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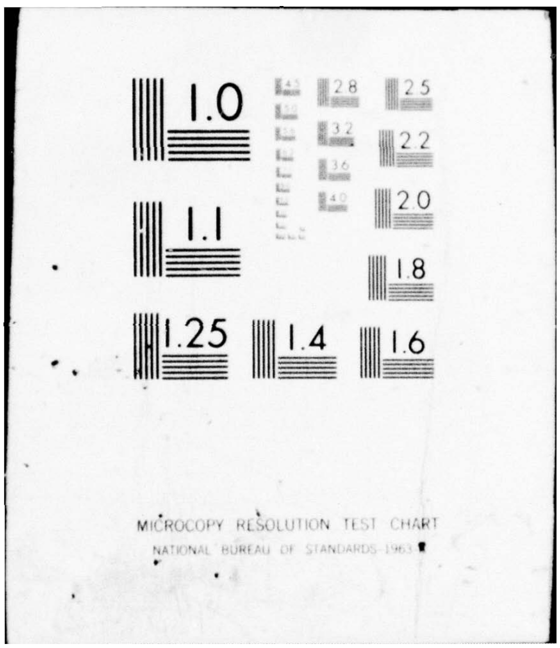
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AERO-ASTRONAUTICS REPORT NO. 132

FINAL REPORT

on

AIR FORCE GRANT No. AF-AFOSR-72-2185

NUMERICAL METHODS IN AEROSPACE SYSTEMS THEORY

by

ANGELO MIELE

RICE UNIVERSITY

1976

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Numerical analysis, numerical methods, computing methods, computing techniques. Differential equations, stiff differential equations, two-point boundary-value

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This document summarizes the research performed at Rice University during the period 1971-76 under Air Force Grant No. AF-AFOSR-72-2185 in several areas of numerical analysis of interest in aerospace systems theory, namely: (i) solution of non-

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19. KEY WORDS (continued)

problems, multipoint boundary-value problems.

Method of particular solutions, quasilinearization algorithm, modified quasilinearization algorithm, restoration algorithm.

Gradient algorithm, conjugate gradient algorithm, sequential gradient-restoration algorithm, sequential conjugate gradient-restoration algorithm.

Optimal control, calculus of variations, differential constraints, nondifferential constraints, bounded state, bounded control, bounded time rate of change of the state, free initial state, transformation techniques.

Aerospace engineering, optimum atmospheric flight trajectories, optimum extra-atmospheric flight trajectories, optimum aerodynamic shapes, optimum structures.

20. ABSTRACT (continued)

linear equations, (ii) solution of differential equations, (iii) mathematical programming problems, and (iv) optimal control problems.

The work summarized here is eminently applicable to these areas of aerospace engineering: (i) optimum atmospheric flight trajectories, (ii) optimum extra-atmospheric flight trajectories, (iii) optimum aerodynamic shapes, and (iv) optimum structures.

Final Report

on

Air Force Grant No. AF-AFOSR-72-2185

Numerical Methods in Aerospace Systems Theory[1,2]

by

ANGELO MIELE[3]

Abstract. This document summarizes the research performed at Rice University during the period 1971-76 under Air Force Grant No. AF-AFOSR-72-2185 in several areas of numerical analysis of interest in aerospace systems theory, namely: (i) solution of nonlinear equations, (ii) solution of differential equations, (iii) mathematical programming problems, and (iv) optimal control problems. The work summarized here is eminently applicable to these areas of aerospace engineering: (i) optimum atmospheric flight trajectories, (ii) optimum extra-atmospheric flight trajectories, (iii) optimum aerodynamic shapes, and (iv) optimum structures.

Key Words. Numerical analysis, numerical methods, computing methods, computing techniques.

Differential equations, stiff differential equations, two-point boundary-value problems, multipoint boundary-value problems.

Method of particular solutions, quasilinearization algorithm, modified quasilinearization algorithm, restoration algorithm.

Gradient algorithm, conjugate gradient algorithm, sequential gradient-

[1]Period October 1, 1971 through August 31, 1976.

[2]This research was supported by the Office of Scientific Research, Office of Aerospace Research, United States Air Force, Grant No. AF-AFOSR-72-2185.

[3]Professor of Astronautics and Mathematical Sciences, Rice University, Houston, Texas.

restoration algorithm, sequential conjugate gradient-restoration algorithm.

Optimal control, calculus of variations, differential constraints, nondifferential constraints, bounded state, bounded control, bounded time rate of change of the state, free initial state, transformation techniques.

Aerospace engineering, optimum atmospheric flight trajectories, optimum extra-atmospheric flight trajectories, optimum aerodynamic shapes, optimum structures.

Contents

I. Introduction

This ducument summarizes the research performed at Rice University in the period 1971-76 under Air Force Grant No. AF-AFOSR-72-2185. The total duration of the grant was 59 months. The grant was successively monitored by Lt. Col. N. P. Callas, Dr. R. G. Pohrer, and Lt. Col. E. H. Ramirez.

The personnel participating in the research effort included the following people:

Faculty Personnel

Prof. A. Miele
Prof. H. Y. Huang

Senior Personnel

Dr. J. N. Damoulakis
Dr. J. C. Heideman
Dr. A. V. Levy
Dr. A. Mangiavacchi
Major G. R. Hennig, USAF

Junior Personnel

Mr. A. K. Aggarwal
Mr. J. R. Cloutier
Mr. A. Esterle
Mr. A. V. Levy
Mr B. P. Mohanty
Mr. S. Naqvi
Mr. J. L. Tietze
Mr. K. H. Well
Mr. A. K. Wu

As a partial result of research performed under this grant, the following advanced degrees were awarded:

MS Degrees

A. K. Aggarwal
J. R. Cloutier
S. Naqvi
A. K. Wu

PhD Degrees

A. K. Aggarwal
J. R. Cloutier
A. V. Levy
S. Naqvi
J. L. Tietze
K. H. Well

II. Research Achievements

The research undertaken under Grant No. AF-AFOSR-72-2185 over the past five years has spanned several areas of numerical analysis, namely: (i) solution of nonlinear equations, (ii) solution of differential equations, (iii) mathematical programming problems, and (iv) optimal control problems. The principal results of this effort, summarized in 26 reports and 17 journal articles are described below.

(a) Development of techniques to handle two-point boundary-value problems characterized by stiff differential equations (Refs.11,12,22 and Refs.30,36).

(b) Comparison of several gradient algorithms for mathematical programming problems (Refs.2 and 37).

(c) Comparison of several gradient algorithms for optimal control problems (Refs.3 and 27).

(d) Development of the sequential gradient-restoration algorithm and the combined gradient restoration algorithm for optimal control problems (Refs.1,4,25 and Refs.28,43).

(e) Sequential gradient-restoration algorithm for optimal control problems with bounded state (Refs.5,6,10 and Refs.31,33).

(f) Modified quasilinearization algorithm for optimal control problems with bounded state (Refs.7,9,10 and Refs.32-33).

(g) Unconstrained approach to the extremization of terminally constrained optimal control problems (Refs.8 and 29).

(h) Sequential gradient-restoration algorithm for optimal control problems with nondifferential constraints (Refs.13-15 and 34).

(i) Modified quasilinearization algorithm for optimal control problems with nondifferential constraints (Refs.16-17 and 35).

(j) Sequential conjugate gradient-restoration algorithm for optimal control problems (Refs.20-21 and Refs.38-39).

(k) Hybrid approach to optimal control problems with bounded state (Refs.18-19 and 40).

(l) New transformation technique for optimal control problems with bounded state (Refs.23-24 and Refs.41-42).

(m) Conversion of optimal control problems with free initial state into optimal control problems with fixed initial state (Ref.26).

(n) Survey of the present state of the art in gradient algorithms for optimal control problems (Refs.25 and 43).

A list of research reports is given in Section VI, and a list of the research papers is given in Section VII. Then, the abstracts of the reports are given in Section VIII, and the abstracts of the papers are given in Section IX.

III. Areas of Application

Attention of USAF technical personnel is called on the fact that the research summarized in Refs.1-43 is not the result of an abstract mathematical formulation, but rather the result of concrete problems of applied mathematics arising in several areas of aerospace engineering, namely: (i) optimum atmospheric flight trajectories, (ii) optimum extra-atmospheric flight trajectories, (iii) optimum aerodynamic shapes, and (iv) optimum structures. For particular examples illustrating (i) and (iv) see the next section. For general examples illustrating (i) and (iii), consult the books listed below.[4,5]

[4] Miele, A., Flight Mechanics, Vol.1, Addison-Wesley Publishing Company, Reading, Massachusetts, 1962.

[5] Miele, A., Editor, Theory of Optimum Aerodynamic Shapes, Academic Press, New York, New York, 1965.

IV. Collaboration with Air Force Personnel

As a result of seminars given by the principal investigator at the USAF Academy, Colorado, and at Wright-Patterson Air Force Base, Ohio, a collaboration has been undertaken with Major G. R. Hennig (USAFA and WPAFB) and with Dr. V. B. Venkayya (WPAFB).

Directly, the collaboration with Major G. R. Hennig has resulted in the development of the sequential gradient-restoration algorithm for optimal control problems with bounded state (Hennig and Miele, Refs.5-6 and Ref.31). Indirectly, the collaboration with Major Hennig has lead to a paper by Major Hennig and his associates on optimum flight trajectories in a three-dimensional space.[6] In this paper, the methodology of the sequential gradient-restoration algorithm has been employed to optimize turning manoeuvers for turbojet-powered aircraft.

More recently, the collaboration with Dr. V. B. Venkayya has led to the employment of the sequential gradient-restoration algorithm in some problems of structural analysis.[7] It is anticipated that this preliminary effort will lead to further collaboration and to subsequent research on optimum structures.

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[6] Humphrey, R.P., Hennig, G.R., Bolding, W.A., and Hegelson, L.A., Optimum Three-Dimensional Minimum Time Turns for an Aircraft, Journal of the Astronautical Sciences, Vol.20, No.2, 1972.

[7] Miele, A., Numerical Determination of Minimum Mass Structures with Specified Natural Frequencies, Rice University, Internal Memorandum, 1976.

V. Research in Progress

At this time, considerable research is in progress on (i) the development of the sequential conjugate gradient-restoration algorithm for optimal control problems involving nondifferential constraints[8] and (ii) the study of transformation techniques for optimal control problems[9]. Completion of this research does not seem possible within the time limits of the current grant. However, completion is expected before the end of this calendar year and will take place within the limits of a subsequent grant being negotiated at this time.

[8] Cloutier, J.R., Mohanty, B.P., and Miele, A., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Rice University, Internal Memorandum, 1976.

[9] Miele, A., Transformation Techniques for Optimal Control Problems, Rice University, Internal Memorandum, 1976.

VI. Reports of the Aero-Astronautics Group

1. MIELE, A., Combined Gradient-Restoration Algorithm for Optimal Control Problems, Rice University, Aero-Astronautics Report No.91, 1971.
2. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Mathematical Programming Problems, Rice University, Aero-Astronautics Report No.94, 1972.
3. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Optimal Control Problems, Rice University, Aero-Astronautics Report No.95, 1972.
4. MIELE, A., Gradient Methods in Optimal Control Theory, Rice University, Aero-Astronautics Report No.98, 1971.
5. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State Variables, Part 1, Theory, Rice University, Aero-Astronautics Report No.101, 1972.
6. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State Variables, Part 2, Examples, Rice University, Aero-Astronautics Report No.102, 1972.
7. MIELE, A., WELL, K. H., and TIETZE, J. L., Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State Variables, Part 1, Theory, Rice University, Aero-Astronautics Report No.103, 1972.
8. HUANG, H. Y., and NAQVI, S., Extremization of Terminally Constrained Control Problems, Rice University, Aero-Astronautics Report No.104, 1972.
9. MIELE, A., WELL, K. H., and TIETZE, J. L., Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State Variables, Part 2, Examples, Rice University, Aero-Astronautics Report No.105, 1972.

10. HUANG, H. Y., and ESTERLE, A., Some Properties of the Sequential Gradient-Restoration Algorithm and the Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State, Rice University, Aero-Astronautics Report No.106, 1972.

11. MIELE, A., AGGARWAL, A. K., and TIETZE, J. L., Solution of a Two-Point Boundary-Value Problem with Jacobian Matrix Characterized by Extremely Large Eigenvalues, Rice University, Aero-Astronautics Report No.107, 1972.

12. MIELE, A., WELL, K. H., and TIETZE, J. L., Multipoint Approach to the Two-Point Boundary-Value Problem, Rice University, Aero-Astronautics Report No.108, 1972.

13. MIELE, A., DAMOULAKIS, J. N., and CLOUTIER, J. R., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 1, Theory, Rice University, Aero-Astronautics Report No.109, 1973.

14. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, Examples, Rice University, Aero-Astronautics Report No.110, 1973.

15. MIELE, A., DAMOULAKIS, J. N., and TIETZE, J. L., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 3, Examples, Rice University, Aero-Astronautics Report No.111, 1973.

16. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A. K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 1, Theory, Rice University, Aero-Astronautics Report No.112, 1973.

17. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A. K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, Examples, Rice University, Aero-Astronautics Report No.113, 1973.

18. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., A Hybrid Approach to Optimal Control Problems with Bounded State, Part 1, Theory, Rice University, Aero-Astronautics Report No.114,1974.

19. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., A Hybrid Approach to Optimal Control Problems with Bounded State, Part 2, Examples, Rice University, Aero-Astronautics Report No.115, 1974.

20. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 1, Theory, Rice University, Aero-Astronautics Report No.116, 1974.

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22. AGGARWAL, A. K., Some Numerical Results on Holt's Two-Point Boundary-Value Problem, Rice University, Aero-Astronautics Report No.118, 1973.

23. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 1, Theory, Rice University, Aero-Astronautics Report No.122, 1974.

24. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 2, Examples, Rice University, Aero-Astronautics Report No.123, 1974.

25. MIELE, A., Recent Advances in Gradient Algorithms for Optimal Control Problems, Rice University, Aero-Astronautics Report No.129, 1975.

26. MIELE, A., MOHANTY, B. P., and WU, A. K., Conversion of Optimal Control Problems with Free Initial State into Optimal Control Problems with Fixed Initial State, Rice University, Aero-Astronautics Report No.130, 1976.

VII. Papers of the Aero-Astronautics Group

27. MIELE, A., TIETZE, J. L., and LEVY, A. V., Summary and Comparison of Gradient-Restoration Algorithms for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol.10, No.6, 1972.

28. MIELE, A., Gradient Methods in Optimal Control Theory, Optimization and Design, Edited by M. Avriel, M. J. Rijckaert, and D. J. Wilde, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

29. HUANG, H. Y., and NAQVI, S., Extremization of Terminally Constrained Control Problems, Journal of the Astronautical Sciences, Vol.20, No.4, 1973.

30. MIELE, A., WELL, K. H., and TIETZE, J. L., Multipoint Approach to the Two-Point Boundary-Value Problem, Journal of Mathematical Analysis and Applications, Vol.44, No.3, 1973.

31. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State, Journal of Optimization Theory and Applications, Vol.12, No.1, 1973.

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34. MIELE, A., DAMOULAKIS, J. N., CLOUTIER, J. R., and TIETZE, J. L., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Journal of Optimization Theory and

Applications, Vol.13, No.2, 1974.

35. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A.K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Journal of Optimization Theory and Applications, Vol.14, No.5, 1974.

36. MIELE, A., AGGARWAL, A. K., and TIETZE, J. L., Solution of Two-Point Boundary-Value Problems with Jacobian Matrix Characterized by Large Positive Eigenvalues, Journal of Computational Physics, Vol.15, No.2, 1974.

37. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Mathematical Programming Problems, Omaggio a Carlo Ferrari, Edited by G. Jarre, Libreria Editrice Universitaria Levrotto e Bella, Torino, Italy, 1974.

38. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 1, Theory, Journal of Optimization Theory and Applications, Vol.15, No.2, 1975.

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40. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., A Hybrid Approach to Optimal Control Problems with Bounded State, Computer and Mathematics with Applications, Vol.1, No.2, 1975.

41. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 1, Theory, Aerotecnica, Missili, e Spazio, Vol.54, No.2, 1975.

42. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 2, Examples, Aerotecnica,

Missili, e Spazio, Vol.54, No.3, 1975.

43. MIELE, A., Recent Advances in Gradient Algorithms for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol.17, Nos.5/6, 1975.

VIII. Abstracts of Reports

1. MIELE, A., Combined Gradient-Restoration Algorithm for Optimal Control Problems, Rice University, Aero-Astronautics Report No.91, 1971.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state and the parameter are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A combined gradient-restoration algorithm is presented: this is an iterative algorithm characterized by variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading toward the optimality condition while simultaneously leading toward constraint satisfaction.

The variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are generated by requiring the first variations of the augmented functional J and the constraint error P to be negative. The procedure leads to a linear, two-point boundary-value problem, which is solved via the method of particular solutions. The descent properties of the algorithm are studied, and schemes to determine the optimum stepsize are discussed.

Key Words. Optimal control, numerical methods, computing methods, gradient algorithm, gradient-restoration algorithm, combined gradient-restoration algorithm.

2. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Mathematical Programming Problems, Rice University, Aero-Astronautics Report No.94, 1972.

Abstract. In this paper, the numerical solution of the basic problem of mathematical programming is considered. This is the problem of minimizing a function f(x) subject to a constraint $\phi(x) = 0$. Here, f is a scalar, x an n-vector, and ϕ a q-vector, with $q < n$.

Six variations of the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are considered, and their relative efficiency (in terms of number of iterations for convergence) is evaluated. The variations being considered are as follows:

(i) SGRA-CR, sequential gradient-restoration algorithm, complete restoration,
(ii) SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration,
(iii) SGRA-OR, sequential gradient-restoration algorithm, optional restoration,
(iv) CGRA-NR, combined gradient-restoration algorithm, no restoration,
(v) CGRA-AR, combined gradient-restoration algorithm, alternate restoration,
(vi) CGRA-OR, combined gradient-restoration algorithm, optional restoration.

Evaluation of these algorithms is accomplished through eight numerical examples. The first two examples pertain to quadratic functions subject to linear constraints. The remaining examples pertain to nonquadratic functions subject to nonlinear constraints. The results indicate that (a) the inclusion of a restoration phase is necessary for rapid convergence and (b) the algorithms with alternate restoration or optional restoration are the most efficient among those considered here.

Key Words. Mathematical programming, numerical methods, computing methods, gradient algorithm, restoration algorithm, gradient-restoration algorithm.

3. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Optimal Control Problems, Rice University, Aero-Astronautics Report No.95, 1972.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state and the parameter are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

Four variations of the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are considered, and their relative efficiency (in terms of number of iterations for convergence) is evaluated. The variations being considered are as follows:

(i) SGRA-CR, sequential gradient-restoration algorithm, complete restoration,

(ii) SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration,

(iii) CGRA-NR, combined gradient-restoration algorithm, no restoration,

(iv) CGRA-AR, combined gradient-restoration algorithm, alternate restoration.

Evaluation of these algorithms is accomplished through six numerical examples. The results indicate that (a) the inclusion of a restoration phase is necessary for rapid convergence and (b) the algorithms with alternate restoration are the most efficient among those considered here.

Key Words. Optimal control, numerical methods, computing methods, gradient algorithm, restoration algorithm, gradient-restoration algorithm.

4. MIELE, A., Gradient Methods in Optimal Control Theory, Rice University, Aero-Astronautics Report No.98, 1971.

Abstract. In this paper, recent work performed at Rice University on gradient methods in optimal control theory is presented from a unified point of view. The problem considered is that of minimizing a functional I with respect to the state $x(\theta)$ and the control $u(\theta)$ which satisfy a vector differential constraint, a vector initial condition, and a vector final condition. The algorithms presented are two: (i) the sequential gradient-restoration algorithm and (ii) the combined gradient-restoration algorithm.

Key Words. Optimal control, numerical methods, computing methods, gradient algorithm, restoration algorithm, sequential gradient-restoration algorithm, combined gradient-restoration algorithm.

5. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State Variables, Part 1, Theory, Rice University, Aero-Astronautics Report No.101, 1972.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state variable inequality constraint, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized. The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

Here, the state variable inequality constraint is handled in a direct manner. A predetermined number and sequence of subarcs is assumed and, for the time interval for which the trajectory of the system lies on the state boundary, the control is determined so that the state boundary is satisfied. The state boundary and the entrance conditions are assumed to be linear in x and π , and the sequential gradient-restoration algorithm

is constructed in such a way that the state variable inequality constraint is satisfied at each iteration of the gradient phase and the restoration phase along all of the subarcs composing the trajectory.

At first glance, the assumed linearity of the state boundary and the entrance conditions appears to be a limitation to the theory. Actually, this is not the case. The reason is that every constrained minimization problem can be brought to the present form through the introduction of additional state variables.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that each of the subarcs composing the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time corner conditions and variable-time terminal conditions are transformed into fixed-time corner conditions and fixed-time terminal conditions. The actual times θ_1, θ_2, τ at which (i) the state boundary is entered, (ii) the state boundary is exited, and (iii) the terminal boundary is reached are regarded to be components of the parameter π being optimized.

Several numerical examples illustrating the theory of this paper are given in Part 2 (see Ref.6). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, sequential gradient-restoration algorithm, anchoring conditions.

6. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State Variables, Part 2, Examples, Rice University, Aero-Astronautics Report No.102, 1972.

Abstract. In Ref.5, Hennig and Miele developed the sequential gradient-restoration algorithm for minimizing a functional subject to differential constraints, a state variable inequality constraint, and terminal constraints. In this report, six numerical examples are presented, four pertaining to the fixed-final-time case and two pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm of Ref.5.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, sequential gradient-restoration algorithm, anchoring conditions.

7. MIELE, A., WELL, K. H., and TIETZE, J. L., Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State Variables, Part 1, Theory, Rice University, Aero-Astronautics Report No.103, 1972.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state variable inequality constraint, and terminal constraints. The problem is to find the state $x(t)$, the control $u(t)$, and the parameter π so the the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

A modified quasilinearization algorithm is developed. Its main property is the descent property in the performance index R, the cumulative error in the constraints and the optimum conditions. Modified quasilinearization differs from ordinary quasilinearization because of the inclusion of the scaling factor (or stepsize) α in the system of variations. The stepsize is determined by a one-dimensional search on the performance index R. Since the first variation δR is negative, the decrease in R is guaranteed if α is sufficiently small. Convergence to the solution is achieved when R becomes smaller than some preselected value.

Here, the state variable inequality constraint is handled in a direct manner. A predetermined number and sequence of subarcs is assumed and, for the time interval for which the trajectory of the system lies on the state boundary, the control is determined so that the state boundary is satisfied. The state boundary and the entrance conditions are assumed to be linear in x and π, and the modified quasilinearization algorithm is constructed in such a way that the state variable inequality constraint is satisfied at each iteration and along all of the subarcs composing

the trajectory.

At first glance, the assumed linearity of the state boundary and the entrance conditions appears to be a limitation to the theory. Actually, this is not the case. The reason is that every constrained minimization problem can be brought to the present form through the introduction of additional state variables.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that each of the subarcs composing the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time corner conditions and variable-time terminal conditions are transformed into fixed-time corner conditions and fixed-time terminal conditions. The actual times θ_1, θ_2, τ at which (i) the state boundary is entered, (ii) the state boundary is exited, and (iii) the terminal boundary is reached are regarded to be components of the parameter π being optimized.

In order to start the algorithm, some nominal functions $x(t)$, $u(t)$, π and nominal multipliers $\lambda(t)$, $\rho(t)$, σ, μ must be chosen. In a real problem, the selection of the nominal functions can be made on the basis of physical considerations. Concerning the nominal multipliers, no useful guidelines have been available thus far. In this paper, an auxiliary minimization algorithm for selecting the multipliers optimally is presented: the performance index R is minimized with respect to $\lambda(t)$, $\rho(t)$, σ, μ. Since the functional R is quadratically dependent on the multipliers, the resulting variational problem is governed by optimality conditions which are linear and, therefore, can be solved without difficulty.

Several numerical examples illustrating the theory of this paper are given in Part 2 (see Ref.9). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, modified quasilinearization algorithm, anchoring conditions.

8. HUANG, H. Y., and NAQVI, S., Extremization of Terminally Constrained Control Problems, Rice University, Aero-Astronautics Report No.104, 1972.

Abstract. In this paper, the problem of extremizing a functional I involving the state x(t), the control u(t), and the parameter π is considered. The admissible state, control, and parameter are required to satisfy a vector differential constraint, a vector initial constraint, and a vector terminal constraint.

This problem is transformed into a mathematically simpler, unconstrained problem of minimizing a new functional, the performance index R. In turn, R involves the state, the control, the parameter, the Lagrange multiplier $\lambda(t)$ associated with the vector differential constraint, and the Lagrange multiplier μ associated with the vector terminal constraint. To obtain the minimum $R = 0$ of the performance index, a gradient algorithm is first developed. In order to achieve simplicity in programming and to bypass the explicit use of second-order derivatives, the gradient algorithm is modified so that it becomes a pure, first-order method. For better convergence property, a conjugate-gradient algorithm is also developed.

Concerning the determination of the stepsize in these algorithms, a one-cycle cubic interpolation scheme is presented. Again, the explicit use of second-order derivatives is avoided here.

Both the gradient algorithm and the conjugate-gradient algorithm are tested through several numerical examples. The results show that, while the gradient algorithm is relatively slow in convergence, the conjugate-gradient algorithm displays better convergence characteristics.

Key Words. Optimal control, numerical methods, computing methods, terminal constraints, performance index, gradient algorithm, conjugate-gradient algorithm.

9. MIELE, A., WELL, K. H., and TIETZE, J. L., Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State Variables, Part 2, Examples, Rice University, Aero-Astronautics Report No.105, 1972.

Abstract. In Ref.7, Miele, Well, and Tietze developed the modified quasilinearization algorithm for minimizing a functional subject to differential constraints, a state variable inequality constraint, and terminal constraints. In this report, six numerical examples are presented, four pertaining to the fixed-final-time case and two pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the modified quasilinearization algorithm of Ref.7.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, modified quasilinearization algorithm, anchoring conditions.

10. HUANG, H. Y., and ESTERLE, A., Some Properties of the Sequential Gradient-Restoration Algorithm and the Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State, Rice University, Aero-Astronautics Report No.106, 1972.

Abstract. In Refs.5-7 and 9, the sequential gradient-restoration algorithm and the modified quasilinearization algorithm were developed for optimal control problems with bounded state. These algorithms have a basic property: for a subarc lying on the state boundary, the state boundary equations are satisfied at every iteration, if they are satisfied at the beginning of the computational process. Thus, the subarc remains anchored on the state boundary. In this paper, the anchoring conditions employed in Refs.5-7 and 9 are derived.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, sequential gradient-restoration algorithm, modified quasilinearization algorithm, anchoring conditions.

11. MIELE, A., AGGARWAL, A. K., and TIETZE, J. L., Solution of a Two-Point Boundary-Value Problem with Jacobian Matrix Characterized by Extremely Large Eigenvalues, Rice University, Aero-Astronautics Report No.107, 1972.

Abstract. This paper treats the nonlinear, two-point boundary-value problem

$$\ddot{x} - k \sinh (kx) = 0, \quad x(0) = 0, \quad x(1) = 1,$$

for relatively large values of k, namely, $k = 5$, $k = 6$, and $k = 10$. Computationally speaking, this is an extremely difficult problem, owing to the fact that the Jacobian matrix is characterized by an extremely large positive eigenvalue: for $k = 10$, the order of magnitude of this positive eigenvalue near the final point is 10^3.

The resulting numerical difficulties are reduced by treating the two-point boundary-value problem as a multipoint boundary-value problem. The modified quasilinearization algorithm of Ref.12 is employed. This is a totally finite- difference approach, which bypasses the integration of the nonlinear equations, which characterizes shooting methods. Solutions for x(t) precise to six significant figures are obtained.

Key Words. Differential equations, stiff differential equations, numerical methods, computing methods, eigenvalues, two-point boundary-value problem, multipoint boundary-value problem, modified quasilinearization algorithm.

12. MIELE, A., WELL, K. H., and TIETZE, J. L., Multipoint Approach to the Two-Point Boundary-Value Problem, Rice University, Aero-Astronautics Report No.108, 1972.

Abstract. This paper treats nonlinear, two-point boundary-value problems of the form $\dot{x} - \phi(x,t) = 0$, in which the Jacobian matrix $\phi_x(x,t)$ is characterized by large positive eigenvalues. The resulting numerical difficulties are reduced by treating the two-point boundary-value problem as a multipoint boundary-value problem. A totally finite-difference approach is employed, thus bypassing the often troublesome integration of the nonlinear equations, which characterizes shooting methods.

The approach employed consists of extending to multipoint boundary-value problems the modified quasilinearization method developed by Miele and Iyer for two-point boundary-value problems. Basic to the method is the consideration of the performance index P, the cumulative error in the differential equations, the boundary conditions, and the interface conditions.

A modified quasilinearization algorithm is generated by requiring the first variation of the performance index δP to be negative. This algorithm differs from the ordinary quasilinearization algorithm because of the inclusion of the scaling factor or stepsize α in the system of variations. The main property of the modified quasilinearization algorithm is the descent property: if the stepsize α is sufficiently small, the reduction in P is guaranteed. Convergence to the desired solution is achieved when the inequality $P \leq \varepsilon$ is met, where ε is a small, preselected number.

The variations per unit stepsize $\Delta x(t)/\alpha = A(t)$ are governed by a system of mn nonhomogeneous, linear differential equations subjected to p initial conditions, q final conditions, and (m-1)n interface conditions,

with $p + q = n$, where n is the dimension of the vector x and m is the number of subintervals. Therefore, the total number of boundary conditions and interface conditions is mn. The above system is solved employing the method of particular solutions: $m(n+1)$ particular solutions are combined linearly, and the coefficients of the combination are determined so that the linear system is satisfied.

Four numerical examples are presented, two dealing with linear systems and two dealing with nonlinear systems. The examples illustrate the effectiveness as well as the rapidity of convergence of the present method.

Key Words. Differential equations, stiff differential equations, numerical methods, computing methods, eigenvalues, two-point boundary-value problem, multipoint boundary-value problem, modified quasilinearization algorithm.

13. MIELE, A., DAMOULAKIS, J. N., and CLOUTIER, J. R., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 1, Theory, Rice University, Aero-Astronautics Report No.109, 1973.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, nondifferential constraints, and terminal constraints. The problem is to find the state x(t), the control u(t),and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

The stepsize of the gradient phase is determined by a one-dimensional search on the augmented functional J, and the stepsize of the restoration phase by a one-dimensional search on the constraint error P. If α_g is the gradient stepsize and α_r is the restoration stepsize, the gradient corrections are of $O(\alpha_g)$ and the restoration corrections are of $O(\alpha_r \alpha_g^2)$. There-

fore, for α_g sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional $\hat{I}$ at the end of any cycle is smaller than the functional I at the beginning of the cycle.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time terminal conditions are transformed into fixed-time terminal conditions. The actual time τ at which the terminal boundary is reached is regarded to be a component of the parameter π being optimized.

The present general formulation differs from that contained in previous work by Miele, Pritchard, and Damoulakis because of the inclusion of the nondifferential constraints to be satisfied everywhere over the interval $0 \leq t \leq 1$. Its importance lies in that (i) many optimization problems arise directly in the form considered here, (ii) problems involving state equality constraints can be reduced to the present scheme through suitable transformations, and (iii) problems involving inequality constraint can be reduced to the present scheme through suitable transformations. The latter statement applies, for instance, to the following situations: (a) problems with bounded control, (b) problems with bounded state, (c) problems with bounded time rate of change of the state, and (d) problems where some bound is imposed on an arbitrarily prescribed function of the parameter, the control, the state, and the time rate of change of the state.

Numerical examples are presented in Parts 2 and 3 for both the fixed-final-time case and the free-final-time case (see Refs.14 and 15). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, non-differential constraints, transformation techniques, sequential gradient-restoration algorithm.

14. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, Examples, Rice University, Aero-Astronautics Report No.110, 1973.

Abstract. In Ref.13, Miele, Damoulakis, and Cloutier developed the sequential gradient-restoration algorithm for minimizing a functional subject to differential constraints, nondifferential constraints, and terminal constraints. In this report, several numerical examples are presented, some pertaining to the fixed-final-time case and some pertaining to the free-final-time case. Both equality constrained optimization problems and problems with bounded control are investigated. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm of Ref.13.

Key Words. Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, sequential gradient-restoration algorithm.

15. MIELE, A., DAMOULAKIS, J. N. and TIETZE, J. L., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 3, Examples, Rice University, Aero-Astronautics Report No.111, 1973.

Abstract. In Ref.13, Miele, Damoulakis, and Cloutier developed the sequential gradient-restoration algorithm for minimizing a functional subject to differential constraints, nondifferential constraints, and terminal constraints. In this report, several numerical examples are presented, some pertaining to the fixed-final-time case and some pertaining to the free-final-time case. Both problems with bounded state and problems with bounded time rate of change of the state are investigated. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm of Ref.13.

Key Words. Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, sequential gradient-restoration algorithm.

16. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A.K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 1, Theory, Rice University, Aero-Astronautics Report No. 112, 1973.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, nondifferential constraints, and terminal constraints. The problem is to find the state $x(t)$, the control $u(t)$, and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

A modified quasilinearization algorithm is developed. Its main property is the descent property in the performance index R, the cumulative error in the constraints and the optimality conditions. Modified quasilinearization differs from ordinary quasilinearization because of the inclusion of the scaling factor or stepsize α in the system of variations. The stepsize is determined by a one-dimensional search on the performance index R. Since the first variation δR is negative, the decrease in R is guaranteed if α is sufficiently small. Convergence to the solution is achieved when R becomes smaller than some preselected value.

In order to start the algorithm, some nominal functions $x(t)$, $u(t)$, π and nominal multipliers $\lambda(t)$, $\rho(t)$, μ must be chosen. In a real problem, the selection of the nominal functions can be made on the basis of physical considerations. Concerning the nominal multipliers, no useful guidelines have been available thus far. In this paper, an auxiliary minimization algorithm for selecting the multipliers optimally is presented: the performance index R is minimized with respect to $\lambda(t)$, $\rho(t)$, μ. Since the functional R is quadratically dependent on the multipliers, the resulting

variational problem is governed by optimality conditions which are linear and, therefore, can be solved without difficulty.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time terminal conditions are transformed into fixed-time terminal conditions. The actual time τ at which the terminal boundary is reached is regarded to be a component of the parameter π being optimized.

The present general formulation differs from that contained in previous work by Miele, Iyer, and Well because of the inclusion of the nondifferential constraints to be satisfied everywhere over the interval $0 \leq t \leq 1$. Its importance lies in that (i) many optimization problems arise directly in the form considered here, (ii) there are problems involving state equality constraints which can be reduced to the present scheme through suitable transformations, and (iii) there are problems involving inequality constraints which can be reduced to the present scheme through the introduction of auxiliary variables.

Numerical examples are presented in Part 2 for both the fixed-final-time case and the free-final-time case (see Ref.17). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, modified quasilinearization algorithm.

17. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A. K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, Examples, Rice University, Aero-Astronautics Report No.113, 1973.

Abstract. In Ref.16, Miele, Mangiavacchi, and Aggarwal developed the modified quasilinearization algorithm for minimizing a functional subject to differential constraints, nondifferential constraints, and terminal constraints. In this report, several numerical examples are presented, some pertaining to the fixed-final-time case and some pertaining to the free-final-time case. Both equality constrained optimization problems and inequality constrained optimization problems are investigated. The numerical results show the rapid convergence characteristics of the modified quasilinearization algorithm of Ref.16.

Key Words. *Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, modified quasilinearization algorithm.*

18. MIELE, A., TIETZE, J. L., and CLOUTIER, J. R., A Hybrid Approach to Optimal Control Problems with Bounded State, Part 1, Theory, Rice University, Aero-Astronautics Report No.114, 1974.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state inequality constraint, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized. The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

The technique employed is of the hybrid type, in an attempt to combine some of the best features of the direct approach and the indirect approach. While a predetermined number and sequence of subarcs are assumed (a feature of direct methods), enforcement of the state inequality constraint is obtained through a Valentine-type representation (a feature of indirect methods).

By properly choosing the analytical form of certain nondifferential constraints to be satisfied by the augmented control along each subarc composing the extremal arc (these nondifferential constraints are arrived at through the Valentine-type transformation), one ensures satisfaction of the state inequality constraint everywhere. Specifically, strict inequality is enforced for the subarcs internal to the state boundary and strict equality is enforced for the subarcs lying on the state boundary.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that each subarc composing the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time corner conditions and variable-time terminal conditions are transformed into fixed-time corner conditions and fixed-time terminal conditions. The actual times θ_1, θ_2, τ at which (i) the state boundary is entered, (ii) the state boundary is exited, and (iii) the terminal manifold is reached are regarded to be components of the parameter π being optimized.

Five numerical examples illustrating the theory are given in Part 2 (see Ref.19) and demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, nondifferential constraints, hybrid approach, sequential gradient-restoration algorithm.

19. MIELE, A., TIETZE, J. L., and CLOUTIER, J.R., A Hybrid Approach to Optimal Control Problems with Bounded State, Part 2, Examples, Rice University, Aero-Astronautics Report No.115, 1974.

Abstract. In Ref.18, Miele, Tietze, and Cloutier developed a modification of the sequential gradient-restoration algorithm for minimizing a functional subject to differential constraints, a state inequality constraint, and terminal constraints. In this report, five numerical examples are presented, some pertaining to the fixed-final-time case and some pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the modification of the sequential gradient-restoration algorithm discussed in Ref.18.

Key Words. Optimal control, numerical methods, computing methods, bounded state, nondifferential constraints, hybrid approach, sequential gradient-restoration algorithm.

20. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 1, Theory, Rice University, Aero-Astronautics Report No.116, 1974.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state and the parameter are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

First, the case of a quadratic functional subject to linear constraints is considered, and a conjugate gradient algorithm is derived. Nominal functions x(t), u(t), π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

Next, the more general case of a nonquadratic functional subject to nonlinear constraints is considered. The algorithm derived for the linear-quadratic case is employed with one modification: a restoration phase is inserted between any two successive conjugate gradient phases.

In the restoration phase, variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are determined by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Thus, a sequential conjugate gradient-restoration algorithm is

constructed in such a way that the differential equations and the boundary conditions are satisfied at the end of each complete conjugate gradient-restoration cycle.

Several numerical examples illustrating the theory of this paper are given in Part 2 (see Ref.21). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, sequential conjugate gradient-restoration algorithm.

21. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 2, Examples, Rice University, Aero-Astronautics Report No.117, 1974.

Abstract. In Ref.20, Heideman and Levy developed the sequential conjugate gradient-restoration algorithm for minimizing a functional subject to differential constraints and terminal constraints. In this report, several numerical examples are presented, some pertaining to a quadratic functional subject to linear constraints and some pertaining to a nonquadratic functional subject to nonlinear constraints. These examples demonstrate the feasibility as well as the rapid convergence characteristics of the sequential conjugate gradient-restoration algorithm of Ref.20.

Key Words. Optimal control, numerical methods, computing methods, sequential conjugate gradient-restoration algorithm.

22. AGGARWAL, A. K., Some Numerical Results on Holt's Two-Point Boundary-Value Problem, Rice University, Aero-Astronautics Report No.118, 1973.

Abstract. This paper treats the nonlinear, two-point boundary-value problem formulated by Holt for relatively large values of the final time τ, namely, $\tau = 11.3$, $\tau = 13.3$, and $\tau = 20.0$. Computationally speaking, this is a difficult problem, owing to the fact that the Jacobian matrix is characterized by relatively large positive eigenvalues.

The resulting numerical difficulties are reduced by treating the two-point boundary-value problem as a multipoint boundary-value problem. The modified quasilinearization algorithm of Ref.12 is employed. This approach bypasses the integration of the nonlinear equations, which characterizes shooting methods.

Key Words. Holt's problem, differential equations, stiff differential equations, numerical methods, computing methods, eigenvalues, two-point boundary-value problem, multipoint boundary-value problem, modified quasilinearization algorithm.

23. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 1, Theory, Rice University, Aero-Astronautics Report No.122, 1974.

Abstract. In this paper, the numerical solution of optimal control problems involving a state inequality constraint of the form $L(x,\theta) \geq 0$ is considered. The approach employed is of the hybrid type, in an attempt to combine some of the best features of the indirect approach and the direct approach. While a predetermined number and sequence of subarcs are assumed (a feature of direct methods), enforcement of the state inequality constraint is obtained through a Valentine-type representation (a feature of indirect methods).

A new transformation technique is developed by applying the Valentine-type representation to the kth derivative of the function $L(x,\theta)$, assumed to have a constant sign along each of the subarcs composing the extremal arc. By properly choosing the analytical form of certain nondifferential constraints to be satisfied by the augmented control along each subarc composing the extremal arc (these nondifferential constraints are arrived at through the Valentine-type transformation), one ensures satisfaction of the state inequality constraint everywhere. Specifically, strict inequality is enforced for the subarcs internal to the state boundary and strict equality is enforced for the subarcs lying on the state boundary.

Since the Valentine-type representation is applied to the kth derivative of the function $L(x,\theta)$, rather than to the function itself, problems with bounded state can be treated without augmenting the dimension of the state vector. This concept is important computationally, in that the computing time per iteration increases with the square of the dimension of the state vector.

The algorithm developed belongs to the class of sequential gradient-restoration algorithms. The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions $x(t)$, $u(t)$, π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

Five numerical examples illustrating the theory are given in Part 2 (see Ref.24). They demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, hybrid approach, transformation techniques, sequential gradient-restoration algorithm.

24. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 2, Examples, Rice University, Aero-Astronautics Report No.123, 1974.

Abstract. In Ref.23, Miele and Cloutier developed a new transformation technique for optimal control problems with bounded state. This transformation technique was employed in conjunction with the sequential gradient-restoration algorithm. In this report, five numerical examples are presented, some pertaining to the fixed-final-time case and some pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the algorithm discussed in Ref.23.

Key Words. Optimal control, numerical methods, computing methods, bounded state, hybrid approach, transformation techniques, sequential gradient-restoration algorithm.

25. MIELE, A., Recent Advances in Gradient Algorithms for Optimal Control Problems, Rice University, Aero-Astronautics Report No.129, 1975.

Abstract. This paper summarizes recent advances in the area of gradient algorithms for optimal control problems, with particular emphasis on the work performed by the staff of the Aero-Astronautics Group of Rice University.

The following basic problem is considered: minimize a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

First, the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are presented. The descent properties of these algorithms are studied, and schemes to determine the optimum stepsize are discussed. Both of the above algorithms require the solution of a linear, two-point boundary-value problem at each iteration. Hence, a discussion of integration techniques is given.

Next, a family of gradient-restoration algorithms is introduced. Not only does this family include the previous two algorithms as particular cases, but it allows one to generate several additional algorithms, namely, those with alternate restoration and optional restoration.

Then, two modifications of the sequential gradient-restoration algorithm are presented in an effort to accelerate terminal convergence. In the first modification, the quadratic constraint imposed on the variations

of the control is modified by the inclusion of a positive-definite weighting matrix (the matrix of the second derivatives of the Hamiltonian with respect to the control). The second modification is a conjugate-gradient extension of the sequential gradient-restoration algorithm.

Next, the addition of a nondifferential constraint, to be satisfied everywhere along the interval of integration, is considered. In theory, this seems to be only a minor modification of the basic problem. In practice, the change is considerable in that it enlarges dramatically the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control theory can be brought into this scheme. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with equality constraints on the time rate of change of the state, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with inequality constraints on the time rate of change of the state.

Finally, the simultaneous presence of nondifferential constraints and multiple subarcs is considered. The possibility that the analytical form of the functions under consideration might change from one subarc to another is taken into account. The resulting formulation is particularly relevant to those problems of optimal control involving bounds on the control or the state or the time derivative of the state. For these problems, one might be unwilling to accept the simplistic view of a continuous extremal arc. Indeed, one might want to take the more realistic view of an extremal arc composed of several subarcs, some internal to the boundary being considered and some lying on the boundary.

The paper ends with a section dealing with transformation techniques. This section illustrates several analytical devices by means of which a great number of problems of optimal control can be reduced to one of the formulations presented here. In particular, the following topics are treated: (i) time normalization, (ii) free initial state, (iii) bounded control, and (iv) bounded state.

Key Words. Survey papers, gradient methods, numerical methods, computing methods, calculus of variations, optimal control, gradient-restoration algorithms, boundary-value problems, bounded control problems, bounded state problems, nondifferential constraints.

26. MIELE, A., MOHANTY, B.P., and WU, A.K., Conversion of Optimal Control Problems with Free Initial State into Optimal Control Problems with Fixed Initial State, Rice University, Aero-Astronautics Report No.130, 1976.

Abstract. This paper considers optimal control problems involving the minimization of a functional subject to differential constraints, non-differential constraints, initial conditions, and final conditions. The initial conditions can be partly fixed and partly free. Transformation techniques are suggested, by means of which problems with free initial state are converted into problems with fixed initial state. Thereby, it becomes possible to employ, without change, some of the algorithms already developed for optimal control problems with fixed initial state (for instance, the sequential gradient-restoration algorithm).

The transformations introduced are two: (i) a linear transformation and (ii) a nonlinear transformation. In the linear-quadratic case, the former preserves unchanged the basic structure of the optimization problem, while this is not the case with the latter.

Three numerical examples are presented, namely, one linear-quadratic example and two nonlinear-nonquadratic examples. After transformations (i) and (ii) are introduced, these examples are solved by means of the sequential ordinary gradient-restoration algorithm and the sequential conjugate gradient-restoration algorithm.

Key Words. Optimal control, numerical methods, computing methods, transformation techniques, sequential ordinary gradient-restoration algorithm, sequential conjugate gradient-restoration algorithm, problems with free initial state.

IX. Abstracts of Papers

27. MIELE, A., TIETZE, J. L., and LEVY, A. V., Summary and Comparison of Gradient-Restoration Algorithms for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol.10, No.6, 1972.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state and the parameter are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

Four types of gradient-restoration algorithms are considered, and their relative efficiency (in terms of number of iterations for convergence) is evaluated. The algorithms being considered are as follows: sequential gradient-restoration algorithm, complete restoration (SGRA-CR), sequential gradient-restoration algorithm, incomplete restoration (SGRA-IR), combined gradient-restoration algorithm, no restoration (CGRA-NR), and combined gradient-restoration algorithm, incomplete restoration (CGRA-IR).

Evaluation of these algorithms is accomplished through six numerical examples. The results indicate that (i) the inclusion of a restoration phase is necessary for rapid convergence and (ii) while SGRA-CR is the most desirable algorithm if feasibility of the suboptimal solutions is required, rapidity of convergence to the optimal solution can be increased if one employs algorithms with incomplete restoration, in particular, CGRA-IR.

Key Words. Optimal control, numerical methods, computing methods, gradient algorithm, restoration algorithm, gradient-restoration algorithm.

28. MIELE, A., Gradient Methods in Optimal Control Theory, Optimization and Design, Edited by M. Avriel, M. J. Rijckaert, and D. J. Wilde, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

Abstract. In this paper, recent work performed at Rice University on gradient methods in optimal control theory is presented from a unified point of view. The problem considered is that of minimizing a functional I with respect to the state $x(\theta)$ and the control $u(\theta)$ which satisfy a vector differential constraint, a vector initial condition, and a vector final condition. The algorithms presented are two: (i) the sequential gradient-restoration algorithm and (ii) the combined gradient-restoration algorithm.

Key Words. Optimal control, numerical methods, computing methods, gradient algorithm, restoration algorithm, sequential gradient-restoration algorithm, combined gradient-restoration algorithm.

29. HUANG, H. Y., and NAQVI, S., Extremization of Terminally Constrained Control Problems, Journal of the Astronautical Sciences, Vol.20, No.4, 1973.

Abstract. In this paper, the problem of extremizing a functional I involving the state x(t), the control u(t), and the parameter π is considered. The admissible state, control, and parameter are required to satisfy a vector differential constraint, a vector initial constraint, and a vector terminal constraint.

This problem is transformed into a mathematically simpler, unconstrained problem of minimizing a new functional, the performance index R. In turn, R involves the state, the control, the parameter, the Lagrange multiplier λ(t) associated with the vector differential constraint, and the Lagrange multiplier μ associated with the vector terminal constraint. To obtain the minimum R = 0 of the performance index, a gradient algorithm is first developed. In order to achieve simplicity in programming and to bypass the explicit use of second-order derivatives, the gradient algorithm is modified so that it becomes a pure, first-order method. For better convergence property, a conjugate-gradient algorithm is also developed.

Concerning the determination of the stepsize in these algorithms, a one-cycle cubic interpolation scheme is presented. Again, the explicit use of second-order derivatives is avoided here.

Both the gradient algorithm and the conjugate-gradient algorithm are tested through several numerical examples. The results show that, while the gradient algorithm is relatively slow in convergence, the conjugate-gradient algorithm displays better convergence characteristics.

Key Words. Optimal control, numerical methods, computing methods, terminal constraints, performance index, gradient algorithm, conjugate-gradient algorithm.

30. MIELE, A., WELL, K. H., and TIETZE, J. L., Multipoint Approach to the Two-Point Boundary-Value Problem, Journal of Mathematical Analysis and Applications, Vol.44, No.3, 1973.

Abstract. This paper treats nonlinear, two-point boundary value problems of the form $\dot{x} - \phi(x,t) = 0$, in which the Jacobian matrix $\phi_x(x,t)$ is characterized by large positive eigenvalues. The resulting numerical difficulties are reduced by treating the two-point boundary value problem as a multipoint boundary value problem. A totally finite-difference approach is employed, thus bypassing the integration of the nonlinear equations, which characterizes shooting methods.

The approach employed consists of extending to multipoint boundary value problems the modified quasilinearization method developed by Miele and Iyer for two-point boundary value problems. Basic to the method is the consideration of the performance index P, which measures the cumulative error in the differential equations, the boundary conditions, and the interface conditions.

A modified quasilinearization algorithm is generated by requiring the first variation of the performance index δP to be negative. This algorithm differs from the ordinary quasilinearization algorithm because of the inclusion of the scaling factor or stepsize α in the system of variations. The main property of the modified quasilinearization algorithm is the descent property: if the stepsize α is sufficiently small, the reduction in P is guaranteed. Convergence to the desired solution is achieved when the inequality $P \leq \varepsilon$ is met, where ε is a small, preselected number.

The variations per unit stepsize $\Delta x(t)/\alpha = A(t)$ are governed by a

system of mn nonhomogeneous, linear differential equations subjected to p initial conditions, q final conditions, and (m-1)n interface conditions, with p + q = n, where n is the dimension of the vector x and m is the number of subintervals. Therefore, the total number of boundary conditions and interface conditions is mn. The above system is solved employing the method of particular solutions: m(n +1) particular solutions are combined linearly, and the coefficients of the combination are determined so that the linear system is satisfied.

Two numerical examples are presented, one dealing with a linear system and one dealing with a nonlinear system. The examples illustrate the effectiveness as well as the rapidity of convergence of the present method.

Key Words. Differential equations, stiff differential equations, numerical methods, computing methods, eigenvalues, two-point boundary-value problem, multipoint boundary-value problem, modified quasilinearization algorithm.

31. HENNIG, G. R., and MIELE, A., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Bounded State, Journal of Optimization Theory and Applications, Vol.12, No.1, 1973.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state inequality constraint, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized. The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

Here, the state inequality constraint is handled in a direct manner. A predetermined number and sequence of subarcs is assumed and, for the time interval for which the trajectory of the system lies on the state boundary, the control is determined so that the state boundary is satisfied. The state boundary and the entrance conditions are assumed to be linear in x and π , and the sequential gradient-restoration algorithm is constructed

in such a way that the state inequality constraint is satisfied at each iteration of the gradient phase and the restoration phase along all of the subarcs composing the trajectory.

At first glance, the assumed linearity of the state boundary and the entrance conditions appears to be a limitation to the theory. Actually, this is not the case. The reason is that every constrained minimization problem can be brought to the present form through the introduction of additional state variables.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that each of the subarcs composing the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time corner conditions and variable-time terminal conditions are transformed into fixed-time corner conditions and fixed-time terminal conditions. The actual times θ_1, θ_2, τ at which (i) the state boundary is entered, (ii) the state boundary is exited, and (iii) the terminal boundary is reached are regarded to be components of the parameter π being optimized.

The numerical examples illustrating the theory demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, sequential gradient-restoration algorithm, anchoring conditions.

32. MIELE, A., WELL, K. H., and TIETZE, J. L., Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State, Journal of Optimization Theory and Applications, Vol.12, No.3, 1973.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state inequality constraint, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

A modified quasilinearization algorithm is developed. Its main property is the descent property in the performance index R, the cumulative error in the constraints and the optimality conditions. Modified quasilinearization differs from ordinary quasilinearization because of the inclusion of the scaling factor or stepsize α in the system of variations. The stepsize is determined by a one-dimensional search on the performance index R. Since the first variation δR is negative, the decrease in R is guaranteed if α is sufficiently small. Convergence to the solution is achieved when R becomes smaller than some preselected value.

Here, the state inequality constraint is handled in a direct manner. A predetermined number and sequence of subarcs is assumed and, for the time interval for which the trajectory of the system lies on the state boundary, the control is determined so that the state boundary is satisfied. The state boundary and the entrance conditions are assumed to be linear in x and π, and the modified quasilinearization algorithm is constructed in such a way that the state inequality constraint is satisfied at each iteration and along all of the subarcs composing the trajectory.

At first glance, the assumed linearity of the state boundary and the entrance conditions appears to be a limitation to the theory. Actually, this is not the case. The reason is that every constrained minimization problem can be brought to the present form through the introduction of additional state variables.

In order to start the algorithm, some nominal functions $x(t)$, $u(t)$, π and nominal multipliers $\lambda(t)$, $\rho(t)$, σ, μ must be chosen. In a real problem, the selection of the nominal functions can be made on the basis of physical considerations. Concerning the nominal multipliers, no useful guidelines have been available thus far. In this paper, an auxiliary minimization algorithm for selecting the multipliers optimally is presented: the performance index R is minimized with respect to $\lambda(t)$, $\rho(t)$, σ, μ. Since the functional R is quadratically dependent on the multipliers, the resulting variational problem is governed by optimality conditions which are linear and, therefore, can be solved without difficulty.

The numerical examples illustrating the theory demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, modified quasilinearization algorithm, anchoring conditions.

33. HUANG, H. Y., and ESTERLE, A., Anchoring Conditions for the Sequential Gradient-Restoration Algorithm and the Modified Quasilinearization Algorithm for Optimal Control Problems with Bounded State, Journal of Optimization Theory and Applications, Vol.12, No.5, 1973.

Abstract. In Refs.31-32, the sequential gradient-restoration algorithm and the modified quasilinearization algorithm were developed for optimal control problems with bounded state. These algorithms have a basic property: for a subarc lying on the state boundary, the state boundary equations are satisfied at every iteration, if they are satisfied at the beginning of the computational process. Thus, the subarc remains anchored on the state boundary. In this paper, the anchoring conditions employed in Refs.31-32 are derived.

Key Words. Optimal control, numerical methods, computing methods, bounded state, transformation techniques, sequential gradient-restoration algorithm, modified quasilinearization algorithm, anchoring conditions.

34. MIELE, A., DAMOULAKIS, J. N., CLOUTIER, J. R., and TIETZE, J. L., Sequential Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Journal of Optimization Theory and Applications, Vol.13, No.2, 1974.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, nondifferential constraints, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

The stepsize of the gradient phase is determined by a one-dimensional search on the augmented functional J, and the stepsize of the restoration phase by a one-dimensional search on the constraint error P. If α_g is the gradient stepsize and α_r is the restoration stepsize, the gradient

corrections are of $O(\alpha_g)$ and the restoration corrections are of $O(\alpha_r \alpha_g^2)$. Therefore, for α_g sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional $\hat{I}$ at the end of any complete gradient-restoration cycle is smaller than the functional I at the beginning of the cycle.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time terminal conditions are transformed into fixed-time terminal conditions. The actual time τ at which the terminal boundary is reached is regarded to be a component of the parameter π being optimized.

The present general formulation differs from that of Miele, Pritchard, and Damoulakis because of the inclusion of the nondifferential constraints to be satisfied everywhere over the interval $0 \leq t \leq 1$. Its importance lies in that (i) many optimization problems arise directly in the form considered here, (ii) problems involving state equality constraints can be reduced to the present scheme through suitable transformations, and (iii) problems involving inequality constraints can be reduced to the present scheme through suitable transformations. The latter statement applies, for instance, to the following situations: (a) problems with bounded control, (b) problems with bounded state, (c) problems with bounded time rate of change of the state, and (d) problems where some bound is imposed on an arbitrarily prescribed function of the parameter, the control, the state, and the time rate of change of the state.

Numerical examples are presented for both the fixed-final-time case and the free-final-time case. These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, sequential gradient-restoration algorithm.

35. MIELE, A., MANGIAVACCHI, A., and AGGARWAL, A. K., Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints, Journal of Optimization Theory and Applications, Vol.14, No. 5, 1974.

Abstract. This paper considers the numerical solution of optimal control problems involving a fuctional I subject to differential constraints, nondifferential constraints, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

A modified quasilinearization algorithm is developed. Its main property is the descent property in the performance index R, the cumulative error in the constraints and the optimality conditions. Modified quasilinearization differs from ordinary quasilinearization because of the inclusion of the scaling factor or stepsize α in the system of variations. The stepsize is determined by a one-dimensional search on the performance index R. Since the first variation δR is negative, the decrease in R is guaranteed if α is sufficiently small. Convergence to the solution is achieved when R becomes smaller than some preselected value.

In order to start the algorithm, some nominal functions x(t), u(t), π and nominal multipliers $\lambda(t)$, $\rho(t)$, μ must be chosen. In a real problem, the selection of the nominal functions can be made on the basis of physical considerations. Concerning the nominal multipliers, no useful guidelines have been available thus far. In this paper, an auxiliary minimization algorithm for selecting the multipliers optimally is presented: the performance index R is minimized with respect to $\lambda(t)$, $\rho(t)$, μ . Since the functional R is quadratically dependent on the multipliers, the resulting variational problem is governed by optimality conditions which are linear

and, therefore, can be solved without difficulty.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time terminal conditions are transformed into fixed-time terminal conditions. The actual time τ at which the terminal boundary is reached is regarded to be a component of the parameter π being optimized.

The present general formulation differs from that contained in previous work by Miele, Iyer, and Well because of the inclusion of the nondifferential constraints to be satisfied everywhere over the interval $0 \leq t \leq 1$. Its importance lies in that (i) many optimization problems arise directly in the form considered here, (ii) there are problems involving state equality constraints which can be reduced to the present scheme through suitable transformations, and (iii) there are some problems involving inequality constraints which can be reduced to the present scheme through the introduction of auxiliary variables.

Numerical examples are presented for the free-final-time case. These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, nondifferential constraints, transformation techniques, modified quasilinearization algorithm.

36. MIELE, A., AGGARWAL, A. K., and TIETZE, J. L., Solution of Two-Point Boundary-Value Problems with Jacobian Matrix Characterized by Large Positive Eigenvalues, Journal of Computational Physics, Vol.15, No.2, 1974.

Abstract. This paper treats the nonlinear, two-point boundary-value problem formulated by Troesch and studied by Roberts and Shipman. Computationally speaking, this is a difficult problem, owing to the fact that the Jacobian matrix is characterized by large positive eigenvalues. The resulting difficulties are reduced by treating the two-point boundary-value problem as a multipoint boundary-value problem. The modified quasilinearization algorithm of Ref. 30 is employed. This approach bypasses the integration of the nonlinear equations, which characterizes shooting methods. Computational results are also presented for another difficult nonlinear, two-point boundary-value problem, namely, the problem formulated by Holt.

Key Words. Troesch's problem, Holt's problem, differential equations, stiff differential equations, numerical methods, computing methods, eigenvalues, two-point boundary-value problem, multipoint boundary-value problem, modified quasilinearization algorithm.

37. MIELE, A., TIETZE, J. L., and LEVY, A. V., Comparison of Several Gradient Algorithms for Mathematical Programming Problems, Omaggio a Carlo Ferrari, Edited by G. Jarre, Libreria Editrice Universitaria Levrotto e Bella, Torino, Italy, 1974.

Abstract. In this paper, the numerical solution of the basic problem of mathematical programming is considered. This is the problem of minimizing a function $f(x)$ subject to a constraint $\phi(x) = 0$. Here, f is a scalar, x an n-vector, and ϕ a q-vector, with $q < n$.

Six variations of the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are considered, and their relative efficiency (in terms of number of iterations for convergence) is evaluated. The variations being considered are as follows:

(i) SGRA-CR, sequential gradient-restoration algorithm, complete restoration,
(ii) SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration,
(iii) SGRA-OR, sequential gradient-restoration algorithm, optional restoration,
(iv) CGRA-NR, combined gradient-restoration algorithm, no restoration,
(v) CGRA-AR, combined gradient-restoration algorithm, alternate restoration,
(vi) CGRA-OR, combined gradient-restoration algorithm, optional restoration.

Evaluation of these algorithms is accomplished through eight numerical examples. The first two examples pertain to quadratic functions subject to linear constraints. The remaining examples pertain to nonquadratic functions subject to nonlinear constraints. The results indicate that (a) the inclusion of a restoration phase is necessary for rapid convergence and (b) the algorithms with alternate restoration or optional restoration are the most efficient among those considered here.

Key Words. Mathematical programming, numerical methods, computing methods, gradient algorithm, restoration algorithm, gradient-restoration algorithm.

38. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 1, Theory, Journal of Optimization Theory and Applications, Vol.15, No.2, 1975.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π . Here, I is a scalar, x and n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state and the parameter are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

First, the case of a quadratic functional subject to linear constraints is considered, and a conjugate-gradient algorithm is derived. Nominal functions x(t), u(t), π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

Next, the more general case of a nonquadratic functional subject to nonlinear constraints is considered. The algorithm derived for the linear-quadratic case is employed with one modification: a restoration phase is inserted between any two successive conjugate-gradient phases.

In the restoration phase, variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are determined by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary

conditions. Thus, a sequential conjugate-gradient-restoration algorithm is constructed in such a way that the differential equations and the boundary conditions are satisfied at the end of each complete conjugate gradient-restoration cycle.

Several numerical examples illustrating the theory of this paper are given in Part 2 (see Ref.39). These examples demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, sequential conjugate gradient-restoration algorithm.

39. HEIDEMAN, J. C., and LEVY, A. V., Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems, Part 2, Examples, Journal of Optimization Theory and Applications, Vol.15, No.2, 1975.

Abstract. In Ref.38, Heideman and Levy developed the sequential conjugate gradient-restoration algorithm for minimizing a functional subject to differential constraints and terminal constraints. In this paper, several numerical examples are presented, some pertaining to a quadratic functional subject to linear constraints and some pertaining to a nonquadratic functional subject to nonlinear constraints. These examples demonstrate the feasibility as well as the rapid convergence characteristics of the sequential conjugate gradient-restoration algorithm of Ref.38.

Key Words. Optimal control, numerical methods, computing methods, sequential conjugate gradient-restoration algorithm.

40. MIELE, A., TIETZE, J. L., and CLOUTIER, J.R., A Hybrid Approach to Optimal Control Problems with Bounded State, Computer and Mathematics with Applications, Vol.1, No.2, 1975.

Abstract. This paper considers the numerical solution of optimal control problems involving a functional I subject to differential constraints, a state inequality constraint, and terminal constraints. The problem is to find the state x(t), the control u(t), and the parameter π so that the functional is minimized, while the constraints are satisfied to a predetermined accuracy.

The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized. The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions x(t), u(t), π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

The technique employed is of the hybrid type, in an attempt to combine some of the best features of both direct and indirect approaches. While a predetermined number and sequence of subarcs are assumed (a feature of direct methods), enforcement of the state inequality constraint is obtained through a Valentine-type representation (a feature of indirect methods).

By properly choosing the analytical form of certain nondifferential

constraints to be satisfied by the augmented control along each subarc composing the extremal arc (these nondifferential constraints are arrived at through the Valentine-type transformation), one ensures satisfaction of the state inequality constraint everywhere. Specifically, strict inequality is enforced for the subarcs internal to the state boundary and strict equality is enforced for the subarcs lying on the state boundary.

To facilitate the numerical solution on digital computers, the actual time θ is replaced by the normalized time t, defined in such a way that each subarc composing the extremal arc has a normalized time length $\Delta t = 1$. In this way, variable-time corner conditions and variable-time terminal conditions are transformed into fixed-time corner conditions and fixed-time terminal conditions. The actual times θ_1, θ_2, τ at which (i) the state boundary is entered, (ii) the state boundary is exited, and (iii) the terminal manifold is reached are regarded to be components of the parameter π being optimized.

The numerical examples illustrating the theory demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, nondifferential constraints, hybrid approach, sequential gradient-restoration algorithm.

41. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 1, Theory, Aerotecnica, Missili, e Spazio, Vol.54, No.2, 1975.

Abstract. In this paper, the numerical solution of optimal control problems involving a state inequality constraint of the form $L(x,\theta) \geq 0$ is considered. The approach employed is of the hybrid type, in an attempt to combine some of the best features of the direct approach and the indirect approach. While a predetermined number and sequence of subarcs are assumed (a feature of direct methods), enforcement of the state inequality constraint is obtained through a Valentine-type representation (a feature of indirect methods).

A new transformation technique is developed by applying the Valentine-type representation to the kth derivative of the function $L(x,\theta)$, assumed to have a constant sign in each of the subarcs composing the extremal arc. By properly choosing the analytical form of certain nondifferential constraints to be satisfied by the augmented control along each subarc composing the extremal arc (these nondifferential constraints are arrived at through the Valentine-type transformation), one ensures satisfaction of the state inequality constraint everywhere. Specifically, strict inequality is enforced for the subarcs internal to the state boundary and strict equality is enforced for the subarcs lying on the state boundary.

Since the Valentine-type representation is applied to the kth derivative of the function $L(x,\theta)$, rather than to the function itself, problems with bounded state can be treated without augmenting the dimension of the state vector. This concept is important computationally, in that the computing time per iteration increases with the square of the dimension of the state vector.

The algorithm developed belongs to the class of sequential gradient-restoration algorithms. The approach taken is a sequence of two-phase processes or cycles, each composed of a gradient phase and a restoration phase. The gradient phase involves a single iteration and is designed to decrease the functional, while the constraints are satisfied to first order. The restoration phase involves one or several iterations and is designed to restore the constraints to a predetermined accuracy, while the norm of the variations of the control and the parameter is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions $x(t)$, $u(t)$, π obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the functionals of any two elements of the sequence are comparable.

Numerical examples illustrating the theory are given in Part 2 (see Ref.42) and demonstrate the feasibility as well as the rapidity of convergence of the technique developed in this paper.

Key Words. Optimal control, numerical methods, computing methods, bounded state, hybrid approach, transformation techniques, sequential gradient-restoration algorithm.

42. MIELE, A., and CLOUTIER, J. R., New Transformation Technique for Optimal Control Problems with Bounded State, Part 2, Examples, Aerotecnica, Missili, e Spazio, Vol.54, No.3, 1975.

Abstract. In Ref.41, Miele and Cloutier developed a new transformation technique for optimal control problems with bounded state. This transformation technique was employed in conjunction with the sequential gradient-restoration algorithm. In this paper, three numerical examples are presented, one pertaining to the fixed-final-time case and two pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the algorithm discussed in Ref.41.

Key Words. Optimal control, numerical methods, computing methods, bounded state, hybrid approach, transformation techniques, sequential gradient-restoration algorithm.

43. MIELE, A., Recent Advances in Gradient Algorithms for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol.17, Nos.5/6, 1975

Abstract. This paper summarizes recent advances in the area of gradient algorithms for optimal control problems, with particular emphasis on the work performed by the staff of the Aero-Astronautics Group of Rice University.

The following basic problem is considered: minimize a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations.

First, the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are presented. The descent properties of these algorithms are studied, and schemes to determine the optimum stepsize are discussed. Both of the above algorithms require the solution of a linear, two-point boundary-value problem at each iteration. Hence, a discussion of integration techniques is given.

Next, a family of gradient-restoration algorithms is introduced. Not only does this family include the previous two algorithms as particular cases, but it allows one to generate several additional algorithms, namely, those with alternate restoration and optional restoration.

Then, two modifications of the sequential gradient-restoration algorithm are presented in an effort to accelerate terminal convergence. In the first modification, the quadratic constraint imposed on the variations of the control is modified by the inclusion of a positive-definite weighting matrix (the matrix of the second derivatives of the Hamiltonian with

respect to the control). The second modification is a conjugate-gradient extension of the sequential gradient-restoration algorithm.

Next, the addition of a nondifferential constraint, to be satisfied everywhere along the interval of integration, is considered. In theory, this seems to be only a minor modification of the basic problem. In practice, the change is considerable in that it enlarges dramatically the number and variety of problems of optimal control which can be treated by gradient-restoration algorithms. Indeed, by suitable transformations, almost every known problem of optimal control theory can be brought into this scheme. This statement applies, for instance, to the following situations: (i) problems with control equality constraints, (ii) problems with state equality constraints, (iii) problems with equality constraints on the time rate of change of the state, (iv) problems with control inequality constraints, (v) problems with state inequality constraints, and (vi) problems with inequality constraints on the time rate of change of the state.

Finally, the simultaneous presence of nondifferential constraints and multiple subarcs is considered. The possibility that the analytical form of the functions under consideration might change from one subarc to another is taken into account. The resulting formulation is particularly relevant to those problems of optimal control involving bounds on the control or the state or the time derivative of the state. For these problems, one might be unwilling to accept the simplistic view of a continuous extremal arc. Indeed, one might want to take the more realistic view of an extremal arc composed of several subarcs, some internal to the boundary being considered and some lying on the boundary.

The paper ends with a section dealing with transformation techniques.

This section illustrates several analytical devices by means of which a great number of problems of optimal control can be reduced to one of the formulations presented here. In particular, the following topics are treated: (i) time normalization, (ii) free initial state, (iii) bounded control, and (iv) bounded state.

Key Words. Survey papers, gradient methods, numerical methods, computing methods, calculus of variations, optimal control, gradient-restoration algorithms, boundary-value problems, bounded control problems, bounded state problems, nondifferential constraints.