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A CLASS OF CONTROLLABLE NONLINEAR SYSTEMS*

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Abstract

Sufficient conditions for global and local controllability of the nonlinear system

$$\frac{d}{dt} x(t) = A(t,x(t),u(t))x(t) + B(t)u(t) + f(t,x(t),u(t))$$

and its perturbed system are given. These conditions extend some previous results through the removal of certain boundedness conditions involving the functions (A,f) and their partial derivatives.

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I. INTRODUCTION

The purpose of this correspondence is to extend the results of [1] by considering a more general class of nonlinear control systems. The results obtained in this work provide sufficient conditions for global and local controllability of perturbed nonlinear systems.

Consider the nonlinear time-varying system

$$\frac{d}{dt} x(t) = A(t, x(t), u(t))x(t) + B(t)u(t) + f(t, x(t), u(t)) \quad (1)$$

$t \in [t_0, t_1]$, where $x(t)$ is a $n \times 1$ state vector; $u(t)$ is a $m \times 1$ input vector; A, B, f are $n \times n$, $n \times m$, $n \times 1$ matrix-valued functions, respectively.

Denote $C_{nm}[t_0, t_1]$ as the Banach space of continuous $R^n \times R^m$ valued functions on $[t_0, t_1]$ with the uniform norm $\| [x(t), u(t)] \| = |x(t)| + |u(t)|$, where $|w(t)| = \max_i \max_{t \in [t_0, t_1]} |w_i(t)|$ and $|w_i(t)|$ is the absolute value of $w_i(t)$, the element of $w(t)$.

Define the norm of a continuous $n \times m$ matrix-valued function $S(t)$ by $\|S(t)\| = \max_i \sum_{j=1}^m \max_{t \in [t_0, t_1]} |S_{ij}(t)|$, where S_{ij} are elements of S .

Given (x_0, x_1) as the initial and final state, respectively of (1). The problem is to find a continuous input function $u(t)$, defined in $[t_0, t_1]$, which steers system (1) from x_0 at t_0 to x_1 at t_1 . The usual definitions of globally and locally completely and totally controllable are assumed [2].

For each fixed element $[z, v] \in C_{nm}[t_0, t_1]$, consider the following system

$$\frac{d}{dt} x(t) = A(t, z(t), v(t))x(t) + B(t)u(t) + f(t, z(t), v(t)) \quad (2)$$

The solution of the system (2) with $x(t_0) = x_0$ is given by

$$\begin{aligned}
 x(t) = & \phi(t, t_0; z, v)x_0 + \int_{t_0}^t \phi(t, s; z, v)B(s)u(s)ds \\
 & + \int_{t_0}^t \phi(t, s; z, v)f(s, z, v)ds
 \end{aligned}
 \tag{3}$$

where $\phi(t, t_0; z, v)$ is the state transition matrix of the homogeneous system $\frac{d}{dt} x(t) = A(t, z(t), v(t))x(t)$, $\phi(t_0, t_0; z, v) = I$ the identity matrix. Define the controllability matrix by

$$G(t_0, t; z, v) = \int_{t_0}^t \phi(t, s; z, v)B(s)B'(s) \phi'(t, s; z, v)ds
 \tag{4}$$

the prime denotes the matrix transpose operation. Obviously, $G(t_0, t; z, v)$ is symmetric and non-negative definite.

II. GLOBAL CONTROLLABILITY RESULT

Theorem 1: The system (1) is globally (a) completely controllable at t_0 , or (b) totally controllable if the following conditions are satisfied.

- (i) $B(t)$ has a continuous first derivative with respect to t ,
- (ii) $A(t, x, u)$, $A_x(t, x, u)$, $A_u(t, x, u)$, $f(t, x, u)$, $f_x(t, x, u)$, and $f_u(t, x, u)$ are continuous and bounded in $[t_0, t_1] \times R^n \times R^m$,
- (iii) there exists a positive constant q such that

$$\inf_{[z, v] \in C_{nm}} \det_{[t_0, t_1]} G(t_0, t_1; z, v) \geq q
 \tag{5}$$

- (a) for some $t_1 > t_0$, or (b) for all t_0 and all $t_1 > t_0$.

Proof: The proof of the theorem is based on the Schauder's fixed point theorem of the following version, "Every continuous map which maps a compact convex subset of a Banach space into itself has a fixed point".

For each fixed element $[z, v] \in C_{nm} [t_0, t_1]$, consider the control function $u(t)$ for $t \in [t_0, t_1]$

$$u(t) = B'(t)\phi'(t_1, t; z, v)G^{-1}(t_0, t_1; z, v)[x_1 - \phi(t_1, t_0; z, v)x_0 - \int_{t_0}^{t_1} \phi(t_1, s; z, v)f(s, z, v)ds] \quad (6)$$

where $\phi(t, t_0; z, v)$ is defined as that in Eq. (3). With this expression, Eq. (3) can be rewritten as

$$x(t) = \phi(t, t_0; z, v)x_0 + \int_{t_0}^t \phi(t, s; z, v)B(s)B'(s)\phi'(t_1, s; z, v)G^{-1}(t_0, t_1; z, v)ds \cdot [x_1 - \phi(t_1, t_0; z, v)x_0 - \int_{t_0}^{t_1} \phi(t_1, s; z, v)f(s, z, v)ds] + \int_{t_0}^t \phi(t, s; z, v)f(s; z, v)ds \quad (7)$$

It should be noted that by hypothesis (iii) $G^{-1}(t_0, t_1; z, v)$ is well-defined in above expression. It is easily seen that $x(t)$ in Eq. (7) satisfies both boundary conditions at $t = t_0$ and $t = t_1$.

Now the right sides of Eqs. (6)-(7) can be viewed as a pair of operators, $P_2([z, v])(t)$ and $P_1([z, v])(t)$, respectively. Define the nonlinear mapping

$$P([z, v])(t) = [P_1([z, v])(t), P_2([z, v])(t)] .$$

Obviously, $P_1([z, v])(t)$ and $P_2([z, v])(t)$ are continuous in t by the uniform continuity of $\phi(t, t_0; z, v)$ in t . Therefore, P maps $C_{nm} [t_0, t_1]$ into $C_{nm} [t_0, t_1]$. It can also be easily verified that, by hypothesis (ii) and the definition of $\phi(t, t_0; z, v)$, P is continuous in $[z, v]$.

Considering the subset of $C_{nm}[t_0, t_1]$

$$S = \left\{ [z, v] \in C_{nm}[t_0, t_1] : \begin{aligned} & \| [z, v] \| \leq K, \\ & \| [z(t), v(t)] - [z(\tilde{t}), v(\tilde{t})] \| \leq K |t - \tilde{t}|, \forall t, \tilde{t} \in [t_0, t_1] \end{aligned} \right\}$$

where K is certain positive constant depending upon the bounds of A, B, f and their partial derivatives, it can be easily shown that the image set $P(S) \subset S$. Besides, S is closed and convex by this construction. Furthermore, each sequence $\{s_i\}_{i=1}^{\infty} \subset S$ constitutes a uniformly bounded equicontinuous family. Hence by the Arzela-Ascoli theorem [3], S is relative compact and therefore compact.

Then, the Schauder's theorem [3] can be applied to conclude that P has a fixed point $[z^*, v^*]$, i.e.,

$$P([z^*, v^*])(t) = [P_1([z^*, v^*]), P_2([z^*, v^*])] = [z^*, v^*]$$

Substitute this fixed point into Eqs. (6)-(7), a direct differentiation of Eq. (7) with respect to t shows that $z^*(t)$ is a solution to the system (1) for the control function $u(t)$ given by $v^*(t)$.

If condition (iii)(a) holds, $v^*(t)$ drives the system (1) from x_0 to x_1 on some interval $[t_0, t_1]$ for all $x_0, x_1 \in R^n$, system (1) is globally completely controllable at t_0 . If condition (iii)(b) holds, we have global total controllability.

Corollary 1: Given the system (1) with conditions (i)-(iii), then the perturbed system

$$\frac{d}{dt} x(t) = [A(t, x, u) + \epsilon \tilde{A}(t, x, u)]x(t) + [B(t) + \epsilon \tilde{B}(t)]u(t) + f(t, x, u), \quad (8)$$

with \tilde{A}, \tilde{B} satisfying the same type of conditions imposed on A, B , is globally controllable provided ϵ is sufficiently small.

Proof: One needs only to show that the determinant of the modified controllability matrix G^* has a positive infimum. By observing that the determinant of $G^*(t_1, t_0; z, v, \epsilon)$ can be expanded about $\epsilon = 0$ [5] into the sum of $\det G(t_1, t_0; z, v)$, $\epsilon H(t_1, t_0; z, v)$ and $o(\epsilon^2)$ where $H(t_1, t_0; z, v)$ is a bounded scalar, we know that by taking ϵ small enough there exists some $q^* > 0$ such that

$$\det G^*(t_1, t_0; z, v, \epsilon) \geq q^* \quad \text{for all } [z, v] \in C_{nm}[t_0, t_1].$$

Therefore, Theorem 1 concludes the result of controllability.

Remarks: (i) If (A, f) do not explicitly depend on $u(t)$, the theorem is applicable to the class of systems with $B = B(t, x(t))$. These results then reduce to the work reported in [4].

(ii) Uniform boundedness of partial derivatives can be slightly weakened by considering a Lipschitz condition on corresponding variables.

III. LOCAL CONTROLLABILITY RESULT

Consider the following subset of $[t_0, t_1] \times R^n \times R^m$,

$$D = \{(t, x, u) : t_0 \leq t \leq t_1, |x| + |u| \leq d, \\ d \text{ is some positive constant}\}.$$

Theorem 2: The system (1) is locally (a) completely controllable at t_0 , or (b) totally controllable if the following conditions are satisfied.

- (i) $B(t)$ has a continuous first derivative with respect to t ,
- (ii) $A(t, x, u)$, $A_x(t, x, u)$, $A_u(t, x, u)$, $f(t, x, u)$, $f_x(t, x, u)$ and $f_u(t, x, u)$ are continuous in D ,

(iii) There exists a positive constant \tilde{q} such that

$$\inf_{[z,v] \in E} \det G(t_1, t_0; z, v) \geq \tilde{q}$$

where

$$E = \{[z,v] \in C_{nm}[t_0, t_1] : \|[z,v]\| \leq d\}$$

and

$$M < \min \left\{ \frac{d}{k_1}, d, \frac{d}{k_2} \right\}$$

where

M is the bound of f in D , k_1 and k_2 are certain positive constants depending upon the bounds of A, B and their partial derivatives in D .

(a) for some $t_1 > t_0$, or (b) for all t_0 and all $t_1 > t_0$.

Proof: The proof is similar to that of Theorem 1. We leave it to the reader.

Corollary 2: Given the system (1) with conditions (i)-(iii) in Theorem 2, the perturbed system

$$\frac{d}{dt} x(t) = [A(t, x, u) + \epsilon \tilde{A}(t, x, u)]x(t) + [B(t) + \epsilon \tilde{B}(t)]u(t) + f(t, x, u) \quad (9)$$

with \tilde{A}, \tilde{B} satisfying the same type of conditions imposed upon A, B is locally controllable provided ϵ is sufficiently small.

Proof: The proof is similar to that of Corollary 1 and follows immediately from Theorem 2.

Example: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + [\sin^2(x_1 + u)]x_2 \\ \dot{x}_2 &= x_1 + u + [\sin^2(x_1 + u)]x_1\end{aligned}$$

In matrix form, we have

$$A(t,x,u) = \begin{bmatrix} 0 & 1 + \sin^2(x_1 + u) \\ 1 + \sin^2(x_1 + u) & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It is easily seen that $B, dB/dt, A, \partial A/\partial x, \partial A/\partial u$ exist and are bounded in R and $R^2 \times R$, respectively. Furthermore, for each fixed $[z,v] \in C_{21}[t_0, t_1]$, the controllability matrix G is

$$G(t_0, t_1; z, v) = \frac{1}{4} \begin{bmatrix} \int_{t_0}^{t_1} (a-b)^2 dy & \int_{t_0}^{t_1} (a^2 - b^2) dy \\ \int_{t_0}^{t_1} (a^2 - b^2) dy & \int_{t_0}^{t_1} (a+b)^2 dy \end{bmatrix}$$

where

$$a = \exp \int_{t_0}^{t_1} [1 + \sin^2(z_1 + v)] dy, \quad b = \exp \int_{t_0}^{t_1} -[1 + \sin^2(z_1 + v)] dy .$$

It can be easily shown that $\det G(t_0, t_1; z, v) \geq q(t_0, t_1)$ for all $[z,v]$ where $q(t_0, t_1) = \frac{1}{16} [e^{2(t_1 - t_0)} - 1 - 2(t_1 - t_0) - 2(t_1 - t_0)^2 - \frac{4}{3}(t_1 - t_0)^3] > 0$ if $t_1 > t_0$. Hence, by Theorem 1 the system is globally totally controllable.

However, the result of [1] cannot be applied here because in this case the non-linearity $f(t,x,u) = [x_2 \sin^2(x_1 + u), x_1 \sin(x_1 + u)]'$ is not bounded in $R \times R^2 \times R$.

IV. CONCLUSION

Sufficient conditions for global and local controllability of a class of nonlinear systems have been given. A natural question arises as to whether or not these results hold by the same fixed point arguments if the system matrices (A, f) are parameterized as $A(t, p(t), q(t))$ and $f(t, p(t), q(t))$ where $[p, q] \in C_{nm}[t_0, t_1]$ so that for each given $[p, q]$ the parametrized linear time-varying system is controllable. The answer is negative because boundedness and continuity of the partial derivatives of relevant matrices are required in addition to the nonsingularity of the controllability matrix.

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