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CUTTING-PLANES FOR COMPLEMENTARITY CONSTRAINTS.(U)

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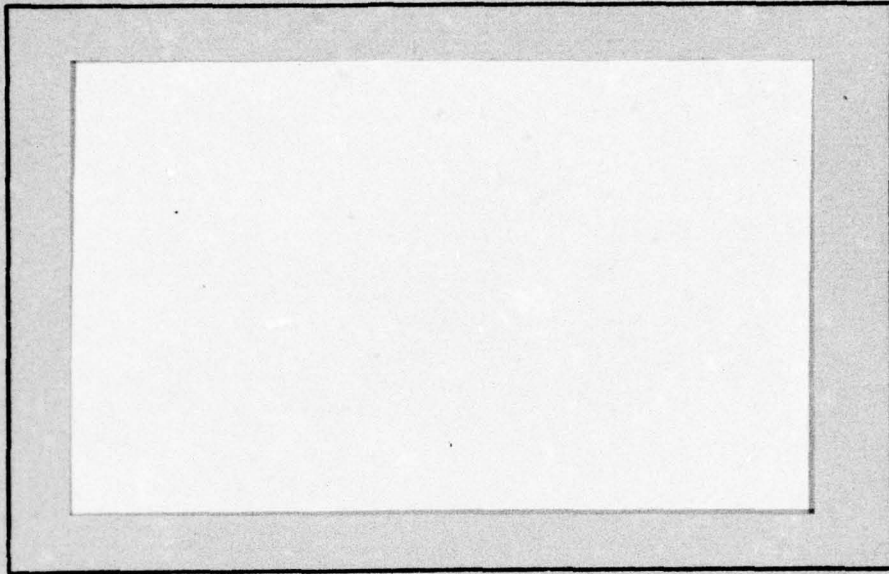


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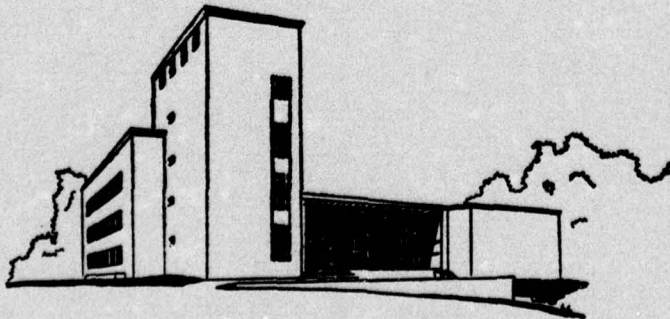
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CUTTING-PLANES FOR  
COMPLEMENTARITY CONSTRAINTS.

by

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Abstract

We describe two simple rules of cutting-plane generation for the complementarity constraints

$$Ax \geq b$$

$$x \geq 0$$

$$(CMP) \quad \sum_{h=1}^t \pi_{K \in J_h} \left( \sum_{k \in K} x_k \right) = 0$$

and we show that these rules generate all (and only) the valid cutting-planes for (CMP), if there is some  $b'$  for which  $\{ x \geq 0 \mid Ax \geq b' \}$  is non-empty and bounded.

In (CMP),  $x = (x_1, \dots, x_r)$ , and  $J_h$  is a set of subsets  $K$  of  $\{ 1, \dots, r \}$ . The problem (CMP) includes the linear complementarity problem and bivalent integer programming, along with many other constraint sets which impose logical restrictions on linear inequalities.

Key Words:

1. Cutting-planes
2. Complementarity
3. Integer programming
4. Disjunctive methods

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CUTTING-PLANES FOR  
COMPLEMENTARITY CONSTRAINTS

by R.G. Jeroslow

We provide a characterization of the set of all valid inequalities for a constraint system (CMP) (see page 39 below) which includes, as special cases, the linear complementarity problem and the constraints of the bivalent integer program, as well as many other constraint sets which impose logical restrictions on linear inequalities. The characterization is derived in terms of "co-propositions" [16], [18], [19] (see Theorem 8 below). Then this characterization is put in an alternate form by means of rules for cutting-plane generation which are of a particularly simple form, yet which are shown to generate precisely the set of valid inequalities for (CMP) when iteratively applied (see Theorem 9 below).

Our main result (Theorem 8) generalizes both a result of Balas [2] and one of Blair [4]. We employ the techniques of the disjunctive approach of cutting-plane theory (see e.g. [1], [6], [12], [13], [16], [24], [30], [31], or [19] for a survey of this topic); for other applications of disjunctive methods to complementarity problems see [1], [13], [24]. These methods combine the theory of linear inequalities with concepts and some elementary results from mathematical logic. When material from logic is needed, it is developed below so that the paper is self-contained. We shall use [27] and [29] as general references for linear inequalities and polyhedra.

Section 1. Motivation and Some Basic Results

In what follows,  $x = (x_1, \dots, x_r)$  denotes a vector in  $R^r$ , and the letters  $A, B, C, \dots$  etc. denote matrices while  $b, d, \dots$  etc. denote vectors in some finite-dimensional real space. The writing of a matrix inequality such as  $Ax \geq b$  entails the compatibility of  $A, x, b$ : i.e., for some integer  $m$ ,  $A$  is  $m$  by  $r$  and  $b$  is  $m$  by  $1$ .

We reserve script letters  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \dots$  from the last half of the alphabet to denote atomic propositions, which for our purposes will always have the form

$$(1) \quad \sum_{j=1}^r a_j x_j \geq a_0$$

of a single linear inequality. Script letters  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$  etc. from the first half of the alphabet are used to denote both atomic propositions (1) and also more complex propositions, which arise by repeatedly placing "v" (for : 'or') or "w" (for : 'and') between propositions already constructed. The "v" used here is in the inclusive sense:  $\mathcal{A} \vee \mathcal{B}$  is true if either one of them is true, or if both are true.

We say that  $Dx \geq d$  is facial for  $Ax \geq b$  when  $\{x \mid Dx \geq d, Ax \geq b\}$  is a face of  $\{x \mid Ax \geq b\}$ . Concerning results on polyhedra and faces, facets, etc. for polyhedra, the reader may wish to consult [27] or [29]. The term "facial" is due to Balas [2], originally to treat the instance that  $Dx \geq d$  has only one constraint.

Again following [2], we shall say that the constraint system

$$(2) \quad Ax \geq b ;$$

(3) for each  $h = 1, \dots, t$  at least one of the conditions

$$A^{h,1}x \geq b^{h,1} \text{ or } \dots \text{ or } A^{h,t(h)}x \geq b^{h,t(h)}$$

holds;

is facial, if for all  $h = 1, \dots, t$  and  $p = 1, \dots, t(h)$  the constraint system  $A^{h,p}x \geq b^{h,p}$  is facial to  $Ax \geq b$ .

To explain the result that we shall strengthen in this paper, define inductively the convex polyhedra

$$(4) \quad K = \{ x | Ax \geq b \}$$

$$(5) \quad K_{h+1} = \text{clconv} \left( \bigcup_{p=1}^{t(h+1)} \left( K_h \cap \{ x | A^{h+1,p}x \geq b^{h+1,p} \} \right) \right), \quad 0 \leq h \leq t-1.$$

In what follows,  $\text{conv } S$  resp.  $\text{clconv } S$  denotes the smallest convex resp closed convex set containing  $S$ .

Theorem : (Balas [2 ])

If  $\{ x | Ax \geq b \}$  is bounded and (2), (3) is facial,

then

$$(6) \quad K_t = \text{clconv} \{ x | (2) \text{ and } (3) \text{ hold} \}.$$

Remark: Half of Theorem 1 is straightforward, for it is easy to prove  $K_t \supseteq \text{clconv} \{ x | (2) \text{ and } (3) \text{ hold} \}$ , without the boundedness or faciality assumptions.

To see that the reverse inclusion has non-trivial content, consider the following constraint system which is an alternate format for the constraints of an integer program:

$$(2)' \quad -2x_1 + 2x_2 = 1$$

$$0 \leq x_1 \leq 1$$

(3)' at least one of the conditions

$$x_1 = 0 \quad \text{or} \quad x_1 = 1$$

holds, and at least one of the conditions

$$x_2 = 0 \quad \text{or} \quad x_2 = 1 \quad \text{or} \quad x_2 = 2$$

holds.

One easily shows that the constraint system has no solutions, since  $x_2 = \frac{1}{2} + x_1$  and  $0 \leq x_1 \leq 1$  forces  $x_2 = 1$  in any solution, which in turn forces  $x_1 = 1 - \frac{1}{2} = \frac{1}{2}$  and hence is impossible.

The constraint  $x_2 = 1$  is not facial, although all the other constraints (i.e.,  $x_1 = 0$ ,  $x_1 = 1$ ,  $x_2 = 0$ ,  $x_2 = 1$ ) are facial, some giving the empty face.

$$\text{We have } K_0 = \{(x_1, x_2) \mid -2x_1 + 2x_2 = 1, 0 \leq x_1 \leq 1\},$$

$$K_1 = K_0, \quad K_2 = K_1 \cap \{(x_1, x_2) \mid x_2 = 1\} = \left\{ \left( \frac{1}{2}, 1 \right) \right\}.$$

Since  $K_2 \neq \emptyset$ , equation (6) fails.

Even if (2), (3) is facial, equation (6) can fail if the boundedness assumption fails; see remarks after Theorem 8 in this regard.

In [16] we introduced the co-propositions and further developed them in [18]. Our notation is from [18], [19].

The co-proposition construction is an assignment, which makes correspond, to a proposition  $\mathcal{A}$  whose atomic letters  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \dots$ , are linear inequalities (1), a polyhedral cone  $CT(\mathcal{A})$  of cuts that are all valid for the set  $\{x \mid \mathcal{A}(x) \text{ is true}\}$ , hence also for  $\text{clconv}\{x \mid \mathcal{A}(x) \text{ is true}\}$ . Here, we have used the notation  $\mathcal{A} = \mathcal{A}(x)$  to emphasize the dependence of the proposition  $\mathcal{A}$  upon  $x \in \mathbb{R}^r$ . The construction generalizes Balas' disjunctive constraints [1], and we describe it next.

If  $\mathcal{P}$  is given by (1), put

$$(7) \quad CT(\mathcal{P}) = \text{cone} \left\{ (a_0, -a_1, \dots, -a_r), (-1, 0, \dots, 0), \right. \\ \left. (0, -1, 0, \dots, 0), \dots, (0, \dots, 0, -1) \right\}$$

where cone  $S$  denotes the smallest convex cone containing the set  $S$ .

(Unit vectors  $(0, -1, 0, \dots, 0), \dots, (0, \dots, 0, -1)$  occur in (7) due to an implicit condition  $x \geq 0$  on the variables  $x$  in (1)). Then inductively set

$$(8) \quad CT(\mathcal{B} \wedge \mathcal{D}) = CT(\mathcal{B}) + CT(\mathcal{D})$$

$$(9) \quad CT(\mathcal{B} \vee \mathcal{D}) = CT(\mathcal{B}) \cap CT(\mathcal{D})$$

to determine  $CT(\mathcal{A})$  for any proposition  $\mathcal{A}$ . In (8), for convex sets  $K, L$  set  $K + L = \{k + l \mid k \in K, l \in L\}$ . In [16] we showed that  $CT(\mathcal{A})$  is always a polyhedral cone.

Associated with  $CT(\mathcal{A})$  is the relaxation  $cp(\mathcal{A})$  that it determines:

$$(10) \quad \text{cp}(\mathcal{A}) = \left\{ x \geq 0 \mid \begin{array}{l} \text{for every } (\pi_0, -\pi_1, \dots, -\pi_r) \in \text{CT}(\mathcal{A}), \\ \sum_{j=1}^r \pi_j x_j \geq \pi_0 \end{array} \right\}$$

As noted in [18, Sec. 2.1] or in [19, Theorem in Sec. 2.1], when each proposition  $\mathcal{A}_h$ ,  $1 \leq h \leq s$ , is the conjunction of the inequalities in the matrix inequality  $A^h x \geq b^h$ , then  $\text{CT}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s)$  consists of all the disjunctive constraints cuts [1] for the logical condition, that at least one system among  $A^h x \geq b^h$ ,  $x \geq 0$  holds as  $h$  varies over  $h = 1, \dots, s$ . However,  $\text{CT}(\mathcal{A})$  is defined for all  $\mathcal{A}$  built up from atomic propositions (1) via the connectives " $\vee$ " and " $\wedge$ ," whether or not they have the special logical form for disjunctive cuts.

In seeking a generalization of Theorem 1, we note that Theorem 1 refers to actual geometric bodies  $K_h$ ,  $0 \leq h \leq t$ , while co-propositions are defined from logical descriptions  $\mathcal{A}$  of these bodies, which are syntactic objects and not geometric in nature. Now suppose we can find a suitable syntactic description  $\mathcal{A}$  of  $K_t$ , for which we can prove

(6)'  $\text{CT}(\mathcal{A})$  contains precisely the valid inequalities

for  $\text{clconv} \{ x \mid (2) \text{ and } (3) \text{ hold} \}$ .

Then since  $K_t \supseteq \text{clconv} \{ x \mid (2) \text{ and } (3) \text{ hold} \}$  is obvious from (4), (5), we will have strengthened (6) of Theorem 1, in view of the fact that  $\text{CT}(\mathcal{A})$  contains only valid cuts for  $K_t$ . In fact, we would also have obtained, as a consequence of (6)', that for the  $\mathcal{A}$  chosen,  $\text{CT}(\mathcal{A})$  is all (and not just some) of the valid cuts for  $K_t$  provided  $K_t \neq \emptyset$ . It is well-known that the disjunctive constraints construction [1] does not necessarily provide all valid cuts (see e.g., [16] or just below eqn. (18)).

Even more, if we can establish (6)' for a suitably chosen sentence  $\mathcal{A}$ , we will obtain a compact description of how to generate all the valid cuts for  $\text{clconv} \{ x \mid (2) \text{ and } (3) \text{ hold} \}$ , by using the inductive clauses (8) and (9) of the co-proposition construction.

Now it turns out, that there is actually more than one sentence  $\mathcal{A}$  for which (6)' is true. In Theorem 8, we exhibit one of these sentences, for which the description of all valid cuts via (7), (8), (9) has a particularly surprising form that we will exhibit in Theorem 9.

In our selection of a sentence  $\mathcal{A}$  possessing the property (6)', we have been guided by a striking discovery of C.E. Blair [ 4 , Chapter 3], in the form of an unusual inductive characterization of the valid cutting planes of a bivalent integer program.

We now summarize some results that we will use later. In what follows,  $S^P$  denotes the polar of the set  $S$  (see e.g., [27], [29]).

Theorem 2:

1) [ 16, Theorem 25]

$$(11) \quad \text{cp}(\mathcal{A}) \supseteq \text{clconv} \{ x \geq 0 \mid \mathcal{A}(x) \text{ is true} \}$$

2) [ 16, Theorem 22]

$$(12) \quad \text{cp}(\mathcal{B} \vee \mathcal{D}) \supseteq \text{clconv} ( \text{cp}(\mathcal{B}) \cup \text{cp}(\mathcal{D}) )$$

with = in place of  $\supseteq$  if both

$$\text{cp}(\mathcal{B}) \neq \emptyset \text{ and } \text{cp}(\mathcal{D}) \neq \emptyset$$

3) [ 16, Theorem 22]

If  $\text{cp}(\mathcal{B}) = \text{cp}(\mathcal{A}) = \emptyset$ ,

$$(13) \quad \text{cp}(\mathcal{B} \vee \mathcal{A}) = \emptyset$$

4) [ 16, Theorem 22]

$$(14) \quad \text{cp}(\mathcal{B} \wedge \mathcal{A}) = \text{cp}(\mathcal{B}) \cap \text{cp}(\mathcal{A})$$

5) [ 16, Theorem 24]

$$(15) \quad \text{cp}(\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{A})) \supseteq \text{cp}((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{A}))$$

6) [ 18, Section 2.1.2, Part 2] or [ 19, Theorem of Section 2.1.2]

If  $\text{cp}(\mathcal{B}) \cap \text{cp}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s)$  is a face of  $\text{cp}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s)$  and if  $\text{cp}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s)$  is a polytype, then

$$(16) \quad \begin{aligned} \text{cp}(\mathcal{B} \wedge (\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s)) &= \text{cp}((\mathcal{B} \wedge \mathcal{A}_1) \vee \dots \vee (\mathcal{B} \wedge \mathcal{A}_s)) \\ &= \text{clconv} \left( \bigcup_{h=1}^s (\text{cp}(\mathcal{B}) \cap \text{cp}(\mathcal{A}_h)) \right) \end{aligned}$$

7) If  $\mathcal{A}$  is the conjunction of the inequalities in the matrix system of inequalities  $Ax \geq b$ , then

$$(17) \quad \text{cp}(\mathcal{A}) = \{ x \geq 0 \mid Ax \geq b \}$$

Proof of 7: From [ 18, Part 2, eqn., (2.1.M), p. 84], or from

[ 19, Lemma in Sec. 2.1], we see that 7) holds if  $Ax \geq b$  contains

only the single inequality (1). For the general case several

applications of Theorem 2(4) above gives (17).

Q.E.D.

Theorem 2(6) is our generalization of a basic lemma [ 2 ,

Lemma 5.1] used by Balas to prove Theorem 1. In our generalization Theorem 8 of Theorem 1, we shall need the principle of Theorem 2(6).

When the equation

$$(18) \quad \text{cp}(\mathcal{D}_1 \vee \dots \vee \mathcal{D}_s) = \text{clconv} \left( \bigcup_{h=1}^s \text{cp}(\mathcal{D}_h) \right)$$

holds, we say that the disjunction  $\mathcal{D}_1 \vee \dots \vee \mathcal{D}_s$  is exact. In [16, p.66] we noted that exactness fails if  $s = 2$ ,  $\mathcal{D}_1$  is  $-x_1 \geq -1$  and  $\mathcal{D}_2$  is  $0 \cdot x_1 \geq 1$ , although by Theorem 2(2) one of the two inclusions implicit in the equality of (18) always holds.

To develop our generalization (6)' of Theorem 1, we shall need to know more about the relationship of  $\text{cp}(\mathcal{D}_1 \vee \dots \vee \mathcal{D}_s)$  to  $\text{clconv} \left( \bigcup_{h=1}^s \text{cp}(\mathcal{D}_h) \right)$ , and in specific, to know more about exactness (18) than given in [18]. This additional information will be given in Theorem 5, which also is a new result of some independent interest.

To state Theorem 5 in a concise form, we shall define the recession cone of a sentence  $\mathcal{A}$ , which is

$$(19) \quad \text{rec}(\mathcal{A}) = \{ v \mid (0, v) \in \text{CT}(\mathcal{A})^P \}$$

The recession cone  $\text{rec}(\mathcal{A})$  has an alternate definition, which is independent of  $\text{CT}(\mathcal{A})$  and the co-proposition construction, and which allows the determination of  $\text{rec}(\mathcal{A})$  in a simple, inductive manner, as our next result reveals.

First, we recall two basic polarity laws for polyhedral cones  $C_1, C_2$ :

$$(20) (c_1 + c_2)^P = c_1^P \cap c_2^P$$

$$(21) (c_1 \cap c_2)^P = c_1^P + c_2^P$$

See e.g. [27], [29].

Lemma 3:

$$1) \text{ rec } (\mathcal{B} \wedge \mathcal{B}) = \text{rec } (\mathcal{B}) \cap \text{rec } (\mathcal{B})$$

$$2) \text{ rec } (\mathcal{B} \vee \mathcal{B}) = \text{rec } (\mathcal{B}) + \text{rec } (\mathcal{B})$$

3) If  $\mathcal{A}$  is  $Ax \geq b$ , then

$$(22) \text{ rec } (\mathcal{A}) = \{x \geq 0 \mid Ax \geq 0\}$$

Proof:

1) From (8), (20) we have

$$(23) \text{CT}(\mathcal{B} \wedge \mathcal{B})^P = \text{CT}(\mathcal{B})^P \cap \text{CT}(\mathcal{B})^P$$

The Lemma 3(1) follows from (19), (23).

2) From (9), (21) we have

$$(24) \text{CT}(\mathcal{B} \vee \mathcal{B})^P = \text{CT}(\mathcal{B})^P + \text{CT}(\mathcal{B})^P$$

Next, note that for any proposition  $\mathcal{A}$ ,

$$(25) (v_0, v) \in \text{CT}(\mathcal{A})^P \text{ implies } v_0 \geq 0$$

Indeed,  $\text{CT}(\mathcal{A}) \neq \emptyset$ , and if  $(\pi_0, -\pi) \in \text{CT}(\mathcal{A})$  we have

$$(26) \pi v \geq \pi_0 v_0$$

by  $(v_0, v) \in \text{CT}(\mathcal{A})^P$ . However, holding  $\pi$  fixed in (26),  $\pi_0$

can be indefinitely decreased, since  $(-1, 0, \dots, 0) \in \text{CT}(\mathcal{A})$  by

induction on the clauses (7), (8), (9). But an indefinite decrease

in  $\pi_0$  of (26) with  $\pi, v$  fixed is possible only if  $v_0 \geq 0$ , proving (25).

Using (24), (25) we have from the definition (19),

$$\begin{aligned}
 (27) \quad v \in \text{rec}(\mathcal{B} \vee \mathcal{D}) &\iff (0, v) \in \text{CT}(\mathcal{B} \vee \mathcal{D})^P \\
 &\iff (0, v) = (u_0, u) + (w_0, w) \\
 &\quad \text{with } u_0 \geq 0, w_0 \geq 0 \\
 &\quad (u_0, u) \in \text{CT}(\mathcal{B})^P \text{ and } (w_0, w) \in \text{CT}(\mathcal{D})^P \\
 &\iff (0, v) = (0, u) + (0, w) \\
 &\quad \text{with} \\
 &\quad (0, u) \in \text{CT}(\mathcal{B})^P \text{ and } (0, w) \in \text{CT}(\mathcal{D})^P \\
 &\iff v = u + w \\
 &\quad \text{with } u \in \text{rec}(\mathcal{B}) \text{ and } w \in \text{rec}(\mathcal{D})
 \end{aligned}$$

for some  $u, w$ . From (27) we have (2) of this lemma.

3) It suffices to prove this result when  $Ax \geq b$  is a single inequality

(1) and use Lemma 3(1) above to complete the proof.

However, if  $\mathcal{O}$  is given by (1), from (7)

$$(28) \quad (0, v) \in \text{CT}(\mathcal{O})^P \iff \text{for all } \theta, \lambda_0, \lambda_1, \dots, \lambda_r \geq 0$$

we have

$$\begin{aligned}
 0 &\geq 0 \cdot (\theta a_0 - \lambda_0) + \sum_{j=1}^r v_j (-\theta a_j - \lambda_j) \\
 \iff v_j &\geq 0, \quad j = 1, \dots, r \\
 \text{and} \quad 0 &\leq \sum_{j=1}^r a_j v_j,
 \end{aligned}$$

But (28) is (22) for the case of one constraint, by the definition (19).

Q.E.D.

Our next result explains the use of the term "recession cone of  $\mathcal{A}$ ," If  $\text{cp}(\mathcal{A}) \neq \emptyset$ ,  $\text{rec}(\mathcal{A})$  is the recession cone of the polyhedron  $\text{cp}(\mathcal{A})$  in the usual sense [27], [29], i.e.,  $v \in \text{rec}(\mathcal{A})$  if and only if

$$(29) \quad \text{for all } x \in \text{cp}(\mathcal{A}) \text{ and } \lambda \geq 0, \text{ we have } x + \lambda v \in \text{cp}(\mathcal{A})$$

If  $\text{cp}(\mathcal{A}) \neq \emptyset$ , since  $\text{cp}(\mathcal{A})$  is polyhedral (29) is known to be equivalent to :

$$(29)' \quad x^0 + \lambda v \in \text{cp}(\mathcal{A}) \text{ for all } \lambda \geq 0$$

where  $x^0 \in \text{cp}(\mathcal{A})$  is an arbitrary element of  $\text{cp}(\mathcal{A})$ .

However, if  $\text{cp}(\mathcal{A}) = \emptyset$ , (29) holds vacuously for all  $v \in \mathbb{R}^r$ , while  $\text{rec}(\mathcal{A})$  defined by (19) usually differs from  $\mathbb{R}^r$ . It is  $\text{rec}(\mathcal{A})$  which provides the "correct" definition of a recession cone for the results of this paper, rather than the conventional definition (29).

Lemma 4: If  $\text{cp}(\mathcal{A}) \neq \emptyset$ , then  $v \in \text{rec}(\mathcal{A})$  if and only if (29) holds.

Proof: From the definition (10),

$$(30) \quad x \in \text{cp}(\mathcal{A}) \iff (1, x) \in \text{CT}(\mathcal{A})^P$$

Let  $x^0 \in \text{cp}(\mathcal{A})$ . Then  $v$  satisfies (29)' if and only if

$$(31) \quad \text{for all } \lambda \geq 0 \text{ we have } (1, x^0 + \lambda v) \in \text{CT}(\mathcal{A})^P.$$

Since  $(1, x^0) \in \text{CT}(\mathcal{A})^P$  by (30), clearly (31) holds if  $v \in \text{rec}(\mathcal{A})$ ,

for then  $(0, v) \in \text{CT}(\mathcal{A})^P$  and  $(1, x^0) + \lambda(0, v) \in \text{CT}(\mathcal{A})^P$ . For the converse,

let (31) hold; then for  $\lambda > 0$  arbitrarily large, we have

$$(32) \quad (1/\lambda, v + x^0/\lambda) \in \text{CT}(\mathcal{A})^P$$

since  $CT(\mathcal{A})^P$  is a cone. Putting  $\lambda \uparrow + \infty$  in (32), we obtain  $(0, v) \in CT(\mathcal{A})^P$  since the polyhedral cone  $CT(\mathcal{A})^P$  is closed. Hence by (19),  $v \in \text{rec}(\mathcal{A})$ .

Q.E.D.

Theorem 5: Suppose that

$$(33) \quad \begin{aligned} cp(\mathcal{D}_i) &\neq \emptyset, \quad i = 1, \dots, u \\ cp(\mathcal{D}_i) &= \emptyset, \quad i = u+1, \dots, s \end{aligned}$$

and that  $1 \leq u < s$ .

Then

$$(34) \quad cp(\mathcal{D}_1 \vee \dots \vee \mathcal{D}_s) = \text{clconv} \left( \bigcup_{h=1}^u cp(\mathcal{D}_h) \right) + \text{rec}(\mathcal{D}_{u+1}) + \dots + \text{rec}(\mathcal{D}_s)$$

Also, exactness (18) holds if and only if

$$(35) \quad \text{rec}(\mathcal{D}_h) \subseteq \text{rec}(\mathcal{D}_1) + \dots + \text{rec}(\mathcal{D}_u) \quad \text{for } h = u+1, \dots, s.$$

Proof: For all  $i = 1, \dots, s$  the non-empty polyhedral cone  $CT(\mathcal{D}_i)^P$  has a finite bases, so put

$$(36) \quad CT(\mathcal{D}_i)^P = \text{cone} \left\{ (0, v^{i,1}), \dots, (0, v^{i,a(i)}), (1, w^{i,1}), \dots, (1, w^{i,b(i)}) \right\},$$

$$i = 1, \dots, s.$$

In (36), we differentiate between elements of the basis for  $CT(\mathcal{D}_i)^P$  with zero or positive first co-ordinate, by (25); by normalization, we take those with positive first co-ordinates to have first co-ordinate unity.

By (30), (33) we have

$$(37) \quad \begin{aligned} b(i) &\geq 1 && \text{for } i = 1, \dots, u \\ b(i) &= 0 && \text{for } i = u+1, \dots, s \end{aligned}$$

We can always assume that  $a(i) \geq 1$ ,  $i = 1, \dots, s$ .

Note also that

$$(38) \quad (0, v) \in \text{CT}(\mathcal{P}_i)^P \iff \text{for some } \lambda_j^i \geq 0, \quad j = 1, \dots, a(i)$$

we have

$$(0, v) = \sum_{j=1}^{a(i)} \lambda_j^i v^{i,j}$$

From (38) and the definition (19),

$$(39) \quad \text{rec}(\mathcal{P}_i) = \text{cone} \left\{ v^{i,1}, \dots, v^{i,a(i)} \right\}$$

From (9), (21), (36), (37) we have

$$(40) \quad \begin{aligned} \text{CT}(\mathcal{P}_1 v \dots v \mathcal{P}_1)^P = \\ \text{cone} \left\{ (0, v^{1,1}), \dots, (0, v^{1,a(1)}), (1, w^{1,1}), \dots, (1, w^{1,b(1)}), \dots \right. \\ (0, v^{u,1}), \dots, (0, v^{u,a(u)}), (1, w^{u,1}), \dots, (1, w^{u,b(u)}) , \\ (0, v^{u+1,1}), \dots, (0, v^{u+1,a(u+1)}), \dots, \\ \left. (0, v^s, 1), \dots, (0, v^s, a(s)) \right\} \end{aligned}$$

Also, (30), (36) show

$$(41) \quad \begin{aligned} \text{cp}(\mathcal{P}_i) = \text{conv} \left\{ w^{i,1}, \dots, w^{i,b(i)} \right\} \\ + \text{cone} \left\{ v^{i,1}, \dots, v^{i,a(i)} \right\}, \quad i = 1, \dots, u \end{aligned}$$

while (30), (40), and  $u \geq 1$  show

$$\begin{aligned}
 (42) \quad \text{cp}(\mathcal{D}_1 v \dots v \mathcal{D}_s) = & \\
 & \text{conv} \{ w^{1,1}, \dots, w^{1,b(1)}, \dots, w^{u,1}, \dots, w^{u,b(u)} \} \\
 & + \text{cone} \{ v^{1,1}, \dots, v^{1,a(1)} \} \\
 & + \dots \\
 & + \text{cone} \{ v^{s,1}, \dots, v^{s,a(s)} \}
 \end{aligned}$$

From (41),

$$\begin{aligned}
 (43) \quad \text{clconv} \left( \bigcup_{h=1}^u \text{cp}(\mathcal{D}_h) \right) = & \text{conv} \{ w^{1,1}, \dots, w^{1,b(1)}, \dots, \\
 & w^{u,1}, \dots, w^{u,b(u)} \} \\
 & + \text{cone} \{ v^{1,1}, \dots, v^{1,a(1)} \} \\
 & + \dots \\
 & + \text{cone} \{ v^{u,1}, \dots, v^{u,a(u)} \} .
 \end{aligned}$$

Then from (42), (43) and (39) we have (34).

Now, by (33) exactness (18) is equivalent to

$$(44) \quad \text{cp}(\mathcal{D}_1 v \dots v \mathcal{D}_s) = \text{clconv} \left( \bigcup_{h=1}^u \text{cp}(\mathcal{D}_h) \right) .$$

Hence from (34) exactness holds if and only if  $\text{rec}(\mathcal{D}_i), i = u+1, \dots, s$  consists of directions of recession for  $\text{clconv} \left( \bigcup_{h=1}^u \text{cp}(\mathcal{D}_h) \right)$ . But

by (43) and Lemma 4, these directions constitute the cone

$$\begin{aligned}
 (45) \quad & \text{cone} \{ v^{1,1}, \dots, v^{1,a(1)} \} \\
 & + \dots \\
 & + \text{cone} \{ v^{u,1}, \dots, v^{u,a(u)} \}
 \end{aligned}$$

by (39), hence the necessary and sufficient condition (35).

Q.E.D.

The condition (35) for exactness is equivalent to the one obtained in [18, Sec. 2.1.1] and [19, Theorem in Sec. 2.1.1], although in [18], [19] we did not give the inductive definition of  $\text{rec } (a)$  that is supplied by Lemma 3, and is useful in Corollary 6 below toward our main result Theorem 8. Recall in [18], [19] that we used polyhedral definitions of  $\text{CT}(\mathcal{D}_h)^P$  in the form

$$(46) \quad \text{CT}(\mathcal{D}_h)^P = \{(x_0, x) \mid Q^h x - q^h x_0 \geq 0, x_0 \geq 0\}, \quad h = 1, \dots, s$$

and gave, as a sufficient condition for exactness,

that

$$(47) \quad \text{if } Q^h x \geq q^h \text{ is inconsistent, then}$$

$$Q^h x \geq 0 \text{ implies } x = \sum \{x^{(p)} \mid Q^p x \geq q^p \text{ consistent}\}$$

for certain  $x^{(p)}$  satisfying

$$Q^p x \geq 0.$$

By (46), we see

$$(48) \quad \text{cp}(\mathcal{D}_h) = \{x \mid Q^h x \geq q^h\},$$

$$(49) \quad \text{rec}(\mathcal{D}_h) = \{x \mid Q^h x \geq 0\}.$$

From (48), (49) we see that (47) is identical with (35). In [18], [19] the condition (47) is derived from a result of [5].

While here our interest in (34) and the exactness result (35) that it supplies, is motivated by an instance of (35) (specifically, Corollary 6(1) below that we need in Theorem 7), exactness (18) is of interest in itself. More precisely, when each  $\mathcal{D}_h$  is the matrix inequality  $A^h x \geq b^h$ , it is not hard to show that exactness (18) holds precisely if  $\text{CT}(\mathcal{D}_1 \vee \dots \vee \mathcal{D}_h)$  contains cuts sufficient to define

$\text{clconv} \left( \bigcup_{h=1}^s \text{cp} (\mathcal{D}_h) \right) = \text{clconv} \left( \bigcup_{h=1}^s \{x \geq 0 \mid A^h x \geq b^h\} \right)$ , i.e., if Balas' disjunctive constraints construction [1], [2] provides sufficiently many (and not just some) valid cuts for the logical condition that at least one of the inequality systems  $A^h x \geq b^h$ ,  $h = 1, \dots, s$  holds. Since Balas' construction is so simple, it is of value to know when it has this property.

Corollary 6: The following hypotheses imply exactness (18):

- 1)  $\text{rec} (\mathcal{D}_h)$  is independent of  $h = 1, \dots, s$
- 2)  $\text{cp} (\mathcal{D}_1 \vee \dots \vee \mathcal{D}_s)$  is a polytope
- 3) The sentences  $\mathcal{D}_h^H$ ,  $h = 1, \dots, s$ , are identical, where  $\mathcal{D}_h^H$  is obtained from  $\mathcal{D}_h$  by changing every right-hand-side  $\pi_0$ , in every atomic letter  $\mathcal{O}$  of (1), to zero.
- 4) Each  $\mathcal{D}_h$ ,  $h = 1, \dots, s$ , has the form  $(A^h x \geq b^h) \wedge \mathcal{D}_h'$ ,  
with

$$(50) \quad \{0\} = \{x \geq 0 \mid A^h x \geq 0\}, \quad h = 1, \dots, s$$

Proof:

- 1) If  $\text{cp} (\mathcal{D}_h) = \emptyset$  for  $h = 1, \dots, s$  we have exactness by Theorem 2(3).  
If  $\text{cp} (\mathcal{D}_h) \neq \emptyset$  for  $h = 1, \dots, s$ , exactness holds by Theorem 2(2).  
Therefore we may assume (33), and then exactness holds since (35) follows from the fact that

(51)  $\text{rec}(\mathcal{A}_1) + \dots + \text{rec}(\mathcal{A}_u) = \text{rec}(\mathcal{A}_h)$  when  
 $\text{rec}(\mathcal{A}_i)$  is independent of  $i = 1, \dots, s$ .

2) If  $\text{cp}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_1) = \emptyset$ , from Theorem 2(2) we have  $\text{cp}(\mathcal{A}_h) = \emptyset$   
 for  $h = 1, \dots, s$ , so exactness holds.

If  $\text{cp}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_s) \neq \emptyset$ , (34) and the fact that it is a  
 polytope gives  $\text{rec}(\mathcal{A}_h) = \{0\}$  for  $h = u+1, \dots, s$ , from which (35)  
 is immediate.

3) It suffices to show that this hypothesis implies the hypothesis  
 of Corollary 6(1). However, this implication is immediate by  
 induction on the number of connectives in  $\mathcal{A}_h$ , using Lemma 3(3)  
 for the ground step and Lemma 3(1) and (2) for the inductive step.

4) It suffices to show, by Corollary 6(1), that (50) gives  $\text{rec}(\mathcal{A}_h) = \{0\}$   
 for  $h=1, \dots, s$ . However from (50) and Lemma 3(1) we indeed have  
 $\text{rec}(\mathcal{A}_h) = \{0\}$

Q.E.D.

Corollary 6(2) was given earlier in [18], [19] and an instance  
 of Corollary 6(3) occurs as [16, Corollary 23].

Lemma 7: If either  $\text{cp}(\mathcal{A}) \neq \emptyset$  or  $\text{rec}(\mathcal{A}) = \{0\}$ , then  $\text{CT}(\mathcal{A})$   
 contains all the valid cuts for  $\text{cp}(\mathcal{A})$ , in the sense that (1) is  
 valid for  $\text{cp}(\mathcal{A})$  if and only if  $(a_0, -a_1, \dots, -a_r) \in \text{CT}(\mathcal{A})$ .

Proof: First, suppose  $\text{cp}(\mathcal{A}) \neq \emptyset$ , and (1) is valid.

If  $(a_0, -a_1, \dots, -a_r) \notin \text{CT}(\mathcal{A})$ , since  $\text{CT}(\mathcal{A})$  is a polyhedral cone containing  $(-1, 0, \dots, 0)$ , there would be a separating hyperplane  $(x_0, x_1, \dots, x_r)$ :

$$(52) \quad \begin{aligned} -x_0 &\leq 0 \\ \pi_0 x_0 - \sum_{j=1}^r \pi_j x_j &\leq 0 \text{ if } (\pi_0, -\pi_1, \dots, -\pi_r) \in \text{CT}(\mathcal{A}), \\ a_0 x_0 - \sum_{j=1}^r a_j x_j &> 0 \end{aligned}$$

If  $x_0 > 0$  in (52), we may assume  $x_0 = 1$  by multiplying all inequalities in (52) by  $1/x_0$ . Then from (10),  $x = (x_1, \dots, x_r) \in \text{cp}(\mathcal{A})$  and from (52) we have  $a_0 > \sum_{j=1}^r a_j x_j$ , a contradiction to the validity of (1).

If  $x_0 = 0$ , (52) becomes

$$(52)' \quad \begin{aligned} 0 &\leq \sum_{j=1}^r \pi_j x_j \text{ if } (\pi_0, -\pi_1, \dots, -\pi_r) \in \text{CT}(\mathcal{A}) \\ 0 &> \sum_{j=1}^r a_j x_j \end{aligned}$$

Since by hypothesis  $\text{cp}(\mathcal{A}) \neq \emptyset$ , there exists an element of  $\text{cp}(\mathcal{A})$ ,

$x^0 = (x_1^0, \dots, x_r^0)$ . From (10),

$$(53) \quad \pi_0 \leq \sum_{j=1}^r \pi_j x_j^0 \text{ if } (\pi_0, -\pi_1, \dots, -\pi_r) \in \text{CT}(\mathcal{A}).$$

From the first inequality of (52)' combined with (53), for any  $\rho \geq 0$

$$(54) \quad \pi_0 \leq \bigvee_{j=1}^r \pi_j(x_j^0 + \rho x_j) \text{ if } (\pi_0, -\pi_1, \dots, -\pi_r) \in \text{CT}(\mathcal{A})$$

From (10), we see that  $x^0 + \rho x \in \text{cp}(\mathcal{A})$  for all  $\rho \geq 0$ .

However, from the second inequality of (82)', for a suitably large  $\rho > 0$  we have .

$$(55) \quad a_0 > \sum_{j=1}^r a_j(x_j^0 + \rho x_j) .$$

This contradicts the validity of (1), and thus proves the desired result.

Next, suppose  $\text{rec}(\mathcal{A}) = \{0\}$ . If  $\text{cp}(\mathcal{A}) \neq \emptyset$ , the previous case applies. Otherwise  $\text{cp}(\mathcal{A}) = \emptyset$ , and we must prove that  $\text{CT}(\mathcal{A}) = \mathbb{R}^{r+1}$ , since every inequality (1) is valid for  $\text{cp}(\mathcal{A})$ .

From the definition (19) of  $\text{rec}(\mathcal{A})$  and from (30),  $\text{rec}(\mathcal{A}) = \{0\}$  and  $\text{cp}(\mathcal{A}) = \emptyset$  imply  $\text{CT}(\mathcal{A})^P = \{(0,0)\}$ . But then  $\text{CT}(\mathcal{A}) = \{(0,0)\}^P = \mathbb{R}^{r+1}$ , as desired.

Q.E.D.

## Section 2. The Main Result

In a manner analogous to the set definitions (4), (5), we define the propositions  $\mathcal{P}_h$ ,  $0 \leq h \leq t$ , by the inductive clauses:

$$(56) \quad \begin{aligned} \mathcal{P}_0 &= (Ax \geq b) \\ \mathcal{P}_{h+1} &= \bigvee_{p=1}^{t(h+1)} \left( \mathcal{P}_h \wedge (A^{h+1,p} x \geq b^{h+1,p}) \right), \quad 0 \leq h \leq (t-1). \end{aligned}$$

Here is our generalization of Theorem 1.

Theorem 8: If  $\{x | Ax \geq b'\}$  is bounded and non-empty for some  $b'$ ,  $x \geq 0$  occurs among the inequalities  $Ax \geq b$  of (2), and the system (2), (3) is facial, then CT  $(\mathcal{O}_t)$  contains all the valid inequalities for  $\text{clconv} \{x | (2) \text{ and } (3) \text{ hold}\}$ .

In fact,

$$(57) \quad \text{cp}(\mathcal{O}_u) = \mathcal{K}_u = \text{clconv} \left( x \left| \begin{array}{l} Ax \geq b \text{ and, for each } h = 1, \dots, u, \text{ at least} \\ \text{one of} \\ A^{h,1}x \geq b^{h,1} \text{ or } \dots \text{ or } A^{h,t(h)}x \geq b^{h,t(h)} \\ \text{holds.} \end{array} \right. \right)$$

for  $u = 0, 1, \dots, t$ .

Proof: First, we prove inductively that

$$(58) \quad \text{rec}(\mathcal{O}_u) = \{0\}, \quad u = 0, 1, \dots, t$$

For  $u = 0$ , (58) follows from the facts that  $\{x | Ax \geq b'\}$  is bounded and non-empty, and a bounded non-empty polyhedron has only 0 as a direction of recession. To go from  $u$  to  $(u+1)$  note in (56) for  $h = u$  that for  $p = 1, \dots, t(u+1)$  we have by the inductive hypothesis

$$(59) \quad \text{rec} \left( \mathcal{O}_u \wedge (A^{u+1,p}x \geq b^{u+1,p}) \right) = \{0\} \cap \text{rec} \left( A^{u+1,p}x \geq b^{u+1,p} \right) \\ = \{0\}$$

from Lemma 3(1). Then again from (56) and Lemma 3(2) we obtain  $\text{rec}(\mathcal{O}_{u+1}) = \{0\}$ , completing the induction.

By (58) and Corollary 6(1), the disjunction  $\mathcal{O}_{u+1}$  is exact for  $0, \dots, t-1$ .

Next, we assign to each proposition  $\mathcal{O}_u$ ,  $u = 0, 1, \dots, t$  its disjunctive normal form  $\mathcal{O}_u'$  [23], [28], which in this context is

defined as follows. We put  $\mathcal{P}'_0 = \mathcal{P}_0$ , and if inductively  $\mathcal{P}'_u$  has

taken the form

$$(60) \quad \mathcal{P}'_u = \bigvee_{j=1}^{n(u)} \mathcal{P}'_{u,j}$$

with  $\mathcal{P}'_{u,j}$  a conjunction of atomic sentences (1),

(initially  $n(0) = 1$ ,  $\mathcal{P}'_{0,1} = \mathcal{P}'_0 = \mathcal{P}_0$ ) we set

$$(61) \quad \mathcal{P}'_{u+1} = \bigvee_{j=1}^{n(u)} \left( \bigvee_{p=1}^{t(u+1)} (\mathcal{P}'_{u,j} \wedge (A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \begin{matrix} u+1,p \end{matrix})) \right)$$

where (61) also exhibits  $\mathcal{P}'_{u+1}$  as the form (60), by re-indexing over pairs  $(j,p)$  with  $j = 1, \dots, n(u)$  and  $p = 1, \dots, t(u+1)$ . One easily notes inductively, that  $\mathcal{P}'_u$  and  $\mathcal{P}_u$  are logically equivalent, i.e.,

$$(62) \quad \mathcal{P}'_u(x) \text{ if and only if } \mathcal{P}_u(x)$$

where in (62) we explicitly note the dependence of the truth of  $\mathcal{P}'_u$  and  $\mathcal{P}_u$  upon the vector  $x$ .

By induction upon the construction (60), (61) of  $\mathcal{P}'_u$ , it is easy to see that  $Ax \geq b$  occurs as part of the conjunction of atomic sentences which constitute  $\mathcal{P}'_{u,j}$  for  $j = 1, \dots, n(u)$ . Since  $\{0\} = \{x \mid Ax \geq 0\}$ , from Corollary 6(4) we see that  $\mathcal{P}'_u$  of (60) is exact. Also, the proposition

$$\bigvee_{p=1}^{t(u+1)} (\mathcal{P}'_{u,j} \wedge (A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \begin{matrix} u+1,p \end{matrix})) \text{ that occurs in } \mathcal{P}'_{u+1} \text{ of (61)}$$

is exact for the same reason, as is any subformula of  $\mathcal{P}'_{u+1}$ .

We now show

$$(63) \quad \text{cp}(\mathcal{P}_u) \supseteq \text{cp}(\mathcal{P}'_u), \quad u = 0, 1, \dots, t$$

by induction on  $u$ . For  $u = 0$ , the result follows from  $\mathcal{P}_0 = \mathcal{P}'_0$ .

To go from  $u$  to  $(u+1)$ , we have by exactness of  $\mathcal{P}_{u+1}$  that

$$(64) \quad \text{cp}(\mathcal{P}_{u+1}) = \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \left( \text{cp}(\mathcal{P}_u) \cap \text{cp} \left( A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \right)^{u+1,p} \right) \right)$$

$$(l.1) \quad \supseteq \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \left( \text{cp}(\mathcal{P}'_u) \cap \text{cp} \left( A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \right)^{u+1,p} \right) \right)$$

$$(l.2) \quad = \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \text{cp} \left( \left( \bigvee_{j=1}^{n(u)} \mathcal{P}'_{u,j} \right) \wedge \left( A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \right)^{u+1,p} \right) \right)$$

$$(l.3) \quad \supseteq \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \text{cp} \left( \bigvee_{j=1}^{n(u)} \left( \mathcal{P}'_{u,j} \wedge \left( A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \right)^{u+1,p} \right) \right) \right)$$

$$(l.4) \quad = \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \bigcup_{j=1}^{n(u)} \text{cp} \left( \mathcal{P}'_{u,j} \wedge \left( A \begin{matrix} u+1,p \\ x \geq b \end{matrix} \right)^{u+1,p} \right) \right)$$

$$(l.5) \quad = \text{cp}(\mathcal{P}'_{u+1})$$

In (64), (l.1) follows from the inductive hypothesis (63); we get (l.2) from Theorem 2(4) and (60); (l.3) is a repeated application of Theorem 2(5); and both (l.4) and (l.5) are consequences of the exactness that was established in the preceding paragraph.

In what follows, we shall also need to know that

$$(65) \quad \text{cp}(\mathcal{P}_t) \subseteq \text{cp}(\mathcal{P}_{t-1}) \subseteq \dots \subseteq \text{cp}(\mathcal{P}_1) \subseteq \text{cp}(\mathcal{P}_0) = \{x \mid Ax \geq b\}.$$

The equality in (65) follows from Theorem 2(7) and the fact that the inequalities  $x \geq 0$  appear among  $Ax \geq b$ . The inclusions are proven by induction on  $u = 0, 1, \dots, t$ . To go from  $u$  to  $(u+1)$ , first note that for  $p = 1, \dots, t(u+1)$

$$(66) \quad \text{cp}(\mathcal{P}_u \wedge (A^{u+1,p}x \geq b^{u+1,p})) \subseteq \text{cp}(\mathcal{P}_u)$$

from Theorem 2(4). Then from the exactness of the disjunction  $\mathcal{P}_{u+1}$  and (66) plus the fact that  $\text{cp}(\mathcal{P}_u)$  is a polyhedron, we have

$$(67) \quad \text{cp}(\mathcal{P}_{u+1}) = \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \text{cp}(\mathcal{P}_u \wedge (A^{u+1,p}x \geq b^{u+1,p})) \right) \subseteq \text{cp}(\mathcal{P}_u),$$

thus completing the induction step. From (65) it follows that  $\text{cp}(\mathcal{P}_u)$  is a polytope, for  $u = 0, 1, \dots, t$ .

This completes the remarks that are introductory to the main part of the proof. We now prove, by induction on  $u = 0, 1, \dots, t$  that

$$\begin{aligned}
 (68) \quad \text{cp}(\mathcal{P}'_u) &= \text{clconv} \left\{ x \mid \mathcal{P}_u(x) \text{ is true} \right\} \\
 &= \text{clconv} \left( x \mid \begin{array}{l} Ax \geq b \text{ and, for each } h = 1, \dots, u \\ \text{at least one of} \\ A^{h,1}x \geq b^{h,1} \text{ or } \dots \text{ or } A^{h,t(h)}x \geq b^{h,t(h)} \\ \text{holds} \end{array} \right)
 \end{aligned}$$

Note that the second inequality in (68) follows directly from the definition (56) of the propositions  $\mathcal{P}_u$ . Hence we will prove only the first equality in (68).

The first equality of (68) is immediate if  $u = 0$ , by Theorem 2(7) and the fact that  $x \geq 0$  occurs among the inequalities  $Ax \geq b$  of (2). To go inductively from  $u$  to  $(u+1)$ , note

$$\begin{aligned}
 (69) \quad \text{clconv} \left\{ x \mid \mathcal{P}_u(x) \text{ is true} \right\} &= \text{cp} (\mathcal{P}_u) & (l.1) \\
 &\supseteq \text{cp} (\mathcal{P}'_u) & (l.2) \\
 &\supseteq \text{clconv} \left\{ x \mid \mathcal{P}'_u(x) \text{ is true} \right\} & (l.3) \\
 &= \text{clconv} \left\{ x \mid \mathcal{P}_u(x) \text{ is true} \right\} & (l.4)
 \end{aligned}$$

In (69), (l.1) follows from the inductive hypothesis; (l.2) is justified by (63); (l.3) is a consequence of Theorem 2(1); and (l.4) follows from (62). Since the chain of equalities and inclusions in (69) begins and ends with the same set, we conclude that all inclusions of (69) are equalities, and in particular that

$$(70) \quad \text{cp} (\mathcal{P}_u) = \text{cp} (\mathcal{P}'_u)$$

for the current index  $u$  of the induction. Next, from (70) we have

for any  $p = 1, \dots, t(u+1)$  that

$$\begin{aligned}
 (71) \quad \text{cp} \left( \mathcal{P}_u \wedge (A^{u+1,p} x \geq b^{u+1,p}) \right) &= \\
 &= \text{cp}(\mathcal{P}_u) \cap \text{cp} \left( A^{u+1,p} x \geq b^{u+1,p} \right) & (\ell.1) \\
 &= \text{cp}(\mathcal{P}_u') \cap \text{cp} \left( A^{u+1,p} x \geq b^{u+1,p} \right) & (\ell.2) \\
 &= \text{cp} \left( \mathcal{P}_u' \wedge (A^{u+1,p} x \geq b^{u+1,p}) \right) & (\ell.3) \\
 &= \text{cp} \left( \left( \bigvee_{j=1}^{n(u)} \mathcal{P}_{u,j}' \right) \wedge (A^{u+1,p} x \geq b^{u+1,p}) \right) & (\ell.4) \\
 &= \text{clconv} \left\{ x \mid \left( \bigvee_{j=1}^{n(u)} \mathcal{P}_{u,j}' \right) \wedge (A^{u+1,p} x \geq b^{u+1,p}) \text{ is true} \right\} & (\ell.5)
 \end{aligned}$$

In (71), ( $\ell.1$ ) follows from Theorem 2(4); ( $\ell.2$ ) is justified by (70); ( $\ell.3$ ) is again a consequence of Theorem 2(4); and ( $\ell.4$ ) is valid by the definition (60). As to ( $\ell.5$ ) of (71), this follows from Theorem 2(6); in fact,  $A^{u+1,p} x \geq b^{u+1,p}$  is a facial constraint relative to  $Ax \geq b$ , and  $\text{cp} \left( \bigvee_{j=1}^{n(u)} \mathcal{P}_{u,j}' \right) = \text{cp}(\mathcal{P}_u')$  is a polytope with

$$(72) \quad \text{cp}(\mathcal{P}_u') \subseteq \text{cp}(\mathcal{P}_u) \subseteq \text{cp}(\mathcal{P}_0) = \{ x \mid Ax \geq b \}$$

$$\text{From (72), } \text{cp} \left( A^{u+1,p} x \geq b^{u+1,p} \right) = \{ x \mid A^{u+1,p} x \geq b^{u+1,p} \}$$

provides a face of  $\text{cp}(\mathcal{P}_u')$ , as required in Theorem 2(6).

Finally, by exactness of  $\mathcal{P}_{u+1}$  in (56) for  $u = h$ , we have

$$\begin{aligned}
(73) \quad \text{cp}(\mathcal{P}_{u+1}) &= \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \text{cp}(\mathcal{P}_u \wedge (A^{u+1,p}_x \geq b^{u+1,p})) \right) \\
(\ell.1) &= \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \{x \mid \mathcal{P}_u \wedge (A^{u+1,p}_x \geq b^{u+1,p}) \text{ is true} \} \right) \\
(\ell.2) &= \text{clconv} \left( \{x \mid \bigvee_{p=1}^{t(u+1)} (\mathcal{P}_u \wedge (A^{u+1,p}_x \geq b^{u+1,p})) \text{ is true} \} \right) \\
(\ell.3) &= \text{clconv} \{x \mid \mathcal{P}_{u+1}(x) \text{ is true} \}
\end{aligned}$$

In (73), ( $\ell.1$ ) is a consequence of (71); ( $\ell.2$ ) is a valid way of interchanging union and disjunction; and ( $\ell.3$ ) follows from the definition (56) with  $u = h$ . We have now completed the induction step for (68) and see that (68) is true.

To establish (57), we need to prove that

$$(74) \quad \text{cp}(\mathcal{G}_u) = K_u, \quad u = 0, 1, \dots, t$$

where  $K_u$  is as defined in (4), (5). The proof of (74) is by induction on  $u$ , and the ground case  $u = 0$  is immediate. To go from  $u$  to  $(u+1)$ , using the exactness of  $\mathcal{F}_{u+1}$  we have

$$(75) \quad \text{cp}(\mathcal{G}_{u+1}) = \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} \text{cp}(\mathcal{G}_u) \cap \text{cp}(A^{u+1,p}_x \geq b^{u+1,p}) \right) \quad (\ell.1)$$

$$= \text{clconv} \left( \bigcup_{p=1}^{t(u+1)} (K_u \cap \{x \mid A^{u+1,p}_x \geq b^{u+1,p}\}) \right) \quad (\ell.2)$$

$$= K_{u+1}. \quad (\ell.3)$$

In (75), (L.1) follows from Theorem 2(4); (L.2) is valid by the inductive hypothesis (74) and Theorem 2(7); and (L.3) is direct from the definition (5) for  $h = u$ . This completes our proof of (57).

By (53) for  $u = t$  and by (10), Lemma 7 shows that  $CT(\mathcal{C}_t)$  contains all the valid inequalities for

$$\text{clconv} \{x \mid \mathcal{O}_t(x) \text{ is true}\} = \text{clconv} \{x \mid (2) \text{ and } (3) \text{ hold}\} . \quad \text{Q.E.D.}$$

We next make several remarks, which are directed at showing how Theorem 8 can be improved in various ways.

Remark 1: The restriction that  $x \geq 0$  occur among  $Ax \geq b$  can be removed, by a change in the definition of the co-proposition  $CT(\mathcal{O})$  for an atomic sentence as in (1).

In place of (7), we can use

$$(7)' \quad CT'(\mathcal{O}) = \text{cone} \{(a_0, -a_1, \dots, -a_r), (-1, 0, \dots, 0)\},$$

the single unit vector  $(-1, 0, \dots, 0)$  being retained since one may always decrease the r.h.s.  $a_0$  of a valid inequality (1), whether or not the variables  $x$  are non-negative. Then the inductive clauses (8), (9) are retained as before, putting the prime on  $CT(\mathcal{A})$  to write  $CT'(\mathcal{A})$ .

In place of (10), we write

$$(10)' \quad \text{cp}'(\mathcal{A}) = \left( x \mid \begin{array}{l} \text{for every } (\pi_0, -\pi_1, \dots, -\pi_r) \in CT'(\mathcal{A}), \\ \sum_{j=1}^r \pi_j x_j \geq \pi_0 \end{array} \right)$$

With these changes, Theorem 2(2)-(6) are proven by precisely the same proofs, as we indicated in [18], [19] when we discussed (7)'.

Theorem 2(1) becomes

$$(11)' \quad \text{cp}(\mathcal{A}) \supseteq \text{clconv} \{ x \mid \mathcal{L}(x) \text{ is true} \}$$

and Theorem 2(7) becomes

$$(17)' \quad \text{cp}(\mathcal{A}) = \{ x \mid Ax \geq b \}$$

Lemma 3(1), (2) are proven exactly as above, while Lemma 3(3) becomes

$$(22)' \quad \text{rec}(\mathcal{A}) = \{ x \mid Ax \geq 0 \}$$

The proof of Lemma 4 need not be changed, and the same holds for the proof of Theorem 5: both these results remain true. Corollary 6(1)-(3) remains valid with the same proof, but in Corollary 6(4) we must change

(50) to

$$(50)' \quad \{0\} = \left\{ x \mid A^h x \geq 0 \right\}, \quad h = 1, \dots, s.$$

Then the proof of Theorem 8 goes through almost unchanged, so the theorem holds even if  $x \geq 0$  is not among the constraints of  $Ax \geq b$ .

Remark 2: Theorem 8 is false if the hypothesis, that  $\{ x \mid Ax \geq b' \}$  is non-empty and bounded for some  $b'$ , is dropped. In fact, one easily proves that

$$\text{cp}(\mathcal{G}_u) \supseteq K_u, \quad u = 0, 1, \dots, t$$

without this hypothesis, and without facial constraints, by induction on  $u$ . But even the result (6) need not be true.

Consider, for instance, the facial constraint system

$$(2)' \quad 1 \geq x_1 \geq 0, \quad x_2 \geq 0$$

(3)' at least one of

$$x_1 \leq 0 \text{ or } (x_1 = 1, x_2 = 0)$$

holds, and

at least one of

$$x_1 \geq 1$$

holds.

From (4), (5) we have

$$(76) \quad \begin{aligned} K_1 &= \text{clconv} \left( \{x \mid x_1 = 0, x_2 \geq 0\} \cup \{x \mid x_1 = 1, x_2 = 0\} \right) \\ &= \{x \mid 0 \leq x_1 \leq 1, x_2 \geq 0\} \\ K_2 &= \{x \mid x_1 = 1, x_2 \geq 0\} \end{aligned}$$

Therefore

$$(77) \quad \begin{aligned} K_2 &\neq \{(1,0)\} \\ &= \text{clconv} \{(1,0)\} = \text{clconv} \{x \mid (2)' \text{ and } (3)' \text{ hold}\}. \end{aligned}$$

Here the failure of (6) is due to the non-trivial direction of recession  $(0,1)$  for  $(2)'$ .

Nevertheless, by a device due to A. Charnes (see e.g. [7], [17]) it is possible to obtain a result very similar to Theorem 8 without hypotheses of boundedness or non-emptiness. The idea is as follows.

We shall always add, to the constraints of (2), (3) for real matrices  $A, b, A^{h,p}, b^{h,p}$  the constraints

$$(78) \quad -M \leq x_j \leq M \quad , \quad j = 1, \dots, r$$

where  $M$  is an infinitely large quantity (or, if  $x \geq 0$  occurs in  $Ax \geq b$ , one may use the simple constraint  $x_1 + \dots + x_r \leq M$ ). We now have a system of constraints in  $R(M)$ , the simple transcendental extension of  $R$  obtained by adjoining  $M$ , and ordered by placing on  $M$  the infinite valuation (for details of this field, see [17]).

In the field  $R(M)$ , the co-propositions  $CT(\mathcal{L})$  or  $CT'(\mathcal{L})$  are defined as before. All previous results can be recovered, with proofs virtually unchanged, except that at points in some proofs the term "bounded" is to be replaced by "is a polytope." In fact, these results are valid in any ordered field, of which  $R(M)$  is one. In a general ordered field, a "polytope" is the empty set or the convex span of finitely many points; or, equivalently, an intersection of half spaces which, if non-empty, has no non-trivial directions of recession.

In particular, Theorem 8 holds in  $R(M)^r$ , with the condition on that there be no non-zero solution to  $Ax \geq 0$  replacing the requirement of boundedness and non-emptiness. But here  $Ax \geq b$  has rows corresponding to the added constraints (78), hence there is no non-zero solution and  $CT(\mathcal{P}_t)$  in  $R(M)$  has all the valid cuts for (2), (3) augmented by (78).

Next, note that the elements  $(\pi_0, -\pi_1, \dots, \pi_r) \in CT(\mathcal{P}_t)$  in  $R(M)^{r+1}$ , which are purely real (i.e.,  $\pi_j$  is real for  $j = 0, 1, \dots, r$ ), provide all the valid cuts for  $\text{clconv} \{ x \mid (2) \text{ and } (3) \text{ hold} \}$ . Indeed, if

$$(79) \quad \sum_{j=1}^r \pi_j x_j \geq \pi_0$$

is in  $CT(\mathcal{O}_t)$ , it is valid for (2), (3), (78). Hence it is valid for any  $x \in R^r$  satisfying (2), (3) as  $x$  surely satisfies (78). For the converse, if (79) is valid for (2), (3) it must also be valid for (2), (3), (78) and hence be in  $CT(\mathcal{O}_t)$ . For if (79) is not valid for (78) also, there exists  $x(M) \in R(M)^r$  which satisfies (2), (3), (78) and such that

$$(80) \quad \sum_{j=1}^r \pi_j x_j(M) < \pi_0$$

But then for large integral  $k$ ,  $x(k) \in R^r$  satisfies (2) and (3) and yet

$$(81) \quad \sum_{j=1}^r \pi_j x_j(k) < \pi_0$$

(see [17] for details), contradicting the validity of (79) for (2), (3).

In summary, to obtain the conclusion of Theorem 8, i.e., that all valid cuts are in  $CT(\mathcal{O}_t)$ , one need only suitably adjoin an infinite quantity in constraints (78) in the construction, and then use only those cutting-planes in which the infinite quantity is absent.

Remark 3: As is evident from the proof of Theorem 8, one can weaken the requirement that the system (2), (3) is facial, and instead require that all of the matrix inequalities  $A^{h+1,p} x \geq b^{h+1,p}$  for  $p = 1, \dots, t(h+1)$  are facial to the polyhedron  $cp(\mathcal{O}_h) = K_h$ , for  $h = 0, \dots, t-1$ .

In order to relate Theorem 8 to previous results, we shall next develop a succinct formulation of one of its consequences, in terms of logical derivations in a certain system of formal deduction. Chvátal was the first to implicitly state results on cutting-planes in this form [ 8 ], and Blair has since used it [ 4 ].

In what follows, we shall draw on concepts and terminology from the fields of mathematical logic called proof theory and model theory. No background in logic is presumed, but the interested reader may wish to see a fuller development in [25], [26].

To present a system of deduction that we shall use, and do so in an informal manner, we proceed as follows.

We shall be studying certain finite tree structures that we shall call derivations, or (equivalently) derivation-trees. To the nodes of these trees shall correspond certain linear inequalities (1). Were we to be entirely formal, the nodes would correspond to certain statements in a formal language that express linear inequalities; but we shall make no such distinction here. The tree shall be spread out at the

"top," and narrow to one node at the "bottom."

Where a given node has several others connected to it by an edge and just above it, we require that the inequality assigned to this node "follow from" the inequalities assigned to the nodes just above, in the sense that it is the conclusion of one of the rules of deduction of the logical system and the inequalities above are the premises of this rule.

For instance, one of our rules of deduction shall be

$$(LC) \frac{a_1 v_1 + \dots + a_n v_n \geq a_0, \quad b_1 v_1 + \dots + b_n v_n \geq b_0}{(\lambda a_1 + \theta b_1) v_1 + \dots + (\lambda a_n + \theta b_n) v_n \geq c_0}$$

and for the application of (LC) we require that  $\lambda, \theta \geq 0$  and that  $c_0 \leq \lambda a_0 + \theta b_0$ . The premisses of (LC) are  $a_1 v_1 + \dots + a_n v_n \geq a_0$  and also  $b_1 v_1 + \dots + b_n v_n \geq b_0$ ; the conclusion of (LC) is

$(\lambda a_1 + \theta b_1) v_1 + \dots + (\lambda a_n + \theta b_n) v_n \geq c_0$ . The parameters of (LC) are  $\theta, \lambda$  and  $c_0$ . The rule (LC) is understood (as is any rule) as "saying" that, if its premisses have already been "deduced," one is entitled to "deduce" its conclusion. For the generic variables  $v_1, \dots, v_n$  one may employ any of the variables  $x_1, x_2, \dots, x_r, y_1, y_2, \dots, z_1, \dots$

When (LC) occurs in a tree, as part of it, that part looks like



where the top two nodes correspond to the premisses, and the bottom node corresponds to the conclusion. An instance of (LC), with the corresponding inequalities underlined, and  $\lambda = \theta = 1$ , is

$$\begin{array}{ccc}
 \underline{2x_1 - x_2 \geq 7} & & \underline{5x_1 + 3x_2 \geq -2} \\
 & \searrow \quad \swarrow & \\
 & \underline{7x_1 + 2x_2 \geq 5} &
 \end{array}$$

which is often abbreviated

$$\begin{array}{ccc}
 \underline{2x_1 - x_2 \geq 7} & & \underline{5x_1 + 3x_2 \geq -2} \\
 & \searrow \quad \swarrow & \\
 & \underline{7x_1 + 2x_2 \geq 5} &
 \end{array}$$

The inequality assigned to the last node of a derivation tree is called the endformula of the tree. The inequalities assigned to the top nodes of the tree are called the assumptions of the derivation. We say that the derivation is a "proof of its endformula from its assumptions."

In various contexts, different inequalities are designated as axioms; if the assumptions of a derivation are all axioms, we shall say that the derivation is a proof of its endformula. A proof may have only one node, that corresponds to an axiom: it proves the axiom. With all this terminology, one evident consequence is this: if there is a proof of every assumption of a derivation, these may be appended above

the assumptions so as to constitute a proof of the endformula of the derivation.

To give an extended illustration of a proof tree, we add a rule, which is one instance of a type of rule we shall consider in results to follow.

Consider the rule

$$\begin{array}{c}
 \text{(CMP)'} \quad \frac{sx_1 + ty_1 \geq \alpha_0 \qquad s'x_1 + t'y_1 \geq \alpha_0}{s'x_1 + ty_1 \geq \alpha_0}
 \end{array}$$

The rule (CMP)' is of course not true in general, but it is valid for problems with a constraint  $x_1 \cdot y_1 = 0$ . Indeed, for such problems, if  $x_1 = 0$  the conclusion becomes the premiss on the left, while if  $y_1 = 0$  the conclusion becomes the premiss on the right.

All the rules, other than (LC), that we shall introduce below, will share the property that the conclusion becomes equivalent to one of the premisses, provided that one of a finite number of alternatives hold. Rules of this type were first suggested by Blair [4] for bivalent programming, and we shall call them disjunctive rules.

For our example, we take as axioms  $2y_1 + z_1 \geq 4$ ,  $y_1 + z_1 \geq 3$ ,  $y_1 + 2z_1 \geq 4$ . Using rules (LC) and (CMP)', we have the following derivation tree  $\Sigma$ , where we indicate the rule used at each node other than a top node:

$$\begin{array}{cccc}
 \underline{2y_1 + z_1 \geq 4} & \underline{y + z_1 \geq 3} & \underline{y_1 + z_1 \geq 3} & \underline{y_1 + 2z_1 \geq 4} \\
 \swarrow & \swarrow & \swarrow & \swarrow \\
 \text{(LC)} & & \text{(LC)} & \\
 \underline{3y_1 + 2z_1 \geq 7} & & \underline{2y_1 + 3z_1 \geq 7} & \\
 \swarrow & & \swarrow & \\
 \text{(CMP)'} & & & \\
 \underline{2y_1 + 2z_1 \geq 7} & & & 
 \end{array}$$

This is a proof of  $2y_1 + 2z_1 \geq 7$ , since all assumption inequalities are axioms.

A model for a given set of axioms and of rules of deduction, is a specific non-empty structure whose elements are vectors, the components of which are designated by variables  $x_1, x_2, \dots, y_1, \dots, z_1, \dots$  etc. in the axioms and rules of deduction, that satisfies the following two conditions: 1) For every substitution of a vector in an axiom (components being substituted for the corresponding variable), a true numerical statement results; 2) For every substitution of a vector in the premisses of a rule of deduction (components being substituted for the corresponding variable) such that all the premisses become true numerical statements, the same substitution in the conclusion of the rule yields a true numerical statement. From this definition of a model, an easy induction on the length of a derivation shows that the endformula of any derivation is true for all substitutions for which all assumption inequalities are true. In specific, the

the endformula of a proof is true for all substitutions in any model for the given axioms and rules of deduction. For a discussion of models in a more general context, see [26].

Clearly, a model for (LC) is  $R^n$ , in which  $(v_1, \dots, v_n)$  varies over all  $n$ -tuples of real numbers. Any  $K^n$ , with  $K$  an ordered field, also provides a model with vectors in  $K^n$ , as the co-efficients  $a_1, \dots, a_n, a_0, b_1, \dots, b_n, b_0$  and the parameters  $\lambda, \theta, c_0$  vary over elements of  $K$ .

From the model  $R^2$ , it is easy to see that  $2y_1 + 2z_1 \geq 7$  cannot be the endformula of any proof that uses rule (LC) alone. Indeed, the assumption inequalities of  $\Sigma$  hold for  $(y_1, z_1) = (1, 2)$ , while the endformula does not.

A stronger endformula statement than that of  $\Sigma$  occurs in this shorter proof  $\Gamma$  from the same axioms:

$$\Gamma \quad \frac{\frac{2y_1 + z_1 \geq 4}{\quad} \quad \frac{y_1 + 2z_1 \geq 4}{\quad}}{y_1 + z_1 \geq 4} \text{ (CMP)'}$$

A model for these axioms, plus the rules of deduction (LC) and (CMP)', is

$$M = \left\{ (y_1, z_1) \in R^2 \mid y_1 z_1 = 0, 2y_1 + z_1 \geq 4, y_1 + z_1 \geq 3, y_1 + 2z_1 \geq 4 \right\},$$

as one easily verifies. Since  $(0, 4) \in M$ , we see that the endformula of  $\Gamma$  cannot be improved, in the sense that there is no proof of

$y_1 + z_1 \geq 4 + \delta$  for any  $\delta > 0$ , from these axioms and rules of deduction.

We next discuss a specific model and provide disjunctive rules of deduction that are clearly valid for it; then we prove a surprising property of these rules.

Consider the following, very general, complementarity constraints, in which  $J_1, \dots, J_t$  are certain non-empty sets of subsets of  $\{1, \dots, r\}$ , so that  $J_h = \{K_{h(1)}, \dots, K_{h(t(h))}\}$ , where each  $K_{h(j)}$  is a non-empty subset of  $\{1, \dots, r\}$ , and  $x = (x_1, \dots, x_r)$ :

$$Ax \geq b$$

$$x \geq 0$$

$$(CMP) \quad \sum_{h=1}^t \prod_{K \in J_h} \left( \sum_{k \in K} x_k \right) = 0$$

A specific instance of (CMP) is

$$Ay + Bz \geq b$$

$$(GLC) \quad y, z \geq 0$$

$$y \cdot z = 0$$

in which  $r = 2s$  for an integer  $s \geq 1$ ,  $y = (x_1, \dots, x_s)$  and

$z = (x_{s+1}, \dots, x_{2s})$ ; also,  $J_h = \{\{j\}, \{j+s\}\}$  for  $h = 1, \dots, t$ .

The problem (GLC) is itself a generalization of the linear complementarity problem, since we do not require that

$$(82) \quad \begin{aligned} A &= \begin{bmatrix} M \\ -M \end{bmatrix}, & (M \text{ a square matrix}) \\ B &= \begin{bmatrix} -I_s \\ I_s \end{bmatrix}, \\ b &= \begin{bmatrix} q \\ -q \end{bmatrix}, \end{aligned}$$

as would be required for linear complementarity constraints [10],[11],[21].

Due to the condition  $x \geq 0$  imposed on the variables in (CMP), the nonlinear constraint is equivalent to the logical constraints

$$(3)'' \quad \text{for } h = 1, \dots, t \text{ at least one of the conditions}$$

$$\left( x_k = 0, k \in K_{h(1)} \right) \text{ or } \dots \text{ or } \left( x_k = 0, k \in K_{h(h')} \right)$$

holds,

where  $h' = t(h)$  in the notation of (2),(3). Therefore, (CMP) is among the class of constraint sets (2),(3), and since  $x \geq 0$  is included in the constraints of (CMP), the logical conditions (3)'' are all facial in Balas' sense [2].

For convenience in stating our rule of deduction, we shall assume that all  $J_h$  have at least two elements. This is not a serious restriction, for if  $J_h = \{K_{h(1)}\}$  then all variables  $x_k, k \in K_{h(1)}$ , may be removed from those among  $x$  by setting them to zero, with the resulting constraint system equivalent to (CMP) (if all variables are thus removed, (CMP) is consistent, if and only if  $0 \geq b$ , in which case the unique solution is  $x = 0$ ).

$$a_{11}x_1 + \dots + a_{1r}x_r \geq a_0, \dots, a_{j1}x_1 + \dots + a_{jr}x_r \geq a_0, \dots, a_{h1}x_1 + \dots + a_{hr}x_r \geq a_0$$


---


$$(h' = t(h))$$

$$a_{11}x_1 + \dots + a_{r1}x_r \geq a_0$$

where  $a_{jk} = a_k$  if  $k \neq h(j)$

Figure 1: The Rule (CMC)<sub>h</sub> for J<sub>h</sub>

Our rules of deduction for (CMP) are given in Figure 1, where the stipulations on the scalar quantities which occur deserve note. The rule  $(CMC)_h$  is clearly valid in that model  $\mathcal{M}$  consisting of all  $x \in R^r$  that satisfy the constraints (CMP), if  $\mathcal{M} \neq \emptyset$ . Indeed, suppose that it is the condition  $x_k = 0, k \in K_{h(j)}$  which is satisfied by a point  $(x_1, \dots, x_r) \in \mathcal{M}$ . Whenever  $a_{j1}x_1 + \dots + a_{jr}x_r \geq a_0$  is also satisfied, then so is  $a_1x_1 + \dots + a_rx_r \geq a_0$ , provided that  $a_k$  is  $a_{jk}$  if  $k \in K_{h(j)}$ , while  $a_k$  can be arbitrary for  $k \in K_{h(j)}$  since  $x_k = 0$ . The stipulation on scalar quantities in  $(CMC)_h$  insures this proviso.

As an example of an application of the rule  $(CMC)_h$ , we have, with  $r = 4$  and  $J_h = \{\{1,2\}, \{1,3,4\}, \{1,4\}\}$ :

$$\underline{x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5, 6x_1 + 7x_2 + 8x_3 + 9x_4 \geq 5, 10x_1 + 7x_2 + 3x_3 + 11x_4 \geq 5}$$

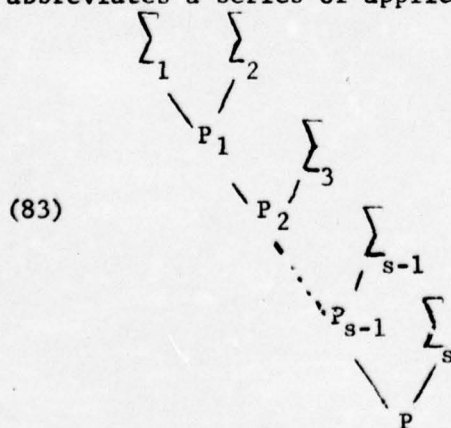
$$12x_1 + 7x_2 + 3x_3 + 4x_4 \geq 5$$

In this example, we have made coefficients distinct whenever possible. E.g.,  $x_1$  has the new co-efficient '12' in the conclusion, since the index '1' appears in  $K_{h(1)} = \{1,2\}$ ,  $K_{h(2)} = \{1,3,4\}$ , and  $K_{h(3)} = \{1,4\}$ , indicating that there are no restrictions on  $a_1$  in  $(CMC)_h$ . Similarly, since  $3 \notin K_{h(1)}$  and  $3 \notin K_{h(3)}$ , while  $3 \in K_{h(2)}$ , the co-efficient of  $x_3$  in the conclusion must match that in the leftmost and rightmost premisses, and these two coefficients must themselves agree (they are all '3').

We shall say that a set of axioms and of rules of deduction is complete for a model  $\mathcal{M}$  of these axioms and rules, if every linear inequality holding for all vectors  $\mathcal{M}$  has a proof via these rules from these axioms.

In this terminology, a basic principle on linear inequalities [27], [29] has the following statement: if  $\mathcal{M} = \{x \in \mathbb{R}^r \mid Ax \geq b\}$  is non-empty, then the axioms  $Ax \geq b$  and rule of deduction (LC) are complete for  $\mathcal{M}$ . As is well-known, from this principle one immediately obtains both the Farkas Lemma and the Duality Theorem of linear programming (see e.g., [27]). The result of Chvátal is a completeness theorem for (LC) plus a second rule specified in [8] (i.e., integer truncation), for models of the form  $\mathcal{M} = \{x \mid Ax \geq b, x \text{ integer}\}$  with axioms  $Ax \geq b$ , such that  $\{x \mid Ax \geq b\}$  is bounded. The result of Blair [4, Chapter 3] is a completeness theorem for two rules of deduction, concerning models of the form  $\mathcal{M} = \{x \mid Ax \geq b, x_j = 0 \text{ or } 1, j = 1, \dots, r\}$ , and we will explicitly present Blair's result in this form in Corollary 11.

A notational abbreviation we shall use, is that (84) below abbreviates a series of applications of (LC) as follows:



where  $\sum_1, \sum_2, \dots, \sum_s$  are derivations and  $P_1, P_2, \dots, P$  are statements of linear inequalities.

Also, a statement  $P$  is said to be valid in a model  $\mathcal{M}$  if  $P$  becomes a true numerical statement for every substitution of a vector of  $\mathcal{M}$  for the variables of  $P$  (components being substituted for the corresponding variable).

**Theorem 9:** The axioms  $Ax \geq b, x \geq 0$  together with rules of deduction (LC) and  $(CMC)_h, h = 1, \dots, t$  are complete for the model  $\mathcal{M}$  of (CMP) consisting of all real vectors  $(x_1, \dots, x_r)$  satisfying (CMP), provided that  $\{x \geq 0 | Ax \geq b\}$  is bounded and non-empty.

In fact with this proviso, there are finitely many derivations  $\sum_1, \dots, \sum_s$  with assumption formula from  $Ax \geq b, x \geq 0$ , such that any linear inequality statement  $P$  that is valid in  $\mathcal{M}$  has a deduction of the form

$$(84) \quad \frac{\sum_1 \dots \sum_s}{P}$$

in which the last rule of deduction is (LC).

Furthermore, in going from any topmost node of  $\sum_q, q = 1, \dots, s$  downward by arcs of the tree to its bottommost node, one encounters, in the following specified order, these rules of deduction: one application of (LC), followed by one application each of  $(CMC)_1, \dots, (CMC)_t$ ,

between each of which is one or more applications of (LC).

Also, if  $\sum$  is a subderivation of  $\sum_q$ ,  $q = 1, \dots, s$  obtained by selecting a node from the tree  $\sum_q$  and detaching the subtree of nodes and arcs above and including this node, and if the last inference in  $\sum$  is an application of the rule  $(CMC)_h$ , then the endformula of  $\sum$  provides a facet or singular inequality of  $K_h$ . Moreover, any facet or singular inequality of  $K_h$  arises as the endformula of exactly one such subderivation  $\sum$  of the derivation (84) with last inference  $(CMC)_h$ .

If  $\mathcal{M} = \emptyset$ , but  $\{x \geq 0 | Ax \geq b'\}$  is bounded and non-empty for some  $b'$ , then from the axioms and rules of deduction cited, any inequality can be proven.

Proof: The proof is by induction on  $t$ .

For  $t = 1$ , only the rule (LC) is to be used. Then the derivations  $\sum_1, \dots, \sum_s$  are each single formulae, designating some inequality among  $Ax \geq b$ ,  $x \geq 0$  which is a facet or singular inequality for the set  $\{x \geq 0 | Ax \geq b\}$ . The derivations  $\sum_1, \dots, \sum_s$  enumerate all such facets and singular inequalities. By the hypothesis that this latter set is non-empty, there is derivation (80) for any valid consequence  $P$ , according to the fundamental results on linear inequalities [27],[29].

The induction step from  $t$  to  $(t+1)$  is as follows.

Let  $Q$  be a valid inequality (1) for  $\mathcal{M}$ . Then by Lemma 7,

$$(85) \quad (a_0, -a_1, \dots, -a_r) \in CT(\mathcal{Q}_{t+1}).$$

Using (8), (9), and (56), we have in the notation (3)",

$$(86) \quad CT(\mathcal{P}_{t+1}) = \bigcap_{p=1}^{h(t+1)} \left( CT(\mathcal{P}_t) + CT(A^{t+1,p} x \geq b^{t+1,p}) \right) \\ = \bigcap_{p=1}^{h(t+1)} \left( CT(\mathcal{P}_t) + CT(x_k = 0, k \in K_{h(p)}) \right)$$

Putting  $h' = h(t+1)$ , (85) and (86) imply

$$(87) \quad (a_0, -a_1, \dots, -a_r) \in CT(\mathcal{P}_t) + CT(x_k = 0, k \in K_{h(p)}) \\ \text{for } p = 1, \dots, h'$$

From (87), for each  $p = 1, \dots, h'$  there exists a vector

$$(a_{p0}, -a_{p1}, \dots, -a_{pr}) \in CT(\mathcal{P}_t) \text{ and unrestricted scalars } \theta_k, k \in K_{h(p)},$$

such that

$$(88) \quad a_k = a_{pk} \quad , \text{if } k \notin K_{h(p)} \\ a_k = a_{kj} + \theta_k, \text{if } k \in K_{h(p)}$$

Since  $0 \notin K_{h(p)}$ , we see that  $a_0 = a_{p0}$  is independent of  $p$ . Then by

(88), Figure 1 is a deduction of the valid inequality  $Q$  of (1) from the valid inequalities

$$(89) \quad \sum_{j=1}^r a_{pj} x_j \geq a_0, \quad p = 1, \dots, h'$$

obtained from vectors  $(a_0, -a_{p1}, \dots, -a_{pr}) \in CT(\mathcal{P}_t)$ .

Assume  $\mathcal{M} \neq \emptyset$ .

By the inductive hypothesis, any valid inequality for  $K_t$  has a proof of the type described in the theorem. Let derivations  $\Delta_1, \dots, \Delta_w$  of this type be given for every facet and every singular inequality of  $K_t$ . Since (89) is valid for  $K_t$ , there is a derivation of  $Q$  of the form

$$(90) \quad \frac{\Delta_1 \dots \Delta_w \text{ (LC)}}{Q_1} \quad \dots \quad \frac{\Delta_1 \dots \Delta_w \text{ (LC)}}{Q_{h'} \text{ (CMC)}}_{t+1}$$

Q

where  $Q_p$  denotes the  $p$ -th inequality of (89).

Let derivations  $\sum_1, \dots, \sum_s$  be obtained as derivations (90), varying  $Q$  over all the finitely many facets and singular inequalities of  $\text{cp}(\mathcal{P}_{t+1})$ , which by Theorem 8 is the set of all elements of  $\mathcal{M}$ . Then clearly there is a derivation (84) for any valid inequality  $P$  of  $\mathcal{M}$  of the desired form, as a valid inequality is obtained by a non-negative combination of facets and singular inequalities in the manner (LC) [27], [29].

In the event  $\mathcal{M} = \emptyset$ , by induction there are proofs  $\Delta_i$  of  $Q_i$ ,  $i = 1, \dots, h'$ , and we obtain a proof

$$(90)' \quad \frac{\Delta_1 \quad \dots \quad \Delta_{h'}}{Q}$$

of  $Q$ . This completes the induction for  $\mathcal{M} = \emptyset$ .

Q.E.D.

The proof of Theorem 9 has been constructive. For instance, if the inequalities  $A^*x \geq b^*$ ,  $x \geq 0$  for  $cp(\mathcal{O}_t)$  have been obtained, then the cutting-planes for  $cp(\mathcal{O}_t)$  are those of

$$(91) \quad CT \left( \bigvee_{p=1}^{h'} \left( (A^*x \geq b^*) \wedge (x_k = 0, k \in K_{h(p)}) \right) \right)$$

which are obtained directly by the disjunctive constraints construction. From [2], [16], [19] we have  $(\pi_0, -\pi_1, \dots, -\pi_r)$  in the co-proposition of (91) if and only if there are vectors  $\lambda^p, \theta^p$  with

$$(92) \quad \begin{aligned} \lambda^p A^* + \theta^p Z_{h(p)} &\leq \pi \\ \lambda^p b^* &\geq \pi_0 & p = 1, \dots, h' \\ \lambda^p &\geq 0 \end{aligned}$$

where  $\pi = (\pi_1, \dots, \pi_r)$  and  $Z_{h(p)}$  is a square matrix of zeroes, except for diagonal entries of unity in the  $(k, k)$  - position for  $k \in K_{h(p)}$ . Since  $\theta^p$  is unconstrained, the first inequalities in (92) simplify in that the occurrence of  $\theta^p Z_{h(p)}$  can be deleted if the entire constraint on the  $k$ -th component is also deleted for  $k \in K_{h(p)}$ .

The system (92) describes a cone of cutting-planes whose extreme rays or lineality vectors, projected on the  $(\pi, \pi_0)$  - coordinates, yield facets or singular inequalities, etc. Extreme rays can be converted to extreme points by various normalizations, and extreme points may be obtained via Phase I of the Simplex Method; similar remarks apply to singular inequalities.

We may apply Theorem 9 to the special case (GLC) discussed above. We note that the rule  $(CMC)_j$  is  $(ii)_j$ , and (LC) is (i), in the next result. If we then disregard the information of Theorem 9 concerning the special structure of the proof of P, we obtain the following corollary, which was announced in [20].

Corollary 10: If

$$\{(x,y) \mid Ax + Bz \geq d', x \geq 0, z \geq 0\}$$

is bounded and non-empty for some  $d'$ , then any valid cutting-plane for for the complementarity constraints

$$(GLC) \quad Ax + Bz \geq d, x \geq 0, z \geq 0, x \cdot z = 0,$$

is obtained by starting from the linear defining inequalities

$$(91) \quad Ax + Bz \geq d, x \geq 0, z \geq 0$$

and applying, finitely often the following two rules (the second for  $j = 1, \dots, r$ ):

(i) Take non-negative combinations of given inequalities, and possibly weaken the right-hand-side.

$(ii)_j$  Having already obtained two inequalities

$$\alpha_1 x_1 + \dots + u x_j + \dots + \alpha_r x_r + \beta_1 z_1 + \dots + t z_j + \dots + \beta_r z_r \geq \alpha_0$$

$$\alpha_1 x_1 + \dots + u' x_j + \dots + \alpha_r x_r + \beta_1 z_1 + \dots + t' z_j + \dots + \beta_r z_r \geq \alpha_0$$

one may deduce

$$\alpha_1 x_1 + \dots + u x_j + \dots + \alpha_r x_r + \beta_1 z_1 + \dots + t' z_j + \dots + \beta_r z_r \geq \alpha_0.$$

Conversely, any inequality thus obtained is valid for the complementarity constraints.

We next derive a completeness result of Blair [ 4 ] from Corollary 10, in a manner that shows that rule  $(ii)_j$  and  $(CMC)_h$  are generalizations of Blair's rule  $(BR)_j$ .

Corollary 11: Any valid cutting-plane for the constraints

$$(IP) \quad Ax \geq d$$

$$x_j = 0 \text{ or } 1, \quad j = 1, \dots, r$$

is obtained by starting from the linear inequalities

$$(92) \quad Ax \geq d$$

$$x_j \geq 0 \quad , \quad j = 1, \dots, r$$

$$-x_j \geq -1 \quad , \quad j = 1, \dots, r$$

and applying, finitely often, the following two rules (the second for  $j = 1, \dots, r$ ):

- (i) Take non-negative combinations of given inequalities, and possibly weaken the right-hand-side.

$(BR)_j$  Having already obtained two inequalities

$$\theta_1 x_1 + \dots + u x_j + \dots + \theta_r x_r \geq P$$

$$\theta_1 x_1 + \dots + w x_j + \dots + \theta_r x_r \geq T$$

one may deduce

$$\theta_1 x_1 + \dots + (w + P - T)x_j + \dots + \theta_r x_r \geq P .$$

Conversely, any inequality thus obtained is valid for (IP).

Proof: The validity of  $(BR)_j$  is immediate, since the conclusion is equivalent to the first hypothesis if  $x_j = 0$  and the second hypothesis if  $x_j = 1$ . It remains only to show that the rules are adequate to obtain any given valid inequality.

Now if (1) is valid for (IP), it is also valid for the following equivalent of (IP):

$$\begin{aligned}
 & Ax \geq d \\
 & x_j + z_j = 1, \quad j = 1, \dots, r \\
 \text{(IP)'} \quad & x_j, z_j \geq 0, \quad j = 1, \dots, r \\
 & \sum_{j=1}^r x_j z_j = 0
 \end{aligned}$$

Since (IP)' is of the form (GLC), Corollary 10 applies, and there is a proof of (1) using only rules (i), (ii)<sub>j</sub>.

This proof involves the additional variables  $z_j$ . We now show how to systematically convert it to a proof of the logical system  $\mathcal{L}'$  of this corollary, in only variables  $x$ , using the rules (i) and (BR)<sub>j</sub>. The validity of our conversion procedure is proven by induction on the number  $\Delta$  of rules of deduction occurring in the proof of (1) in the logical system  $\mathcal{L}$  of Corollary 10. Our inductive hypothesis is that, if the inequality

$$(93) \quad a_1 x_1 + \dots + a_r x_r + a'_1 z_1 + \dots + a'_r z_r \geq a_0$$

is provable in the logical system  $\mathcal{L}$  with  $\leq \Delta$  occurrences of rules of deduction, then in the logical system  $\mathcal{L}'$  of this corollary the converted inequality

$$(94) \quad (a_1 - a'_1) x_1 + \dots + (a_r - a'_r) x_r \geq a_0 - \sum_{j=1}^r a'_j$$

is provable.

For  $\Delta = 0$ , the proof is trivial. If the inequality (1) is among the axioms  $Ax \geq d$ , it is also an axiom in (92). If (1) is

$x_j + z_j \geq 1$ , we must provide a proof of  $(1-1) x_j \geq 1 - 1$ , or  $0 \cdot x_j \geq 0$ ; but (LC) yields  $0 \cdot x_j \geq 0$  by using zero as parameters. If (1) is  $-x_j - z_j \geq -1$ , again we obtain a formal proof of  $0 \cdot x_j \geq 0$ . If (1) is  $x_j \geq 0$ , then this inequality is also an axiom (92). Finally, if (1) is  $z_j \geq 0$ , then  $(0-1)x_j \geq 0 - 1$  or  $-x_j \geq -1$  is also an axiom of (92).

The inductive step from  $\Delta$  to  $(\Delta + 1)$  is as follows.

If the last rule used in the proof is (LC), then applying (LC), with the same parameters, to the conversions (94) of the hypotheses (93) of the rule, will yield the conversion of the conclusion of the rule; we leave details to the reader.

If the last rule used in the proof is  $(ii)_j$ , let the hypotheses of this rule be as in  $(ii)_j$  of Corollary 10. These hypotheses have proofs using  $\leq \Delta$  applications of rules, and hence by induction there are proofs in the logical system of this corollary of the two conversions of these hypotheses, i.e., of

$$(95) \quad (\alpha_1 - \beta_1)x_1 + \dots + (u-t)x_j + \dots + (\alpha_r - \beta_r)x_r \geq \alpha_0 - \sum_{k \neq j} \beta_k - t$$

and

$$(\alpha_1 - \beta_1)x_1 + \dots + (u'-t')x_j + \dots + (\alpha_r - \beta_r)x_r \geq \alpha_0 - \sum_{k \neq j} \beta_k - t'$$

With the inequalities of (95) as hypotheses,

$$P = \alpha_0 - \sum_{k \neq j} \beta_k - t', \quad N = \alpha_0 - \sum_{k \neq j} \beta_k - t, \quad \theta_j = \alpha_j - \beta_j \text{ for } j \neq k,$$

by one application of  $(BR)_j$  we obtain the conclusion

$$(96) \quad (\alpha_1 - \beta_1)x_1 + \dots + (u-t')x_j + \dots + (\alpha_r - \beta_r)x_r \geq \alpha_0 - \sum_{k \neq j} \beta_k - t'$$

Note that (96) is the conversion of the conclusion of rule (ii)<sub>j</sub>. Therefore, by adding on top of these hypotheses (95) of (BR)<sub>j</sub> their proofs in the logical system of this corollary, we obtain a proof of the conversion of the endformula. This completes our induction.

If an inequality (93) is entirely in the variables  $x_1, \dots, x_r$  (i.e., if  $a_1' = \dots = a_r' = 0$ ), then its conversion (94) is itself. This completes the proof.

Q.E.D.

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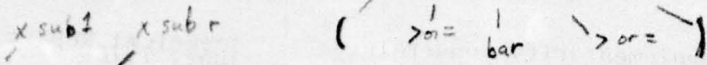
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and we show that these rules <sup>→ which →</sup> generate all (and only) the valid cutting-planes for (CMP), if there is some  $b'$  for which  $\{ x \geq 0 \mid Ax \geq b' \}$  is non-empty and bounded.



In (CMP),  $x = (x_1, \dots, x_r)$ , and  $J_h$  is a set of subsets  $K$  of  $\{ 1, \dots, r \}$ . The problem (CMP) includes the linear complementarity problem and bivalent integer programming, along with many other constraint sets which impose logical restrictions on linear inequalities.

