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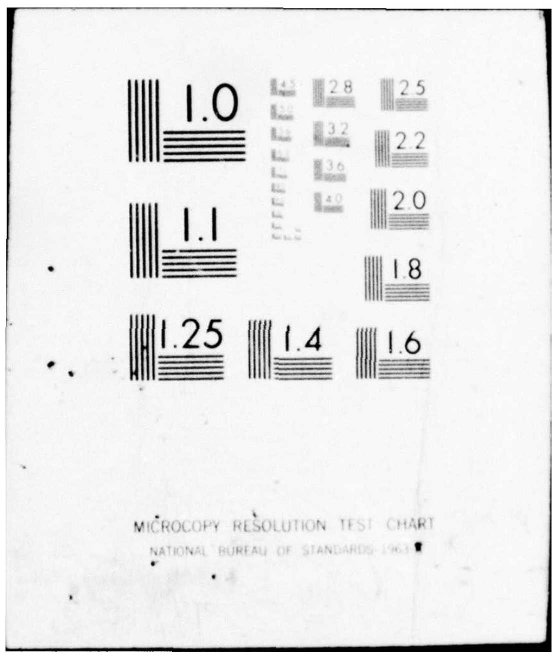
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Edward R. Fadell and Paul H. Rabinowitz*

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ABSTRACT

A bifurcation theorem is proved for odd potential operators. The operator equation (*) $f'(u) \equiv Lu + H(u) = \lambda u$ (LAMBDA) is treated where λ (LAMBDA) $\in$ (is a member of) $\mathbb{R}$ and $u \in$ (is a member of) E, a real Hilbert space. A sharp description is given of the structure of the set of solutions of (*) near a bifurcation point as a function of λ (LAMBDA). A crucial role is played here by a notion of topological index alternative to other indices used in critical point theory and the properties of this index are developed in some detail.

AMS (MOS) Subject Classifications: 47H15, 49F99, 55C99, 58E05

Key Words: bifurcation, odd potential operator, topological index, critical point, deformation theorem, minimax

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BIFURCATION FOR ODD POTENTIAL OPERATORS AND AN ALTERNATIVE TOPOLOGICAL INDEX

Edward R. Fadell and Paul H. Rabinowitz*

§ 1. Introduction

In several recent papers [1-5], bifurcation theorems have been proved for potential operators. The purpose of this study is to prove a sharper result of this nature for odd potential operators. In doing so we will employ a topological index alternative to the notions of genus, Ljusternik-Schnirelman category, etc., which may also be of use in other problems.

To describe our work more fully, let E be a real Hilbert space and Ω a neighborhood of 0 in E. Suppose f is a twice continuously Frechet differentiable real valued map on Ω, i.e. $f \in C^2(\Omega, \mathbb{R})$ with $f(0) = 0$. Some standard remarks are in order. The Frechet derivative of f at $u \in \Omega$, $f'(u)$, is a linear map from E to $\mathbb{R}$ so $f'(u) \in E'$, the dual space of E. Since E is self dual we can and will interpret the map $u \to f'(u)$ as a map from E to E. We further assume $f'(u) = Lu + H(u)$ where L is linear and $H(u) = o(\|u\|)$ at $u = 0$.

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For $\lambda \in \mathbb{R}$, consider the equation

$$f'(u) = \lambda u . \tag{1.1}$$

A solution of (1.1) is a pair $(\lambda, u) \in \mathbb{R} \times E$. Our above assumptions imply $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ are solutions of (1.1) and they shall be referred to as the trivial solutions of (1.1). A trivial solution $(\mu, 0)$ is called a bifurcation point if every neighborhood of $(\mu, 0)$ contains nontrivial solutions. It is well-known and easily shown that a necessary condition for $(\mu, 0)$ to be a bifurcation point is that $\mu \in \sigma(L)$, the spectrum of L. Under mild additional hypotheses, this necessary condition is also sufficient. (See e.g. [5] for references).

In some applications, e.g. to buckling problems in elasticity theory, solutions of (1.1) represent the possible equilibrium states of a physical system depending on a parameter λ. It is therefore of interest to study the solution set of (1.1) as a function of λ. Moreover in such problems it is often the case that f is even and therefore solutions of (1.1) occur in pairs $(\lambda, \pm u)$. Our goal here is to give lower bounds for the number of nontrivial solutions of (1.1) near a bifurcation point as a function of λ when f is even. Our main result is:

<u>Theorem 1.2</u>: Let E be a real Hilbert space, Ω a neighborhood of 0 in E, and $f \in C^2(\Omega, \mathbb{R})$ where f is even and $f'(u) = Lu + H(u)$ with L linear and $H(u) = o(\|u\|)$ at $u = 0$. Suppose $\mu \in \sigma(L)$ is an isolated eigenvalue of L of multiplicity $n < \infty$. Then either (i) $(\mu, 0)$

is not an isolated solution of (1.1) in $\{\mu\} \times E$ or (ii) there exist left and right neighborhoods, $\mathcal{I}_\ell$ and $\mathcal{I}_r$, of μ in $\mathbb{R}$ and integers $k, m \geq 0$ such that $k + m \geq n$ and if $\lambda \in \mathcal{I}_\ell$ (resp. $\mathcal{I}_r$), (1.1) possesses at least k (resp. m) distinct pairs of nontrivial solutions. Moreover as $\lambda \to \mu$, these solutions converge to $(\mu, 0)$.

Remark 1.3: Either $\mathcal{I}_\ell$ or $\mathcal{I}_r$ may be empty. A characterization of k and m will be given in the course of the proof of the theorem.

Theorem 1.2 improves earlier results in this direction due to Clark [3] and Rabinowitz [5]. Other work on (1.1) for f even has been carried out by Böhme [1] and Marino [2] who studied the solutions of (1.1) near $(\mu, 0)$ as a function of $\rho = \|u\|$. They showed in particular that under the hypotheses of Theorem 1.2, for each $\rho > 0$, there are at least n distinct pairs of solutions $(\lambda(\rho), \pm u(\rho))$ of (1.1) having $\|u(\rho)\| = \rho$ and $(\lambda(\rho), u(\rho)) \to (\mu, 0)$ as $\rho \to 0$. Thus Theorem 1.2 is a natural complement to the Böhme-Marino result. We suspect that there is a better approach to (1.1) by means of which both Theorem 1.2 and the ρ dependent result may be obtained simultaneously.

The proof of Theorem 1.2 will be given in § 2. In brief the main steps are: (1) Use a standard argument to reduce the problem of solving (1.1) near $(\mu, 0)$ to that of determining the critical points (with respect to v) of a function $g(\lambda, v)$ defined near $(\mu, 0)$ in $\mathbb{R} \times \mathbb{R}^n$; (2) Work in an appropriately defined neighborhood, Q, of 0 in $\mathbb{R}^n$ to construct several families of sets Γ_j in $\overline{Q}$ and study their properties;

(3) Minimax $g(\lambda, v)$ over each of these families of sets thereby producing a set of numbers; (4) Verify that each of these minimax values is a critical value of $g(\lambda, \cdot)$ and that we obtain the required number of critical points.

To define the sets in (2), a notion of topological index is introduced which plays a crucial role there and is of independent interest. To avoid unduly interrupting the proof of Theorem 1.2 in § 2, we state a lemma in § 2 which asserts the existence of an index with the properties we require and delay the definition of the index and development of its properties to § 3. The relationship of this index to others that have been employed earlier in critical point theory such as Ljusternik-Schnirelman category [6], coindex [7], genus [8] [9], and the indices of Yang [10-11] will also be discussed in § 3.

The authors acknowledge with thanks several helpful conversations with Charles Conley. In particular we are indepted to him for a suggestion which led to the final form of Theorem 3.14.

§2. The Main Theorem

In this section we will carry out the proof of Theorem 1.2. To begin, observe that although E may be infinite dimensional, we can reduce (1.1) to a finite dimensional problem in a standard fashion using the method of Lyapunov-Schmidt. This has been done already e.g. in [5] but since it is brief we will include it here. Let $N \equiv N(L - \mu I)$, the null space of $L - \mu I$ and let $N^{\perp}$ denote its orthogonal complement in E. Since N is n dimensional, we can identify it with $\mathbb{R}^n$.

If $u \in E$, $u = v + w$ with $v \in N$ and $w \in N^{\perp}$. Letting P and $P^{\perp}$ denote respectively the orthogonal projectors of E onto N and $N^{\perp}$, we see (1.1) is equivalent to the pair of equations:

$$
(2.1) \qquad \begin{cases} \text{(i)} \ (\mu - \lambda)v + PH(v + w) = 0 \\ \text{(ii)} \ (L - \lambda I)w + P^{\perp}H(v + w) \equiv F(\lambda, v, w) = 0 . \end{cases}
$$

Note that $F(\mu, 0, 0) = 0$ and the Frechet derivative of F with respect to w at $(\mu, 0, 0)$, $F_w(\mu, 0, 0) = L - \mu I$ which is an isomorphism from $N^{\perp}$ to $N^{\perp}$. *Consequently by the implicit function theorem,* (2.1) (ii) can be solved for $w = \varphi(\lambda, v)$ in a neighborhood, $\mathcal{O}$, of $(\mu, 0) \in \mathbb{R} \times N$ with $\varphi \in C^1(\mathcal{O}, N^{\perp})$. Since f is even in u, it follows that $\varphi(\lambda, v)$ is odd in v. Moreover since $H(u) = o(\|u\|)$ at $u = 0$, (2.1) (ii) shows $\varphi(\lambda, v) = -(L - \lambda I)^{-1}P^{\perp}H(v + \varphi(\lambda, v)) = o(\|v\|)$ at $v = 0$ uniformly for λ near μ (where the inverse is relative to $N^{\perp}$). Thus solving (1.1) for (λ, u) near $(\mu, 0)$ in $\mathbb{R} \times E$ is equivalent to solving (2.1) (i) for (λ, v) near $(\mu, 0)$ in $\mathbb{R} \times N$.

The next step in the proof is to define

$$g(\lambda, v) = f(v + \varphi(\lambda, v)) - \frac{\lambda}{2}(\|v\|^2 + \|\varphi(\lambda, v)\|^2) . \tag{2.2}$$

Note that g is even in v since f is even and φ is odd in v. A simple computation shows that for fixed λ, critical points of g are solutions of (2.1) (i). Thus to prove Theorem 1.2, it suffices to determine lower bounds for the number of critical points of $g(\lambda, \cdot)$ near $v = 0$ for λ fixed near μ.

From (2.2),

$$g_v(\lambda, v) = (\mu - \lambda)v + PH(v + \varphi(\lambda, v)) . \tag{2.3}$$

The right hand side of (2.3) is continuously differentiable. Hence $g(\lambda, v)$ is a C^2 function of v near $v = 0$ even though $\varphi(\lambda, v)$ and $f(v + \varphi(\lambda, v))$ are only continuously differentiable in v. Consider the ordinary differential equation:

$$\begin{cases} \dfrac{d\psi}{dt} = -g_v(\mu, \psi) \\ \psi(0, x) = x \end{cases} \tag{2.4}$$

for x near 0 in N. If $v = 0$ is not an isolated critical point of $g(\mu, v)$, then we obtain (i) of Theorem 1.2. Thus now and henceforth we can assume there is a neighborhood, V, of 0 in N such that 0 is the unique critical point of $g(\mu, v)$ in V.

<u>Lemma 2.5</u>: There is a constant $c > 0$ and a symmetric open neighborhood Q of 0, $Q \subset V$ such that $\overline{Q}$ is compact and

1° If $x \in Q$, $|g(\mu, x)| < c$.

2° If $x \in Q$, then $\psi(t,x) \in Q$ for all t satisfying $|g(\mu, \psi(t,x))| < c$.

3° If $x \in \partial Q$, $|g(\mu,x)| = c$ or $\psi(t,x) \in \partial Q$ for all t such that $|g(\mu, \psi(t,x))| \leq c$.

Proof: The proof of Lemma 2.5 can be found in [5]. Q is simply the union of all orbit segments $\psi(t,x)$, for x appropriately chosen near 0, which lie in $g(\mu,\cdot)^{-1}(-c,c)$ for c sufficiently small.

Remark 2.6: For future reference observe that if $x \in \bar{Q}$, the orbit $\psi(t,x)$ can only leave $\bar{Q}$ by crossing $g(\mu,\cdot)^{-1}(-c)$. Note also that $\{x \in \bar{Q} \mid g(\mu,x) = c\}$ may be empty. This occurs when $g(\mu,\cdot)$ has an isolated local maximum at $v = 0$. Similarly $\{x \in \bar{Q} \mid g(\mu,x) = -c\}$ may be empty.

Given the existence of Q, we obtain a standard sort of "deformation theorem". For $z \in \mathbb{R}$, let $A_{\lambda z} = \{x \in \bar{Q} \mid g(\lambda,x) \leq z\}$ and $K_{\lambda z} = \{x \in A_{\lambda z} \mid g(\lambda,x) = z,\ g_v(\lambda,x) = 0\}$.

Lemma 2.7: If $z \in \mathbb{R}$, $\varepsilon_1 > 0$, and U is any neighborhood of $K_{\lambda z}$, then there exists an $\varepsilon \in (0,\varepsilon_1)$ and an $\eta \in C([0,1] \times \bar{Q}, \bar{Q})$ such that:

1° $\eta(t,v)$ is odd in v.

2° $\eta(t,v) = v$ if $v \notin g(\lambda,\cdot)^{-1}[z-\varepsilon_1, z+\varepsilon_1]$.

3° $\eta(t,v)$ is a homeomorphism of $\bar{Q}$ to $\eta(t,\bar{Q})$ for each $t \in [0,1]$.

4° $\eta(1, A_{\lambda, z+\varepsilon} \setminus U) \subset A_{\lambda, z-\varepsilon}$.

5° If $K_{\lambda z} = \phi$, $\eta(1, A_{\lambda, z+\varepsilon}) \subset A_{\lambda, z-\varepsilon}$.

Proof: This lemma is the same as Lemma 1.19 in [5]. It is in the proof of this lemma that the special features of Q play a role.

Next we require a suitable notion of index. We identify N with $\mathbb{R}^n$ and set $B_\rho(y) = \{x \in \mathbb{R}^n \mid |x - y| < \rho\}$. Let $\mathcal{E}$ denote the set of compact subsets of $\mathbb{R}^n \setminus \{0\}$ which are symmetric with respect to the origin. $\mathbb{N}$ will denote the non-negative integers.

Lemma 2.8: There exists an index theory, i.e. a mapping $\mathcal{E} \to \mathbb{N}$, $A \to \text{Index } A$, possessing the following properties:

1° If $A = \phi$, Index $A = 0$; if $A \neq \phi$, Index $A \geq 1$; if $A = \{x, -x\}$, Index $A = 1$.

2° If $A, B \in \mathcal{E}$ and there is an odd map $\psi \in C(A, B)$, then Index $A \leq$ Index B. If ψ is also a homeomorphism of A onto B, then Index A = Index B.

3° Index $(A \cup B) \leq$ Index A + Index B.

4° If $A \in \mathcal{E}$, there exists a $\delta > 0$ and a uniform neighborhood of A, $N_\delta(A) = \{x \in \mathbb{R}^n \mid |x - A| \leq \delta\}$ such that Index $N_\delta(A)$ = Index A.

5° If U is a symmetric bounded open neighborhood of 0 in $\mathbb{R}^n$, Index $\partial U = n$.

6° Let $\rho > 0$, $K \in \mathcal{E}$ with $K \cap \overline{B_\rho(0)} = \phi$. Let $\tau > 0$ and suppose $\theta : K \times [0, \tau] \to \mathbb{R}^n \setminus \{0\}$ is an imbedding (i.e., θ is a one-one mapping) such that $\theta(x, 0) = x$, $x \in K$ and $\theta(\cdot, t)$ is odd on K for each t. Then, if $\theta(K \times \{\tau\}) \subset B_\rho(0)$,

$$\text{Index}(\theta(K \times [0, \tau]) \cap \partial B_\rho(0)) = \text{Index}\, K .$$

We remark that it is the need for an index theory satisfying 6° that requires us to go beyond the usual indices used in critical point theory, in particular, genus or Ljusternik-Schirelman category. We leave the precise definition of Index and the verification of its basic properties until § 3 and proceed now to complete the proof of Theorem 1.2 making use of Lemma 2.8.

Let $S^+ = \{x \in V\setminus\{0\} \mid \psi(x,t) \subset V$ for all $t > 0\}$ and $S^- = \{x \in V\setminus\{0\} \mid \psi(x,t) \subset V$ for all $t < 0\}$. It is not difficult to see that either S^+ or S^- is nonempty [5]. In fact both are nonempty unless $v = 0$ is an isolated local maximum or minimum for $g(\mu, \cdot)$. Let $T^+ = S^+ \cap \partial Q$ and $T^- = S^- \cap \partial Q$. The proof of Theorem 1.2 is now a consequence of the following three results.

Theorem 2.9: Suppose Index $T^- = k > 0$. Then there is a left neighborhood $\mathcal{I}_\ell$ of μ such that for each $\lambda \in \mathcal{I}_\ell$, $g(\lambda, \cdot)$ possesses at least k distinct pairs of nontrivial critical points. These points converge to 0 as $\lambda \to \mu^-$.

Corollary 2.10: Suppose Index $T^+ = m > 0$. Then there is a right neighborhodd $\mathcal{I}_r$ of μ such that for each $\lambda \in \mathcal{I}_r$, $g(\lambda, \cdot)$ possesses at least m distinct pairs of nontrivial critical points. These points converge to 0 as $\lambda \to \mu^+$.

Lemma 2.11: Index T^- + Index $T^+ \geq n$.

To establish Theorem 2.9, we require several families of sets, Γ_j, which are constructed next. Suppose Index $T^- = k$. For $K \subset T^-$ we define $\Phi(K) = \{\psi(t,x) \mid (t,x) \in (-\infty, 0) \times K\}$, i.e. we cone K over 0 using the flow ψ. Now let $\mathfrak{F} = \{\chi \in C(\overline{Q}, \overline{Q}) \mid \chi$ is odd, one to one, and $\chi(v) = v$ if $v \in T^-\}$. For $1 \leq j \leq k$, define

$$G_j = \{\chi(\Phi(K)) \mid \chi \in \mathfrak{F},\ K \subset T^-,\ \text{and Index } K \geq j\}.$$

Observe that $\theta \in \mathfrak{F}$ and $A \in G_j$ implies that $\theta(A) \in G_j$. Finally for $1 \leq j \leq k$, define

$$\Gamma_j = \{\overline{A \setminus Y} \mid A \in G_q \text{ for some } q,\ j \leq q \leq k,\ Y \in \mathcal{E}, \text{ and Index } Y \leq q - j\}.$$

Lemma 2.12: The sets Γ_j possess the following properties:

1^o $\Gamma_{j+1} \subset \Gamma_j$, $1 \leq j \leq k - 1$.

2^o If $\chi \in \mathfrak{F}$ and $B \in \Gamma_j$, then $\chi(B) \in \Gamma_j$.

3^o If $B \in \Gamma_j$ and $Z \in \mathcal{E}$ with Index $Z \leq s < j$, then $B \setminus Z \in \Gamma_{j-s}$.

Proof: 1^o is obvious. To verify 2^o, let $B \in \Gamma_j$. Therefore $B = \overline{A \setminus Y}$ with $A \in G_q$, $Y \in \mathcal{E}$, and Index $Y \leq q - j$. If $\chi \in \mathfrak{F}$, then $\chi(\overline{A\setminus Y}) = \overline{\chi(A \setminus Y)} = \overline{\chi(A) \setminus \chi(Y)}$. But $\chi(A) \in G_q$ by an above remark, $\chi(Y) \in \mathcal{E}$, and Index $\chi(Y) =$ Index Y by 2^o of Lemma 2.8. Hence $\chi(B) \in \Gamma_j$. Finally to prove 3^o, let $B = \overline{A \setminus Y}$ as in 2^o. Therefore $\overline{B \setminus Z} = \overline{\overline{A\setminus Y}\setminus Z} = \overline{A \setminus (Y \cup Z)}$. Since $A \in G_q$ and Index $(Y \cup Z) \leq q - j + s = q - (j - s)$ by 3^o of Lemma 2.8, it follows that $\overline{B \setminus Z} \in \Gamma_{j-s}$.

Proof of Theorem 2.9: Define

$$c_j = \inf_{A \in \Gamma_j} \max_{v \in A} g(\lambda, v), \quad 1 \leq j \leq k. \tag{2.13}$$

By 1° of Lemma 2.12, $c_1 \le c_2 \le \cdots \le c_k$. We will further show: (i) $c_1 > 0$; (ii) c_j is a critical value of $g(\lambda, \cdot)$ with a corresponding critical point in Q. (Since $c_1 > 0$, this critical point is nontrivial). (iii) If $c_{j+1} = \cdots = c_{j+p} \equiv d$, (i.e. d is what we might call a degenerate critical value of $g(\lambda, \cdot)$), then Index $K_{\lambda d} \ge p$. (iv) As $\lambda \to \mu^-$, any critical points corresponding to c_j, $1 \le j \le k$, converge to $v = 0$. By 1° and 2° of Lemma 2.8, if Index $A > 1$, A contains infinitely many distinct pairs of points. Hence Theorem 2.9 is a consequence of (ii) - (iv).

To prove (i), observe first from (2.2) that

$$(2.14) \qquad g(\lambda, v) = \frac{\mu - \lambda}{2} \|v\|^2 + \frac{1}{2}((L - \lambda I)\varphi(\lambda, v), \varphi(\lambda, v)) + h(v + \varphi(\lambda, v))$$

where $(\cdot, \cdot)$ denotes the inner produce in E, $h' = H$, and $h(0) = 0$. Since $\varphi(\lambda, v) = o(\|v\|)$ at $v = 0$ uniformly for λ near μ and $h(u) = o(\|u\|^2)$ at $u = 0$, the dominating term in g for v near 0 is $\frac{\mu - \lambda}{2} \|v\|^2$. Therefore there is a $\rho > 0$, ρ depending on λ, such that for $\lambda < \mu$ and $0 < \|v\| \le \rho$,

$$(2.15) \qquad g(\lambda, v) \ge \frac{\mu - \lambda}{4} \|v\|^2 .$$

We can further assume $B_\rho(0) \cap \partial Q = \phi$. Now choose any $B \in \Gamma_1$. Then $B = \overline{\chi(\Phi(K)) \setminus Y}$ where $K \subset T^-$, Index $K = q \ge 1$, $Y \in \mathcal{E}$, and Index $Y \le q - 1$. For τ, depending on χ and K, sufficiently large, $\chi(\psi(-\tau, K)) \subset B_\rho(0)$. By 6° of Lemma 2.8,

$$(2.16) \qquad \text{Index}\ \chi(\psi([-\tau, 0] \times K)) \cap \partial B_\rho(0) = \text{Index}\ K = q .$$

Now 2° and 3° of Lemma 2.8 together with (2.16) show

$$\text{Index } B \cap \partial B_\rho(0) = \text{Index}\overline{[\chi(\Phi(K)) \cap \partial B_\rho(0)] \setminus Y} \geq \tag{2.17}$$

$$\geq \text{Index } \chi(\Phi(k)) \cap \partial B_\rho(0) - \text{Index } Y \geq q - (q-1) > 0 .$$

Therefore 1° of Lemma 2.8 and (2.17) yield that $B \cap \partial B_\rho(0) \neq \phi$. Hence

$$\max_{v \in B} g(\lambda, v) \geq \min_{\|v\| = \rho} g(\lambda, v) \geq \frac{\mu - \lambda}{4} \rho^2 \tag{2.18}$$

via (2.15). Thus $c_1 \geq \frac{1}{4}(\mu - \lambda)\rho^2 > 0$ by (2.18).

To prove (ii), suppose that c_j is not a critical value of $g(\lambda, \cdot)$. Then by Lemma 2.7 with $z = c_j$ and $\varepsilon_1 < z$, there is an $\varepsilon \in (0, \varepsilon_1)$ and a mapping $\theta(v) = \eta(1, v) \in C(\overline{Q}, \overline{Q})$ such that θ is odd in v and

$$\theta(A_{\lambda, c_j + \varepsilon}) \subset A_{\lambda, c_j - \varepsilon} . \tag{2.19}$$

For λ near μ, $g(\lambda, \cdot) < 0$ on T^-. Hence by 2° and 3° of Lemma 2.7, $\theta(v) = v$ for $v \in T^-$ and θ is 1 - 1 on $\overline{Q}$. Therefore $\theta \in \mathfrak{F}$. Choose $B \in \Gamma_j$ so that

$$\max_{v \in B} g(\lambda, v) \leq c_j + \varepsilon . \tag{2.20}$$

By 2° of Lemma 2.12, $\theta(B) \in \Gamma_j$. Consequently

$$\max_{v \in \theta(B)} g(\lambda, v) \geq c_j . \tag{2.21}$$

But (2.21) contradicts (2.19) - (2.20) so c_j is a critical value of $g(\lambda, \cdot)$.

A similar argument establishes (iii). Suppose $\text{Index } K_{\lambda d} < p$. By 4° of Lemma 2.8, there is a $\delta > 0$ so that $\text{Index } N_\delta(K_{\lambda d}) = \text{Index } K_{\lambda d} < p$. Invoking Lemma 2.7 again with $z = d$ and $\varepsilon_1 < d$, there exists an $\varepsilon \in (0, \varepsilon_1)$ and an odd map $\theta(v) = \eta(1, v) \in C(\overline{Q}, \overline{Q})$ such that $\theta \in \mathfrak{F}$ and

$$\theta(A_{\lambda, d+\varepsilon} \setminus N_\delta(K_{\lambda, d})) \subset A_{\lambda, d-\varepsilon} . \tag{2.22}$$

Choose $B \in \Gamma_{j+p}$ so that

$$\max_{v \in B} g(\lambda, v) \le d + \varepsilon = c_{j+p} + \varepsilon . \tag{2.23}$$

By 3° of Lemma 2.12, $\overline{B \setminus N_\delta(K_{\lambda d})} \in \Gamma_{j+1}$ and by 2° of the same lemma, $\theta(B \setminus N_\delta(K_{\lambda d})) \equiv M \in \Gamma_{j+1}$. Therefore

$$\max_{v \in M} g(\lambda, v) \ge d = c_{j+1} . \tag{2.24}$$

But (2.24) contradicts (2.22) - (2.23).

Finally to prove (iv), observe that $g(\lambda, v) \to g(\mu, v)$ uniformly for $v \in \overline{Q}$ as $\lambda \to \mu$. Moreover $\overline{\Phi(T^{-})} \in \Gamma_j$ for $1 \le j \le k$ and if $v \in \Phi(T^{-})$, $g(\mu, v) < 0$. Since $0 \in \overline{\Phi(T^{-})}$,

$$\max_{v \in \overline{\Phi(T^{-})}} g(\mu, v) = 0 .$$

Therefore as $\lambda \to \mu^{-}$,

$$0 < c_j(\lambda) \le \max_{v \in \overline{\Phi(T^{-})}} g(\lambda, v) \to 0 .$$

Thus if $v_j(\lambda)$ is a critical point of $g(\lambda, \cdot)$ in Q with $g(\lambda, v_j(\lambda)) = c_j(\lambda)$, we can find a sequence $\lambda_s \to \mu$ so that $v_j(\lambda_s) \to v$ with $g(\mu, v) = 0$ and $g_v(\mu, v) = 0$. But 0 is the unique critical point of $g(\mu, \cdot)$ in $\overline{Q}$. Hence as $\lambda \to \mu$, $v_j(\lambda) \to 0$. The proof of Theorem 2.9 is now complete.

Proof of Corollary 2.10: Replace $g(\lambda, v)$ by $-g(\lambda, v)$. The result is then immediate from Theorem 2.9.

Proof of Lemma 2.11: The proof is based on that of Lemma 2.7 of [5]. Let $\rho > 0$ with $B_\rho(0) \subset V$. By Lemma 2.5 with V replaced by $B_\rho(0)$, we can find a neighborhood Q_b of 0 having the same properties as Q with c replaced by b. If $v \in \partial Q_b \cap g(\mu, \cdot)^{-1}(-b)$, there is a unique $\tau(v) > 0$ so that $g(\mu, \psi(\tau(v), v)) = -c$. Moreover the map $\theta(v) = \psi(\tau(v), v)$ is odd and is in $C(\partial Q_b \cap g(\mu, \cdot)^{-1}(-b), \partial Q \cap g(\mu, \cdot)^{-1}(-c))$ with $\theta(S^- \cap \partial Q_b) = T^-$. Since Index $T^- = k$, by 2^o and 4^o of Lemma 2.8, there is a $\delta > 0$ so that Index $(N_\delta(T^-) \cap \partial Q) = k$. We claim for ρ sufficiently small, $\theta(\partial Q_b \cap g(\mu, \cdot)^{-1}(-b)) \subset N_\delta(T^-) \cap \partial Q$. For otherwise there exist sequences $\rho_m \to 0$, $b_m \to 0$, and $x_m \in B_{\rho_m}(0)$ such that $g(\mu, x_m) = b_m > 0$ and $\psi(\tau(x_m), x_m) \in \partial Q$ but $x_m \notin N_\delta(T^-)$. Clearly along a subsequence $\psi(\tau(x_m), x_m) \to y \in \partial Q$ and $y \notin T^-$. But since $x_m \to 0$, $\tau(x_m) \to \infty$. Therefore $y \in T^-$, a contradiction, and we can find ρ as above.

By 2^o of Lemma 2.8,

$$\text{(2.25)} \qquad \text{Index } S^- \cap \partial Q_b = \text{Index } T^- = k \leq$$

$$\text{Index}(\partial Q_b \cap g(\mu, \cdot)^{-1}(-b)) \leq \text{Index } N_\delta(T^-) \cap \partial Q = k .$$

Hence all inequalities in (2.25) are equalities. Similarly

$$\text{(2.26)} \qquad \text{Index}(\partial Q_b \cap g(\mu, \cdot)^{-1}(b)) = m = \text{Index } T^+ .$$

If $v \in \partial Q_b \backslash g(\mu, \cdot)^{-1}(-b)$, there is a unique $t(v) \leq 0$ so that $g(\mu, \psi(t(v), v)) = b$. It follows that $\xi(v) = \psi(t(v), v)$ is a continuous odd map of $\partial Q_b \backslash g(\mu, \cdot)^{-1}(-b)$ onto $\partial Q_b \cap g(\mu, \cdot)^{-1}(b)$. Hence by 2^o of Lemma 2.8 again,

$$\text{(2.27)} \quad \mathrm{Index}(\partial Q_b \setminus \overline{g(\mu,\cdot)^{-1}(-b)}) \le$$

$$\mathrm{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(b)) = m \le \mathrm{Index}(\partial Q_b \setminus g(\mu,\cdot)^{-1}(-b)) .$$

Thus we have equality in (2.27). Combining (2.25), (2.27) and 3^o and 5^o of Lemma 2.8 yields

$$\text{(2.28)} \quad n = \mathrm{Index}\ \partial Q_b \le \mathrm{Index}(\partial Q_b \setminus \overline{g(\mu,\cdot)^{-1}(-b)}) + $$
$$+ \mathrm{Index}(\partial Q_b \cap g(\mu,\cdot)^{-1}(-b)) = m + k$$

and the proof of Lemma 2.11 is complete.

§ 3. Definition and Properties of Index

The concepts of (Ljusternik-Schnirelmann) category as well as that of genus (called B-index by Yang [10] and coindex by Conner-Floyd [7]) have played a useful role in problems involving the existence of critical points. We develop here an alternative notion which is equivalent in a restricted category to the index introduced by Yang [11], and which has the properties usually enjoyed by these notions as well as one important additional one (Theorem 3.12 below). These properties were used in § 2 and summarized in Lemma 2.8, with 6° corresponding to Theorem 3.12 below.

We work with the category $\mathcal{C}$ of compact metric spaces which admit a free $\mathbb{Z}_2$-action. More precisely, an object of $\mathcal{C}$ is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a fixed point free homeomorphism of period 2. The morphisms of $\mathcal{C}$ are equivarient maps. i.e. given (X, T) and (X', T') in $\mathcal{C}$ a morphism $f : (X, T) \to (X', T')$ is a (continuous) map $f : X \to X'$ such that $f(Tx) = T'f(x)$, for $x \in X$. Thus, compact symmetric subsets of a normed linear space are then objects in $\mathcal{C}$ and odd maps between such subsets are morphisms in $\mathcal{C}$. A fortiori, then the category $\mathcal{E}$ of symmetric subsets of some $\mathbb{R}^n \setminus 0$ is included in $\mathcal{C}$.

Given $(X, T) \in \mathcal{C}$, $\tilde{X} = X/T$ is the corresponding orbit space and the map $q : X \to \tilde{X}$ which identifies x and Tx is a 2-fold covering map.

As usual, we will denote by S^∞, the direct limit of the sequence of spheres of ascending dimension $S^1 \subset S^2 \subset S^3 \subset \cdots$, i.e., $S^\infty = \bigcup_{k=1}^{\infty} S^k$. S^∞ admits the antipodal action and P^∞, the corresponding infinite dimensional projective space, is on one hand the orbit space S^∞/T, and on the other, the direct limit of the projective spaces $P^1 \subset P^2 \subset P^3 \subset \cdots$. It is easy to see that there exist equivariant maps $f : X \to S^\infty$ (in fact into S^N for N large) and any such map induces a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & S^\infty \arrow[d] \\ \tilde{X} \arrow[r, "\tilde{f}"] & P^\infty \end{tikzcd}$$

where the vertical maps are the 2-fold covering maps and $\tilde{f}$ is naturally induced by $\tilde{f}$. We call any such $(f, \tilde{f})$ a <u>classifying map</u> for $(X, q, \tilde{X})$.

<u>Remark 3.1</u>: Both S^∞ and P^∞ receive the weak (= direct limit, = inductive) topology. For example, $U \subset S^\infty$ is open if, and only if $U \cap S^k$ is open in S^k for all $k = 1, 2, \ldots$. It then follows easily that every compact subset of $S^\infty(P^\infty)$ lies in some $S^k(P^k)$ for k sufficiently large.

<u>Remark 3.2</u>: We employ Čech cohomology with $\mathbb{Z}_2$ coefficients and the notation $H^q(X)$ stands for $H^q(X, \mathbb{Z}_2)$. We also use the fact that the $\mathbb{Z}_2$ cohomology of the real projective space P^n is the polynomial ring over $\mathbb{Z}_2$ on one indeterminate $u \in H^1(P^n)$, truncated by the relation

$u^{n+1} = 0$. Recall also that the inclusion map $i : P^n \to P^{n+1}$ indices an isomorphism $i^* : H^q(P^{n+1}) \to H^q(P^n)$ for $q \leq n$.

We now give the definition of index which we will employ. Let (X, T) denote an object of $\mathcal{C}$, as above, and let $(f, \tilde{f})$ denote a classifying map and N chosen so that $f(X) \subset S^N$. Then set $\varphi(f, \tilde{f})$ equal to the max k such that $\tilde{f}^*(u^k) \neq 0$ where

$$\tilde{f}^* : H^k(P^N) \to H^k(\tilde{X})$$

is induced by $\tilde{f} : \tilde{X} \to P^N$. Observe that $\varphi(f, \tilde{f})$ is independent of N and that $\varphi(f, \tilde{f}) \leq \dim X$.

Proposition-Definition 3.3: Set

$$\text{index } X = \varphi(f, \tilde{f})$$

for any classifying map $(f, \tilde{f})$, [or alternatively for any equivariant map $f : X \to S^\infty$]. Then, index X is independent of the choice of $(f, \tilde{f})$.

Proof: In order to prove independence of $(f, \tilde{f})$ let $(g, \tilde{g})$ denote another classifying map and choose N such that

$$\begin{array}{ccc} X & \xrightarrow{f} & S^N \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{f}} & P^N \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & S^N \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{g}} & P^N \end{array} .$$

We imbed X in the Hilbert cube Q^ω. If $\eta : X \to Q^\omega$ is such an imbedding, then $\zeta : X \to Q^\omega \times Q^\omega$ defined by $\zeta(x) = (x, Tx)$ is an *equivariant imbedding using the action* $S(u, v) = (v, u)$ on $Q^\omega \times Q^\omega$. Recall now that $\eta(X)$ in Q^ω can be approximated by polyhedra in the following sense: for every $\varepsilon > 0$ there is a set K_ε such $\eta(X) \subset \operatorname{int} K_\varepsilon \subset K_\varepsilon \subset U_\varepsilon \subset Q^\omega$ where U_ε is the ε-neighborhood of X, and K_ε is homeomorphic to $P_\varepsilon \times Q^\omega$ where P_ε is a *finite polyhedron*. A simple modification of this yields the following

Lemma 3.4: For every $\varepsilon > 0$ there is an invariant set $K_\varepsilon \subset Q^\omega \times Q^\omega$, (i.e. $(u, v) \varepsilon K_2 \Longleftrightarrow (v, u) \varepsilon K_\varepsilon$) on which S acts freely such that

$$\zeta(X) \subset \operatorname{int} K_\varepsilon \subset K_\varepsilon \subset U_\varepsilon \subset Q^\omega \times Q^\omega$$

where U_ε is the ε-nghd of $\zeta(X)$ in $Q^\omega \times Q^\omega$ and K_ε is homeomorphic to $P_\varepsilon \times Q^\omega$ where P_ε is a finite polyhedron.

Now, using the above lemma we may identify X with $\zeta(X)$ and T with S, so that $X \subset Q^\omega \times Q^\omega$. We may extend the equivariant maps f and g to a neighborhood V of X in $Q^\omega \times Q^\omega$ and hence to equivariant maps

$$F : K_\varepsilon \to S^N \qquad G : K_\varepsilon \to S^N$$

where $X \subset K_\varepsilon \subset V$ and K_ε is homeomorphic to $P_\varepsilon \times Q^\omega$, as in the above lemma. Now, we may appeal to the fact that $S^\infty \to P^\infty$ is a universal principal $\mathbb{Z}_2$-bundle to prove that $\tilde{F} \sim \tilde{G} : \tilde{K}_\varepsilon \to P^\infty$. Alternatively, working separately on the components of K_ε, one shows that

$$\tilde{F}_\# = \tilde{G}_\# : \pi_1(\tilde{K}_\varepsilon) \to \pi_1(P^\infty),$$

where $\tilde{F}_\#$ and $\tilde{G}_\#$ are the homomorphisms induced by $\tilde{F}$ and $\tilde{G}$, and then this forces $\tilde{F} \sim \tilde{G}$ since P^∞ is a $K(\mathbb{Z}_2, 1)$ (see [12], pg. 427). *Hence, for a large positive integer* M we have

$$\tilde{f} \sim \tilde{g} : \tilde{X} \to P^M$$

and hence $\tilde{f}^* = \tilde{g}^* : H^*(P^M) \to H^*(\tilde{X})$ and thus $\varphi(f, \tilde{f}) = \varphi(g, \tilde{g})$.

Remark 3.5: We adopt the convention that the index of the null set is -1 and if X is a non-empty set in $\mathcal{C}$ with $\tilde{f}^*(u) = 0$ above, then index $X = 0$. Also, notice that $\tilde{f}^*(u^k) = 0$ implies $\tilde{f}^*(u^\ell) = 0$ for $\ell > k$. We might also note here that a more inclusive notation would be index (X, T) rather than index X, since T plays a vital role. However, T is not usually displayed, by convention.

We now investigate the basic properties of this index.

Proposition 3.6: index $X \leq \dim X$.

Proof: This is immediate because $H^q(X) = 0$ for $q > \dim X$, where $\dim X$ refers to the covering dimension of X [13].

Proposition 3.7: If $g : X \to Y$ is equivariant, i.e. if g is a morphism of the category $\mathcal{C}$, then index $X \leq$ index Y.

Proof: Let $(f, \tilde{f})$ denote a classifying map for Y. Then, we have the diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Y \arrow[r, "f"] \arrow[d] & S^\infty \arrow[d] \\
\tilde{X} \arrow[r, "\tilde{g}"] & \tilde{Y} \arrow[r, "\tilde{f}"] & P^\infty
\end{tikzcd}$$

where $(h = fg, \tilde{h} = \tilde{f}\tilde{g})$ is a classifying map for X. If index $Y = k$, then for $j > k$

$$\tilde{h}^*(u^j) = \tilde{g}^*\tilde{f}^*(u^j) = 0$$

and hence index $X \leq k =$ index Y.

Corollary 3.8: If $X \subset Y$, then index $X \leq$ index Y.

Proposition 3.9: Let $K_1 \supset K_2 \supset \cdots \supset K_p \supset K_{p+1} \supset \cdots$ denote a descending sequence of compacta in Γ with $X = \bigcap K_p$ and all receiving their free $\mathbb{Z}_2$-action by restricting that of K_1. Then, for some p_0, index K_p = index X, $p \geq p_0$.

Proof: We know that index $X \leq$ index K_p for every p, since $X \subset K_p$. Therefore, it suffices to show that for some p_0, index $K_p \leq$ index X for $p \geq p_0$. Given an equivariant map $f : X \to S^N \subset S^\infty$, we may extend f to a neighborhood (in K_1) of X and hence we may assume without loss that f extends to $F : K_1 \to S^N \subset S^\infty$. Let $F_p = F|K_p$ and consider the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{i}_p"] \arrow[dr, "\tilde{i}_{p+1}"'] & \tilde{K}_p \arrow[r, "\tilde{F}_p"] & P^N \\
& \tilde{K}_{p+1} \arrow[u, "\tilde{j}_{p+1}"] \arrow[ur, "\tilde{F}_{p+1}"'] &
\end{tikzcd}$$

where $i_p : X \subset K_p$ and $j_{p+1} : K_{p+1} \subset K_p$ are inclusion maps. Then, we have an induced diagram

$$H^q(X) \xleftarrow{\alpha} \varinjlim H^q(K_p) \xleftarrow{\beta} H^q(P^N)$$

where $\alpha = \varinjlim \tilde{i}_p^*$ is an isomorphism using the continuity property of Čech theory, $p = \varinjlim \tilde{F}_p^*$ and $\alpha \circ \beta = f^* : H^q(P^N) \to H^q(\tilde{X})$. Suppose now that index $X = k$. Then, since $\tilde{f}^*(u^{k+1}) = 0$ and α is an isomorphism it follows that $\tilde{F}_{p_0}^*(u^{k+1}) = 0$ for some p_0 and hence for every $p \geq p_0$. Thus, index $K_p \leq k$ for all $p \geq p_0$ and the result follows.

Corollary 3.10: If $X \in \mathcal{C}$ is a subset of $\mathbb{R}^n \setminus \{0\}$, there is a symmetric polyhedron K in $\mathbb{R}^n \setminus 0$ such that $X \subset$ interior K and index X = index K. K may be chosen within any neighborhood of X and in fact K may be chosen as a smooth n-manifold with boundary.

Proof: Given a neighborhood W of X choose a sufficiently fine smooth triangulation of $\mathbb{R}^n \setminus \{0\}$ and let K denote a regular neighborhood of an appropriate subpolyhedron containing X.

Corollary 3.11: If $(X, T) \in \mathcal{C}$, then X may be equivariantly imbedded in $Q^\omega \times Q^\omega$ using the flip action $S(u, v) = (v, u)$ on $Q^\omega \times Q^\omega$. Identifying X with its image in $Q^\omega \times Q^\omega$ and T with S, there is a compact invariant set $K \subset Q^\omega \times Q^\omega$ such that $X \subset \operatorname{int} K$, index X = index K and K is homeomorphic to $\underline{P} \times Q^\omega$ where $\underline{P}$ is a finite polyhedron. K may be chosen within any neighborhood of X.

Proof: Apply Lemma 3.4.

Proposition 3.12: Suppose $X = A \cup B$, with A, B, and X in $\mathcal{C}$ and where A and B receive their free $\mathbb{Z}_2$-actions from X. Then,

$$\text{index } X \leq \text{index } A + \text{index } B + 1 .$$

Proof: We will make use of the cup product in Čech theory over $\mathbb{Z}_2$ (see [14], p. 288)

$$H^p(X, A) \otimes_{\mathbb{Z}_2} H^q(X, B) \to H^{p+q}(X, A \cup B) .$$

Suppose index $A = p$, index $B = q$ and index $X = k$. Let $(f, \tilde{f})$ be a classifying map for X, with $(f_1, \tilde{f}_1)$ and $(f_2, \tilde{f}_2)$ serving as classifying maps for A and B, respectively, where $f_1 = f|A$ and $f_2 = f|B$. Then, for N sufficiently large, we have the diagram

$$\begin{tikzcd} & H^m(P^N) \arrow[dl, "\tilde{f}_2^*"'] \arrow[d, "\tilde{f}^*"] \arrow[dr, "\tilde{f}_1^*"] & \\ -H^m(\tilde{B}) & \arrow[l, phantom] \arrow[r, phantom] H^m(\tilde{X}) \arrow[ul, "\tilde{i}^*"] \arrow[ur, "\tilde{j}^*"'] & H^m(\tilde{A}) \end{tikzcd}$$

and exact sequences for pairs

$$\cdots \to H^m(\tilde{X}, \tilde{A}) \xrightarrow{\alpha^*} H^m(\tilde{X}) \xrightarrow{\tilde{i}^*} H^m(\tilde{A}) \xrightarrow{\delta} H^{m+1}(\tilde{X}, \tilde{A}) \to \cdots$$

$$\cdots \to H^m(\tilde{X}, \tilde{B}) \xrightarrow{\beta^*} H^m(\tilde{X}) \xrightarrow{\tilde{j}^*} H^m(\tilde{B}) \xrightarrow{\delta} H^{m+1}(\tilde{X}, \tilde{B}) \to \cdots .$$

Since

$$0 = \tilde{f}_1^*(u^{p+1}) = \tilde{i}^*\tilde{f}^*(u^{p+1})$$

$$0 = \tilde{f}_2^*(u^{q+1}) = \tilde{j}^*\tilde{f}^*(u^{q+1})$$

we have $x \in H^{p+1}(\tilde{X}, \tilde{A})$, $y \in H^{q+1}(\tilde{X}, \tilde{B})$ such that

$$\alpha^*(x) = \tilde{f}^*(u^{p+1}), \quad \beta^*(y) = \tilde{f}^*(u^{q+1}) .$$

Now, using the naturality of the cup product;

$$\begin{tikzcd} H^{p+1}(\tilde{X}, \tilde{A}) \otimes H^{q+1}(\tilde{X}, \tilde{B}) \arrow[r] \arrow[d] & H^{p+q+2}(\tilde{X}, \tilde{A} \cup \tilde{B}) \arrow[d] \\ H^{p+1}(\tilde{X}) \otimes H^{q+1}(\tilde{X}) \arrow[r] & H^{p+q+2}(\tilde{X}) \end{tikzcd}$$

we see that $x \cup y = 0$ implies

$$0 = \tilde{f}^*(u^{p+1}) \cup \tilde{f}(u^{q+1}) = \tilde{f}^*(u^{p+q+2}) .$$

Therefore, $k \leq p + q + 1$ and the proof is complete.

Proposition 3.13: If U is a bounded symmetric open set in $\mathbb{R}^{n+1}$ containing the origin with boundary $B = \partial U$, then

$$\text{index } B = n .$$

Proof: One considers, as usual, the odd map $f : B \to S^n$ which takes x to $x/\|x\|$. This map induces an injection

$$\tilde{f}^* : H^q(P^n) \to H^q(\tilde{B}), \quad q \leq n .$$

The proof that $\tilde{f}^*$ is an injection is more or less classical and may be effected by using the transfer map (see Dold [14, p. 309]) as follows. First, we may assume that f is extended to an odd map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f^{-1}(S^n) = B$. If we let N^{n+1} denote $\mathbb{R}^{n+1} \setminus \{0\}$ with antipodes identified then f induces $\tilde{f} : N^{n+1} \to N^{n+1}$ with $\tilde{f}^{-1}(P^n) = \tilde{B}$ and $\tilde{f}_*(o_{\tilde{B}}) = o_{P^n}$ where $o_{\tilde{B}} \in H_{n+1}(N^{n+1}, N^{n+1} \setminus \tilde{B})$, $o_{P^n} \in H_{n+1}(N^{n+1}, N^{n+1} \setminus P^n)$ are fundamental classes over $\mathbb{Z}_2$. Then, according to [14], there is a transfer map (over $\mathbb{Z}_2$)

$$\tilde{f}_! : H^q(\tilde{B}) \to H^q(P^n)$$

which acts as a right inverse for $\tilde{f}^* : H^q(P^n) \to H^q(\tilde{B})$. Thus, $\tilde{f}^*$ is an injection and this forces index $B \geq n$. Finally, since index $B \leq \dim B = n$, we have the desired result.

We now proceed to verify an important additional geometric property of index as defined above and which corresponds to 6° of Lemma 2.8.

<u>Theorem 3.14</u>: Assume the following:

(i) M^{n-1} is a compact connected symmetric manifold in $\mathbb{R}^n\setminus\{0\}$ separating $\mathbb{R}^n$ into components U and V.

(ii) A is a symmetric compact subset of U.

(iii) $\varphi : A \times [0, \tau] \to \mathbb{R}^n\setminus\{0\}$ is a symmetric imbedding ($\varphi(-x,t) = -\varphi(x,t)$) such that $\varphi(a, 0) = a$, $a \in A$ and $\varphi(A \times \tau) \subset V$.

Then, if we set $C = M^{n-1} \cap \varphi(A \times [0, \tau])$, we have index C = index A.

The proof of this theorem will make use of the following result.

<u>Proposition 3.15</u>: Suppose N^n is a manifold and $X \subset N^n$ is a compact subset of N^n separating N^n, say $N^n\setminus X = U|V$, so that $\bar{U} \cap \bar{V} = X$. Let A denote a compact space, $I = [0,1]$, and $\varphi : A \times I \to N^n$ an imbedding such that $\varphi(A \times \{0\}) \subset U$ and $\varphi(A \times \{1\}) \subset V$. If we set

$$C = \varphi(A \times I) \cap X, \quad g = \mathrm{proj}_1 \circ \varphi^{-1} : \varphi(A \times I) \to A,$$

and $g_0 = g|C$, then

$$g_0^* : H^q(A) \to H^q(C)$$

is injective (one to one) for all $q \geq 0$ (any coefficients).

<u>Proof</u>: There is no loss in identifying A and $\varphi(A \times \{0\})$ and also assuming that $\varphi(a, 0) = a$, $a \in A$. We introduce the notation:

$$B = \varphi(A \times 1)$$

$$A' = \bar{U} \cap \varphi(A \times I)$$

$$B' = \bar{V} \cap \varphi(A \times I)$$

and notice that

$$A' \cup B' = \varphi(A \times I), \qquad A' \cap B' = C .$$

Furthermore, the inclusion maps

$$A \xrightarrow{\alpha} A' \cup B', \qquad B \xrightarrow{\beta} A' \cup B'$$

are homotopy equivalences and g_0 serves as a homotopy inverse for α. We also introduce the inclusion maps,

$$i_1 : A' \to A' \cup B', \qquad i_2 : B' \to A' \cup B'$$

$$j_1 : A' \cap B' \to A, \qquad j_2 : A' \cap B' \to B'$$

$$k_1 : A \to A', \qquad k_2 : B \to B'.$$

Then, $i_1 \cdot k_1 = \alpha$ and $i_2 \cdot k_2 = \beta$ implies the induced maps i_1^* and i_2^* on cohomology are both injections. Consider now the Mayer-Vietoris sequence for $A' \cup B'$,

$$\longrightarrow H^q(A' \cup B') \xrightarrow{\zeta} H^q(A') \oplus H^q(B') \xrightarrow{\eta} H^q(A' \cap B') \longrightarrow$$

where $\zeta = (i_1^*, -i_2^*)$ and $\eta = j_1^* + j_2^*$. This forces $j_1^* : H^q(A') \to H^q(A' \cap B')$ to be an injection as follows. Suppose $j_1^*(a') = 0$. Then, for some $y \in H^q(A' \cup B')$ we have

$$\zeta(y) = (a', 0) = (i_1^*(y), -i_2^*(y))$$

and hence $i_2^*(y) = 0$. This forces $y = 0$ and hence $a' = 0$. Now, consider the retraction $g_1 = gi_1$ of A' to A. Since $g_1 k_1 = id_{A'}$, g_1^* is an injection and hence the diagram

$$\begin{tikzcd} H^q(A') \arrow[rr, "j_1^*"] & & H^q(A' \cap B') \\ & H^q(A) \arrow[ul, "g_1^*"] \arrow[ur, "g_0^*"'] & \end{tikzcd}$$

shows that g_0^* is an injection.

<u>Proof of Theorem 3.14</u>: Let N^n denote $\mathbb{R}^n \backslash \{0\}$ with antipodal points identified and apply Proposition 3.15 in N^n with $X = M^{n-1}$ as follows. Set

$$g = \mathrm{proj}_1 \cdot \varphi^{-1} : \varphi(A \times I) \to A$$
$$C = M^{n-1} \cap \varphi(A \times I) .$$

Let $\tilde{A}$, $\tilde{C}$, $\tilde{g}$ denote the corresponding objects in N^n and by Proposition 3.15

$$\tilde{g}_0^* : H^q(\tilde{C}) \to H^q(\tilde{A})$$

is injective. Take a classifying map $(f, \tilde{f})$ for A and we obtain a diagram

$$\begin{tikzcd} C \arrow[r, "g_0"] \arrow[d] & A \arrow[r, "f"] \arrow[d] & S^N \arrow[d] \\ \tilde{C} \arrow[r, "\tilde{g}_0"] & \tilde{A} \arrow[r, "\tilde{f}"] & P^N \end{tikzcd}$$

and $\tilde{f}^*(u^k) \neq 0$ if, and only if, $\tilde{g}_0^* \tilde{f}^*(u^k) \neq 0$ and the theorem follows.

<u>Remark 3.16</u>: Proposition 3.15 may also be employed to give an alternative proof of Proposition 3.13.

We indicated at the beginning of this section that this notion of index is equivalent in a restricted category to that introduced by Yang in [11]. We develop this further now.

Let $\mathcal{K}$ denote the category whose objects are pairs (X, T) with X a compact Hansdorff space and T a fixed point free involution on X, and whose morphisms are equivariant maps. The following definition is an equivalent formulation of Yang's index (see [11], § 3.6).

Definition 3.17: Given $(X, T) \in \mathcal{K}$, the Yang index of (X, T), denoted by Yang index X, is the largest integer n such that for any equivariant map $f : X \to Y$, with $(Y, S) \in \mathcal{K}$ arbitrary,

$$f_* : H_n(\tilde{X}) \to H_n(\tilde{Y})$$

is non-trivial, using Čech homology with $\mathbb{Z}_2$-coefficients, where $\tilde{X}$ and $\tilde{Y}$ are the orbit spaces X/T, Y/S, respectively.

Proposition 3.18: For $(X, T) \in \mathcal{C}$

$$\text{Yang index } X = \text{index } X .$$

Proof: The proof will make use of duality in Čech theory ([13], [15]) which takes the following form. On the category of compact spaces X, there are natural transformations φ and ψ

$$H^q(X) \xrightarrow{\varphi} [H^q(X)]^* \xrightarrow{\psi} H_q(X)$$

which are isomorphisms for each X, where $[H^q(X)]^*$ is the dual over $\mathbb{Z}_2$ of $H^q(X)$. We also make use of the fact that if $(Y, T) \in \mathcal{K}$, there is a finite complex K which admits a free $\mathbb{Z}_2$-action and an equivariant map $h : Y \to K$. K is, in fact, the nerve of an appropriate finite cover of Y and h a barycentric mapping (see [11]).

Now, suppose $(X, T) \in \mathcal{C}$ and $(Y, S) \in \mathcal{K}$ and let $f : X \to Y$ be an equivariant map. Then we have a diagram

$$\begin{array}{ccccc} H^q(\tilde{X}) & \xrightarrow{\varphi} & [H^q(\tilde{X})]^* & \xrightarrow{\psi} & H_q(\tilde{X}) \\ \uparrow f^* & & \downarrow (f^*)^* & & \downarrow f_* \\ H^q(\tilde{Y}) & \xrightarrow{\varphi} & [H^q(\tilde{Y})]^* & \xrightarrow{\psi} & H_q(\tilde{Y}) \end{array} .$$

If $f_* \neq 0$ for every Y, then this is so far $Y = S^N$ and $f^*(u^q) \neq 0$ for $\tilde{Y} = P^N$. Thus, index $X \geq$ Yang index X. On the other hand, to show index $X \leq$ Yang index X, suppose Y is chosen so that $f_* = 0$. First choose K as above and an equivariant map $g : Y \to K$ and then an equivariant map $h : K \to S^N$ for N sufficiently large. Now, $(h\, g\, f)_* = h_* g_* f_* = 0$ and hence $(h\, g\, f)^* = 0$, where

$$(h\, g\, f)^* : H^q(P^N) \to H^q(\tilde{X}) .$$

This shows, index $X \leq$ Yang index X and the proof is complete.

Let us recall the notion of genus which may be derived from Yang's notion of B-index (or the notion of coindex of Conner-Floyd). Given $(X, T) \in \mathcal{K}$, B-index X is the minimum k such that X admits an equivariant map $f : X \to S^k$. Then, we have, for $(X, T) \in \mathbf{C}$,

$$\text{Yang index } X = \text{index } X \leq \text{B-index } X .$$

Furthermore, for any symmetric compact subset X in a linear space, we have (directly from definitions)

$$\text{genus } X = \text{B-index } X + 1 .$$

It is, therefore, convenient to increase the index by 1 and define the notion of Index X as follows.

<u>Definition 3.19</u>: For $(X, T) \in \mathbf{C}$, set

$$\text{Index } X = \text{index } X + 1 .$$

Remark 3.20: Clearly then

$$\text{Index } X \leq \text{genus } X$$

and we note that in [10] Yang has an example of a symmetric imbedding of a polyhedron K in $\mathbb{R}^4$ such that

$$\text{Yang index } K = 1, \quad \text{B-index } K = 2 .$$

Since Yang index K = index K (by Theorem 3.17) we see that

$$\text{Index } K < \text{genus } K$$

so that the Index we have introduced may be strictly less than genus.

Finally one can translate the above relationships to those between Ljsternik-Schnirelman category and Index using the equivalence between genus and category in the appropriate setting (see [9]).

Lemma 2.8 was stated in terms of "Index". Basically the propositions we proved for "index" remain valid for "Index" with minor arithmetic changes. For example,

(3.5)' $X \neq \phi$ implies Index $X \geq 1$ and Index $(\phi) = 0$.

(3.6)' Index $X \leq \dim X + 1$.

(3.12)' Index $(A \cup B) \leq$ Index A + Index B .

(3.13)' Index $B = n + 1$, where B is the boundary of a symmetric bounded open neighborhood of 0 in $\mathbb{R}^{n+1}$, e.g. Index $S^n = n + 1$, $n \geq 0$.

Thus, the material in this section constitutes a proof of Lemma 2.8.

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A bifurcation theorem is proved for odd potential operators. The operator equation (*) $f'(u) \equiv Lu + H(u) = \lambda u$ is treated where $\lambda \in \mathbb{R}$ and $u \in E$, a real Hilbert space. A sharp description is given of the structure of the set of solutions of (*) near a bifurcation point as a function of λ. A crucial role is played here by a notion of topological index alternative to other indices used in critical point theory and the properties of this index are developed in some detail.