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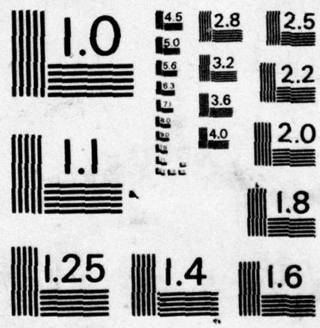

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BOUNDS FOR TRUNCATION ERROR IN SAMPLING EXPANSIONS OF FINITE ENERGY BAND-LIMITED SIGNALS.

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Subject: Bounds for Truncation Error in Sampling Expansions of Finite Energy Band-Limited Signals

Abstract:

-pi(r) < OR=omega < OR= pi(r)

An upper bound is established for the magnitude of the truncation error incurred when a real-valued, finite energy signal which is band-limited to $-\pi r \leq \omega \leq \pi r$ $(o < r < 1)$ is approximated by 2N+1 terms from its Shannon sampling series expansion. The sampling expansion is associated with the band $[-\pi, \pi]$, and consequently involves samples taken at the integer points. The bound obtained is of the form

(-pi, pi)

$$\frac{K(r,t)}{N} \sqrt{E}$$

K(r,t)/N (SQUARE ROOT OF E)

where E is the signal energy. This bound is of the same asymptotic form as the bounds derived by Yao and Thomas [1] and Brown [2]. The bound derived here is tighter than the Yao-Thomas bound for values of r near unity, and is tighter than the bound obtained by Brown for all value of r.

The attached paper is being submitted for publication in the IEEE Transactions on Information Theory.

I. Introduction

It is well known that a band-limited function having finite energy can be represented exactly for all times by an infinite series involving samples of the function. Several authors [1], [2] have obtained bounds on the truncation error that occurs when the function is approximated by a finite number of terms from the sampling theorem expansion, rather than the infinite series. In this paper another bound is derived which represents an improvement over the previous bounds.

In the analysis that follows, we will consider real-valued, band-limited functions with finite energy. Without loss of generality, we can assume that the function is band-limited to $(-\pi,\pi)$. That is, the function has a representation of the form.

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{i\omega t} d\omega \qquad (-\infty < t < \infty) \tag{1}$$

where $F(\omega)$ is a complex-valued function having the property that

$$\int_{-\pi}^{\pi} |F(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} f^2(t) dt = 2\pi E < \infty. \tag{2}$$

By the Shannon sampling theorem, $f(t)$ also has the representation

$$f(t) = \sum_{-\infty}^{\infty} f(n) \frac{\sin\pi(t-n)}{\pi(t-n)} \qquad (-\infty<t<\infty). \tag{3}$$

The problem that will concern us here is to estimate the truncation error incurred when f(t) is represented by a finite number of terms, rather than the infinite series given by (3). We will define the truncation error by

$$e_N^f(t) \equiv f(t) - \sum_{-N}^{N} f(n) \frac{\sin\pi(t-n)}{\pi(t-n)}. \quad (-N \leq t \leq N) \tag{4}$$

It is known [3] that for this class of functions, the magnitude of the truncation error is bounded by

$$\left| e_N^f(t) \right| \leq K(t) \left(\frac{E}{N}\right)^{1/2}.$$

Thus, the truncation error goes to zero asymptotically like $N^{-1/2}$ as $N \to \infty$.

Yao and Thomas [1] examined the bounds on the truncation error when the additional assumption was made that $F(\omega)$ vanishes in $(-\pi, -\pi r)$ and $(\pi r, \pi)$, where $o < r < 1$. That is, they considered the case where

$$f(t) = \frac{1}{2\pi} \int_{-\pi r}^{\pi r} F(\omega) e^{i\omega t} \, d\omega, \qquad (o < r < 1) \tag{5}$$

and the truncation error is defined by (4), involving samples taken at the integer points. Hence, the signal is sampled faster than required for reconstruction by the sampling theorem. Using a contour integral to represent the truncation error and then applying known results concerning the growth of the entire function f(z) in the complex z-plane, Yao and Thomas obtained a bound of the form

$$\left| e_N^f(t) \right| \le \frac{K(r,t)\sqrt{E}}{N} .$$

Thus, by introducing the guardband (or equivalently, by sampling in excess of the Shannon rate), Yao and Thomas obtained a bound which behaves asymptotically like N^{-1} for large N.

Brown [2] also examined this class of functions, and using only results from real variable theory, obtained a bound with the same asymptotic behavior as the Yao-Thomas bound. The bound obtained by Brown is an improvement on the Yao-Thomas bound for values of r near unity; that is, for samples taken nearly at the Shannon rate.

In this report, a bound is obtained which is an improvement on the bound obtained by Brown for all values of r, the guard-band coefficient. Like the analysis used by Brown, only real variable results are used, and no application is made of the rather deep results concerning the theory of entire functions.

II Analysis

Consider a real-valued function, with a representation given by (1). Defining the truncation error by (4) and using the representation for f(t) given in (3), we have

$$e_N^f(t) = \sum_{-\infty}^{\infty} f(n)\,\frac{\sin\pi(t-n)}{\pi(t-n)} - \sum_{-N}^{N} f(n)\,\frac{\sin\pi(t-n)}{\pi(t-n)}$$

$$= \sum_{|n|>N} f(n)\,\frac{\sin\pi(t-n)}{\pi(t-n)} . \qquad (6)$$

Note that for $t=k$, where k is an integer,

$$e_N^f(t) = \sum_{|n|>N} f(n) \frac{\sin\pi(k-n)}{\pi(k-n)}$$

$$= \begin{cases} o, & |k| \leq N \\ f(k), & |k| > N. \end{cases}$$

In what follows, we will assume $|t| < N$, and that t is not equal to an integer. From (6) we have that

$$e_N^f(t) = -\frac{\sin\pi t}{\pi} \sum_{|n|>N} \frac{(-1)^n f(n)}{n-t}$$

Thus,

$$\left|e_N^f(t)\right| \leq \frac{|\sin\pi t|}{\pi} \sum_{|n|>N} \frac{|f(n)|}{|n-t|}$$

Using the Schwarz inequality,

$$\left|e_N^f(t)\right| \leq \frac{|\sin\pi t|}{\pi} \left[\sum_{|n|>N} f^2(n)\right]^{1/2} \left[\sum_{|n|>N} \frac{1}{(n-t)^2}\right]^{1/2}$$

$$\leq \frac{|\sin\pi t|}{\pi} \left[\sum_{-\infty}^{\infty} f^2(n)\right]^{1/2} \left[\sum_{|n|>N} \frac{1}{(n-t)^2}\right]^{1/2} . \tag{7}$$

But $\sum_{-\infty}^{\infty} f^2(n) = \int_{-\infty}^{\infty} f^2(t)dt = E$, from [3] , and (8)

$$\sum_{|n|>N} \frac{1}{(n-t)^2} = \sum_{N+1}^{\infty} \frac{1}{(n+t)^2} + \sum_{N+1}^{\infty} \frac{1}{(n-t)^2}$$

$$\leq \int_N^{\infty} \frac{1}{(x+t)^2}\, dx + \int_N^{\infty} \frac{1}{(x-t)^2}\, dx$$

$$= \frac{2N}{N^2-t^2} . \tag{9}$$

Substituting (8) and (9) into (7), we have

$$\left|e_N^f(t)\right| \leq \frac{|\sin\pi t|}{\pi}\left[\frac{2NE}{N^2-t^2}\right]^{1/2} . \tag{10}$$

This result, which is a slight generalization of a result derived earlier by Balahrishman [3], provides a bound for the magnitude of the truncation error, in terms of the energy of the band-limited signal, which goes to zero asymptotically like $N^{-1/2}$ as N goes to infinity.

We now make the additional assumption that f(t) is band-limited to $(-\pi r, \pi r)$ with $o<r<1$. Then, f(t) has the representation given by (5). Defining the truncation error by (4) and using the representation for f(t) given by (5), we find

$$e_N^f(t) = \frac{1}{2\pi}\int_{-\pi r}^{\pi r} F(\omega)\left[e^{i\omega t} - \sum_{-N}^{N} e^{i\omega n}\frac{\sin\pi(t-n)}{\pi(t-n)}\right]dw \tag{11}$$

Expanding $e^{i\omega t}$ in a Fourier series on $(-\pi,\pi)$, we obtain

$$e^{i\omega t} = \sum_{-oo}^{oo} e^{i\omega n}\frac{\sin\pi(t-n)}{\pi(t-n)} . \quad (-\pi < \omega < \pi)$$

Using the Dirichlet integral [4],

$$T_N(\omega) = \sum_{-N}^{N} e^{i\omega n}\frac{\sin\pi(t-n)}{\pi(t-n)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2})(u-\omega)}{\sin\frac{(u-\omega)}{2}}e^{iut}\,du. \tag{12}$$

But

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2})(u-\omega)}{\sin\left(\frac{u-\omega}{2}\right)}\,e^{i\omega t}\,du = \frac{e^{i\omega t}}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2})(u-\omega)}{\sin\left(\frac{u-\omega}{2}\right)}\,du$$
$$= e^{i\omega t} \qquad (13)$$

Substituting (12) and (13) into (11) we obtain

$$e_N^f(t) = \frac{1}{2\pi}\int_{-\pi r}^{\pi r} F(\omega)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2})(u-\omega)}{\sin\left(\frac{u-\omega}{2}\right)}\left[e^{i\omega t} - e^{iut}\right]du\right\}dw \qquad (14)$$

We will now restrict our attention to t in the range $[-N, N]$. As noted earlier, the truncation error is zero for t equal to an integer and thus, without loss of generality, we can assume that

$$t = m + b \qquad (15)$$

where m is an integer such that

$$-N \leq m \leq N \qquad (16)$$

and

$$o < |b| \leq \frac{1}{2} . \qquad (17)$$

Then,

$$e^{i\omega t} - e^{iut} = e^{i(m+b)\omega} - e^{i(m+b)u}$$
$$= e^{i(m+b)\omega}\left\{1 - e^{im(u-\omega)}\left[e^{\frac{ib(u-\omega)}{2}}\right]^2\right\} .$$

But

$$\left[e^{\frac{ib(u-\omega)}{2}}\right]^2 = \left[\cos\frac{b(u-\omega)}{2} + i\sin\frac{b(u-\omega)}{2}\right]^2$$
$$= 1-2\sin^2\frac{b(u-\omega)}{2} + 2i\sin\frac{b(u-\omega)}{2}\cos\frac{b(u-\omega)}{2} .$$

Thus,

$$e^{i\omega t}-e^{iut}=e^{i(m+b)\omega}\left\{1-e^{im(u-\omega)}+2e^{im(u-\omega)}\left[\sin^2\frac{b(u-\omega)}{2}\right.\right.$$

$$\left.\left.-i\sin\frac{b(u-\omega)}{2}\cos\frac{b(u-\omega)}{2}\right]\right\} \tag{18}$$

Substituting (18) into (14) and making the change of variable $y = \frac{u-\omega}{2}$ in the integration over u, we obtain

$$e_N^f(t) = \frac{1}{\pi^2}\int_{-\pi r}^{\pi r} F(\omega)\, e^{i(m+b)\omega}\left[I_1 + I_2 - i\, I_3\right] d\omega, \tag{19}$$

where

$$I_1 = \frac{1}{2}\int_{-\frac{(\pi+\omega)}{2}}^{\frac{\pi-\omega}{2}} \frac{\sin(2N+1)y}{\sin y}\left(1 - e^{i2my}\right) dy, \tag{20}$$

$$I_2 = \int_{-\frac{(\pi+\omega)}{2}}^{\frac{\pi-\omega}{2}} \frac{\sin\,(2N+1)y}{\sin y}\, e^{i2my}\sin^2 by\, dy \tag{21}$$

and

$$I_3 = \int_{-\frac{(\pi+\omega)}{2}}^{\frac{\pi-\omega}{2}} \frac{\sin(2N+1)y}{\sin y}\, e^{i2my}\sin by\cos by\, dy. \tag{22}$$

Note that

$$I_1 = \frac{1}{2}\int_{-\frac{(\pi+\omega)}{2}}^{\pi-\omega} \frac{\sin(2N+1)y}{\sin y}\left(1 - e^{i2my}\right) dy$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)y}{\sin y}\left(1-e^{i2my}\right)dy,$$

since m is an integer and the integrand is periodic with period π. Also, $\frac{\sin (2N+1)y}{\sin y}$ is an even function and hence,

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin (2N+1) y}{\sin y} (1 - \cos 2m y) dy$$

$$= \int_0^{\pi/2} \frac{\sin (2N+1)y}{\sin y} dy - \frac{1}{2}\left\{ \int_0^{\frac{\pi}{2}} \frac{\sin\left[2(N+m)+1\right] y}{\sin y} dy + \int_0^{\frac{\pi}{2}} \frac{\sin\left[2(N-m)+1\right] y}{\sin y} dy \right\}$$

$$= 0. \tag{23}$$

The problem that concerns us now is to evaluate I_2 and I_3. The approach used will be to show that for $o < |b| \leq \frac{1}{2}$, $-\pi r \leq \omega \leq \pi r$ and $o<r<1$, $\frac{\sin^2 b y}{\sin y}$ is monotonic on $\left[-(\frac{\pi+\omega}{2}), \frac{\pi-\omega}{2}\right]$ and $\frac{\sin by \cos by}{\sin y}$ is monotonic on $\left[-(\frac{\pi+\omega}{2}), o\right]$ and also on $\left[o, \frac{\pi-\omega}{2}\right]$. The second mean value theorem for integrals is then used to evaluate I_2 and I_3.

$$I_2 = \int_{-(\frac{\pi+\omega}{2})}^{\frac{\pi-\omega}{2}} \frac{\sin (2N+1)y}{\sin y} e^{i2my} \sin^2 by\, dy$$

$$= \int_{-(\frac{\pi+\omega}{2})}^{\frac{\pi-\omega}{2}} \sin (2N+1)y \cos 2my \left(\frac{\sin^2 by}{\sin y}\right) dy$$

$$+ i\int_{-(\frac{\pi+\omega}{2})}^{\frac{\pi-\omega}{2}} \sin (2N+1) y \sin 2my \left(\frac{\sin^2 by}{\sin y}\right) dy.$$

We are interested in ω in the interval $\left[-\pi r, \pi r\right]$. Hence,

$$-\pi < -\frac{\pi}{2}(1+r) \leq -\left(\frac{\pi+\omega}{2}\right) \leq -\frac{\pi}{2}(1-r) < 0$$

and

$$0 < \frac{\pi}{2}(1-r) \leq \frac{\pi-\omega}{2} \leq \frac{\pi}{2}(1+r) < \pi.$$

But it is shown in Appendix I that $\frac{\sin^2 by}{\sin y}$ is monotone increasing on $(-\pi,\pi)$. Hence, $\frac{\sin^2 by}{\sin y}$ is monotone increasing on $\left[-\left(\frac{\pi+\omega}{2}\right), \frac{\pi-\omega}{2}\right]$, and by the second mean value theorem for integrals (4),

$$I_2 = \frac{-\sin^2 b\left(\frac{\pi+\omega}{2}\right)}{\sin\left(\frac{\pi+\omega}{2}\right)} \int_{-\left(\frac{\pi+\omega}{2}\right)}^{\beta_1} \sin(2N+1)y \cos 2my \, dy$$

$$+ \frac{\sin^2 b\left(\frac{\pi-\omega}{2}\right)}{\sin\left(\frac{\pi-\omega}{2}\right)} \int_{\beta_1}^{\frac{\pi-\omega}{2}} \sin(2N+1)y \cos 2my \, dy$$

$$-i \frac{\sin^2 b\left(\frac{\pi+\omega}{2}\right)}{\sin\left(\frac{\pi+\omega}{2}\right)} \int_{-\left(\frac{\pi+\omega}{2}\right)}^{\beta_2} \sin(2N+1)y \sin 2my \, dy$$

$$+i \frac{\sin^2 b\left(\frac{\pi-\omega}{2}\right)}{\sin\left(\frac{\pi-\omega}{2}\right)} \int_{\beta_2}^{\frac{\pi-\omega}{2}} \sin(2N+1)y \sin 2my \, dy, \qquad (24)$$

where

$$-\left(\frac{\pi+\omega}{2}\right) \leq \beta_1 \leq \frac{\pi-\omega}{2} \quad \text{and}$$

$$-\left(\frac{\pi+\omega}{2}\right) \leq \beta_1 \leq \frac{\pi-\omega}{2} .$$

Also,

$$I_3 = \int_{-\left(\frac{\pi+\omega}{2}\right)}^{\frac{\pi-\omega}{2}} \frac{\sin(2N+1)y}{\sin y} e^{i2my} \sin by \cos by \, dy$$

$$= \int_{-\left(\frac{\pi+\omega}{2}\right)}^{o} \sin(2N+1)y \cos 2my \left(\frac{\sin 2by}{2\sin y}\right) dy$$

$$+ \int_{o}^{\frac{\pi-\omega}{2}} \sin(2N+1)y \cos 2my \left(\frac{\sin 2by}{2\sin y}\right) dy$$

$$+i \int_{-\left(\frac{\pi+\omega}{2}\right)}^{o} \sin(2N+1)y \sin 2my \left(\frac{\sin 2by}{2\sin y}\right) dy$$

$$+i \int_{o}^{\frac{\pi-\omega}{2}} \sin(2N+1)\, y \sin 2my \left(\frac{\sin 2by}{2 \sin y}\right) dy .$$

It is shown in Appendix I that for $b > o$, $\frac{\sin 2by}{2\sin y} \geq o$ and monotone decreasing for $-(\frac{\pi+\omega}{2}) \leq y \leq o$, and $\frac{\sin 2by}{2\sin y} \geq o$ and monotone increasing for $o \leq y \leq \frac{\pi-\omega}{2}$. Also, for $b < o$, $\frac{\sin 2by}{2\sin y} \leq o$ and monotone increasing for $-(\frac{\pi+\omega}{2}) \leq y \leq o$ and $\frac{\sin 2by}{2\sin y} \leq o$ and monotone decreasing for $o \leq y \leq \frac{\pi-\omega}{2}$.

Thus, the Bonnet form of the second mean value theorem [4] can be applied, and we find

$$
\begin{aligned}
I_3 = & \frac{\sin 2b(\frac{\pi+\omega}{2})}{2\sin(\frac{\pi+\omega}{2})} \int_{-(\frac{\pi+\omega}{2})}^{\beta_3} \sin(2N+1)y \cos 2my\, dy \\
& + \frac{\sin 2b(\frac{\pi-\omega}{2})}{2\sin(\frac{\pi+\omega}{2})} \int_{\beta_4}^{\frac{\pi-\omega}{2}} \sin(2N+1)y \cos 2my\, dy \\
& +i \frac{\sin 2b(\frac{\pi+\omega}{2})}{2\sin(\frac{\pi+\omega}{2})} \int_{-(\frac{\pi+\omega}{2})}^{\beta_5} \sin(2N+1)y \sin 2my\, dy \\
& +i \frac{\sin 2b(\frac{\pi-\omega}{2})}{2\sin(\frac{\pi-\omega}{2})} \int_{\beta_6}^{\frac{\pi-\omega}{2}} \sin(2N+1)y \sin 2my\, dy, \qquad (25)
\end{aligned}
$$

where

$$-(\frac{\pi+\omega}{2}) \leq \beta_3 \leq o$$

$$o \leq \beta_4 \leq \frac{\pi-\omega}{2} ,$$

$$-\left(\frac{\pi+\omega}{2}\right) \leq \beta_5 \leq o ,$$

and
$$o \leq \beta_6 \leq \frac{\pi-\omega}{2} .$$

The integrals appearing in (24) and (25) are evaluated in Appendix II, and we obtain

$$|I_2 - iI_3| \leq \frac{(1+\sqrt{2})|\sin b\pi|}{2\cos\frac{\omega}{2}} \left(\frac{1}{2N+2m+1} + \frac{1}{2N-2m+1}\right) (|\sin b\omega| + |\cos b\omega|) \qquad (26)$$

From (19) and (23),

$$e_N^f(t) = \frac{1}{\pi^2} \int_{-\pi r}^{\pi r} F(\omega)\, e^{i(m+b)\omega} (I_2 - i\, I_3)\, d\omega .$$

Applying the Schwarz inequality,

$$|e_N^f(t)| \leq \frac{1}{\pi^2} \left[\int_{-\pi r}^{\pi r} |F(\omega)|^2 d\omega\right]^{1/2} \left[\int_{-\pi r}^{\pi r} |I_2 - i\, I_3|^2 d\omega\right]^{1/2} . \qquad (27)$$

But

$$\int_{-\pi r}^{\pi r} |I_2 - i\, I_3|^2 d\omega \leq \left[\frac{(1+\sqrt{2})|\sin b\pi|}{2}\left(\frac{1}{2N+2m+1} + \frac{1}{2N-2m+1}\right)\right]^2 \int_{-\pi r}^{\pi r} \frac{(|\sin b\omega| + |\cos b\omega|)^2}{\cos^2\frac{\omega}{2}} d\omega \qquad (28)$$

Using the Schwarz inequality for sums,

$$(|\sin b\omega| + |\cos b\omega|)^2 \leq (|\sin b\omega|^2 + |\cos b\omega|^2)(1+1)$$

$$= 2 , \text{ for all real } \omega .$$

Hence,

$$\int_{-\pi r}^{\pi r} \frac{(|\sin b\omega| + |\cos b\omega|)^2}{\cos^2\frac{\omega}{2}}\, d\omega \le 2\int_{-\pi r}^{\pi r} \sec^2\frac{\omega}{2}\, d\omega$$

$$= 8\tan\frac{\pi r}{2},$$

and thus

$$\int_{-\pi r}^{\pi r} \left|I_2 - iI_3\right|^2 d\omega \le 2\left[(1+\sqrt{2})\left|\sin b\pi\right|\left(\frac{1}{2N+2m+1} + \frac{1}{2N-2m+1}\right)\left(\tan\frac{\pi r}{2}\right)^{1/2}\right]^2 \tag{29}$$

Also,

$$\int_{-\pi r}^{\pi r} \left|F(\omega)\right|^2 d\omega = 2\pi E. \tag{30}$$

Substituting (29) and (30) into (27), we obtain

$$\left|e_N^f(t)\right| \le \frac{(\pi E\tan\frac{\pi r}{2})^{1/2}\,(1+\sqrt{2})\left|\sin b\pi\right|}{\pi^2}\left(\frac{1}{N+m+1/2} + \frac{1}{N-m+1/2}\right).$$

Recall that $t = m+b$, and note that $\left|\sin b\pi\right| = \left|\sin(m-t)\pi\right| = \left|\sin\pi t\right|$,

so that

$$\left|e_N^f(t)\right| \le \frac{(\pi E\tan\frac{\pi r}{2})^{1/2}\,(1+\sqrt{2})\left|\sin\pi t\right|}{\pi^2}\left(\frac{1}{N+m+1/2} + \frac{1}{N-m+1/2}\right), \tag{31}$$

where m is the nearest integer to t, and $|t| < N$.

Equation (31) is our desired result. This bound, involving the square-root of the signal energy, has the same asymptotic behavior as the bounds derived earlier by Yao and Thomas [1] and Brown [2]. That is, all three bounds are of the form

$$\left| e_N^f (t) \right| \le \sqrt{E} \; \frac{K(r,t)}{N} ,$$

However, the functions K (r,t), are different for the three different cases.

The bound obtained by Yao and Thomas is

$$\left| e_N^f (t) \right| \le \frac{2(rE)^{1/2} \left|\sin\pi t\right|}{\pi^2 (1-r)} \left(\frac{1}{N_1} + \frac{1}{N_2} \right) ,$$

where N_1 and N_2 are the number of samples to the left and to the right of the sample nearest to t. In our notation.

$N_2 = N - m$ and

$N_1 = N + m.$

Or,

$$\left| e_N^f (t) \right| \le \frac{2(rE)^{1/2} \left|\sin\pi t\right|}{\pi^2 (1-r)} \left(\frac{1}{N+m} + \frac{1}{N-m} \right)$$

If we define $R_1(r)$ as the ratio of the bound obtained here to the bound obtained by Yao and Thomas, we find

$$R_1(r) = \frac{\dfrac{(\pi E \tan \frac{\pi r}{2})^{1/2} (1+\sqrt{2}) \left|\sin \pi t\right|}{\pi^2} \left(\dfrac{1}{N+m+1/2} + \dfrac{1}{N-m+1/2} \right)}{\dfrac{2 (rE)^{1/2} \left|\sin \pi t\right|}{\pi^2 (1-r)} \left(\dfrac{1}{N+m} + \dfrac{1}{N-m} \right)}$$

$$< \frac{(1+\sqrt{2})\ (1-r)}{2} \left(\frac{\pi}{r} \tan \frac{\pi r}{2} \right)^{1/2} .$$

The ratio $R_1(r)$ for $o < r < 1$ is shown in Figure 1.

Note that

$$\lim_{r \to o} R_1(r) = \pi\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right) = 2.68, \text{ while}$$

$$\lim_{r \to 1} R_1(r) = o.$$

Note also that for values of $r > 0.73$, the bound on the truncation error given here is tighter than the Yao-Thomas bound. That is, for samples taken nearly at the Shannon rate, the bound obtained here is an improvement on the Yao-Thomas bound.

The bound given by Brown in [2] is

$$\left| e_N^f(t) \right| \leq \frac{2\sqrt{2}}{\pi^{3/2}} \left|\sin \pi t\right| \sqrt{E} \left(\tan \frac{\pi r}{2}\right)^{1/2} \left(\frac{1}{N_1} + \frac{1}{N_2}\right),$$

where N_1 and N_2 are the same as in the Yao-Thomas bound. Forming the ratio of the bound obtained here to the bound obtained by Brown, we find

$$R_2(r) = \frac{\dfrac{\left(\pi E \tan \frac{\pi r}{2}\right)^{1/2} (1+\sqrt{2}) \left|\sin \pi t\right|}{\pi^2} \left(\dfrac{1}{N+m+1/2} + \dfrac{1}{N-m+1/2}\right)}{\dfrac{2\sqrt{2}}{\pi^2} \left|\sin \pi t\right| \left(\pi E \tan \frac{\pi r}{2}\right)^{1/2} \left(\dfrac{1}{N+m} + \dfrac{1}{N-m}\right)}$$

$$< \frac{1+2}{2\sqrt{2}} = 0.85.$$

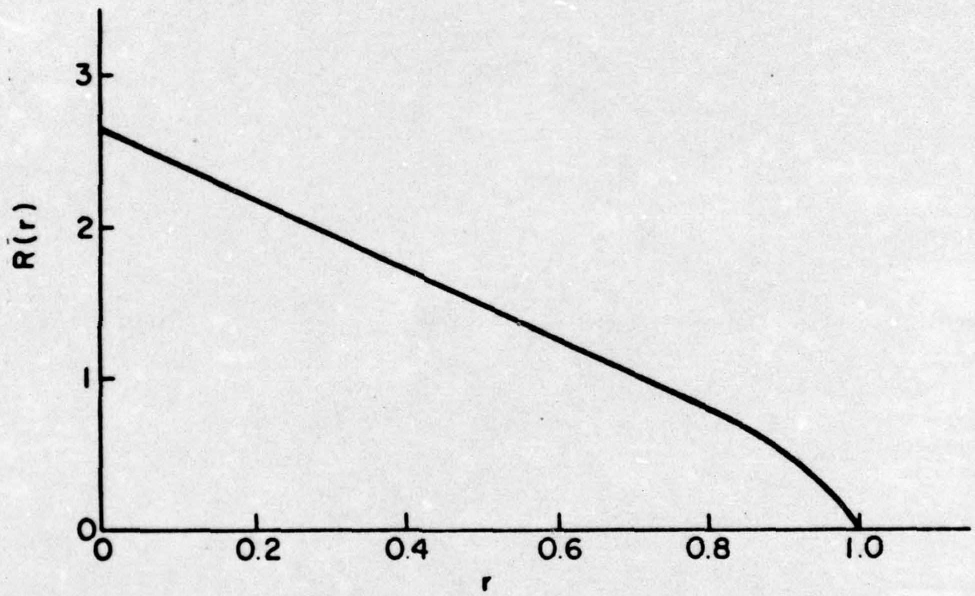

Figure 2.1 Ratio Between Derived Truncation Error Bound and Yao-Thomas Bound

Hence, the bound obtained here is an improvement on the bound obtained by Brown for all values of the guard-band coefficient, r.

April 18, 1974
HSP:pgs
Serial No. R-1018

APPENDIX I

Let $g(y) = \frac{\sin^2 |b| y}{\sin y}$, for $-\pi < y < \pi$, $o < |b| \leq \frac{1}{2}$. We want to show that $g(y)$ is monotone increasing on $(-\pi,\pi)$. Note that $g(y)$ is an odd function of y, and hence it is sufficient to consider $o \leq y < \pi$. Also, $g(y) > o$ for $o < y < \pi$ and $g(o) = 0$. Thus, we only need to consider $o < y < \pi$.

$$g'(y) = \frac{2|b| \sin |b| y \cos |b| y \sin y - \cos y \sin^2 |b| y}{\sin^2 y}$$

$$= \frac{\sin |b| y}{\sin^2 y} \left[2|b| \cos |b| y \sin y - \cos y \sin |b| y \right].$$

Case 1: $o < y \leq \frac{\pi}{2}$. Then $o < |b| y \leq |b| \frac{\pi}{2} \leq \frac{\pi}{4}$, and hence,

$\sin |b| y \geq o \qquad \cos |b| y \geq o$

$\sin y \geq o \qquad \cos y \geq o.$

$$g'(y) \geq \frac{\sin |b| y}{\sin^2 y} \left[2 |b| \sin y \cos |b| y - |b| y \cos y \right],$$

since $\sin |b| y \leq |b| y$. Thus,

$$g'(y) \geq \frac{\sin |b| y}{\sin^2 y} |b| y \left[2 \frac{\sin y}{y} \cos |b| y - \cos y \right]$$

$$\geq \frac{\sin |b| y}{\sin^2 y} |b| y \left[2 \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} \cos |b| y - \cos y \right]$$

$$> \frac{\sin |b| y}{\sin^2 y} |b| y \left[\cos |b| y - \cos y \right]$$

$> o,$

since $o < |b|y < y \leq \frac{\pi}{2}$ and cos y is monotone decreasing on $\left[o, \frac{\pi}{2}\right]$.

Case 2: $\frac{\pi}{2} < y < \pi$. Then,

$o < \frac{|b|\pi}{2} < |b|y < |b|\pi \leq \frac{\pi}{2}$, and therefore

$\sin y > o$ $\qquad$ $\cos y < o$

$\sin|b|y > o$ $\qquad$ $\cos|b|y > o.$

$$g'(y) = \frac{\sin|b|y}{\sin^2 y}\left[2|b|\sin y \cos|b|y - \sin|b|y \cos y\right]$$

$> o.$

Hence, for $o < y < \pi$, $g'(y)>o$ and consequently g(y) is monotone increasing on $-\pi < y < \pi$.

Now consider

$$h(y) = \frac{\sin by \ \cos by}{\sin y} = \frac{\sin 2by}{2 \sin y} .$$

Note that h(y) is an even function of y, and therefore if h(y) is monotone increasing for $o \leq y < \pi$, it will be monotone decreasing on $-\pi < y \leq o$. Also, if h(y) is monotone increasing for $b > o$, it will be monotone decreasing for $b < o$. Thus, it is sufficient to consider $o \leq y < \pi$, $b > o$.

$$h'(y) = \frac{2b \cos 2by \sin y - \sin 2by \cos y}{2 \sin^2 y}$$

Case 1: $o \leq y < \frac{\pi}{2}$, $o \leq 2by < \frac{\pi}{2}$.

Note that $h'(o) = o$, and hence it is sufficient to consider $o < y < \frac{\pi}{2}$.

$$h'(y) = \frac{2b \tan y - \tan 2by}{2 \tan y \dfrac{\sin y}{\cos 2by}} .$$

But $2 \tan y \dfrac{\sin y}{\cos 2by} > o$ for $o < y < \frac{\pi}{2}$, $o < 2by < \frac{\pi}{2}$. Thus, we only need consider $2b \tan y - \tan 2by$, and

$$2b \tan y - \tan 2by = 2by\left[\frac{\tan y}{y} - \frac{\tan 2by}{2by}\right] \geq o,$$

since $\dfrac{\tan y}{y}$ is monotone increasing on $o < y < \frac{\pi}{2}$. Hence,

$$h'(y) \geq o \quad \text{for} \quad o \leq y < \frac{\pi}{2} .$$

Case 2: $\frac{\pi}{2} \leq y < \pi$. If $o < 2by \leq \frac{\pi}{2}$, then

$$\cos 2by \geq o \qquad \sin y > o$$

$$\cos y \leq o \qquad \sin 2by > o \text{ , and thus}$$

$$h'(y) = \frac{2b \cos 2by \sin y - \sin 2by \cos y}{2 \sin^2 y}$$

$$> o .$$

For $\frac{\pi}{2} < 2by < \pi$, $\frac{\pi}{2} \leq y < \pi$, we have $\cos 2by \geq \cos y$, since $2by \leq y$, and hence,

$$h'(y) \geq \frac{\cos y}{2 \sin^2 y}\left[2b \sin y - \sin 2by\right]$$

$$= \frac{2by \cos y}{2 \sin^2 y}\left[\frac{\sin y}{y} - \frac{\sin 2by}{2by}\right].$$

But $\frac{\sin 2\,by}{2\,by} \geq \frac{\sin y}{y}$, since $2\,by \leq y$ and $\frac{\sin y}{y}$ is monotone decreasing on $o \leq y \leq \pi$. Also, $\cos y \leq o$, and thus,

$$h'(y) \geq o\ , \text{ for } \frac{\pi}{2} \leq y < \pi.$$

We have shown that $h'(y) \geq o$ for $o \leq y < \pi$, and consequently $h(y)$ is monotone increasing on $o \leq y < \pi$ for $b > o$.

Appendix II

We wish to evaluate $|I_2 - I_3|$, where I_2 and I_3 are given by (24) and (25).

$$\int_a^b \sin(2N+1)y \cos 2\,my\, dy = \frac{1}{2}\int_a^b \left[\sin(2N+2m+1)y + \sin(2N-2m+1)y\ dy\right]$$

$$= -\frac{1}{2}\left\{\frac{\cos(2N+2m+1)\,b - \cos(2N+2m+1)\,a}{2N+2m+1} + \frac{\cos(2N-2m+1)\,b - \cos(2N-2m+1)\,a}{2N-2m+1}\right\}$$

Also,

$$\int_a^b \sin(2N+1)y \sin 2\,my\, dy = \frac{1}{2}\left\{\frac{\sin(2N-2m+1)b - \sin(2N-2m+1)a}{2N-2m+1} - \frac{\sin(2N+2m+1)b - \sin(2N+2m+1)a}{2N+2m+1}\right\}.$$

Using these integrals and simplifying by means of elementary trigonometric identities, we find that

$$I_2 = \frac{1}{2\cos\frac{\omega}{2}}\left\{\frac{\sin b\,\pi \sin b\,\omega \ \sin(N+m+\frac{1}{2})\pi \left[e^{-i(N+m+\frac{1}{2})\omega}\right]}{2N+2m+1}\right.$$

$$+\frac{\left[1-\cos b\pi \cos b\omega\right]\left[\cos\,(2N+2m+1)\beta_1 + i\sin\,(2N+2m+1)\beta_2\right]}{2\ N+2m+1}$$

$$+\frac{\sin b\pi \sin b\omega \ \sin\,(N-m+\frac{1}{2})\pi\left[ie^{-i(N-m+\frac{1}{2})\omega}\right]}{2\ N-2m+1}$$

$$\left.+\frac{\left[1-\cos b\pi \cos b\omega\right]\left[\cos\,(2N-2m+1)\beta_1 + i\sin\,(2N-2m+1)\beta_2\right]}{2N-2m+1}\right\}$$

and

$$I_3 = \frac{1}{2\cos\frac{\omega}{2}}\Bigg(- \frac{\sin b\pi \cos b\omega \sin (N+m+1/2)\pi \left[ie^{-i(N+m+1/2)\omega}\right]}{2\ N+2m+1}$$

$$+ \frac{\sin b\pi \cos b\omega \sin (N-m+1/2)\pi \left[ie^{i(N-m+1/2)\omega}\right]}{2\ N-2m+1}$$

$$- \left[\frac{\sin b\pi \cos b\omega + \cos b\pi \sin b\omega}{2}\right] X$$

$$\left[\frac{\cos (2N+2m+1)\beta_3 + i \sin(2N+2m+1)\beta_5}{2N+2m+1} + \frac{\cos (2N-2m+1)\beta_3 - i \sin (2N-2m+1)\beta_5}{2N-2m+1}\right]$$

$$+ \left[\frac{\sin b\pi \cos b\omega - \cos b\pi \sin b\omega}{2}\right] X$$

$$\left[\frac{\cos (2N+2m+1)\beta_4 + i \sin (2N+2m+1)\beta_6}{2N+2m+1} + \frac{\cos (2N-2m+1)\beta_4 - i \sin (2N-2m+1)\beta_6}{2N-2m+1}\right]$$

Forming $I_2 - iI_3$, collecting terms and taking magnitude, we get

$$\left|I_2 - iI_3\right| \leq \frac{1}{2\cos\frac{\omega}{2}}\Bigg(\frac{1}{2N+2m+1}\Bigg[\left|\sin b\pi\right| + 2\ (1 - \cos b\pi \cos b\omega)$$

$$+ \frac{\sqrt{2}}{2}\left|\sin b\pi \cos b\omega + \cos b\pi \sin b\omega\right|$$

$$+ \frac{\sqrt{2}}{2}\left|\sin b\pi \cos b\omega - \cos b\pi \sin b\omega\right|\Bigg]$$

$$+ \frac{1}{2N-2m+1}\Bigg[\left|\sin b\pi \sin b\omega\right| + \sqrt{2}\ (1 - \cos b\pi \cos b\omega)$$

$$+ \left|\sin b\pi \cos b\omega\right| + \frac{\sqrt{2}}{2}\left|\sin b\pi \cos b\omega + \cos b\pi \sin b\omega\right|$$

$$+ \frac{\sqrt{2}}{2}\left|\sin b\pi \cos b\omega - \cos b\pi \sin b\omega\right|\Bigg]\Bigg).$$

But $\left|\sin b\pi\right| = \left|\sin b\pi\right|\left[\sin^2\left|b\omega\right| + \cos^2\left|b\omega\right|\right]$

$$\leq \left|\sin b\pi\right|\left[\left|\sin b\omega\right| + \left|\cos b\omega\right|\right]$$

$$= \left|\sin b\pi \sin b\omega\right| + \left|\sin b\pi \cos b\omega\right| .$$

Also,

$$\left|\sin b\pi \cos b\omega + \cos b\pi \sin b\omega\right| = \left|\sin b(\pi+\omega\right| = \sin\left|b\right|(\pi+\omega),$$

and $\left|\sin b\pi \cos b\omega - \cos b\pi \sin b\omega\right| = \left|\sin b(\pi-\omega)\right| = \sin\left|b\right|(\pi-\omega).$

Or,

$$\left|\sin b\pi \cos b\omega + \cos b\pi \sin b\omega\right| + \left|\sin b\pi \cos b\omega - \cos b\pi \sin b\omega\right|$$

$$= 2\left|\sin b\pi\right|\cos b\omega$$

Also,

$$1 - \cos b\pi \cos b\omega \leq 1 - \cos^2 b\pi = \sin^2 b\pi.$$

Making use of these inequalitities, we find

$$\left|I_2 - iI_3\right| \leq \frac{(1+2)\left|\sin b\pi\right|}{2\cos\frac{\omega}{2}}\left[\frac{1}{2N-2m+1} + \frac{1}{2N-2m+1}\right]\left[\left|\sin b\omega\right| + \left|\cos b\omega\right|\right],$$

which is our desired result.

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Serial No. R-1018

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References

[1] K. Yao and J. B. Thomas, "On truncation error bounds for sampling representations of band-limited signals," IEEE Trans. Aerospace and Electronic Systems, Vol. AES-2, pp. 640-647, November 1966.

[2] J. L. Brown, Jr. "Bounds for truncation error in sampling expansions of band-limited signals," IEEE Trans. on Information Theory, Vol. IT-15, pp. 440-444, July 1969.

[3] A. V. Balakrishnan, Communication Theory, New York: McGraw-Hill, 1968.

[4] John M. H. Olmsted, Real Variables. New York: Appleton-Century - Crofts, 1959.

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