1976

Force-annihilation conditions for variable-coefficient lanchester-type equations of modern warfare, I: Mathematical Theory

Taylor, James G.

Monterey, California. Naval Postgraduate School

NPS-55Tw76081

NAVAL POSTGRADUATE SCHOOL

Monterey, California

FORCE-ANNIHILATION CONDITIONS FOR VARIABLE-COEFFICIENT LANCHESTER-TYPE EQUATIONS OF MODERN WARFARE, I: MATHEMATICAL THEORY

by

James G. Taylor and Craig Comstock

Partial Report for Period

April 1976 - July 1976

Prepared for:
Office of Naval Research, Arlington, Virginia 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, CA 93940

Rear Admiral Isham Linder
Superintendent

Jack R. Borsting
Provost

This work was supported by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research.

This report was prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS-55Tw76081	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Force-Annihilation Conditions for Variable-Coeffieicent Lanchester-Type Equations of Modern Warfare, I: Mathematical Theory		5. TYPE OF REPORT & PERIOD COVERED Technical Report - Partial Report for April - July 76
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James G. Taylor Craig Comstock		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61152N,RR 000-01-10 N0001476WR60052
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Arlington, VA 22217		12. REPORT DATE August 1976
		13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lanchester Theory of Combat
Combat Dynamics
Deterministic Combat Attrition

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper develops a mathematical theory for predicting force annihilation from initial conditions without explicitly computing force-level trajectories for deterministic Lanchester-type "square-law" attrition equations for combat between two homogeneous forces with temporal variations in fire effectiveness (as expressed by the Lanchester attrition-rate coefficients). It introduces a cannonical auxiliary parity-condition problem for the determination of a single parity- condition parameter ("the enemy force

(continued on next page)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

equivalent of a friendly force of unit strength") and new exponential-like general Lanchester functions. Prediction of force annihilation within a fixed finite time would involve the use of tabulations of the quotient of two Lanchester functions. These force-annihilation results provide further information on the mathematical properties of hyperbolic-like general Lanchester functions: in particular, the parity-condition parameter is related to the range of the quotient of two such hyperbolic-like general Lanchester functions. Different parity-condition parameter results and different new exponential-like general Lanchester functions arise from different mathematical forms for the attrition-rate coefficients. This theory is applied to general power attrition-rate coefficients: exact force-annihilation results are obtained when the so-called offset parameter is equal to zero, while upper and lower bounds for the parity-condition parameter are obtained when the offset parameter is positive.

1. Introduction.

Deterministic Lanchester-type equations of warfare (see TAYLOR and BROWN[25], WEISS[27]) play an important role in military operations research for developing insights into the dynamics of combat (see, for example, BONDER and FARRELL [4], or BONDER and HONIG[5]), even though combat between two opposing military forces is a far more complex random process. The classic Lanchester theory of combat (see DOLANSKY[9]) considered constant attrition-rate coefficients. New operations research techniques for forecasting temporal variations in fire effectiveness (caused by, for example, changes in force separation, combatant postures, target acquisition rates, firing rates, etc.) have generated interest in variable-coefficient combat formulations. Unfortunately, the resultant differential equations are not well studied.

In this paper we present a mathematical theory for predicting battle outcome from initial conditions without explicitly computing force-level trajectories for variable coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces (see Note 1). The determination of conditions on initial values that predict (in the sense of necessary and/or sufficient conditions) force annihilation in such Lanchester-type combat leads to some new mathematical problems in the theory of ordinary differential equations. This force annihilation problem may be viewed as either a problem of determining the asymptotic behavior of the solution (depending on given initial conditions) or a problem of determining the range of the quotient of two linearly independent solutions to, for example, the X force-level equation[25] (see Note 2). In either case, the classic ordinary differential equation theories (see, for example, HILLE[10], INCE[11], OLVER[17]) are inadequate to supply all the answers sought. We show that questions of force annihilation can be reduced to the study of certain "exponential-like" Lanchester functions and may be simply answered by examining certain inequalities involving the initial conditions and by possibly consulting tabulations of new special functions that are suggested here. Our general results apply to a wide class of attrition-rate coefficients (namely, those that yield continuous force-level trajectories).

Thus, in this paper we provide a general theoretical framework for determining force annihilation without explicitly computing force-level trajectories for variable-coefficient Lanchester-type equations of modern warfare. [Other modes of battle termination are briefly discussed.] We introduce a canonical auxiliary parity-condition problem for such determinations. New exponential-like general Lanchester functions arise from the solution to this problem, and tabulations of these would facilitate force-annihilation prediction. Different mathematical forms for attrition-rate coefficients lead to different auxiliary parity-condition problems. Our theory is applied to general power attrition-rate coefficients: exact force-annihilation results are obtained for cases of "no offset" (modelling, for example, weapon systems with the same maximum effective range); and although qualitative results are obtained, future computational work is required for quantitative results for cases of "offset" (modelling for example, weapon systems with different maximum effective ranges).

2. Lanchester's Classic Formulation.

F. W. LANCHESTER[13] hypothesized in 1914 that combat between two military forces could be modelled by

$$dx/dt = -ay, \qquad dy/dt = -bx, \tag{1}$$

with initial conditions

$$x(t=0) = x_0, \qquad y(t=0) = y_0, \tag{2}$$

where $t = 0$ denotes the time at which the battle begins, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and a and b are nonnegative constants which are today called Lanchester attrition-rate coefficients and represent each side's fire effectiveness. Lanchester (see McCLOSKEY[16] for his influence on operations research) considered this model (1) in order to provide insight into the dynamics of combat under "modern conditions" and justify the principle of concentration (see Note 3). We will accordingly refer to (1) as Lanchester's equations of modern warfare. Various sets of physical circumstances have been hypothesized to yield them: for example, (a) both sides use aimed fire and target acquisition times are constant (see Weiss[27]), or (b) both sides use area fire and a constant density defense (see BRACKNEY[6]). Other forms of Lanchester-type equations appear in the literature, but we will not consider these here (see Dolansky[9] and TAYLOR[22]).

From (1) Lanchester deduced his classic square law

$$b(x_0^2 - x^2(t)) = a(y_0^2 - y^2(t)). \tag{3}$$

Consider now a battle terminated by either force level reaching a given "breakpoint" (see Note 4): for example, Y wins when $x_f = x(t_f) = x_{BP} = f_X^{BP} x_0$ but $y_f > y_{BP} = f_Y^{BP} y_0$, where t_f, x_f, y_f denote final values and x_{BP} denotes X's breakpoint which is given fraction f_X^{BP} of his initial strength. It follows from (3) that

$$\text{Y wins} \Leftrightarrow x_0 < [\{1-(f_Y^{BP})^2\}/\{1-(f_X^{BP})^2\}]^{1/2} \sqrt{a/b}\, y_0, \tag{4}$$

which for a fight-to-the-finish becomes the classic result

$$\text{Y wins fight-to-the-finish} \Leftrightarrow x_0 < \sqrt{a/b}\, y_0. \tag{5}$$

Since, unfortunately, no relationship similar to (3) holds in general for variable attrition-rate coefficients, we observe that (4) may also be obtained from the time history of the X force level

$$x(t) = \{(x_0 - y_0\sqrt{a/b})\exp(\sqrt{ab}\, t) + (x_0 + y_0\sqrt{a/b})\exp(-\sqrt{ab}\, t)\}/2, \tag{6}$$

via determining the time for X to reach his breakpoint (i.e. $x(t=t_X^{BP}) = x_{BP}$)

$$t_X^{BP} = (1/\sqrt{ab})\ln(\{-x_{BP} + \sqrt{x_{BP}^2 + y_0^2 a/b - x_0^2}\}/\{y_0\sqrt{a/b} - x_0\}), \tag{7}$$

and requiring $t_X^{BP} < t_Y^{BP}$. The key result for obtaining (7) is that one of the two linearly independence solutions to the X force-level equation $d^2x/dt^2 - abx = 0$ is the reciprocal of the other. For a fight-to-the-finish (7) becomes

$$t_X^a = \{1/(2\sqrt{ab})\}\ln(\{y_0\sqrt{a/b} + x_0\}/\{y_0\sqrt{a/b} - x_0\}), \tag{8}$$

where t_X^a denotes the time to annihilate the X force. We observe that (5) is an immediate consequence of (8).

In many applications (see Section 3 below), one is interested in whether the battle will be terminated within a given time t_g. In this case $x_0 < \sqrt{a/b}\, y_0$ is a necessary condition for X to be annihilated and annihilation occurs when $t_X^a \leq t_g$. Thus, determination of whether force annihilation will occur within a given time involve consulting a tabulation of a transcendental function, here the natural logarithm (see also (10) below). Similar results hold for other fixed force-level breakpoints.

The time history of the X force level may also be written as

$$x(t) = x_0 \cosh \sqrt{ab}\, t - y_0 \sqrt{a/b} \sinh \sqrt{ab}\, t. \tag{9}$$

Taylor and Brown[25] take (9) as their point of departure for a mathematical theory for solving variable-coefficient formulations. (7) does not follow directly from (9), but (5) does via $x(t=t_X^a) = 0$ and

$$t_X^a = (1/\sqrt{ab}) \tanh^{-1}(\sqrt{b/a}\, x_0/y_0), \tag{10}$$

since the range of the hyperbolic tangent is $[0,1]$ for nonnegative arguments.

The purpose of this paper is to generalize the above to the general case of variable attrition-rate coefficients. We use (6) as our point of departure and base our development on the observation that (5) follows directly from (6), since the second term in brackets is always positive and goes to zero as $t \to +\infty$. [We will ignore the physical impossibility of negative force levels in developing results like (5).] Thus, by (6) $\lim_{t \to \infty} x(t) = -\infty \Leftrightarrow x_0 < \sqrt{a/b}\, y_0$, whence follows (5).

3. Variable Attrition-Rate Coefficients.

The pioneering work of S. Bonder[4],[5] on methodology for the evaluation of military systems (in particular, mobile systems such as tanks) has generated interest in variable-coefficient Lanchester-type equations and has led to improved operations research techniques for the prediction of such coefficients (see BONDER[2],[3]; background and further references are given in Taylor and Brown[25]). Thus, we consider

$$dx/dt = -a(t)y, \qquad dy/dt = -b(t)x, \tag{11}$$

where $a(t)$ and $b(t)$ denote time-dependent Lanchester attrition-rate coefficients.

These coefficients depend on such variables as firing doctrine, firing rate, rate of target acquisition, force separation, tactical posture of targets, etc. (see reference 4). Without loss of generality, we may take $a(t) = k_a g(t)$ and $b(t) = k_b h(t)$, where $g(t)$ and $h(t)$ denote time-varying factors such that $a(t)/b(t) = k_a/k_b$ = constant for $g(t) = h(t)$. We will also refer to (11) as the equations for a square-law attritions process, since an "instantaneous" square law holds even when $a(t)/b(t)$ is not constant (see TAYLOR and PARRY[26]; also references 21, 22, and 24).

A large class of combat situations of interest can be modelled with the following attrition-rate coefficients (see reference 4)

$$a(t) = k_a(t+C)^{\mu} \quad \text{and} \quad b(t) = k_b(t+C+A)^{\nu}, \tag{12}$$

where $A,C \geq 0$. We will refer to these coefficients as general power attrition-rate coefficients. The modelling roles of A and C are discussed in Taylor and Brown.[25] We will refer to C as the starting parameter, since it allows us to model (with $\mu,\nu \geq 0$) battles which begin within the maximum effective ranges of the two systems. We will refer to A as the offset parameter, since it allows us to model (again, with $\mu,\nu \geq 0$) battles between weapon systems with different effective ranges. For example, let us consider Bonder's[4] constant-speed attack on a static defensive position (see also references 22 and 25). Then we have

$$dx/dt = -\alpha(r)y = -\alpha_0(1-r/R_\alpha)^{\mu}y, \qquad dy/dt = -\beta(r)x = -\beta_0(1-r/R_\beta)^{\nu}x, \tag{13}$$

where $\mu,\nu \geq 0$ and R_α denotes the maximum effective range of Y's weapon system (i.e. $\alpha(r) = 0$ for $r > R_\alpha$), and these parameters are given by (we assume that $R_\beta \geq R_\alpha \geq R_0$)

$$A = (R_\beta - R_\alpha)/v, \quad \text{and} \quad C = (R_\alpha - R_0)/v, \tag{14}$$

where R_0 denotes the battle's opening range and $v > 0$ denotes the constant attack speed. By considering (14) and Figure 1, the reader should have no trouble in understanding our terminology for A and C. In this model (13) μ, for example, is used to model the range dependence of Y's attrition-rate coefficient (see Figure 2). Negative values of μ and ν are discussed in reference 25. Range is related to time by $r(t) = R_0 - vt$. Since zero force separation is reached at time $t_g = R_0/v$, the

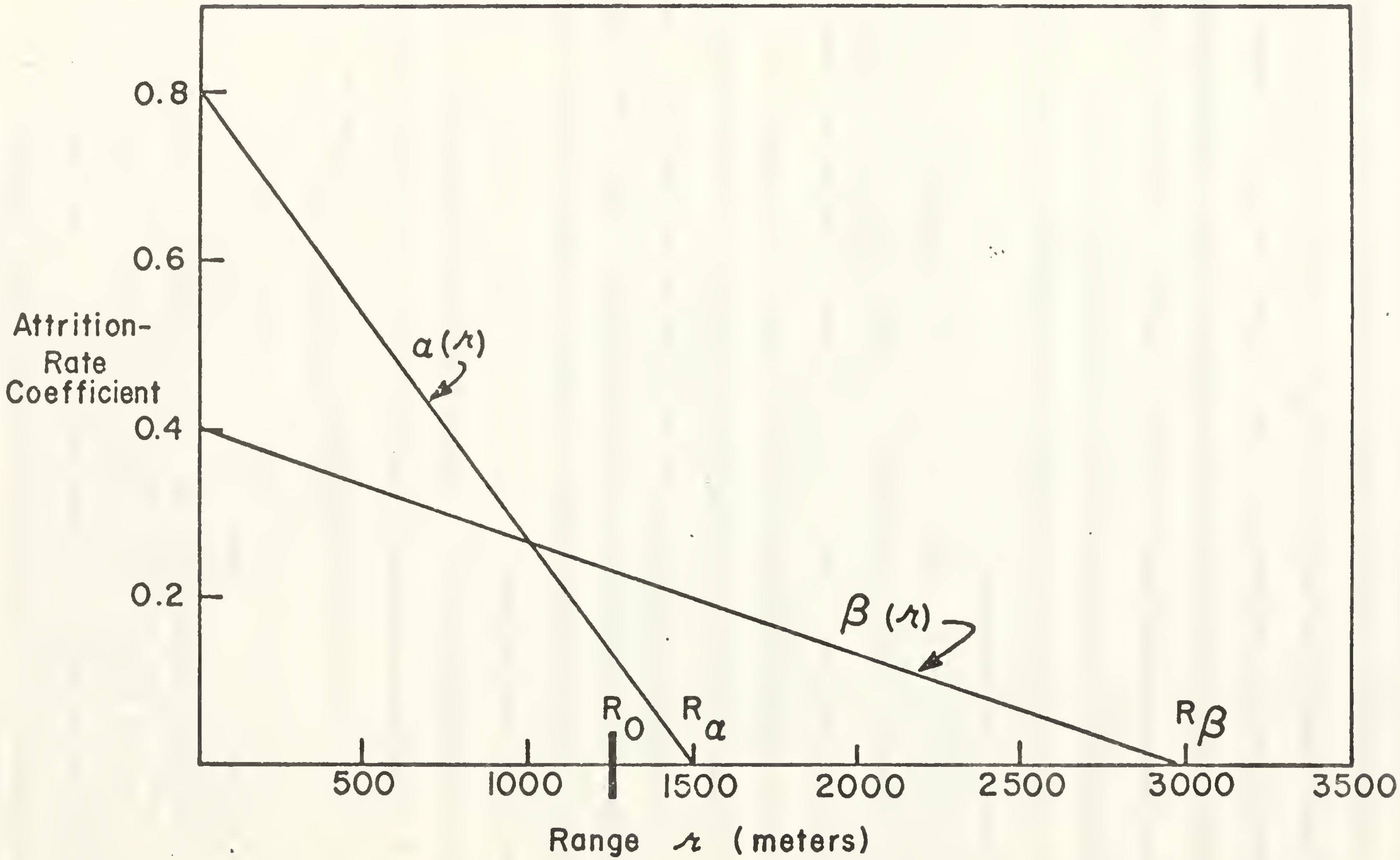

Figure 1. Explanation of starting parameter C and offset parameter A for power attrition-rate coefficients modelling constant-speed attack. [Notes: 1. The maximum effective ranges of the two weapon systems are denoted as R_α and R_β. 2. The opening range of battle is denoted as R_0 and (as shown) $R_0 < \text{minimum}(R_\alpha, R_\beta)$. 3. The starting parameter is given by $C = (R_\alpha - R_0)/v$. 4. The offset parameter is given by $A = (R_\beta - R_\alpha)/v$.]

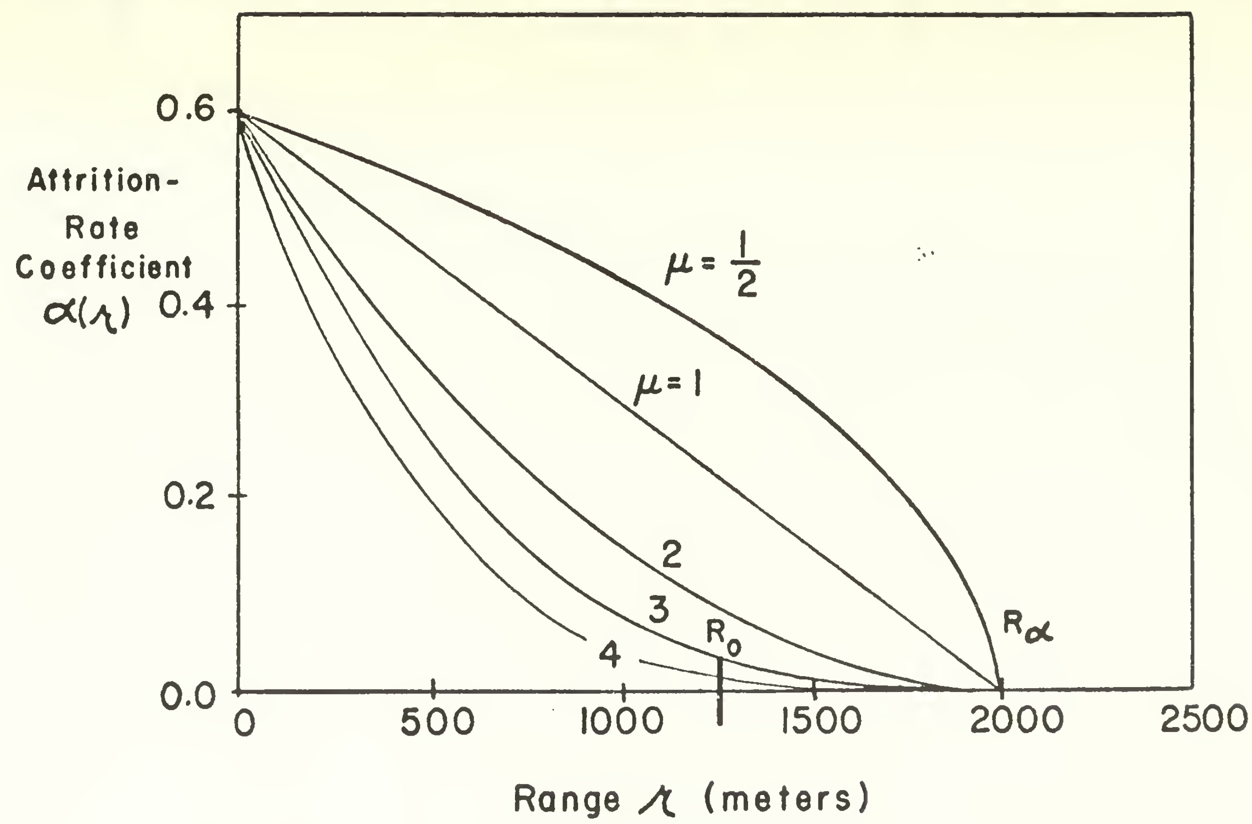

Figure 2. Dependence of the attrition-rate coefficient $\alpha(r)$ on the exponent μ for constant maximum effective range of the weapon system and constant kill capability at zero range. [Notes: 1. The maximum effective range of the system is denoted $R = 2000$ meters. 2. $\alpha(r=0) = \alpha_0 = 0.6X$ casualties/(unit time × number of Y units) denotes the Y force weapon system kill rate at zero force separation (range). 3. The opening range of battle is denoted as $R_0 = 1250$ meters and (as shown) $R_0 < R_\alpha$.]

above model (13) provides interest in predicting battle termination within a given time t (see Section 2 above).

Almost all previous work on the variable-coefficient equations (11) has developed infinite series solutions for force-level trajectories or represented these by tabulated functions (see reference 25, in particular Section 3). Relatively little attention has been given to determining the qualitative behavior of solutions to (11) (such as prediction of battle outcome) without explicitly computing battle trajectories. Bonder and Farrell[4] (see Note 5), however, have considered force annihilation within a given time. Using comparison techniques from the theory of ordinary differential equations (see, for example, CODDINGTON and LEVINSON[8]), they obtained a rather strong sufficient condition for the special case of the model (13) with $\mu = \nu = 1$ and $A > 0$.

4. A Mathematical Theory for Conditions of Force Annihilation.

Motivated by the constant-coefficient results (see Note 6), we introduce the exponential-like general Lanchester functions E_X^+, E_X^-, E_Y^+, and E_Y^-, defined by

$$\begin{aligned} dE_X^+/dt &= \sqrt{k_b/k_a}\, a(t) E_Y^+ \quad &&\text{with} \quad E_X^+(t=t_0) = 1/Q, \\ dE_Y^+/dt &= \sqrt{k_a/k_b}\, b(t) E_X^+ \quad &&\text{with} \quad E_Y^+(t=t_0) = 1, \end{aligned} \tag{15}$$

and

$$\begin{aligned} dE_X^-/dt &= -\sqrt{k_b/k_a}\, a(t) E_Y^- \quad &&\text{with} \quad E_X^-(t=t_0) = 1, \\ dE_Y^-/dt &= -\sqrt{k_a/k_b}\, b(t) E_X^- \quad &&\text{with} \quad E_Y^-(t=t_0) = Q, \end{aligned} \tag{16}$$

where $t_0 = \max(t_0^X, t_0^Y)$, and t_0^X denotes the largest finite singular point (see Taylor and Brown[25]; also p. 69 of Ince[11]) (see Note 7) on the t-axis for the X force-level equation

$$d^2x/dt^2 - \{[1/a(t)]da/dt\}dx/dt - a(t)b(t)x = 0. \tag{17}$$

By assumption, then, $a(t)$ and $b(t)$ are positive continuous functions $\forall\ t > t_0$.

Both $E_X^+(t)$ and $E_X^-(t)$ satisfy (17). Since they are a fundamental system of solutions, we may use them to construct all solutions to (17). Thus, the solution to to (17) which satisfies the initial conditions (2) is given by

$$x(t) = \{[x_0 E_Y^-(t=0) - \sqrt{k_a/k_b}\, y_0 E_X^-(t=0)]E_X^+(t)$$

$$+ [x_0 E_Y^+(t=0) + \sqrt{k_a/k_b}\, y_0 E_X^+(t=0)]E_X^-(t)\}/2, \qquad (18)$$

and from (11) we obtain

$$y(t) = \{[y_0 E_X^-(t=0) - \sqrt{k_b/k_a}\, x_0 E_Y^-(t=0)]E_Y^+(t)$$

$$+ [y_0 E_X^+(t=0) + \sqrt{k_b/k_a}\, x_0 E_Y^+(t=0)]E_Y^-(t)\}/2, \qquad (19)$$

where we have made use of the easily verifiable fact that (see reference 25)

$$E_X^+(t)E_Y^-(t) + E_X^-(t)E_Y^+(t) = 2 \qquad \forall\, t. \qquad (20)$$

Without further restrictions on $a(t)$ and $b(t)$, the systems (15) and (16) might not have a solution: it might be that one or more of the general Lanchester functions is unbounded at t_0 (see Section 4 of reference 25). For example, unboundedness occurs for power attrition-rate coefficients with $\mu + \nu + 2 = 0$ and $A = 0$ (see equation (37) of Taylor and Brown[25]). Thus, we introduce

CONDITION (A): $\int_{t_0}^{t} a(s)ds$ and $\int_{t_0}^{t} b(s)ds$ are bounded for all finite $t \geq t_0$.

Then, we have

THEOREM 1: Condition (A) is a necessary and sufficient condition for (15) and (16) to have a continuous solution for all finite $t \geq t_0$.

PROOF: Sufficiency follows by Theorem 6.4.2 on p. 226 of Hille[10] (see also p. 64 of LEE and MARKUS[15]). To prove necessity, it suffices to consider $E_X^+(t)$. If a continuous solution exists, then from (15) we have $dE_X^+/dt \geq \sqrt{k_b/k_a}\, a(t)$, whence $E_X^+(t) \geq 1/Q + \sqrt{k_b/k_a} \int_{t_0}^{t} a(s)ds$, and the theorem follows. Q.E.D. Let us henceforth assume that Condition (A) holds.

Let us now consider how to choose the general Lanchester functions E_X^- and E_X^+ so that they play the roles of a decaying exponential and an increasing exponential, respectively, in the solution (18) to the variable-coefficient X force-level equation (17) [whence our notation of E_X^- and E_X^+]. We recall the constant-coefficient result (6) and its consequence (5) obtained using $\lim_{t\to+\infty} \exp(-\sqrt{ab}\, t) = 0$ and $\exp(-\sqrt{ab}\, t) > 0$. For any positive value of Q it is clear that $E_X^+(t)$, which satisfies (15), grows without bound just as an increasing exponential does. We will now show how to choose E_X^- so that it corresponds to a decaying exponential. Similar statements hold for E_Y^- and E_Y^+. Considering (16), we see that we should choose E_X^- and E_Y^- to remain positive for all t so that by (16) they continuously decrease. Furthermore, we will be able to specify such behavior for E_X^- and E_Y^- by our selection of the parameter Q in the initial conditions for (16).

The solution $E_X^-(t)$, $E_Y^-(t)$ to (16) depends continuously on the parameter Q of the initial conditions (see, for example, Hille[10]). We denote this dependence by $E_X^-(t;Q)$, $E_Y^-(t;Q)$. Let $Q^* = Q^*(a(t),b(t))$ denote the value of Q such that

$$E_X^-(t;Q=Q^*),\ E_Y^-(t;Q=Q^*) > 0 \quad \text{for all finite } t \geq t_0. \tag{21}$$

It follows from (16) that

$$\lim_{t\to+\infty} E_X^-(t;Q^*) = \lim_{t\to+\infty} E_Y^-(t;Q^*) = 0. \tag{22}$$

It is intuitively obvious that such a Q^* exists, and we will prove its existence in some particular cases below. As we shall see, knowledge of Q^* provides valuable information about the qualitative behavior of force-level trajectories for the Lanchester-type equations (11). Let us refer to the problem of determining Q^* such that (22) holds as the auxiliary parity-condition problem. Unless explicitly stated otherwise, for convenience we will denote, for example, $E_X^+(t;Q^*)$ as $E_X^+(t)$.

Comment 1: For a constant ratio of attrition-rate coefficients, i.e.

$$a(t) = k_a h(t), \qquad \text{and} \qquad b(t) = k_b h(t), \tag{23}$$

where $h(t)$ denotes the common time-varying factor of the two coefficients, it readily follows from the results given in references 4 and 20 that $Q^* = 1$ and

$$E_X^+(t) = E_Y^+(t) = \exp\{\psi(t)\}, \qquad \text{and} \qquad E_X^-(t) = E_Y^-(t) = \exp\{-\psi(t)\}, \tag{24}$$

where $\psi(t) = \sqrt{k_a k_b} \int_{t_0}^{t} h(s)ds.$

Comment 2: By Theorem 1 of Taylor and Brown[25], the solution (18) simplifies to the form of (6) only if (23) holds.

The above exponential-like general Lanchester functions may be related to Taylor and Brown's[25] hyperbolic-like general Lanchester functions. Let

$$C_X(t) = x_1(t), \quad S_X(t) = x_2(t), \quad C_Y(t) = y_1(t), \quad \text{and} \quad S_Y(t) = y_2(t), \tag{25}$$

where x_1, x_2, y_1, and y_2 denote the general Lanchester functions introduced by Taylor and Brown. Then, similar to the well-known relationships between the hyperbolic and exponential functions, we have

$$C_X(t) = \{Q^*E_X^+(t) + E_X^-(t)\}/2, \qquad S_X(t) = \{Q^*E_X^+(t) - E_X^-(t)\}/(2Q^*),$$
$$C_Y(t) = \{Q^*E_Y^+(t) + E_Y^-(t)\}/(2Q^*), \qquad S_Y(t) = \{Q^*E_Y^+(t) - E_Y^-(t)\}/2. \tag{26}$$

The determination of Q^* will be slightly simplified for general power attrition-rate coefficients (12) by considering a modified auxiliary parity-condition problem. For this purpose we introduce the new independent variable

$$s = K\sqrt{k_b/k_a} \int_{t_0}^{t} a(\sigma)d\sigma, \tag{27}$$

and define $s_0 = s(t=0) \geq 0$ for $t_0 \leq 0$. K is an, at present, undetermined parameter. It will be chosen so that a more convenient canonical system of differential equations arises in the modified auxiliary parity-condition problem. By Condition (A), the transformation (27) is well defined for $t \geq t_0$. It has an inverse $t(s)$, since $a(t) >$ $\forall t > t_0$. Letting

$$e_X^+(s) = KE_X^+(t(s)), \quad e_Y^+(s) = E_Y^+(t(s)), \quad e_X^-(s) = E_X^-(t(s)), \quad \text{and} \quad e_Y^-(s) = E_Y^-(t(s))/K,$$

the substitution (27) transforms (15) through (17) into

$$de_X^+/ds = e_Y^+ \qquad \text{with} \quad e_X^+(s=0) = 1/Z,$$
$$de_Y^+/ds = I(s)e_X^+ \qquad \text{with} \quad e_Y^+(s=0) = 1, \tag{29}$$

$$de_X^-/ds = -e_Y^- \qquad \text{with} \quad e_X^-(s=0) = 1,$$
$$de_Y^-/ds = -I(s)e_X^- \qquad \text{with} \quad e_Y^-(s=0) = Z, \tag{30}$$

and

$$d^2x/ds^2 - I(s)x = 0, \tag{31}$$

where for any Q

$$Z = Q/K, \tag{32}$$

and

$$I(s) = (\{b(t)/k_b\}/\{a(t)/k_a\})/K^2, \tag{33}$$

is the invariant of the normal form (31) (see p. 119 of KAMKE[12]) and $t = t(s)$ by (27). The parameter K will be chosen to simplify the form of $I(s)$. In our later work the equation (31) will be easier to analyze than (17).

We will refer to the problem of determining $Z* = Z*(a(t),b(t))$ such that

$$e_X^-(s;Z=Z*),\ e_Y^-(s;Z=Z*) > 0 \quad \text{for all finite} \quad s \geq 0, \tag{34}$$

as the modified auxiliary parity-condition problem. By (32) we then have that $Q* = KZ*$

We also observe that the solution to (31) which satisfies (2) may be expressed in terms of these exponential-like general Lanchester functions for $s \geq s_0$ as

$$x(s) = \{[x_0Ke_Y^-(s=s_0) - \sqrt{k_a/k_b}\ y_0e_X^-(s=s_0)]e_X^+(s)/K$$
$$+ [x_0Ke_Y^+(s=s_0) + \sqrt{k_a/k_b}\ y_0e_X^+(s=s_0)]e_X^-(s)/K\}/2, \tag{35}$$

and similarly

$$y(s) = \{[y_0e_X^-(s=s_0) - \sqrt{k_b/k_a}\ x_0Ke_Y^-(s=s_0)]e_Y^+(s)$$
$$+ [y_0e_X^+(s=s_0) + \sqrt{k_b/k_a}\ x_0Ke_Y^+(s=s_0)]e_Y^-(s)\}/2, \tag{36}$$

where by (20) and (28)

$$e_X^+(s)e_Y^-(s) + e_X^-(s)e_Y^+(s) = 2 \qquad \forall s. \tag{37}$$

From our choice of $Q*$ such that (21) and (22) hold, we can immediately infer the behavior of the solution (18) to (11) as $t \to +\infty$ and similarly for $y(t)$ (19). Thus, we have

THEOREM 2: $\lim_{t\to+\infty} x(t) = -\infty$ if and only if

$$x_0 E_Y^-(t=0;Q^*) < \sqrt{k_a/k_b}\, y_0 E_X^-(t=0;Q^*).$$

Equivalently, we may state

THEOREM 3: Consider combat between two homogeneous forces described by (11). Assume that (11) applies for all time and that Y "wins" when $x(t_f) = 0$ with $y(t_f) > 0$. Then, Y wins if and only if $x_0 E_Y^-(t=0;Q^*) < \sqrt{k_a/k_b}\, y_0 E_X^-(t=0;Q^*)$.

Thus, we see that tabulations of such new exponential-like general Lanchester functions $E_X^-(t;Q^*)$ and $E_Y^-(t;Q^*)$ would facilitate force-annihilation prediction. By our choice of t_0 only one such tabulation is necessary for given attrition-rate coefficients $a(t)$ and $b(t)$ (see Note 7). Alternatively, we may express the force-annihilation condition of Theorem 3 in terms of Taylor and Brown's[25] hyperbolic-like general Lanchester functions (see equations (25) and (26)). We have then

THEOREM 3': Consider combat between two homogeneous forces described by (11). Assume that (11) applies for all time and that Y "wins" when $x(t_f) = 0$ with $y(t_f) > 0$. Then, Y wins if and only if $x_0\{Q^* C_Y(t=0) - S_Y(t=0)\} < \sqrt{k_a/k_b}\, y_0\{C_X(t=0) - Q^* S_X(t=0)\}$.

As an immediate corollary to Theorem 3 we have

COROLLARY 3.1: For $t_0 = 0$, Y wins if and only if $x_0 Q^* < \sqrt{k_a/k_b}\, y_0$.

One shortcoming of our above development is that Theorem 3 and its corollaries are basically existence theorems for the annihilation of one side by the other at some unknown future point in time. As the enemy initial force level decreases towards parity (i.e. equality holding in (38)), the time required to annihilate a force becomes larger and larger. There is, in fact, no limit to how large it may become. Moreover, there is a large class of tactically significant battles (see Bonder's[4] constant-speed attack on a static defensive position discussed in Section 3 above) which has a built-in time limit, denoted as t_g. Hence, it would be desirable to have a method for determining from initial conditions (without explicitly computing the entire force-level trajectories) whether or not force annihilation will occur within a given finite time t_g. For our general model (11), Theorem 3 tells us that

$$x_0 E_Y^-(t=0;Q^*) < \sqrt{k_a/k_b}\, y_0 E_X^-(t=0;Q^*), \tag{38}$$

is a necessary condition for $x(t_X^a) = 0$ with $t_X^a \le t_g$, where t_X^a denotes the time at which the X force is annihilated. Motivated by the well-known constant-coefficient result (8), one intuitively sees that determining whether or not $t_X^a \le t_g$ will require the appropriate tabulations of new transcendents (i.e. new functions). Different such functions arise from different fundamental systems chosen to construct the solution to, for example, the X force-level equation (17). Let us now investigate which fundamenta system of solutions is the most useful.

We begin our investigation by outlining for the new exponential-like general Lanchester functions how to determine from initial conditions (without explicitly computing the force-level trajectories) whether or not force annihilation will occur within a given finite time t_g. Looking at (18) and setting $x(t) = 0$, we see that we must solve for $E_X^+(t)/E_X^-(t) = \eta(t)$ (see Note 8). For any other fundamental system (i.e. pai of solutions, we must still solve for such a quotient (cf. the constant-coefficient results (8) and (10)). Thus, our force-annihilation determination requires the use of tabulations of the quotient of two linearly independent solutions to, for example, the X force-level equation. Our question is now which quotient is the most useful (i.e. which fundamental system yields the most useful quotient). We will show that the quotient of two exponential-like general Lanchester functions is not numerically satisfactory for such determinations, whereas the quotient of two hyperbolic-like general Lanchester functions is.

Motivated by the result for a constant ratio of attrition-rate coefficients (23) that $\eta(t) = \exp\{2\psi(t)\}$, we use the following notation for the $\eta(t)$ of the X force-level equation

$$E^+_{2X}(t) = E^+_X(t)/E^-_X(t), \tag{39}$$

with $E^+_{2Y}(t)$ being similarly defined. Assuming that (38) holds, we see from (18) that if $x(t=t^a_X) = 0$, then

$$E^+_{2X}(t^a_X) = \{\sqrt{k_a/k_b}\, y_0 E^+_X(t=0;Q^*) + x_0 E^+_Y(t=0;Q^*)\}/$$
$$\{\sqrt{k_a/k_b}\, y_0 E^-_X(t=0;Q^*) - x_0 E^-_Y(t=0;Q^*)\}.$$

We will prove below that $E^+_{2X}(t) = E^+_{2X}(t;Q^*)$ is a strictly increasing function of t with initial value $1/Q^*$ at $t = t_0$. Consequently, the inverse function $E^{+\ -1}_{2X}(\xi) = L^+_{2X}(\xi)$ is a well-defined function of $\xi\ \forall\ \xi \in [1/Q^*,+\infty)$. Hence, we may write that the time to annihilate X is given by (see Note 9)

$$t^a_X = E^{+\ -1}_{2X}(\{\sqrt{k_a/k_b}\, y_0 E^+_X(t=0;Q^*) + x_0 E^+_Y(t=0;Q^*)\}/$$
$$\{\sqrt{k_a/k_b}\, y_0 E^-_X(t=0;Q^*) - x_0 E^-_Y(t=0;Q^*)\}). \tag{40}$$

Considering the above developments, one may show that except when (23) holds, it is not possible to determine in the manner of (40) the battle-termination time when X's breakpoint is positive, i.e. $x_{BP} > 0$.

We now show that $E^+_{2X}(t)$ (defined by (39)) is a strictly increasing function. We readily compute using (15), (16), and (20) that

$$dE^+_{2X}/dt = 2\sqrt{k_b/k_a}\, a(t)/\{E^-_X(t)\}^2 \qquad \text{with} \quad E^+_{2X}(t=t_0) = 1/Q^*, \tag{41}$$

whence follows the monotonicity.

[Sometimes it is convenient to express results in terms of the modified exponential-like general Lanchester functions $e^+_X(s)$, $e^-_X(s)$, $e^+_Y(s)$, and $e^-_Y(s)$, which are based on the transformed time variable s defined by (27). The above results are readily extended to this case. In particular, $e^+_{2X}(s) = e^+_X(s)/e^-_X(s)$ is a strictly increasing function of s for $s \geq 0$, $e^+_{2X}(s=0) = K/Q^*$, and $\lim_{s\to+\infty} e^+_{2X}(s) = +\infty$.]

Thus, (I) determination of $Q^* = Q^*(a(t),b(t))$, and (II) tabulation of $E^+_{2X}(t)$ and $E^+_{2Y}(t)$ would allow one to determine whether or not force annihilation occurs in such battles with finite time limit without explicitly computing the entire force-level trajectories (see Figures 3, 4, and 5 of reference 25). Unfortunately, there is a serious drawback to considering $E^{+}_{2X}(t)$ and $E^{+}_{2Y}(t)$: accurate tabulations are difficult (in fact, essentially impossible for large values of t) to generate, since both functions are basically increasing exponentials so that any error in their initial value $1/Q^*$ (which, in general, can be only numerically approximately determined) becomes tremendously magnified over time.

We may develop numerically satisfactory functions for prediction of force annihilation within a given finite time, however, by considering the hyperbolic-like general Lanchester functions of Taylor and Brown[25]. Let us therefore define

$$T_X(t) = S_X(t)/C_X(t). \tag{42}$$

The functions $T_X(t)$ and $T_Y(t)$ are analogous to the hyperbolic tangent, to which they reduce for a constant ratio of attrition-rate coefficients. Considering (25) and equation (16) of reference 25, we see that, for example, $T_X(t)$ does not depend on Q^*, since $S_X(t)$ and $C_X(t)$ do not. Thus, $T_X(t)$ and $T_Y(t)$ are numerically suitable for determining whether or not force annihilation will occur within a given finite time.

Using the results given in Table I of Taylor and Brown[25], we readily compute that

$$dT_X/dt = \sqrt{k_b/k_a}\, a(t)/\{C_X(t)\}^2 \qquad \text{with} \quad T_X(t=t_0) = 0. \tag{43}$$

Hence, $T_X(t)$ is a strictly increasing function, and its inverse T_X^{-1} is well defined. Let us now establish an upper bound for $T_X(t)$. By (26) and (42), we have

$$T_X(t) = (1/Q^*)\{Q^*E^{+}_X(t) - E^{-}_X(t)\}/\{Q^*E^{+}_X(t) + E^{-}_X(t)\},$$

whence it follows that for $t \geq t_0$

$$0 \leq T_X(t) < 1/Q^*, \qquad \text{with} \lim_{t\to+\infty} T_X(t) = 1/Q^*. \tag{44}$$

Thus, our current investigation has yielded important information about the asymptotic behavior of hyperbolic-like general Lanchester functions.

To determine t_X^a such that $x(t_X^a) = 0$, we write the solution to (17) which satisfies the initial conditions (2) as[25] $x(t) = x_0\{C_Y(t=0)C_X(t) - S_Y(t=0)S_X(t)\}$ $- y_0\sqrt{k_a/k_b}\,\{C_X(t=0)S_X(t) - S_X(t=0)C_X(t)\}$ and find that when (38) holds, t_X^a is given by

$$t_X^a = T_X^{-1}(\{x_0C_Y(t=0) + y_0\sqrt{k_a/k_b}\,S_X(t=0)\}/$$
$$\{y_0\sqrt{k_a/k_b}\,C_X(t=0) + x_0S_Y(t=0)\}). \tag{45}$$

We observe that by (26) the argument of the inverse function T_X^{-1} in (45) belongs to the range of T_X (see (44)) when (38) holds (see Theorem 3'). For $t_0 = 0$, (45) simplifies to $t_X^a = T_X^{-1}(\sqrt{k_b/k_a}\,x_0/y_0)$. Whether or not force annihilation occurs within a given finite time t_g then depends on whether or not $t_X^a \le t_g$. Thus, (I) determination of Q^*, and (II) tabulation of $T_X(t)$ and $T_Y(t)$ would allow one to determine (without explicitly computing the force-level trajectories) the time at which a side is annihilated.

5. Application to Power Attrition-Rate Coefficients.

Let us now apply the above general theory to (11) with the general power attrition-rate coefficients (12). We observe that in this case $t_0 = -C$, where $C \ge 0$. In order that Condition (A) holds we must have $\mu,\nu > -1$.

As we have seen in Section 4, our theory of force-annihilation prediction depends on knowing Q^*, the solution to the auxiliary parity-condition problem. For the power attrition-rate coefficients (12), it is more convenient, however, to determine Q^* via the modified auxiliary parity-condition problem (30) (see also (34)). Hence, we apply the transformations (27) and (28) to (16). For the coefficients (12), equations (30) take the form

$$\begin{aligned} d\bar{e}_X/ds &= -\bar{e}_Y & &\text{with } \bar{e}_X(s=0) = 1, \\ d\bar{e}_Y/ds &= -s^\beta(1+\gamma/s^\alpha)^\nu \bar{e}_X & &\text{with } \bar{e}_Y(s=0) = Z, \end{aligned} \tag{46}$$

where the parameter K in (27) is given by $K = (\sqrt{k_ak_b}/(\mu+1))^{2p-1}$; and we have $p = (\mu+1)/(\mu+\nu+2)$, $q = 1-p$, $\alpha = 1/(\mu+1)$, $\beta = (\nu-\mu)/(\mu+1)$, and

$\gamma = A(\sqrt{k_a k_b}/(\mu+1))^{2/(\mu+\nu+2)}$. For $\mu,\nu > -1$, we have $0 < p, q < 1$.

After we have solved the above modified auxiliary parity-condition problem (i.e. determined $Z = Z^*$ for (46) such that (34) holds), we have all the information required to determine without explicitly computing the entire force-level trajectories whether or not force annihilation occurs in battles modelled with (12). We may apply Theorem 3 via (28) (possibly using (26)) to see who can be annihilated and use results such as (40) (or equivalent ones for hyperbolic-like power Lanchester functions (see Taylor and Brown[25])) to see if force annihilation occurs in a given finite time (for example, for battles modelled by (13)). When $C = 0$ for the coefficients (12) (e.g. for the model (13), $R_0 = R_\alpha \leq R_\beta$ from (14)), we have by Corollary 3.1

COROLLARY 3.2: For combat between two homogeneous forces modelled by (11) and (12) with $C = 0$, the X force can be annihilated if and only if $Z^* x_0 < (\sqrt{k_a k_b}/(\mu+1))^{1-2p}\sqrt{k_a/k_b}\, y_0$, where $Z^* = Z^*(\gamma,\mu,\nu)$ is such that (34) holds for (46).

We will next give exact analytic results for cases of no offset (i.e. $A = 0 \Rightarrow \gamma = 0$) and discuss the difficulties of determining Z^* when there is offset (i.e. $A,\gamma > 0$). Moreover, results for the special case of no offset help provide a lower bound for Z^* in the general case.

6. Results for Power Attrition-Rate Coefficients with No Offset.

When the offset parameter $A = 0$, equations (46) become for $\mu,\nu > -1$

$$\begin{aligned} d\bar{e}_X/ds &= -\bar{e}_Y \quad &\text{with } \bar{e}_X(s=0) = 1, \\ d\bar{e}_Y/ds &= -s^\beta \bar{e}_X \quad &\text{with } \bar{e}_Y(s=0) = Z. \end{aligned} \tag{47}$$

Solving (47) by successive approximations, we obtain

$$\bar{e}_X(s;Z) = \sum_{k=0}^{\infty} p^{2k} s^{k(\beta+2)} / \{\prod_{j=1}^{k} j(j-p)\} - Z \sum_{k=0}^{\infty} p^{2k} s^{k(\beta+2)+1} / \{\prod_{j=1}^{k} j(j+p)\}, \tag{48}$$

which is not the most useful result for large s. Without explicitly computing $e_X^-(s;Z)$ it is impossible to determine from (48) how $e_X^-(s;Z)$ behaves for increasing s. However, let us write $e_X^-(s;Z)$ as [observe that e_X^- satisfies the generalized Airy equation (see SWANSON and HEADLEY[19]) $d^2e_X^-/ds^2 - s^\beta e_X^- = 0$, which is well known to be reducible to Bessel's equation]

$$e_X^-(s;Z) = \Gamma(q)p^{-q}\{A_\beta(s) + [1-Zp^{q-p}\Gamma(p)/\Gamma(q)][ps^{1/2}I_p(S)\}, \tag{49}$$

for any arbitrary Z, where $A_\beta(s)$ denotes the generalized Airy function of the first kind of order β (see Swanson and Headley[19]), $I_p(S)$ denotes the modified Bessel function of the first kind of order p, and $S = 2ps^{(\beta+2)/2}$. Also,

$$e_Y^-(s;Z) = \Gamma(q)p^{-q}\{(2p/\pi)(\sin p\pi)\, s^{(\beta+1)/2}K_q(S) - [1-Z_p{}^{q-p}\Gamma(p)/\Gamma(q)]dF/ds(s)\}, \tag{50}$$

where $K_q(S)$ denotes the modified Bessel function of the third kind (also called Macdonald's function) of order q and $F(s) = ps^{1/2}I_p(S)$.

The behavior of $A_\beta(s)$ for $s \geq 0$ is readily seen from (see Swanson and Headley[19]; $A_\beta(s) = (2p/\pi)(\sin p\pi)\, s^{1/2}K_p(S)$. It is readily seen (see p. 119 and p. 123 of LEBEDEV[14] or pp. 250-251 of Olver[17]) that $K_p(S)$ is strictly decreasing and positive, and $\lim_{s\to+\infty} s^\nu K_p(S) = 0$, where ν is any real number. It is then clear (see also p. 1404 of reference 19) that $A_\beta(s)$, $I_p(S) > 0\ \forall\, s \geq 0$, $\lim_{s\to+\infty} A_\beta(s) = 0$, and $\lim_{s\to+\infty} I_p(S) = +\infty$. Consequently, the requirement that (34) holds (i.e. that e_X^- and e_Y^- behave like strictly decaying exponentials) means that the second term in the expressions (49) and (50) must vanish. Thus, for $\gamma = 0$ we have

$$Z^*(\gamma=0,\mu,\nu) = p^{p-q}\Gamma(q)/\Gamma(p). \tag{51}$$

Hence, Theorem 3 becomes (see Note 10)

THEOREM 4: Consider combat between two homogeneous forces described by (11) with attrition-rate coefficients (12) with $A = 0$. Assume that the model applies for all time and that Y "wins" when $x(t_f) = 0$ with $y(t_f) > 0$. Then Y wins if and only if $x_0e_Y^-(s=s_0;Z^*) < \sqrt{k_a/k_b}\, y_0(\sqrt{k_ak_b}/(\mu+1))^{q-p}e_X^-(s=s_0;Z^*)$, where $Z^* = Z^*(\gamma=0,\mu,\nu) = p^{p-q}\Gamma(q)/\Gamma(p)$, $\bar{e}_X(s;Z^*) = p^{-p}\Gamma(q)A_\beta(s)$, and $\bar{e}_Y(s;Z^*) = (2p/\pi)(\sin p\pi)\, p^{-q}\Gamma(q)s^{(\beta+1)/2}K_q(S)$. For $C = 0$, we have $s_0 = 0$ so that Y wins if and only if $x_0\Gamma(q) < \sqrt{k_a/k_b}\; y_0\Gamma(p)(\sqrt{k_ak_b}/(\mu+\nu+2))^{q-p}$.

We observe that the infinite series form (48) is not of any value for determining asymptotic properties of the solution $\bar{e}_X$ to (47) (and consequently Z^*), although it is useful for computational purposes.

7. Offset Linear Attrition-Rate Coefficients.

When the offset parameter $A > 0$, explicit analytic results for Z^* are apparently not possible. Before considering the general case of $\mu,\nu > -1$ and $\gamma > 0$, we will find it instructive to consider the special case of offset linear attrition-rate coefficients (i.e. $\gamma > 0$ with $\mu = \nu = 1$) studied by Bonder and Farrell[4]. This examination will show us why analytic results for Z^* are elusive in cases of positive offset.

For $\gamma > 0$ with $\mu = \nu = 1$, equations (30) and (31) become

$$d^2x/ds^2 - (1+\gamma/\sqrt{s})x = 0, \tag{52}$$

with initial conditions $x(s=0) = x_0$ and $dx/ds(s=0) = -\sqrt{k_a/k_b}\; y_0$, and

$$\begin{aligned} d\bar{e}_X/ds &= -\bar{e}_Y && \text{with } \bar{e}_X(s=0) = 1, \\ d\bar{e}_Y/ds &= -(1+\gamma/\sqrt{s})\bar{e}_X && \text{with } \bar{e}_Y(s=0) = Z. \end{aligned} \tag{53}$$

Let us now consider solving the above modified auxiliary parity-condition problem (i.e. determining $Z = Z^*$ for (53) such that (34) holds). Solving (53) by successive approximations, we may write

$$\bar{e}_X(s) = h(s) - Zw(s), \qquad \text{and} \qquad \bar{e}_Y(s) = ZH(s) - W(s), \tag{54}$$

where $h(s) = h(s;\gamma) = h(\lambda=s,\mu=\gamma/\sqrt{s})$, $w(s)$, $H(s)$, and $W(s)$ denote the auxiliary offset linear Lanchester functions introduced by Taylor and Brown[25]. Infinite series representations of these functions are given in reference 25. Subsequent research has shown that these hyperbolic-like Lanchester functions possess the properties given in

Table I. Unfortunately, information on the asymptotic behavior of the auxiliary offset linear Lanchester functions for large $s > 0$, which is needed to solve the modified auxiliary parity-condition problem (53), is lacking and apparently not obtainable by the standard methods involving integral representation (see Ince[11] or Olver[17]). Consequently, we have not been able to develop an explicit analytic expression for $Z^*(\gamma>0,\mu=1,\nu=1)$, although we give upper and lower bounds for Z^* in the next section for general $\mu,\nu > -1$. Additionally, we should point out that there are computational difficulties in searching for Z^* via its definition (34): (a) one doesn't know how large to take s for "satisfactory" results, and (b) numerical difficulties in evaluating $\bar{e}_X(s;Z)$ and $\bar{e}_Y(s;Z)$ as given by (54) occur for large values of s (since we are taking the difference of two very large numbers and, at least on a digital computer, can retain only a limited number of significant digits in these numbers).

Equation (52) is deceptively simple looking. Using variation of parameters, we may also express its solution as

$$x(s) = x_0 \cosh s - y_0 \sqrt{k_a/k_b} \sinh s + \gamma \int_0^s \sinh(s-\sigma) x(\sigma)/\sqrt{\sigma}\, d\sigma . \tag{55}$$

Although (55) is a simple looking expression, this Volterra integral equation is, unfortunately, no easier to solve than (52) and leads to the same results as given by Taylor[22] and Taylor and Brown[25]. However, in the next section we show an easy way to obtain valuable information about Z^* directly from (53).

8. Bounds on Z^* for Power Attrition-Rate Coefficients With Positive Offset.

We will now develop upper and lower bounds for $Z^*(\gamma>0,\mu,\nu)$. These bounds establish the existence of Z^* (and consequently Q^*) for general power attrition-rate coefficients (12) by the continuous dependence of solutions to (46) on the initial conditions.

The following two lemmas will be used to obtain an upper bound for $Z^*(\gamma,\mu,\nu)$ for $\mu,\nu > -1$.

TABLE I. Properties of the Auxiliary Offset Linear Lanchester Functions

1. $dh/ds = W$, $dW/ds = (1+\gamma/\sqrt{s})h$

2. $dw/ds = H$, $dH/ds = (1+\gamma/\sqrt{s})w$

3. $h(s)H(s) - w(s)\,W(s) = 1 \quad \forall s$

4. $h(s=0) = H(s=0) = 1$

5. $w(s=0) = W(s=0) = 0$

6. $h(s;\gamma=0) = H(s;\gamma=0) = \cosh s$

7 $w(s;\gamma=0) = W(s;\gamma=0) = \sinh s$

<u>Note</u>: We use the notation $h(s) = h(s;\gamma) = h(\lambda=s,\mu=\gamma/\sqrt{s})$ (<u>see</u> Taylor and Brown[25]), and similarly for $w(s)$, $H(s)$, and $W(s)$.

LEMMA 1: For $\delta \geq 1$ and $x,y > 0$, $2^{\delta-1}\{x^\delta + y^\delta\} \geq (x+y)^\delta$.

<u>PROOF</u>: For $\delta \geq 1$, $f(x) = x^\delta$ is a convex function. A well-known theorem for convex functions says that $\{f(x)+f(y)\}/2 \geq f([x+y]/2)$, whence follows the lemma. <u>Q.E.D.</u>

LEMMA 2: For $\delta \leq 1$ and $x,y > 0$, $x^\delta + y^\delta \geq (x+y)^\delta$.

<u>PROOF</u>: Dividing by $(x+y)^\delta$, we need to show that $[x/(x+y)]^\delta + [y/(x+y)]^\delta \geq 1$. If x and $y > 0$, then $x/(x+y)$ and $y/(x+y)$ are < 1. Hence, for any $\delta \leq 1$ we have $[x/(x+y)]^\delta \geq x/(x+y)$ so that $[x/(x+y)]^\delta + [y/(x+y)]^\delta \geq [x/(x+y)] + [y/(x+y)] = 1$. <u>Q.E.D.</u>

Using the above lemmas, we now prove Theorem 5. The upper bound to be given in Theorem 5 might be improved upon, although we feel that for computation determination of Z^* by interval search, it is not essential to have a better bound.

THEOREM 5: For $\mu > -1$ and $\nu \geq 1$, we have $Z^*(\gamma,\mu,\nu) < 1 + 2^{\nu-1}(\mu+1)^2/[(\nu+1)(\mu+\nu+2)] + \gamma^\nu 2^{\nu-1}(\mu+1)^2/(\mu+2)$; while for $\mu > -1$ and $-1 < \nu \leq 1$, we have $Z^*(\gamma,\mu,\nu) < 1 + (\mu+1)^2/[(\nu+1)(\mu+\nu+2)] + \gamma^\nu(\mu+1)^2/(\mu+2)$.

<u>PROOF</u>: Recalling (34), we have from the first equation of (46) that $e_X^-(s;Z^*) < 1$ for $s > 0$; and from the second equation of (46) we then obtain for $s > 0$ and $\nu \geq 1$

$$de_Y^-/ds > -s^\beta(1+\gamma/s^\alpha)^\nu \geq -2^{\nu-1}s^\beta(1+\gamma^\nu/s^{\alpha\nu}), \tag{56}$$

the latter inequality being a consequence of Lemma 1. From (30) we obtain

$$e_Y^-(s) > Z^* - 2^{\nu-1}s^{(\nu+1)/(\mu+1)} \cdot (\mu+1)/(\nu+1) - 2^{\nu-1}\gamma^\nu(\mu+1)s^{1/(\mu+1)}. \tag{57}$$

Using (57) and considering the first equation of (46), we obtain $e_X^-(s;Z^*) < U(s;Z^*)$, where

$$U(s;Z^*) = 1 + 2^{\nu-1}\{(\mu+1)^2/(\mu+2)\}\{s^{(\mu+\nu+2)/(\mu+1)} \cdot (\mu+2)/[(\nu+1)(\mu+\nu+2)] + \gamma^\nu s^{(\mu+2)/(\mu+1)}\} - Z^*s.$$

Since we must have $0 < e_X^-(s;Z^*) < U(s;Z^*)$ for all $s \geq 0$, it follows that for $s = 1$ we must have $U(s=1;Z^*) > 0$, whence follows the theorem for $\nu \geq 1$. Lemma 2 and similar arguments are used to prove the theorem for $-1 < \nu \leq 1$. <u>Q.E.D.</u>

Let us now consider the development of a lower bound for $Z^*(\gamma>0,\mu,\nu)$. Before proving the key lemma (Lemma 3) for the proof of Theorem 6, we discuss some preliminary considerations.

As shown by Taylor and Parry[26], force annihilation for square-law attrition processes sometimes may be predicted by considering the force-ratio equation. For equations (46), the "force ratio" $u = \bar{e}_X/\bar{e}_Y$ satisfies the Riccati equation

$$du/ds = s^{\beta}(1+\gamma/s^{\alpha})^{\nu}u^2 - 1, \tag{58}$$

with initial condition $u(s=0) = 1/Z$. We observe that $u(s_f) = 0$ if X is annihilated (i.e. $\bar{e}_X(s_f) = 0$ but $\bar{e}_Y(s_f) > 0$), and $u(s_f) = +\infty$ if Y is annihilated. For $Z = Z^* = Z^*(\gamma,\mu,\nu)$ such that (34) holds, we have a "draw," and $u^*(s) = u(s;Z=Z^*) > 0$ and finite for all finite $s \geq 0$. Since (58) has a singularity at $s = 0$, we apply the transformation $\tau = (s^{\alpha}+\gamma)^{\mu+1}$ to obtain for $\tau \geq \tau_0 = \gamma^{\mu+1}$

$$du/d\tau = \tau^{\beta}u^2 - (1-\gamma/\tau^{\alpha})^{\mu}, \tag{59}$$

with initial condition $u(\tau=\gamma^{\mu+1}) = 1/Z$.

We now develop a lower bound for $Z^*(\gamma>0,\mu,\nu)$ by comparing results for $\gamma > 0$ with those for $\gamma = 0$. For $\gamma = 0$, we denote u as w and obtain for $\tau \geq 0$

$$dw/d\tau = \tau^{\beta}w^2 - 1. \tag{60}$$

Corresponding to $Z = Z^* = Z^*(\gamma=0,\mu,\nu)$, we have via (49) and (50)

$$w^*(\tau) = \tau^{-\beta/2}K_p(T)/K_q(T) > 0, \tag{61}$$

where $T = 2p\tau^{(\beta+2)/2}$. Since $K_{\nu}(x)$ is finite and > 0 for all $\nu, x > 0$ (<u>see</u> Lebedev[14], p. 136), we readily verify that $w^*(\tau)$ is finite for all $\tau > 0$ and that $w^*(\tau=0) = p^{q-p}\Gamma(p)/\Gamma(q)$. Let us observe that $w(\tau_1) > w^*(\tau_1) = w(\tau_1;Z^*) \Rightarrow$ we have $w(\tau) > w^*(\tau)\ \forall\ \tau \geq 0$ and $w(\tau_2) = +\infty$ for some finite $\tau_2 > \tau_1$, since $D = w - w^*$ satisfies $dD/d\tau = \tau^{\beta}(w+w^*)D$. Consequently, $w(\tau;Z)$ corresponds to $Z < Z^*$ and $w(\tau;Z)$ becomes infinite at some finite time (<u>see</u> equations (49) and (50)). We now state and prove the key lemma for developing a lower bound for Z^*

LEMMA 3: Let $w^*(\tau)$ be given by (61) and let $u(\tau) = u(\tau;Z)$ satisfy (59) for $\tau \geq \tau_0 = \gamma^{\mu+1}$. Then if $u(\tau_1) \geq w^*(\tau_1)$, it follows that $u(\tau) > 0 \ \forall \tau \geq \tau_1$ and $u(\tau_2) = +\infty$ for some finite $\tau_2 > \tau_1$. Consequently, $Z^*(\gamma>0,\mu,\nu) > Z$.

PROOF: Consider $\tilde{D} = u - w$. It satisfies for $\tau \geq \tau_0$ via (59) and (60) the equation $d\tilde{D}/d\tau = \tau^{\beta}(u+w)\tilde{D} + \{1-(1-\gamma/\tau^{\alpha})^{\mu}\}$. If $\tilde{D}(\tau_1) = u(\tau_1) - w(\tau_1) \geq 0$, then $d\tilde{D}/d\tau(\tau_1) > 0$ and $\tilde{D}(\tau) > 0 \ \forall \tau > \tau_1$, Thus, when $u(\tau_1) \geq w^*(\tau_1)$, we can find $w(\tau_2)$ for $\tau_2 \geq \tau_1$ such that $u(\tau_2) > w(\tau_2) > w^*(\tau_2)$, whence follows the lemma from $u(\tau) > w(\tau;\hat{Z})$ $\forall \tau \geq \tau_2$ with $\hat{Z} < Z^*(\gamma=0,\mu,\nu)$ and the above observation that $w(\tau;\hat{Z})$ becomes infinite. Q.E.D.

Letting $\tau_1 = \tau_0$, we obtain

THEOREM 6: $Z^*(\gamma>0,\mu,\nu) > 1/w^*(\gamma^{\mu+1}) = \tau_0^{\beta/2} K_q(T_0)/K_p(T_0)$, where $\tau_0 = \gamma^{\mu+1}$ and $T_0 = 2p\tau_0^{(\beta+2)/2}$.

Since $w^*(\tau=0) \geq w^*(\tau)$ for $\beta \geq 0$, we have as an immediate corollary

COROLLARY 6.1: For $\nu \geq \mu$, $Z^*(\gamma>0,\mu,\nu) > p^{p-q}\Gamma(q)/\Gamma(p)$.

Let us observe that for $\beta > 0$, the lower bound given in Corollary 6.1 is weaker than that in Theorem 6.

9. Future Computational Work.

As we have seen above in Section 4, force-annihilation prediction depends on knowing the parity-condition parameter Q^*, which may be called "the Y equivalent of an X force of unit strength." We have explicitly determined Q^* (via determining Z^* for the modified auxiliary parity-condition problem (46)) for the power attrition-rate coefficients (12) in the case of no offset, i.e. $A = 0$. Tabulations of, for example, the new modified exponential-like general Lanchester functions $\bar{e}_X(s;Z^*)$ and $\bar{e}_Y(s;Z^*)$ would facilitate force-annihilation prediction (see Theorem 4). It remains to determine Q^* for cases of positive offset, i.e. $A > 0$. As discussed in Section 7, analytic results for Z^* in the modified auxiliary parity-condition problem (46) with $\gamma = A \cdot (\sqrt{k_a k_b}/(\mu+1))^{2/(\mu+\nu+2)} > 0$ are apparently not possible by the usual analytical methods so we must turn to numerical methods. It appears that a large number of cases of tactical interest (see Taylor and Brown[25]) would be covered by determining Z^* for $\mu,\nu = 0,1,2,3$ and for a range of values of $\gamma > 0$. One would be interested in, for example, plotting Z^* versus γ for fixed values of μ and ν.

Since we have developed upper and lower bounds for Z^* when $\gamma > 0$ (see Theorems 5 and 6), we can use standard one-dimensional search techniques (see, for example, WILDE[28]) to calculate an approximate value of Z^* with any predetermined degree of accuracy (of course, depending on how much computation we wish to do). Since (34) must hold for all $s \geq 0$, we must determine how long (i.e. for how large a value of s) to carry out computations of $\bar{e}_X(s;\hat{Z})$ and $\bar{e}_Y(s;\hat{Z})$ in the modified auxiliary parity-condition problem (46) for a given trial value of Z^* (denoted as $\hat{Z}$) to see whether it is too large or too small. Although $\bar{e}_X < 0$ or $\bar{e}_Y < 0$ clearly terminates calculations for a given $\hat{Z}$, it would be desirable to be able to terminate computations before this occurs, especially when $\hat{Z}$ is near Z^*. In the future we will show that by considering the Ricatti equation (59) one can "cut off" conputations for a given value of $\hat{Z}$ well before either of the two annihilation conditions is actually reached. This acceleration in the termination of trial computations is particularly useful when force annihilation occurs for large s (i.e. for $\hat{Z}$ near Z^*). Additionally, the qualitative behavior of solutions to (46) (or (11), for that matter) is probably best understood by considering the Ricatti equation (59) (or, equivalently, (58)).

As discussed at the end of Section 4, prediction of force annihilation within a given finite time involves the use of tabulations of the quotient of two linearly-independent general Lanchester functions. We have indicated in Section 4 that the hyperbolic-tangent-like Lanchester functions (e.g. $T_X(t)$ as defined by (42)) are to be preferred for reasons of the accuracy of their numerical computation. Thus, there is a need for tabulations of $T_X(t)$, whose range is $[0,1/Q^*)$ for $t \varepsilon [0,+\infty)$. For the power attrition-rate coefficients (12) with no offset, the power Lanchester functions (also called Lanchester-Clifford-Schläfli (or LCS) functions), however, were inappropriately defined in Taylor and Brown[25] to yield such tabulations. Thus, our newer theory of force-annihilation prediction, which also involves tabulations of canonical solutions (i.e. canonical Lanchester functions) to variable-coefficient Lanchester-type equations of modern warfare, has suggested some refinements in the definition of Taylor and Brown's[25] auxiliary power Lanchester functions (see Note 11). It would be desirable then to redefine the LCS functions to fit within the framework of Section 4 and to develop tabulations of their quotient. If this were to be done, linear combat models with power attrition-rate coefficients (no offset) could be analyzed with somewhat the same ease as constant-coefficient linear models (i.e. (1)).

10. Extension to a More General Model.

In this section we show that all our above force-annihilation results (except those for force-annihilation within a fixed finite time) also apply to a special case of a more general model. Moreover, comparison techniques may be used to extend these results in weakened form to the general case of this more comprehensive model.

Let us consider the following Lanchester-type equations

$$\begin{aligned} dx/dt &= -a(t)y - \beta(t)x \qquad \text{with } x(t=0) = x_0, \\ dy/dt &= -b(t)x - \alpha(t)y \qquad \text{with } y(t=0) = y_0, \end{aligned} \tag{62}$$

where $a(t)$, $b(t)$, $\alpha(t)$, and $\beta(t) > 0$. We may think of these equations as modelling, for example, aimed-fire combat between two homogeneous infantry forces with superimposed effects of supporting weapons (which are not subject to attrition and deliver area fire against the enemy infantry) (see Taylor and Parry[26] for a further discussion of this model (62)). In this case, $\alpha(t)$ and $\beta(t)$ are attrition-rate coefficients which reflect the fire effectiveness of the supporting weapons[26]. Then, the force ratio $u = x/y$ satisfies the generalized Riccati equation

$$du/dt = b(t)u^2 + \{\alpha(t) - \beta(t)\}u - a(t) \quad \text{with} \quad u(t=0) = x_0/y_0. \tag{63}$$

For equal effectivenesses of the supporting fires [i.e. $\alpha(t) = \beta(t)$], equation (63) simplifies to

$$du/dt = b(t)u^2 - a(t), \tag{64}$$

which is the same Riccati equation satisifed by the force ratio for the model (11). Hence, when $\alpha(t) = \beta(t) \ \forall \ t \geq 0$, a battle's outcome (in terms of the force ratio) is the same for the two models (11) and (62), although the battle ends more quickly for (62). Thus, in this special case all our above results on force annihilation without time limitation (e.g. Theorems 2 through 4 and their corollaries) developed for (11) [in general or with the coefficients (12)] also apply to the more general model (62). Furthermore, comparison techniques (see, for example, Hille[10]) may be used to extend these results in weakened form to (62). Consequently, we see that the force-annihilation results developed in this paper are indeed of a fundamental nature.

11. Summary.

We have presented a mathematical theory for predicting force annihilation for variable-coefficient Lanchester-type equations of "modern warfare" for combat between two homogeneous forces without explicitly computing force-level trajectories (see Note 12). This theory both generalizes and complements Taylor and Brown's[25] theory of canonical solution forms. Our force-annihilation theory provides guidance for certain parameter determinations and development of tabulations of Lanchester functions (beyond those suggested in reference 25) that would allow one to parametrically analyze variable-coefficient models with somewhat the same facility as constant-coefficient ones.

We have shown that force annihilation can be predicted from initial conditions, without explicitly computing force-level trajectories, by knowing a parity-condition parameter Q^*, which is the solution to a canonical auxiliary parity-condition problem. In general, this prediction would be facilitated by having tabulations of certain

Lanchester functions available. In particular, to apply Theorem 3 it would be convenient to have tabulations of the new exponential-like general Lanchester functions $\bar{E}_X$ and $\bar{E}_Y$. However, if one wishes to invoke the equivalent Theorem 3', then tabulations of the hyperbolic-like general Lanchester functions C_X, S_X, C_Y, and S_Y would be desirable to have. The parity-condition parameter Q^* was shown to be related to the range of the quotient of two hyperbolic-like general Lanchester functions introduced by Taylor and Brown[25]. Consequently, our force-annihilation theory not only provides new information about the mathematical properties of hyperbolic-like Lanchester functions but also provides guidance for selecting canonical Lanchester functions.

We applied our general theory to the specific case of general power attrition-rate coefficients. Considering a modified auxiliary parity-condition problem, we explicitly determined Q^* (via Z^* of the modified problem) for power attrition-rate coefficients with no offset and gave upper and lower bounds for Z^* for cases of positive offset. Consequently, in the future one-dimensional search techniques may be used to numerically determine approximate values of Z^* as a function of the offset parameter A. We finally showed that certain of our force-annihilation results also applied to a more general linear differential equation combat model.

These results may be used in the analysis of the dynamic combat interactions between two homogeneous forces with time- (or range-) dependent weapon system capabilities. There is interest today in such analytic models because of improved operations research techniques for predicting Lanchester attrition-rate coefficients, in particular their temporal variations (see references 2 through 5 and 25). Further discussion of such applications may be found in Bonder and Farrell[4], Bonder and Honig[5], Taylor[22], and Taylor and Brown[25].

NOTES

1. Bonder and Honig[5] point out, however, that force annihilation may not be the appropriate criterion for evaluating many military operations, expecially when force annihilation does not occur. See pp. 192-242 of Bonder and Farrell[4] for a detailed Lanchester-type analysis of an attack scenario for which other "end of battle conditions" play the major role in the evaluation process. Nevertheless, it is of interest (especially for developing insights into the dynamics of combat) to be able to easily predict (without computing force-level trajectories) the occurrence of force annihilation. Such results are not only of intrinsic interest but also are useful in the optimization of combat dynamics, i.e. the combining of Lanchester-type models with generalized control theory (i.e. optimal control/differential game theory)(see Taylor[21],[23]).

2. Previous work by Bonder and Farrell[4], Taylor[22], and Taylor and Brown[25] shows that new transcendental functions arise even in the case of linear attrition-rate coefficients reflecting weapon systems with different effective ranges (i.e. the coefficients (12) with $\mu = \nu = 1$ and $A > 0$). For example, the differential equation (52) could not be found among the 445 linear second order equations tabulated in Kamke[12]. Moreover, even when one can express a solution in terms of previously known transcendents, the appropriate tabulations (see, for example, ABRAMOWITZ and STEGUN[1]) may not exist (see Section 5 of reference 25). As the equations of mathematical physics have provided interest in many previously studied transcendents, so does the Lanchester theory of combat with time-varying fire effectivenesses provide interest in new transcendents.

3. The influential 19th-century German military philosopher, Carl von Clausewitz (1780-1831), stated in his classic work On War (Vom Kriege) (see p. 276 of reference 7), "The best Strategy is always to be very strong, first generally then at the decisive point. ...There is no more imperative and no simpler law for Strategy than to keep the forces concentrated."

4. As pointed out in reference 26, the entire topic of modelling battle termination is a problem area in contemporary defense planning studies, and there is far from universal agreement as to even which variables should be taken as the significant variables for modelling this complex process. For further references see Taylor[23].

5. Bonder and Farrell[4] take range (i.e. force separation) to be the independent variable in their work, while Taylor[22] and Taylor and Brown[25] take time as we have done in this paper.

6. Recalling the constant coefficient result (6), we consider

$$d\{\exp(\sqrt{ab}\ t)\}/dt = a\sqrt{b/a}\ \exp(\sqrt{ab}\ t) = b\sqrt{a/b}\ \exp(\sqrt{ab}\ t),$$

and

$$d\{\exp(-\sqrt{ab}\ t)\}/dt = -a\sqrt{b/a}\ \exp(-\sqrt{ab}\ t) = -b\sqrt{a/b}\ \exp(-\sqrt{ab}\ t),$$

to obtain motivation for (15) and (16).

7. We take $a(t)$ and $b(t)$ to be analytic in the (finite) complex plane except for a finite number of singularities on the real axis. The singularities of (17) then occur at the zeros and singularities of $a(t)$ and at the singularities of $a(t)b(t)$. Consequently, t_0 belongs to the set of points consisting of the zeros and singularities of $a(t)$ and $b(t)$ (see Taylor and Brown[25]). We define t_0 this way in order to reduce the number of tabulations of exponential general Lanchester functions required for force-annihilation analyses (see Theorem 3). For example, for the general power attrition-rate coefficients (12) we have $t_0 = -C$, and for fixed $A \geq 0$, μ, and ν only a single tabulation of $\bar{e}_X$ and $\bar{e}_Y$ is required to handle all problems with $C \geq 0$ (see, for example, Theorem 4).

8. It is well known (see, for example, pp. 647-650 of Hille[10] or p. 120 of Kamke[12]) that the quotient of two linearly independent solutions to (31), which is equivalent to (17), satisfies Schwarz's (third order) differential equation (see SCHWARZ[18])

$$\{\eta,s\} = -2I(s),$$

where η denotes the quotient of two linearly independent solutions to (31) [i.e. $\eta = e_X^+(s)/e_X^-(s)$], $\{\eta,s\} = \eta''' - (3/2)(\eta'')^2/\eta'$ denotes the Schwarzian derivative of η with respect to s, η' denotes $d\eta/ds$, etc., and $I(s)$ denotes the invariant of the normal form (31). For numerical computation of η, however, there is no advantage to consider this third order equation, and it is preferable to calculate η from, for example, $\eta(s) = e_X^+(s)/e_X^-(s)$, where e_X^+ and e_X^- also satisfy (29) and (30).

9. Recalling our development of (7), we see that, except for the special case in which (23) holds, this same approach fails to yield the time for X to reach his breakpoint (assumed to be positive) [i.e. t_X^{BP} such that $x(t=t_X^{BP}) = x_{BP} > 0$]. Consequently, it is apparently impossible to predict in the manner described in the main text the outcome of a fixed force-level breakpoint battle with positive breakpoints unless (23) holds.

10. For the case of power attrition-rate coefficients with no offset (i.e. $A = 0$ in (12)), the second annihilation condition given in Theorem 4 (i.e. the one for $C = 0$) and an equivalent form of the first (i.e. the one for $C \geq 0$) may be developed by inspection when one expresses, for example, the time history of the X force level (which satisfies (17)) in terms of the so-called generalized Airy functions (see Swanson and Headley[19]). For example, for $C = 0$ we have

$$x(t) = (p^p/(2p))\{X+Y\}A_\beta(T) + (p^p/(2\sqrt{p}))\{X-Y\}B_\beta(T),$$

where A_β and B_β denote the generalized Airy functions of the first and second kinds of order β, $X = x_0\Gamma(q)$, $Y = y_0\sqrt{k_a/k_b}\,\Gamma(p)(\sqrt{k_ak_b}/(\mu+\nu+2))^{1-2p}$, and $T = (\sqrt{k_ak_b}/(\mu+1))^{2p}t^{\mu+1}$. The result given in Theorem 4 for $C = 0$ follows from the properties of the generalized Airy functions (i.e. $A_\nu(\xi), B_\nu(\xi) > 0\ \forall\ \xi \geq 0$, $\lim_{\xi\to+\infty} A_\nu(\xi) = 0$, and $\lim_{\xi\to+\infty} B_\nu(\xi) = +\infty$) and the above representation for $x(t)$. Unfortunately, this result does not generalize to other cases of interest, although it did motivate our general theory of force annihilation developed in this paper. The generalized Airy functions may be considered to be generalizations of the exponential function (see p. 446 of reference 1 or p. 393 of reference 17 for plots of the standard (i.e. $\beta = 1$) Airy functions) and arise in the study of the asymptotic behavior of solutions to certain differential equations.

11. Although theoretically results for power attrition-rate coefficients with no offset are expressible in terms of "known" transcendental functions, new Lanchester functions were introduced by Taylor and Brown[25] because of lack of tabulations of these in many cases of interest.

12. In his well-known survey paper on the Lanchester theory of combat, Dolansky[9] suggested the development of outcome-predicting relations without solving in detail as one of several problems for future research.

ACKNOWLEDGMENT

This research was supported by the Office of Naval Research as part of the Foundation Research Program at the Naval Postgraduate School.

REFERENCES

[1] M. Abramowitz and I. Stegun (Editors), Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, No. 55, Washington, D. C., 1964.

[2] S. Bonder, "The Lanchester Attrition-Rate Coefficient," Opns. Res. 15, 221-232 (1967).

[3] S. Bonder, "The Mean Lanchester Attrition Rate," Opns. Res. 18, 179-181 (1970).

[4] S. Bonder and R. Farrell (Editors), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2 (U), Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, Sept. 1970.

[5] S. Bonder and J. Honig, "An Analytic Model of Ground Combat: Design and Application," Proceedings U. S. Army Operations Research Symposium 10, 319-394 (1971).

[6] H. Brackney, "The Dynamics of Military Combat," Opns. Res. 7, 30-44 (1959).

[7] C. von Clausewitz, On War, edited with an introduction by A. Rapoport, Penguin Books, Ltd., Harmondsworth, Middlesex, England, 1968.

[8] E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[9] L. Dolansky, "Present State of the Lanchester Theory of Combat," Opns. Res. 12, 344-358 (1964).

[10] E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Massachusetts, 1969.

[11] E. Ince, Ordinary Differential Equations, Longmans, Green and Co., London, 1927 (reprinted by Dover Publications, Inc., New York, 1956).

[12] E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Band 1, Gewöhnliche Differentialgleichungen, 3. Auflage, Akademische Verlagsgesellschaft, Leipsig, 1944 (reprinted by Chelsea Publishing Co., New York, 1971).

[13] F. W. Lanchester, "Aircraft in Warfare: The Dawn of the Fourth Arm - No. V., The Principle of Concentration," Engineering 98, 422-423 (1914) (reprinted on pp. 2138-2148 of The World of Mathematics, Vol. IV, J. Newman (Editor), Simon and Schuster, New York, 1956).

[14] N. N. Lebedev, Special Functions and Their Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1965 (reprinted by Dover Publications, Inc., New York, 1972).

[15] E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley, New York, 1967.

[16] J. McCloskey, "Of Horseless Carriages, Flying Machines, and Operations Research: A Tribute to Frederick William Lanchester (1868-1946)," Opns. Res. 4, 141-147 (1956)

[17] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.

[18] H. A. Schwarz, "Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt," J. reine u. angew. Math. 75, 292-335 (1872) (also pp. 211-259 in Gesammelte Mathematische Abhandlungen, Zweiter Band, J. Springer, Berlin, 1890 (reprinted by Chelsea Publishing Co., New York, 1972)).

[19] C. Swanson and V. Headley, "An Extension of Airy's Equation," SIAM J. Appl. Math. 15, 1400-1412 (1967).

[20] J. Taylor, "A Note on the Solution to Lanchester-Type Equations with Variable Coefficients," Opns. Res. 19, 709-712 (1971).

[21] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[22] J. Taylor, "Solving Lanchester-Type Equations for 'Modern Warfare' with Variable Coefficients," Opns. Res. 22, 756-770 (1974).

[23] J. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Processes," presented at the Symposium on the State-of-the-Art of Mathematics in Combat Models, June 1973 (also Tech. Report NPS55Tw74031, Naval Postgraduate School, Monterey, Calif., March 1974).

[24] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21, 683-704 (1974).

[25] J. Taylor and G. Brown, "Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Opns. Res. 24, 44-69 (1976).

[26] J. Taylor and S. Parry, "Force-Ratio Considerations for Some Lanchester-Type Models of Warfare," Opns. Res. 23, 522-533 (1975).

[27] H. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conference on Operational Research, John Wiley, New York, 1957.

[28] D. Wilde, Optimum Seeking Methods, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

INITIAL DISTRIBUTION LIST

	No. of Copies
Advanced Research Projects Agency Department of Defense Washington, DC 20301 (Technical Library)	1
Chairman, Joint Chiefs of Staff Studies Analysis and Gaming Agency Washington, DC 20350	1
Defense Documentation Center Cameron Station Alexandria, VA 22314	12
Chief of Naval Operations Department of the Navy Washington, DC 20350	2
Headquarters, Marine Corps (AX) Washington, DC 20380	1
Office of Naval Research Department of the Navy Arlington, VA 22217	
(Code 431)	2
(Code 432)	1
(Mr. Alvin F. Andrus, Code 230)	1
(Dr. Toke Jayachandran, Code 431)	1
(Mr. James F. Smith, Code 431)	1
Naval War College Newport, Rhode Island 02840 (Technical Library)	1
Naval Ordnance Laboratory Silver Spring, MD 20910 (Technical Library)	1
Naval Research Laboratory Washington, DC 20390 (Code 2029)	1
Director, Army Research Office of the Chief for Research and Development Department of the Army Washington, DC 20310	1

	No. of Copies
Dr. Wilbur B. Payne, Director U. S. Army TRADOC Systems Analysis Activity White Sands Missile Range, NM 88002	2
Mr. David C. Hardison Deputy Under Secretary of the Army (Operations Research) 2E621, The Pentagon Washington, DC 20310	1
Center for Naval Analyses 1400 Wilson Boulevard Arlington, VA 22202 (Technical Library)	1
Institute for Defense Analyses 400 Army-Navy Drive Arlington, VA 22202	1
Professor Robert M. Thrall Department of Mathematical Sciences Rice University Houston, TX 77701	1
Professor Joseph Engel Systems Engineering Department Box 4348 College of Engineering University of Illinois at Chicago Circle Chicago, IL 60680	1
Mr. H. K. Weiss P. O. Box 2668 Palos Verdes Peninsula Palos Verdes, CA 90274	1
Mr. K. C. Bowen Head of Research Defense Operational Analysis Establishment Parvis Road West Byfleet, Surrey KT 14 6LY ENGLAND	2
Library (Code 55) Department of Operations Research Naval Postgraduate School Monterey, CA 93940	2
Library (Code 0212) Naval Postgraduate School Monterey, CA 93940	2

	No. of Copies
Dean of Research Code 023 Naval Postgraduate School Monterey, CA 93940	2
Professor James G. Taylor, Code 55Tw	10
Professor Craig Comstock, Code 53Zk	10
Professor Samuel H. Parry, Code 55Py	1
Professor Michael G. Sovereign, Code 55Zo	1
Professor Gerald G. Brown, Code 55 Bw Department of Operations Research Naval Postgraduate School Monterey, CA. 93940	1
CDR Karl H. Eulenstein Commander-in-Chief Pacific Code J021, Box 13 FPO, San Francisco 96610	1
Dr. Alexander J. Slafoksky Scientific Advisor DC/S for RD&S U. S. Marine Corps Headquarters (AX) Washington, DC 20380	1
Marine Corps Development Center Marine Corps Development and Education Command Quantico, VA 22134	
(COL G. H. Polakoff)	1
(LTC K. P. Harrison)	1