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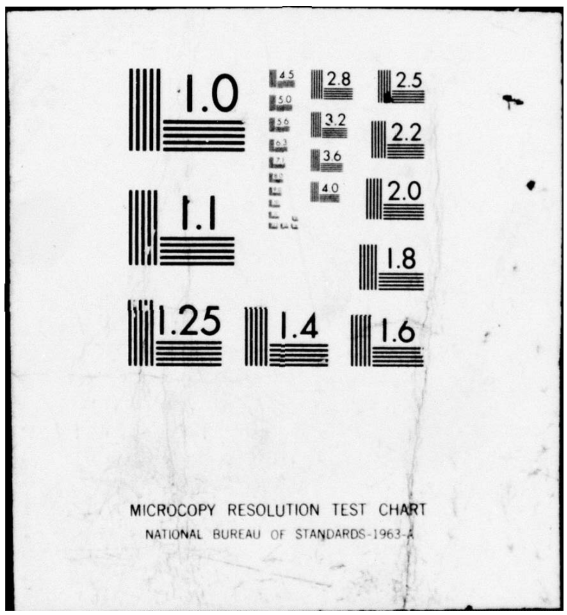
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UNIMODULAR AND TOTALLY UNIMODULAR MATRICES

Research Report 76-21

by

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August, 1976

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The constraint set $(x \mid Ax = b, x \geq 0)$ has all integer extreme points for any integral b iff every basis of A is unimodular. This condition is of obvious importance for integer linear programs, but it is not easily determined. A useful means of testing for unimodularity of basis is implicit in the sample result presented here.

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Abstract

The constraint set $\{x \mid Ax = b, x \geq 0\}$ has all integer extreme points for any integral b iff every basis of A is unimodular. This condition is of obvious importance for integer linear programs, but it is not easily determined. A useful means of testing for unimodularity of bases is implicit in the simple result presented here.

A square matrix* M is <u>unimodular</u> iff $|\det M| = 1$. An $n \times m$ matrix A is <u>totally unimodular</u> iff every non-singular submatrix of A is unimodular. The importance of these concepts to integer programming is established by the following fundamental characterizations (Hoffman and Kruskal [4], Dantzig and Veinott [1]):

Property 1: every basis of A is unimodular iff all extreme points of $\{x \mid Ax = b, x \geq 0\}$ are integral for any integral b.

Property 2: A is totally unimodular iff all extreme points of $\{x \mid Ax \leq b, x \geq 0\}$ are integral for any integral b.

Clearly every totally unimodular matrix has all bases unimodular. However, a matrix all of whose bases are unimodular is not necessarily totally unimodular (e.g. Garfinkel and Nemhauser [2]).

A standard way of approaching an integer linear program is to first determine whether the solution set has integer extreme points. If all extreme points are integer, then at least the problem can be solved using the simplex method of linear programming. From the preceeding results, determining integer extreme points is tantamount to establishing for the constraint matrix either unimodularity of bases or total unimodularity, as appropriate to the constraints. Total unimodularity is the more easily determined property since several very general sufficient conditions are known (e.g. Iri [5], Hoffman and Kruskal [4]). Less readily applicable, but still potentially helpful are the several algebraic and graph-theoretical characterizations of total unimodularity (e.g. Padberg [6]). Unimodularity of bases, however, is a more general condition as well as a more useful property since any

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*Lower case letters represent vectors and upper case letters represent matrices. All vectors and matrices in this discussion have integer entries.

system of inequalities may be enlarged to a system of equalities by the addition of slack and surplus variables. But at the same time, unimodularity of bases is difficult to test for since every basis must be examined. The following observations help to determine integer extreme points by relating unimodularity of bases to total unimodularity.

<u>Theorem</u>: Let the nxm ($n < m$) matrix $A = [B, N]$ be of full rank and let B be a unimodular basis of A. Then all bases of A are unimodular if and only if $B^{-1}N$ is totally unimodular.

<u>Proof</u>: From Property 1, all bases of A are unimodular iff $\{(x_B, x_N) \mid Bx_B + Nx_N = b, (x_B, x_N) \geq 0\}$ has integer extreme points for all integer b, or equivalently $\{(x_B, x_N) \mid Ix_B + B^{-1}Nx_N = B^{-1}b, (x_B, x_N) \geq 0\}$ has integer extreme points for all integer b. Since B is unimodular, $B^{-1}b$ is integer for all integer b. Also, any ($m \times 1$) integer vector $\bar{b}$ can be expressed as $\bar{b} = B^{-1}b$ for some integer b. Therefore $\{(x_B, x_N) \mid Ix_B + B^{-1}Nx_N = \bar{b}, (x_B, x_N) \geq 0\}$ has integer extreme points for all integer $\bar{b}$ iff A has all unimodular bases. Now by the correspondence of extreme points, $\{x_N \mid B^{-1}Nx_N \leq \bar{b}, x_N \geq 0\}$ has integer extreme points iff A has all unimodular bases. Hence from Property 2, $B^{-1}N$ is totally unimodular iff A has all unimodular bases.

Q.E.D.

Thus to establish that A has all unimodular bases, it is sufficient to show that some basis B is unimodular and the $n\mathrm{x}(m - n)$ matrix $B^{-1}N$ is totally unimodular. Note also that any problem with a constraint matrix having all unimodular bases may be transformed to a problem with a totally unimodular constraint matrix.

Using a similar proof technique, one may easily derive the following results that are also of use in determining whether a problem has integer extreme points: (i) a square nonsingular matrix B is totally unimodular iff B^{-1} is totally unimodular. (ii) a matrix A is totally unimodular iff every basis of A is totally unimodular.

These results may also be deduced from the work of Heller [3] who defined and studied unimodular sets of vectors.

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