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A SURVEY OF M-MATRIX CHARACTERIZATIONS. I. NONSINGULAR M-MATRIC--ETC(U)

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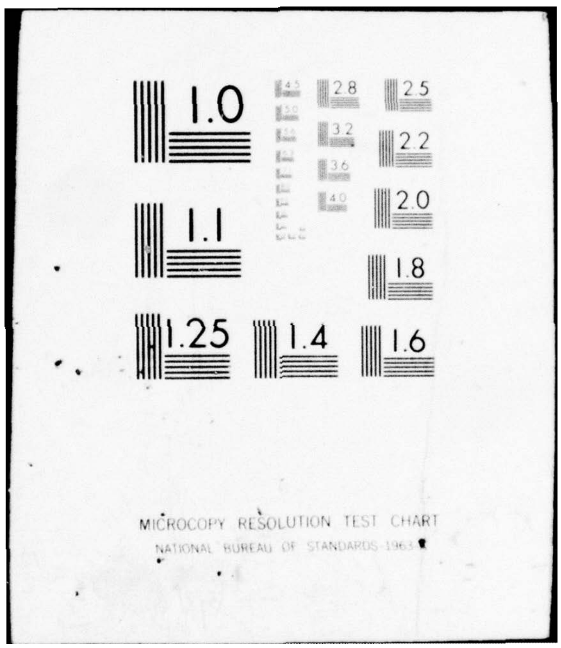
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A SURVEY OF M-MATRIX CHARACTERIZATIONS I: NONSINGULAR M-MATRICES

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I: NONSINGULAR M-MATRICES

R. J. Plemmons[1)2)]

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ABSTRACT

The purpose of this survey is to classify systematically a widely ranging list of characterizations of nonsingular M-matrices from the economics and mathematics literatures. These characterizations are grouped together in terms of their relationships to the properties of (1) positivity of principal minors (2) inverse-positivity and splittings (3) stability and (4) semi-positivity and diagonal dominance.

A list of forty equivalent conditions is given for a square matrix A with nonpositive off-diagonal entries to be a nonsingular M-matrix. These conditions are grouped into classes in order to identify those that are equivalent for arbitrary real matrices A.

In addition, other remarks relating nonsingular M-matrices to certain complex matrices are made and the recent literature on these general topics is surveyed.

AMS (MOS) Subject Classifications: 1502, 65F10

Key Words and Phrases: M-Matrix, Positivity of Principal Minors, Inverse-positivity and Splittings, Stability, Semi-positivity and Diagonal Dominance, Iterative Methods for Systems of Linear Equations

Work Unit Numbers 2 (Matrix Theory) and 7 (Numerical Analysis)

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A SURVEY OF M-MATRIX CHARACTERIZATIONS

I: NONSINGULAR M-MATRICES

R. J. Plemmons[1,2]

I. <u>INTRODUCTION</u>. Very often problems in the biological, physical and social sciences can be reduced to problems involving matrices which, due to certain constraints, have some special structure. One of the most common situations is where the matrix A in question has nonpositive off-diagonal and nonnegative diagonal entries, that is, A is of the type

$$A = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} & \cdot & \cdot & \cdot \\ -a_{21} & a_{22} & -a_{23} & \cdot & \cdot & \cdot \\ -a_{31} & -a_{32} & a_{33} & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{pmatrix}$$

where the a_{ij} are nonnegative. Such matrices usually occur in relation to systems of linear or nonlinear equations or eigenvalue problems in a wide variety of areas including finite difference or finite element methods for partial differential equations, input-output production and growth models in economics, linear complementarity problems in operations research and Markov processes in probability and statistics.

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We adopt the traditional notation here (of Fiedler and Ptak [1962]) by letting $Z^{n,n}$ denote the set of all $n \times n$ real matrices $A = (a_{ij})$ with $a_{ij} \leq 0$ for all $i \neq j$, $1 \leq i, j \leq n$.

Our purpose is to give a systematic treatment of the characterizations of a class of matrices in $Z^{n,n}$ first studied systematically by Ostrowski [1937].

Definition. An $n \times n$ matrix A that can be expressed in the form $A = sI - B$ where $B = (b_{ij})$ with $b_{ij} \geq 0$, $1 \leq i, j \leq n$, and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B, is called an M-matrix.

We shall be primarily concerned in this survey with nonsingular M-matrices and it is easy to see that this is the class of those A given in the Definition for which $s > \rho(B)$.

It should be mentioned that the theory of nonnegative matrices developed by Perron [1907a, b] and Frobenius [1908, 1909, 1912] provided a certain essential background for Ostrowski's work and the work of others on M-matrices. For example, it is easy to see from the Perron-Frobenius theory of nonnegative matrices that if A is a nonsingular M-matrix then the diagonal elements a_{ii} of A must be positive. Also, for any representation $A = sI - B$, $B \geq 0$ it follows that $s > \rho(B)$. In particular then, the class of nonsingular M-matrices forms a proper subclass of the class of M-matrices which forms a proper subclass of matrices in $Z^{n,n}$.

No attempt will be made here to systematically trace the history of the development of the theory of M-matrices. It appears however, that the term M-matrix was first used by Ostrowski [1937, 1956] in reference to the work of Minkowski [1900, 1907] who proved that if $A \in Z^{n,n}$ has all of its row sums positive, then the determinant of A is positive. Papers following the early work of Ostrowski have primarily been produced by two groups of researchers, one in mathematics, the other in economics.

The mathematicians have mainly had in mind the applications of M-matrices to the establishment of bounds on eigenvalues and on the establishment of convergence criteria for iterative methods for the solution of large sparse systems of linear equations. Meanwhile, the economists have studied M-matrices in connection with gross substitutability, stability of a general equilibrium and Leontief's input-output analysis in economic systems (Leontief [1941]). With this in mind it should be mentioned that the terms productive matrix, Leontief matrix and Minkowski matrix have all been used in the economics literature to describe what we call a nonsingular M-matrix. Some of the contributions of individual researchers in these areas will be mentioned later in the paper.

Because the concept of an M-matrix has been applied to such diverse areas of the sciences, much of the theory has been developed at least partially in isolation. As a result much of the terminology and notation have been different and there is often formal overlap in the literature (this is pointed out quite vividly by Varga [1976]). One of the purposes of this survey is to collect the various characterizations of nonsingular M-matrices that have been developed and applied by various researchers into a list that might make it possible for the reader to get an overall picture of the theory. The first systematic effort to characterize M-matrices was by Fiedler and Ptak [1962]. An initial survey of the theory of M-matrices was made by Poole and Boullion [1974]. Varga [1976] has surveyed the role of diagonal dominance in the theory of M-matrices. In addition, Schröder [1976] has surveyed some of the properties of nonsingular M-matrices using operator theory and partially ordered linear spaces. Meanwhile, Kaneko [1976] has compiled a list of characterizations and applications of nonsingular M-matrices in terms of linear complementarity problems in Operations Research. However, our objectives and our approach are quite different in this survey.

Section II is devoted to characterizing in a systematic way those matrices in $Z^{n,n}$ that are nonsingular M-matrices. Characterizations of certain real and complex matrices related to M-matrices are also given and an attempt is made to indicate where in the literature these characterizations may be found.

No proofs are given in this survey. The proofs of the various results can be found in the original papers.

The following notation will be used.

ϕ - The empty set.

0 - A vector or matrix of all zeroes.

I - An identity matrix.

S - A signature matrix, that is, a diagonal matrix with diagonal entries +1 or -1.

$\mathbb{C}$ - The set of complex numbers.

$\mathbb{R}$ - The set of real numbers.

$\mathbb{C}^n$ - The vector space of n-vectors over $\mathbb{C}$.

$\mathbb{R}^n$ - The vector space of n-vectors over $\mathbb{R}$.

$\mathbb{R}^n_+$ - The nonnegative orthant, that is, vectors in $\mathbb{R}^n$ having all nonnegative components. If $x \in \mathbb{R}^n_+$ we write $x \geq 0$ if $x \neq 0$, and $x \geqq 0$ if $x \geq 0$ or $x = 0$. If x has all positive components we write $x > 0$.

$\mathbb{C}^{n,n}$ - The set of all $n \times n$ matrices over $\mathbb{C}$.

$\mathbb{R}^{n,n}$ - The set of $n \times n$ matrices over $\mathbb{R}$.

$Z^{n,n}$ - The set of all $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $a_{ij} \leq 0$ if $i \neq j$, $1 \leq i, j \leq n$.

Let $A \in \mathbb{R}^{n,n}$.

$A \geq 0$ - Indicates that each component of A is nonnegative and that $A \neq 0$.

$A \geqq 0$ - Indicates that $A \geq 0$ or $A = 0$.

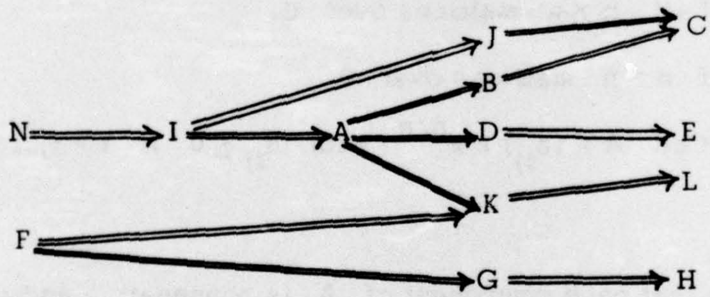
A^t - The transpose of A.

A^{-1} - The inverse of A.

$\rho(A)$ - The spectral radius of A.

II. CHARACTERIZATIONS. For practical purposes in characterizing nonsingular M-matrices, it is evident that we can often begin by assuming that $A \in Z^{n,n}$. However, many of the statements of these characterizations are equivalent without this assumption. We have attempted here to group together all such statements into certain categories. Moreover, certain other implications follow without assuming that $A \in Z^{n,n}$ and we point out such implications in the following inclusive theorem.

Theorem 1. Let $A \in \mathbb{R}^{n,n}$. Then for each fixed letter C representing one of the conditions below, conditions C_i are equivalent for each i. Moreover letting C then represent any of the equivalent conditions C_i, the following implications hold:



Finally, if $A \in Z^{n,n}$ then each of the following conditions is equivalent to the statement: A is a nonsingular M-matrix.

Positivity Of Principal Minors

A_1. All the principal minors of A are positive.

A_2. Every real eigenvalue of each principal submatrix of A is positive.

A_3. $A + D$ is nonsingular for each positive diagonal matrix D.

A_4. For each $x \neq 0$ there exists a positive diagonal matrix D such that

$$x^t ADx > 0 .$$

A_5. For each $x \neq 0$ there exists a nonnegative diagonal matrix D such that

$$x^t ADx > 0 .$$

A_6. A does not reverse the sign of any vector. That is, if $x \neq 0$ then for some subscript i,

$$x_i (Ax)_i > 0 .$$

A_7. For each signature matrix S there exists an $x > 0$ such that

$$SASx > 0 .$$

B_8. The sum of all the $k \times k$ principal minors of A is positive for $k = 1, 2, \ldots, n$.

C_9. Every real eigenvalue of A is positive.

C_{10}. $A + \alpha I$ is nonsingular for each scalar $\alpha \geq 0$.

D_{11}. All the leading principal minors of A are positive.

D_{12}. There exist lower and upper triangular matrices L and U respectively, with positive diagonals such that

$$A = LU .$$

E_{13}. There exists a strictly increasing sequence of subsets $\phi \neq S_1 \subset \cdots \subset S_n = \{1, \ldots, n\}$ such that the determinant of the principal submatrix of A formed by choosing row and column indices from S_i is positive for $i = 1, \ldots, n$.

E_{14}. There exists a permutation matrix P and lower and upper triangular matrices L and U respectively, with positive diagonals, such that

$$PAP^t = LU .$$

Inverse-Positivity And Splittings

F_{15}. A is inverse-positive. That is, A^{-1} exists and

$$A^{-1} \geq 0 .$$

F_{16}. A is monotone. That is

$$Ax \geqq 0 \implies x \geqq 0 \quad \text{for all} \quad x \in \mathbb{R}^n .$$

F_{17}. A has a convergent regular splitting. That is, A has a representation

$$A = M - N, \quad M^{-1} \geq 0, \; N \geqq 0$$

with $M^{-1}N$ convergent. That is, $\rho(M^{-1}N) < 1$.

F_{18}. A has a convergent weak regular splitting. That is, A has a representation

$$A = M - N, \quad M^{-1} \geq 0, \; M^{-1}N \geqq 0$$

with $M^{-1}N$ convergent.

F_{19}. A has a weak regular splitting and there exists $x > 0$ with $Ax > 0$.

F_{20}. There exist inverse-positive matrices M_1 and M_2 with

$$M_1 \leq A \leq M_2 .$$

F_{21}. There exists an inverse-positive matrix M, with $M \geqq A$, and a nonsingular M-matrix B such that $A = MB$.

F_{22}. There exists an inverse-positive matrix M and a nonsingular M-matrix B such that $A = MB$.

G_{23}. Every weak regular splitting of A is convergent.

H_{24}. Every regular splitting of A is convergent.

<u>Stability</u>

I_{25}. There exists a positive diagonal matrix D such that

$$AD + DA^t$$

is positive definite.

I_{26}. A is diagonally similar to a matrix whose symmetric part is positive definite. That is, there exists a positive diagonal matrix E such that for $B = E^{-1}AE$, the matrix

$$(B + B^t)/2$$

is positive definite.

I_{27}. For each nonzero positive semi-definite matrix P, the matrix PA has a positive diagonal element.

I_{28}. Every principal submatrix of A satisfies condition I_{25}.

J_{29}. A is <u>positive stable</u>. That is, the real part of each eigenvalue of A is positive.

J_{30}. There exists a symmetric positive definite matrix W such that

$$AW + WA^t$$

is positive definite.

J_{31}. A + I is nonsingular and

$$G = (A + I)^{-1}(A - I)$$

is convergent.

J_{32}. A + I is nonsingular and for

$$G = (A + I)^{-1}(A - I),$$

there exists a positive definite symmetric matrix W such that

$$W - G^t WG$$

is positive definite.

Semi-Positivity And Diagonal Dominance

K_{33}. A is <u>semi-positive</u>. That is, there exists $x > 0$ with $Ax > 0$.

K_{34}. There exists $x \geq 0$ with $Ax > 0$.

K_{35}. There exists a positive diagonal matrix D such that AD has all positive row sums.

L_{36}. There exists $x > 0$ with $Ax \geq 0$ such that if $(Ax)_{i_0} = 0$, then there exist indices $1 \leq i_1, \ldots, i_r \leq n$, with $a_{i_k i_{k+1}} \neq 0$ for $0 \leq k \leq r - 1$ and $(Ax)_{i_r} > 0$.

M_{37}. There exists $x > 0$ with $Ax \geq 0$ and

$$\sum_{j=1}^{i} a_{ij} x_j > 0$$

for each $i = 1, 2, \ldots, n$.

N_{38}. There exists $x > 0$ such that for each signature matrix S,

$$SASx > 0 .$$

N_{39}. A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant. That is,

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}| d_j$$

for $i = 1, 2, \ldots, n$.

N_{40}. A has all positive diagonal elements and there exists a positive diagonal matrix D such that $D^{-1}AD$ is strictly diagonally dominant.

We remark again that Theorem 1 identifies those conditions characterizing nonsingular M-matrices that are equivalent for an arbitrary matrix in $\mathbb{R}^{n,n}$, as well as certain implications that hold for various classes of conditions. For example, A_1 through A_7 are equivalent and $A \Longrightarrow B$ for an arbitrary matrix in $\mathbb{R}^{n,n}$. We remark also that it follows from the work of Schneider [1953, 1965] that a matrix $A \in Z^{n,n}$ is a nonsingular M-matrix if and only if each irreducible principal submatrix of A is a nonsingular M-matrix and thus satisfies one of the equivalent conditions in Theorem 1. It should also be pointed out that some of the classes have left-right duals with A replaced by A^t.

The problem of giving proper credit to those originally responsible for the various characterizations listed in Theorem 1 is difficult, if not impossible. The situation is complicated by the fact that many of the characterizations are implicit in the work of Perron [1907a, b] and Frobenius [1908, 1909, 1912] and in the work of Ostrowski [1937, 1956], but were not given there in their present form. Another complicating factor is that the diversification of the applications of M-matrices has led to certain conditions being derived independently. We attempt in this survey only to give references to the literature where the various proofs can be found.

First of all Condition A_1, which is known as the Hawkins-Simon [1949] condition in the economics literature, was taken by Ostrowski [1937] as the definition for $A \in Z^{n,n}$ to be a nonsingular M-matrix. He then proceeded to show the equivalence of his definition with ours; namely that A has a representation $A = sI - B$, $B \geqq 0$ and $s > \rho(B)$. Condition A_2 is also in Ostrowski [1937]. Condition A_3 was shown to be equivalent to A_1 in Wilson [1971], while Conditions A_4, A_5 and A_6 were listed in Fiedler and Ptak [1962]. Condition A_6 is also in Gale and Nikaido [1965] and A_7 was shown to be equivalent to A_1 in Moylan [1976].

Next, Condition B_8 can be bound in Johnson [1974] and C_9, C_{10}, D_{11}, D_{12}, E_{13} and E_{14} are in Fiedler and Ptak [1962].

Condition F_{15} is in the original paper by Ostrowski [1937], Condition F_{16} was shown to be equivalent to F_{15} by Collatz [1952], Condition F_{17} is implicit in the work of Varga [1962] on regular splittings, Condition F_{18} is in the work of Schneider [1965] while F_{19} and F_{20} are in the work of Price [1968]. Conditions F_{21}, F_{22} and G_{23} are essentially in the work of Schneider [1965] and H_{24} is in Varga [1962].

Next, Condition I_{25} is in the work of Tartar [1971] and of Araki [1975]. The equivalence of I_{26}, I_{27} and I_{28} to I_{25} is in Barker, Berman and Plemmons [1976]. The stability Condition J_{29} is in the work of Ostrowski [1937] while its equivalence with J_{30} is the Lyapunov [1892] theorem. The equivalence of J_{30} with J_{31} is in Taussky [1961] and the equivalence of J_{32} with J_{31} is the Stein [1952] theorem.

Conditions K_{33}, K_{34} and K_{35} are in Schneider [1953] and Fan [1958]. Condition L_{36} in a slightly different form is in Bramble and Hubbard [1964] while this particular form is in Varga [1976]. Condition M_{37} is in the work of Beauwens [1976] and N_{38} is in that of Moylan [1976]. Conditions N_{39} and N_{40} are essentially in Schneider [1965].

Now let $\mathcal{C}$ represent any of the equivalent Conditions $\mathcal{C}_i$ and let $A \in \mathbb{R}^{n,n}$. That $N \Longrightarrow I$ was established in Barker, Berman and Plemmons [1976]. That $I \Longrightarrow J$ is in Lyapunov [1892] and that $I \Longrightarrow A$ is also in Barker, Berman and Plemmons [1976]. The implications $A \Longrightarrow B \Longrightarrow C$, $A \Longrightarrow D \Longrightarrow E$ and $J \Longrightarrow C$ are immediate. That $A \Longrightarrow K$ can be found in Nikaido [1968] and the implication $K \Longrightarrow L$ is immediate. That $F \Longrightarrow K$ can be found in Schneider [1953]. Finally, the implication $F \Longrightarrow G$ is in Varga [1962] and the implication $G \Longrightarrow H$ is immediate.

It is not known by the author whether $H \Longrightarrow G$. The implication will follow if it can be shown that if $A \in \mathbb{R}^{n,n}$ has a weak regular splitting then it must also have some regular splitting. It is also not known by the author whether $M \Longrightarrow L$.

Next, we consider necessary and sufficient conditions for an arbitrary matrix $A \in \mathbb{R}^{n,n}$ to be a nonsingular M-matrix. In the following theorem, we do not assume that A has nonpositive off-diagonal entries.

Theorem 2. Let $A \in \mathbb{R}^{n,n}$, $n \geq 2$. Then each of the following conditions is equivalent to the statement: A is a nonsingular M-matrix.

1. $A + D$ is inverse-positive for each nonnegative diagonal matrix D.

2. $A + \alpha I$ is inverse-positive for each scalar $\alpha \geq 0$.

3. Each principal submatrix of A is inverse-positive.

4. Each principal submatrix of A of orders 1, 2 and n of A is inverse-positive.

In view of Conditions F_{15} in Theorem 1, the proof of Theorem 2 reduces to that of showing that any $A \in \mathbb{R}^{n,n}$ satisfying one of these conditions can have no positive off-diagonal elements. That this is true if Conditions 1 or 2 hold is given in Wilson [1971] and if Conditions 3 or 4 hold is in Cottle and Veinott [1972].

We now relate some of the characterizations of nonsingular M-matrices to matrices over the complex field.

For $A \in \mathbb{C}^{n,n}$ we define its comparison matrix $\mathcal{M}(A) = (m_{ij}) \in \mathbb{R}^{n,n}$ given by

$$m_{ii} = |a_{ii}|, \; m_{ij} = -|a_{ij}|, \; i \neq j, \; 1 \leq i, \; j \leq n$$

and we define

$$\Omega(A) = \{B = (b_{ij}) \in \mathbb{C}^{n,n} : |b_{ij}| = |a_{ij}|, \; 1 \leq i, \; j \leq n\},$$

to be the set of equimodular matrices associated with A.

Our objective is to characterize matrices A for which $\mathcal{M}(A)$ is a nonsingular M-matrix. Such matrices will be called <u>H-matrices</u> after Ostrowski [1937]. We first use Theorem 1 to characterize certain real H-matrices.

<u>Theorem 3</u>. Let $A \in \mathbb{R}^{n,n}$ have all positive diagonal elements. Then $\mathcal{M}(A)$ is an M-matrix if and only if A satisfies one of the equivalent Conditions N_{38}, N_{39} or N_{40} of Theorem 1.

That N_{38} is equivalent to $\mathcal{M}(A)$ being a nonsingular M-matrix is in Moylan [1976], while N_{39} and N_{40} are in Schneider [1953].

In order to characterize complex H-matrices, we introduce the following splittings of $A \in \mathbb{C}^{n,n}$. Suppose the diagonals of A are all nonzero and let

$$A = M - N = D - L - U$$

where $D = \mathrm{diag}(a_{11}, \ldots, a_{nn})$ and where $-L$ and $-U$ are respectively the lower and upper parts of A. Different choices of M in this splitting lead to certain well known iteration matrices $T = M^{-1}N$. For $\omega > 0$ we define

$$J_{\omega}(A) = \omega D^{-1}(L + U) + (1 - \omega)I$$

$$\mathcal{L}_{\omega}(A) = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$$

$$\mathcal{S}_{\omega}(A) = (D - \omega U)^{-1}[(1 - \omega)D + \omega L](D - \omega L)^{-1}[(1 - \omega)D + \omega U)]$$

respectively the (point) <u>Jacobi overrelaxation iteration matrix</u>, the <u>successive overrelaxation iteration matrix</u> and the <u>symmetric successive overrelaxation</u>

iteration matrix associated with A. These matrices from the iteration matrices for the widely used JOR, SOR and SSOR iteration procedures for solving systems of linear equations. We are interested in conditions under which these iteration matrices are convergent. It turns out that the theory of M-matrices plays a fundamental role in such investigations.

Our final characterization theorem is for nonsingular H-matrices in $C^{n,n}$.

Theorem 4. Let $A \in C^{n,n}$ have all *nonzero* diagonals. Then each of the following conditions is equivalent to the statement: A is a nonsingular H-matrix.

1. For each $B \in C^{n,n}$, $\mathcal{M}(B) \geq \mathcal{M}(A)$ implies B is nonsingular.
2. For each $B \in \Omega(A)$,

$$0 < \omega < \frac{2}{1 + \rho(J_1(B))}$$

implies that

$$\rho(J_\omega(B)) < 1 .$$

3. For each $B \in \Omega(A)$,

$$0 < \omega < \frac{2}{1 + \rho(J_1(B))}$$

implies that

$$\rho(\mathcal{L}_\omega(B)) < 1 .$$

4. For each $B \in \Omega(A)$,

$$0 < \omega < \frac{2}{1 + \rho(J_1(B))}$$

implies that

$$\rho(\mathcal{S}_{\omega}(B)) < 1 \, .$$

Conditions 2 and 3 in Theorem 4 are in the work of Varga [1976] in relation to characterizations of diagonal dominance and Condition 4 is in Alefeld and Varga [1976]. Finally, Condition 1 is given in Ostrowski [1956]. A related result is in Camion and Hoffman [1966], who used the theory of the alternative to show that for each $B \in C^{n,n}$, $\mathcal{M}(B) = \mathcal{M}(A)$ implies B is nonsingular if and only if there exists a permutation matrix P such that $\mathcal{M}(PA)$ is an M-matrix.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The purpose of this survey is to classify systematically a widely ranging list of characterizations of nonsingular M-matrices from the economics and mathematics literatures. These characterizations are grouped together in terms of their relationships to the properties of (1) positivity of principal minors (2) inverse-positivity and splittings (3) stability and (4) semi-positivity and

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20. ABSTRACT - Cont'd.

diagonal dominance.

A list of forty equivalent conditions is given for a square matrix A with nonpositive off-diagonal entries to be a nonsingular M-matrix. These conditions are grouped into classes in order to identify those that are equivalent for arbitrary real matrices A.

In addition, other remarks relating nonsingular M-matrices to certain complex matrices are made and the recent literature on these general topics is surveyed.