i\left(\omega_0-\omega-\underline{K}\cdot\underline{V}\right)\right]
\end{aligned}
\tag{H.23}
$$

When the diffusion effect is small relative to the finite transit time effect, this formula reduces to

$$
S(\underline{K},\omega) = \text{const.}\left(\frac{\sigma}{V}\right)\exp\left[-\frac{(\omega_0-\omega-K\cdot V)^2}{\dfrac{V^2}{2\sigma^2}}\right]
\tag{H.24}
$$

Note again that $2\sigma/V$ is the effective transit time of the particle in the volume, and $V/2\sigma$ is the effective bandwidth of the spectrum.

Appendix I

Intensity of Scattered Light From Acoustic Signals (Edwards et al. Model)

The intensity of scattered light in a medium of dielectric constant ε, magnetic permeability μ, is given by the Poynting vector, $\underline{S} = \frac{c_E}{4\pi} \underline{E} \times \underline{H}$, or

$$\underline{S} = v \mathcal{W} \underline{\Delta}, \quad v = \frac{c_E}{\sqrt{\varepsilon\mu}}, \quad \mathcal{W} = \frac{\varepsilon}{4\pi} E^2 \qquad \text{(I.1)}$$

in which $c_E = 3 \times 10^8 \, \mathrm{M s^{-1}}$

(dimension: E, volt/meter

ε, coulomb/(meter x volt)

$\mathcal{W}$, volt coulomb/meter^3

μ, meter volt/coulomb

C, v meter/sec

$\underline{S}$, coulomb volt/(sec x meter^2))

Thus in MKS units,

$$|\underline{S}| = \frac{c_E}{4\pi} \sqrt{\frac{\varepsilon}{\mu}}^{\,-1} E_s^2 \qquad \text{(I.2)}$$

Now the scattered electric field is a random function of particle motion. We therefore will obtain the quantity E^2 by finding the autovariance of the scattered field $(= R_E(\tau))$ between two moments in time, t_1, t_2 (note $\tau = t_1 - t_2$), using an appropriate spatial averaging to average out the initial position $r_{0n}(0)$, and random motion $\Delta r_{0n}(t)$ over all space. Thus, according to Eq. (H6),

$$R_E(\tau) = A A^* \mathcal{R}e \, e^{-i\Omega_s\tau} \int_{-\infty}^{\infty} \sum_n P_1(\Delta r_{0n}(\tau), \tau) \exp -i \left[(\underline{K} - \underline{K}') \cdot \Delta \underline{r}_{0n}(\tau) \right]$$
$$\times R_{\Delta M} (\Delta r_{0n}(\tau) + \underline{v}\tau + \underline{h}_1 \sin\omega_s\tau) \exp -i \left[(\underline{K} - \underline{K}') \cdot (\underline{v}\tau + \underline{h} \sin\omega_s\tau) \right] d^3 \Delta \underline{r}_{0n} \qquad \text{(I.3)}$$

in which

$$R_{\Delta M}\left(\Delta r_{om}(\tau) + \underline{v}\tau + \underline{h}\sin\omega_s\tau\right) =$$

$$\int_{-\infty}^{\infty} P_r\left(r_{om}(0)\right)\left[\langle \Delta M\left(\underline{r}_{om}(0)\right)\Delta M\left(\underline{r}_{om}(0) + \Delta\underline{r}_{om}(\tau) + \underline{v}\tau + \underline{h}\sin\omega_s\tau\right)\rangle\left(\Delta V(r_{om})\right)^2\right] d^3\underline{r}_{om}(0)$$

The magnitude of intensity, averaged over random time history of particle motion (that is, ensemble averaged through definitions of probability functions $P_r(\Delta r_{om})$ and $P_r(r_{om}(0))$ and averaged over time shift τ, is,

$$|\underline{S}| = \frac{c_E}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\; R_E(0) \tag{I.4}$$

The spectral intensity density $\mathcal{H}_E(\Omega')$ of the scattered light is

$$\mathcal{H}_E(\Omega') = \int_{-\infty}^{\infty} R_E(\tau)e^{i\Omega'\tau}d\tau \tag{I.5}$$

(dimensions: $\frac{V^2 s}{M^2}$)

(see Eq. H.9). The actual power spectrum of the scattered electric field, calculated over a finite interval T, is

$$\frac{2\pi}{T}\left|E_s(\Omega')\right|^2 = \int_{-T/2}^{T/2} R_E(\tau)\,e^{i\Omega'\tau}d\tau \tag{I.6}$$

in which the dimensions of $\left|E_s(\Omega')\right|^2$ are $V^2 s^2 M^{-2}$.