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EXTREMES OF MOVING AVERAGES OF STABLE PROCESSES

by

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## EXTREMES OF MOVING AVERAGES OF STABLE PROCESSES

by

Holger Rootzén\*

**ABSTRACT:** In this paper we study extremes of non-normal stable moving average processes, i.e. of stochastic processes of the form  $X(t) = \int a(\lambda-t)Z(\lambda)$  or  $X(t) = \int a(\lambda-t)dZ(\lambda)$ , where  $Z(\lambda)$  is stable with index  $\alpha < 2$ . The extremes are described as a marked point process, consisting of the point process of (separated) exceedances of a level together with marks associated with the points, a mark being the normalized sample path of  $X(t)$  around an exceedance. It is proved that this marked point process converges in distribution as the level increases to infinity. The limiting distribution is that of a Poisson process with independent marks which have random heights but otherwise are deterministic. As a byproduct of the proof for the continuous-time case, a result on sample path continuity of stable processes is obtained.

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## 1. INTRODUCTION

A distribution is stable  $(\gamma, \alpha, \beta)$  if it has the characteristic function

$$(1.1) \quad \phi(u) = \exp\{-\gamma^\alpha |u|^\alpha (1 - i\beta h(u, \alpha)u/|u|)\},$$

with  $0 \leq \gamma$ ,  $0 < \alpha \leq 2$ ,  $|\beta| \leq 1$  and with  $h(u, \alpha) = \tan(\pi\alpha/2)$  for  $\alpha \neq 1$ ,  $h(u, 1) = 2\pi^{-1} \log|u|$ . Here  $\gamma$  is a scale parameter,  $\alpha$  is the *index*, and  $\beta$  is the *symmetry parameter* of the stable distribution. If  $\beta=0$ , then the distribution is symmetric, while the distribution is said to be *completely asymmetric* if  $|\beta|=1$  and  $\alpha < 2$ . A stochastic process  $\{X(t); t \in T\}$  is stable with index  $\alpha$  if for  $n=1, 2, \dots$  and for arbitrary real numbers  $a_1, \dots, a_n$  and  $t_1, \dots, t_n \in T$ , the random variable  $a_1 X(t_1) + \dots + a_n X(t_n)$  is stable with index  $\alpha$ . In particular, as can be seen from (1.1), a collection of independent stable random variables with index  $\alpha$  is a stable process.

In this paper, further use of the linear structure is made by restricting attention to the subclass consisting of moving averages, i.e. to stationary processes of the form  $X(t) = \int_{\lambda} a(\lambda-t)Z(\lambda)$ , where  $\{Z(\lambda)\}_{\lambda=-\infty}^{\infty}$  is an independent, stationary stable sequence, or of the form  $X(t) = \int a(\lambda-t)dZ(\lambda)$  where  $\{Z(\lambda); -\infty < \lambda < \infty\}$  has independent, stationary stable increments. If  $\alpha=2$  the process is normal, and it can be represented as a moving average iff its spectral distribution is absolutely continuous. Presently no simple characterization of the class of moving averages of stable processes is known for  $\alpha < 2$ . Of course normal processes are extensively analyzed but, partly because of the common linear structure, also stable processes with  $\alpha < 2$  constitute a class of probability models that is amenable to analysis.

The subject of the present study is the asymptotic distribution of extremes of moving averages of stable processes with  $\alpha < 2$ . As can be seen from e.g. Leadbetter et al. (1976), a suitable framework for dealing with extremes of stationary processes is the theory of point processes as given in e.g. Kallenberg (1975) used to study the process of exceedances of a high level. Here we will go one step further and adjoin a *mark* to each point in the process of exceedances, the mark being the entire sample path of the process, normalized and centered at (a point close to) the upcrossing. The main results are that both when  $X(t)$  has discrete parameter (or "time") and when  $X(t)$  has continuous parameter, the marked point process converges in distribution. The limiting distribution is that of a Poisson process (possibly with multiple points) with independent marks that are distributed as  $a(-t)$  multiplied by a certain random variable.

One of the main differences between the normal distribution (stable with  $\alpha=2$ ) and stable distributions with  $\alpha < 2$  is that the tails of the latter decrease much slower. This leads to a radically different behavior of extremes. For the normal distribution the tails are of the order  $e^{-x^2/2}/x$ , while for stable distributions with  $\alpha < 2$  the tails decrease as  $x^{-\alpha}$ . This affects extremes of moving averages in two different ways. To fix ideas, consider e.g. maxima of a process  $X(t) = \sum a(\lambda-t)Z(\lambda)$  with discrete parameter. First, extremes increase much slower when  $\alpha=2$  than when  $\alpha < 2$ , viz. as  $(\log n)^{1/2}$  compared with  $n^{1/\alpha}$ . Secondly, when the independent sequence  $\{Z(\lambda)\}$  is normal, many of the  $Z(\lambda)$ 's,  $0 \leq \lambda \leq n$ , will be almost as large as the largest one, and  $X(t)$  will be large when many rather large  $Z(\lambda)$ 's are added. This entails that the limiting distribution

of  $M_n = \max_{1 \leq t \leq n} X(t)$  only depends on  $\sum a(\lambda)^2$  and that it is the same as if  $\{X(t)\}$  were an independent sequence with the same marginal distributions. On the other hand, when  $\alpha < 2$  the maximum of  $Z(\lambda)$  will be much larger than the typical values, and  $X(t)$  will be large when one very large  $Z(\lambda)$  is multiplied by a large  $a(\lambda)$ . In this case the limiting distribution of  $M_n$  depends on  $\max_{-\infty < \lambda < \infty} a(\lambda)$  and on  $\min_{-\infty < \lambda < \infty} a(\lambda)$  and is in general not the same as if  $X(t)$  were an independent sequence with the same marginals.

In an earlier paper (Rootzén (1974)), the limiting distribution of maxima of moving averages of symmetric stable sequences with  $\alpha < 2$  was obtained, but apart from that there do not seem to be any results published on extremes of stationary stable processes with  $\alpha < 2$ .

The plan of this paper is as follows. Section 2 deals with convergence in distribution of marked point processes. In Section 3 rather complete asymptotic results on extremes of moving averages of stable sequences are obtained. Section 4 contains preliminaries concerning moving averages of continuous parameter stable processes. In particular some conditions that ensure sample path continuity are found, which may be of independent interest. Finally, in Section 5 results for continuous parameter processes corresponding to those of Section 3 are established for  $\alpha \neq 1$ , under some restrictions on  $a(\lambda)$ .

## 2. CONVERGENCE IN DISTRIBUTION OF MARKED POINT PROCESSES

We are interested in the times  $0 \leq t_1 < t_2 < \dots$  of occurrence of extreme values of a stochastic process  $\{X(t)\}$  and in the behavior of the sample

paths of  $\{X(t)\}$  near the  $t_i$ 's, and will describe them as a marked point process. In this section we introduce some notation and develop techniques needed in the remaining sections to prove convergence in distribution of marked point processes. Unfortunately the notation is somewhat cumbersome, but nevertheless we think it is well warranted, considering the completeness of results it makes possible.

Write  $N$  for the space of integer-valued and locally finite Borel measures on  $R^+$  and define a metric on  $N$  in the following way: Let  $F = \{f_i\}_{i=1}^{\infty}$  be a sequence of functions in  $C_c = \{f: R^+ \rightarrow R^+; f \text{ is continuous with compact support}\}$  such that any  $f \in C_c$  can be uniformly approximated by functions in  $F$ . For  $\mu, \nu \in N$  put  $\rho(\mu, \nu) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(\mu, \nu)$ , where  $\rho_i(\mu, \nu) = \min(1, |\int f_i d\mu - \int f_i d\nu|)$ . Then  $\rho$  is a metric on  $N$  that generates the topology of vague convergence ( $\mu_n \in N$  converges vaguely to  $\mu \in N$  if  $\int f d\mu_n \rightarrow \int f d\mu$  for all  $f \in C_c$ ); see e.g. Bauer (1972), p. 241. A point process in  $R^+$  is defined to be a (Borel measurable) random variable with values in  $(N, \rho)$ . As soon as we have (Borel measurable) random variables in a metric space we may of course consider convergence in distribution, using the theory of convergence in distribution in metric spaces as given in e.g. Billingsley (1968). For further information on convergence in distribution of point processes see [6]. We regard the times  $0 \leq t_1 < t_2 < \dots$  of occurrence of extremes of  $\{X(t)\}$  as a point process  $N$  by putting  $N(B) = \#\{t_i \in B\}$  for any Borel set  $B \subset R^+$ .

With each of the  $t_i$ 's we associate a mark  $Y_i$ , where  $Y_i$  is the entire sample path of  $\{X(t)\}$ , normalized and centered to show the behavior close to  $t_i$ . If  $\{X(t)\}$  is a discrete parameter process,  $Y_i$  is a

random variable in a space  $R^\infty = \{(\dots x_{-1}, x_0, x_1, \dots); x_i \in R, i=0, \pm 1, \dots\}$ . We consider  $R^\infty$  as a metric space with the metric  $\delta(x, y) = \sum_{i=-\infty}^{\infty} 2^{-|i|} \delta_i(x, y)$  that generates the product topology, where  $\delta_i(x, y) = \min(1/3, |x_i - y_i|)$ . If  $X(t)$  has continuous parameter we will impose conditions that make  $Y_i$  a random variable in  $D(-\infty, \infty)$ , the space of functions on  $(-\infty, \infty)$  which are right continuous and have lefthand limits. On  $D(-\infty, \infty)$  we use (a slight modification of) the metric given by Lindvall (1973). Since there is no risk of confusion we will use the same notation as for the metric on  $R^\infty$ . Thus for  $x, y \in D(-\infty, \infty)$  we let  $\delta(x, y) = \sum_{i=-\infty}^{\infty} \delta_i(x, y)$  where  $\delta_i(x, y) = \min(1/3, h(c_i x, c_i y))$  and  $h(c_i x, c_i y)$  is the quantity given on p. 113-115 of Lindvall's paper, modified to  $D(-\infty, \infty)$  instead of  $D(0, \infty)$  in the way proposed on p. 121.

The marked point process is the vector  $\eta = (N, Y_1, Y_2, \dots)$  with values in  $S = N \times R^\infty \times R^\infty \times \dots$  (or in  $S = N \times D(-\infty, \infty) \times D(-\infty, \infty) \times \dots$ ) which we again consider as a metric space given a product metric  $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} d_i(x, y)$ , where for  $x = (v, x_1, x_2, \dots)$  and  $y = (v, y_1, y_2, \dots)$  we put  $d_0(x, y) = \rho(v, \mu)$  and  $d_i(x, y) = \delta(x_i, y_i)$ ,  $i \geq 1$ . Our aim is to prove convergence in distribution of marked point processes, and to this end we need the following simple criterions, which we state as lemmas for easy reference.

LEMA 2.1.<sup>1</sup> Let  $\eta_n = (N_n, Y_{n1}, Y_{n2}, \dots)$  and  $\eta = (N, Y_1, Y_2, \dots)$  be random variables in the product space  $(S, d)$ . Suppose that  $N_n, Y_{n1}, Y_{n2}, \dots$  are

<sup>1</sup> The results of this section are formulated in terms of a discrete parameter,  $n$ , which tends to infinity. They of course remain valid if the parameter tends to infinite in a continuous manner, and they will be used accordingly in Section 5.

independent for each  $n$  and that  $N, Y_1, Y_2, \dots$  are independent. Then  $\eta_n \xrightarrow{d} \eta$  if (and only if)  $N_n \xrightarrow{d} N$  and  $Y_{ni} \xrightarrow{d} Y_i$ ,  $i \geq 1$ .

PROOF. Since  $S$  is a product of separable spaces this follows as on p. 21 of [3].  $\square$

LEMMA 2.2. Let  $\eta_n^k = (N_n^k, Y_{n1}^k, Y_{n2}^k, \dots)$ ,  $n \geq 1$ ,  $k \geq 1$ , be random variables in  $(S, d)$ . Suppose that for  $k \geq 1$ ,  $\eta_n^k \xrightarrow{d} \eta^k$  as  $n \rightarrow \infty$  and that  $\eta^k \xrightarrow{d} \eta$  as  $k \rightarrow \infty$ . Suppose further that

$$(2.1) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(d_i(\eta_n^k, \eta_n) > \epsilon) = 0, \quad \forall \epsilon > 0, \quad i = 0, 1, \dots$$

Then  $\eta_n \xrightarrow{d} \eta$  as  $n \rightarrow \infty$ .

PROOF. For given  $\epsilon > 0$  choose  $i$  to make  $2^{-i} < \epsilon/2$  and thus  $\sum_{j=i+1}^{\infty} 2^{-j} d_j(\eta_n^k, \eta_n) < \epsilon/2$ . Then

$$P(d(\eta_n^k, \eta_n) > \epsilon) \leq \sum_{j=0}^i P(d_j(\eta_n^k, \eta_n) > \epsilon/(2(i+1)))$$

and by (2.1) we thus have

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(d(\eta_n^k, \eta_n) > \epsilon) = 0, \quad \forall \epsilon > 0,$$

which by Theorem 4.2 of [3] proves the lemma.  $\square$

For  $i \geq 1$ ,  $d_i(\eta_n^k, \eta_n) = \delta(Y_{ni}^k, Y_{ni})$  and repeating the above argument once more we see that (2.1) holds for  $i \geq 1$  if

$$(2.2) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(\delta_j(Y_{ni}^k, Y_{ni}) > \epsilon) = 0, \quad \forall \epsilon > 0, \quad j \geq 1.$$

In the discrete-parameter case (2.2) is easy to check, but when the parameter

is continuous further simplification is needed.

LEMMA 2.3. Suppose that for each  $i \geq 1$  there are random variables  $\{\epsilon_n^k\}$  with  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|\epsilon_n^k| > x) = 0$ ,  $\forall x > 0$ , and such that

$$(2.3) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left(\sup_{-\ell \leq t \leq \ell} |Y_{ni}^k(t) - Y_{ni}^k(t + \epsilon_n^k)| > x\right) = 0, \quad \forall x > 0, \quad \ell > 0,$$

and that furthermore  $\lim_{u \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left(\sup_{-\ell \leq t \leq \ell} |Y_{ni}(t)| > u\right) = 0$ ,  $\forall \ell > 0$ . Then (2.2) holds and thus (2.1), for  $i \geq 1$ .

PROOF. Using the time transformation  $\lambda(t) = \{1 - e^{-\epsilon} - e^{-\epsilon}/t\}$  it is seen that  $\delta_j(y, z) \leq \sup_{-j-2 \leq t \leq j+2} \{|y(t) - z(t+\epsilon)| + \epsilon|y(t)|\} + \epsilon$  if  $0 \leq \epsilon \leq 1$ , and the lemma follows.  $\square$

It is also possible to give a simpler condition for

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(d_0(\eta_n^k, \eta_n) > \epsilon) = 0.$$

LEMMA 2.4. Let  $0 \leq t_{n1}^k \leq t_{n2}^k \leq \dots$  be the atoms (repeated according to their multiplicities) of  $N_n^k$  and similarly let  $0 \leq t_{n1} \leq t_{n2} \leq \dots$  be the atoms of  $N_n$ . Suppose that

$$(2.4) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|t_{ni}^k - t_{ni}| > \epsilon) = 0, \quad \forall \epsilon > 0, \quad i=1, 2, \dots$$

and that in addition  $N_n^k \xrightarrow{d} N^k$  as  $n \rightarrow \infty$  and that  $N^k \xrightarrow{d} N$  as  $n \rightarrow \infty$ . Then (2.1) holds for  $i=0$ .

PROOF. As above, it is enough to prove

$$(2.5) \quad P = \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(\rho_j(N_n^k, N_n) > \epsilon) = 0, \quad \forall \epsilon > 0, \quad j=1, 2, \dots$$

where  $\rho_j(N_n^k, N_n) \leq |\int f_j dN_n^k - \int f_j dN_n|$  with  $f_j \in C_c$ . Suppose that the

support of  $f_j$  is contained in  $[0, T]$ . For  $\delta > 0$  there is a  $K$  with  $P(N([0, T+1]) > K) > \delta$  and thus  $\limsup_{k \rightarrow \infty} P(N^k([0, T]) > K) < \delta$  and  $\limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(N_n^k([0, T]) > K) > \delta$ . Furthermore take a step function  $g(t) = \sum_{i=1}^{m-1} a_i I(\tau_i < t \leq \tau_{i+1})$ , with  $0 \leq \tau_1 < \dots < \tau_m \leq T$  that approximates  $f_j$  uniformly, with  $\sup_{0 \leq t} |f_j(t) - g(t)| < \epsilon / (2K)$ . Letting  $0 \leq t_1 \leq t_2 \leq \dots$  be the atoms of  $N$ , we assume without loss of generality that

$$(2.6) \quad P(t_i = \tau_j) = 0, \quad j=1, \dots, m, \quad i \geq 1.$$

On the set  $\{N_n^k([0, T]) \leq K, N_n([0, T]) \leq K\}$  we have

$$\left| \int f_j dN_n^k - \int f_j dN_n \right| < \left| \int g dN_n^k - \int g dN_n \right| + 2K\epsilon / (2K).$$

Hence

$$P \leq \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \{P(N_n^k([0, T]) > K) + P(N_n([0, T]) > K) + P(|\int g dN_n^k - \int g dN_n| > 0)\} \\ \leq 2\delta,$$

since  $P(|\int g dN_n^k - \int g dN_n| > 0) \rightarrow 0$  by (2.6) and the hypothesis of the lemma. However,  $\delta > 0$  is arbitrary, so (2.5) follows.  $\square$

Finally, it should perhaps be stressed that the convergence  $\eta_n \xrightarrow{d} \eta$  that is to be proved in the following sections only says something about the sample paths *near* extremes: Namely,  $Y_{ni} \xrightarrow{d} Y_i$  where  $Y_{ni}, Y_i$  are random variables in  $(\mathbb{R}^\infty, \delta)$  if and only if  $(Y_{ni}(-k), \dots, Y_{ni}(k)) \xrightarrow{d} (Y_i(-k), \dots, Y_i(k))$  as  $n \rightarrow \infty$ , for each  $k \geq 1$ . Similarly, since the limits of  $Y_{ni} \in D(-\infty, \infty)$  which we obtain are continuous, the convergence in  $D(-\infty, \infty)$  is equivalent to convergence in  $D(-T, T)$  for each  $T > 0$ .

### 3. EXTREMES IN DISCRETE TIME

Let  $\{Z(\lambda)\}_{\lambda=-\infty}^{\infty}$  be a sequence of independent stable  $(1, \alpha, \beta)$  random variables. It is immediate that  $\sum_{-\infty}^{\infty} a(\lambda)Z(\lambda)$  converges in distribution if and only if

$$(3.1) \quad \sum_{\lambda=-\infty}^{\infty} |a(\lambda)|^{\alpha} < \infty \text{ and in addition, for } \alpha=1, \beta \neq 0, \\ \left| \sum_{\lambda=-\infty}^{\infty} a(\lambda) \log |a(\lambda)| \right| < \infty.$$

Moreover, if (3.1) is satisfied then, since the  $Z(\lambda)$ 's are independent,  $\sum_{-\infty}^{\infty} a(\lambda)Z(\lambda)$  converges with probability one also. The limiting distribution is stable with index  $\alpha$ , scale parameter  $(\sum_{-\infty}^{\infty} |a(\lambda)|^{\alpha})^{1/\alpha}$  and with symmetry parameter  $\beta \sum_{-\infty}^{\infty} \{a^+(\lambda)^{\alpha} - a^-(\lambda)^{\alpha}\} / \sum_{-\infty}^{\infty} |a(\lambda)|^{\alpha}$ , where  $a^+(\lambda) = \max(0, a(\lambda))$  and  $a^-(\lambda) = \max(0, -a(\lambda))$ . Further, if  $\alpha=1$  the distribution is translated by an amount  $-\beta 2\pi^{-1} \sum_{-\infty}^{\infty} a(\lambda) \log |a(\lambda)|$ .

Given  $\{a(\lambda)\}_{\lambda=-\infty}^{\infty}$  satisfying (3.1) a moving average process  $\{X(t)\}_{t=-\infty}^{\infty}$  is obtained by putting  $X(t) = \sum_{\lambda=-\infty}^{\infty} a(\lambda-t)Z(\lambda)$ . Let  $x > 0$  be fixed, take a sequence  $\{h(n)\}_{n=1}^{\infty}$  with  $h(n) \rightarrow \infty$  and  $h(n)/n \rightarrow 0$  but otherwise arbitrary, and define the separated exceedances of  $xn^{1/\alpha}$  recursively by putting  $t_{n1} = \inf\{t \geq h(n); X(t) > xn^{1/\alpha}\}$  and  $t_{ni} = \inf\{t \geq t_{n, i-1} + h(n); X(t) > xn^{1/\alpha}\}$ , for  $i \geq 2$ . The reason for using *separated* exceedances is that we want to count several exceedances which are "a fixed distance apart" as one event only. At the end of the section, also ordinary exceedances will be considered. The time-normalized point process  $N_n$  of separated exceedances is then defined by  $N_n(B) = \#\{t_{ni}/n \in B\}$  for Borel sets  $B \subset \mathbb{R}^+$ . Further, for a given sequence  $\{\tau_{ni}\}_{i=1}^{\infty}$  the mark  $Y_{ni}$  at exceedance no  $i$  is defined as

$$Y_{ni}(t) = X(t + \tau_{ni})/n^{1/\alpha}, \quad t=0, \pm 1, \dots,$$

and we then have a marked point process  $\eta_n = (N_n, Y_{n1}, Y_{n2}, \dots)$ .

In order to find the limiting distribution of  $\eta_n$  it is convenient to consider completely asymmetric processes first. Let  $c_\alpha = \pi^{-1} \Gamma(\alpha) \sin(\alpha\pi/2)$  and put  $A = \max_{-\infty < \lambda < \infty} a^+(\lambda)$ . Further let  $Y_i$  have the distribution of the vector  $(\dots, a(1)Z, a(0)Z, a(-1)Z, \dots)$  in  $R^\infty$ , where  $Z$  is a random variable with distribution function  $F(z) = 1 - x^\alpha A^{-\alpha} z^{-\alpha}$ ,  $z \geq xA^{-1}$ . Then the limiting distribution is that of

$$(3.2) \quad (N, Y_1, Y_2, \dots) \text{ where the components are independent, } N \text{ is a Poisson process with intensity } \mu = 2c_\alpha A^\alpha x^{-\alpha} \text{ and where the } Y_i \text{'s have the distribution given above.}$$

LEMMA 3.1. Suppose that  $\{Z(\lambda)\}_{-\infty}^\infty$  are independent and stable  $(1, \alpha, 1)$ , that  $\{a(\lambda)\}_{-\infty}^\infty$  satisfies (3.1) with  $A = \max_{-\infty < \lambda < \infty} a(\lambda) > 0$ , and that  $X(t) = \sum_{-\infty}^\infty a(\lambda-t)Z(\lambda)$ . Then there are time points  $\{\tau_{ni}; n \geq 1, i \geq 1\}$  with  $\{t_{ni} - \tau_{ni}\}_{n=1}^\infty$  tight for each  $i \geq 1$  (i.e.  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|t_{ni} - \tau_{ni}| > k) = 0, i \geq 1$ ) such that if  $\eta_n$  is the marked point process of separated exceedances of  $xn^{1/\alpha}$  defined above, then  $\eta_n \xrightarrow{d} \eta$  with the distribution of  $\eta$  given by (3.2).

REMARK. It would seem more natural to center the marks at the  $t_{ni}$ 's instead of at some  $\tau_{ni}$ 's which are not explicitly defined in terms of  $X(t)$ , but unfortunately the limiting distribution then becomes much more complicated. However, using the entire observed structure of the sample path near extremes it is possible to find the centering. For instance, if the maximum of  $\{a(\lambda)\}$  is unique then we may take  $\tau_{ni}$  as the time

point when  $\{X(t); t \in [t_{ni}, t_{ni} + h(n)]\}$  first assumes its maximum. (The validity of this claim is verified at the end of the proof of the lemma.)

PROOF. The essential facts we will use are that  $X$  is a moving average and the following simple estimates of the tails of  $F_{\alpha\beta}$ , the stable  $(1, \alpha, \beta)$  distribution (see Bergström (1953)):

$$(3.3) \quad \begin{aligned} 1 - F_{\alpha 1}(z) &\sim 2c_{\alpha} z^{-\alpha} && \text{as } z \rightarrow \infty \\ F_{\alpha 1}(z) &= o(|z|^{-\alpha}) && \text{as } z \rightarrow -\infty \end{aligned}$$

(where  $f \sim g$  means  $f = g(1+o(1))$ ) and

$$(3.4) \quad F_{\alpha\beta}(z) + (1 - F_{\alpha\beta}(z)) \leq k_{\alpha} z^{-\alpha}, \quad z > 0$$

for some constant  $k_{\alpha}$ .

Define  $0 \leq \tau_{n1} < \tau_{n2} < \dots$  as the times when  $Z(\lambda) > xA^{-1}n^{1/\alpha}$ , put  $N'_n(B) = \#\{\tau_{ni}/n \in B\}$  for Borel sets  $B \in \mathbb{R}^+$ , and let  $Y'_{ni}(t) = Z(\tau_{ni})/n^{1/\alpha}$  for  $t=0$  and  $Y'_{ni}(t) = 0$  for  $t \neq 0$ . Further, let  $\zeta$  have the distribution obtained from (3.2) by putting  $a(0)=A$ ,  $a(\lambda)=0$ ,  $\lambda \neq 0$  in the definition of  $Y_i$ . The first step is to prove that for  $\zeta_n = (N'_n, Y'_{n1}, Y'_{n2}, \dots)$  we have

$$(3.5) \quad \zeta_n \xrightarrow{d} \zeta \text{ as } n \rightarrow \infty.$$

Obviously  $\zeta_n$  has independent components, so according to Lemma 2.1 it is sufficient to prove that each of the components converges. From (3.3) we have

$$P(Z(0) > xA^{-1}n^{1/\alpha}) \sim 2c_{\alpha} A^{\alpha} x^{-\alpha} n^{-1}$$

which by Theorem 3.2 of Leadbetter (1976) proves that  $N'_n \xrightarrow{d} N$ . Furthermore,

for  $z \geq xA^{-1}$ ,

$$\begin{aligned} P(Y'_{ni}(0) \leq z) &= 1 - P(Y'_{ni}(0) > z) \\ &= 1 - P(Z(0) > n^{1/\alpha} z | Z(0) > n^{1/\alpha} xA^{-1}) \\ &\sim 1 - \frac{nx^\alpha A^{-\alpha}}{nz^\alpha} = 1 - x^\alpha A^{-\alpha} z^{-\alpha} \end{aligned}$$

and it follows that  $Y'_{ni} \xrightarrow{d} Y'_i$  and thus that (3.5) holds.

The next step is to prove that  $\{t_{ni} - \tau_{ni}\}_{n=1}^{\infty}$  is tight for  $i \geq 1$ , i.e.

$$(3.6) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|t_{ni} - \tau_{ni}| > k) = 0, \quad i=1,2,\dots$$

Now, putting  $A_{ni} = \{|t_{ni} - \tau_{ni}| > k\}$ , we have  $P(A_{ni}) \leq P(A_{n,i-1}^* A_{ni}) + P(A_{n,i-1})$ , (defining  $A_{n0}^* = \Omega$ ), and by recursion (3.6) follows if we prove

$$(3.7) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(A_{n,i-1}^* A_{ni}) = 0, \quad i=1,2,\dots$$

Let  $N$  be a positive integer, put  $B_n = \{\tau_{ni} > nN\}$  and put  $C_n = \{z(t) \in n^{1/\alpha} A^{-1}(x-2\epsilon, x] \text{ for some } t \in (0, nN)\}$ . Taking  $x/3 > \epsilon > 0$  we have, for  $n$  such that  $h(n) > 2k$ , that

$$\begin{aligned} A_{n,i-1}^* \{t_{ni} < \tau_{ni} - k\} &\subset \{X(t_{ni}) > n^{1/\alpha} x, Z(t) \leq n^{1/\alpha} xA^{-1} \text{ for } |t - t_{ni}| \leq k, t_{ni} < nN - k\} \cup B_n \\ &\subset \{X(t_{ni}) > n^{1/\alpha} x, Z(t) \leq n^{1/\alpha} A^{-1}(x-2\epsilon) \text{ for } |t - t_{ni}| \leq k, \\ &\quad t_{ni} < nN - k\} \cup B_n \cup C_n. \end{aligned}$$

Let  $D_n$  be the event that  $z(t) < -n^{1/\alpha} \epsilon (2k+1)^{-1} (\max_{\lambda} a^{-1}(\lambda))^{-1}$  for some  $t \in (0, nN)$ , and let  $E_n$  be the event that there are time points  $t', t''$  ( $0 < t', t'' < nN$ ) with  $|t' - t''| \leq 2k+1$  and  $z(t'), z(t'') > n^{1/\alpha} \epsilon (2k+1)^{-1} A^{-1}$ .

Further introduce  $X_k(t) = \sum_{\lambda=-k+t}^{k+t} a(\lambda-t)z(\lambda)$  and write  $F_n$  for the event that  $\sup\{|X(t)-X_k(t)|; 0 < t \leq nN\}$  exceeds  $n^{1/\alpha} \epsilon$ . Then

$$\begin{aligned} \{X(t_{ni}) > n^{1/\alpha} x, Z(t) \leq n^{1/\alpha} A^{-1}(x-2\epsilon) \text{ for } |t-t_{ni}| \leq k, t_{ni} < nN-k\} \\ \subset \{X_k(t_{ni}) > n^{1/\alpha}(x-\epsilon), Z(t) \leq n^{1/\alpha} A^{-1}(x-2\epsilon) \text{ for } |t-t_{ni}| \leq k, t_{ni} < nN-k\} \cup F_n \\ \subset D_n \cup E_n \cup F_n, \end{aligned}$$

where the last inclusion follows from the fact that if  $D_n^*$  occurs, if  $X_k(t_{ni}) = \sum_{\lambda=-k}^k a(\lambda)z(\lambda+t_{ni}) > n^{1/\alpha}(x-\epsilon)$ , and if  $z(\lambda+t_{ni}) \leq n^{1/\alpha} A^{-1}(x-2\epsilon)$  for  $|\lambda| \leq k$ , then for at least two values of  $\lambda$  with  $|\lambda| \leq k$  the summands  $a^+(\lambda)z(\lambda+t_{ni})$ , which are not larger than  $Az(\lambda+t_{ni})$ , have to exceed  $n^{1/\alpha} \epsilon(2k+1)^{-1}$ . It follows that

$$(3.8) \quad A_{n,i-1}^* \{t_{ni} < \tau_{ni} - k\} \subset B_n \cup C_n \cup D_n \cup E_n \cup F_n.$$

We proceed to estimate the probabilities of the events in the right-hand side of (3.8). From (3.5)

$$P(B_n) = P(N_n'(0, N) \leq i-1) \rightarrow \sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N}$$

as  $n \rightarrow \infty$  and, using Boole's inequality and (3.3),

$$\begin{aligned} P(C_n) &\leq nNP\{Z(0) \leq n^{1/\alpha} A^{-1}(x-2\epsilon, x)\} \\ &\rightarrow N \cdot 2 \cdot c_\alpha A^{-\alpha} \left\{ (x-2\epsilon)^{-\alpha} - x^{-\alpha} \right\} \end{aligned}$$

as  $n \rightarrow \infty$ , and similarly

$$P(D_n) \leq nNP\left\{Z(0) < -n^{1/\alpha} \epsilon(2k+1)^{-1} \left(\max_{\lambda} a^-(\lambda)\right)^{-1}\right\} \rightarrow 0.$$

Again by Boole's inequality and by independence and (3.3) we have

$$\begin{aligned}
P(E_n) &\leq ([nN/(2k+1)]+1)P(\{t', t''; 0 \leq t' \leq t'' \leq 4k+2, z(t') > n^{1/\alpha} \epsilon (2k+1)^{-1} A^{-1}, \\
&\quad z(t'') > n^{1/\alpha} \epsilon (2k+1)^{-1} A^{-1}\}) \\
&\leq ([nN/(2k+1)]+1) (2k+1) (4k+3) \{P(z(0) > n^{1/\alpha} \epsilon (2k+1)^{-1} A^{-1})\}^2 \\
&\sim ([nN/(2k+1)]+1) (2k+1) (4k+3) \{2c_\alpha \epsilon^{-\alpha} (2k+1)^\alpha A^\alpha n^{-1}\}^2 \\
&\rightarrow 0
\end{aligned}$$

as  $n \rightarrow \infty$ . Finally,  $X(t) - X_k(t)$  is stable with index  $\alpha$  and scale parameter  $\{\sum_{|\lambda| > k} |a(\lambda)|^\alpha\}^{1/\alpha}$  so (3.4) gives

$$\begin{aligned}
P(F_n) &\leq nNP(|X(0) - X_k(0)| > n^{1/\alpha} \epsilon) \\
&\leq nNk_\alpha \epsilon^{-\alpha} \sum_{|\lambda| > k} |a(\lambda)|^\alpha n^{-1} \\
&= k_\alpha N \epsilon^{-\alpha} \sum_{|\lambda| > k} |a(\lambda)|^\alpha.
\end{aligned}$$

Hence, by (3.3),

$$\begin{aligned}
\limsup_{n \rightarrow \infty} P(A_{n,i-1}^* \{t_{ni} < \tau_{ni} - k\}) &\leq \\
&\sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N} + 2c_\alpha A^\alpha N ((x-2\epsilon)^{-\alpha} - x^{-\alpha}) + k_\alpha N \epsilon^{-\alpha} \sum_{|\lambda| > k} |a(\lambda)|^\alpha
\end{aligned}$$

so

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(A_{n,i-1}^* \{t_{ni} < \tau_{ni} - k\}) \leq \sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N} + 2c_\alpha A^\alpha N ((x-2\epsilon)^{-\alpha} - x^{-\alpha})$$

and since  $\epsilon > 0$  and  $N$  are arbitrary (subject to  $x/3 > \epsilon > 0$ ) we get

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(A_{n,i-1}^* \{t_{ni} < \tau_{ni} - k\}) = 0.$$

Similarly, we can show

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(A_{n,i-1}^* \{\tau_{ni} < t_{ni} - k\}) = 0,$$

the main change of the proof being that in the definition of  $E_n$  we have to consider  $t', t''$  satisfying  $|t' - t''| \leq h(n) + 2k$  (instead of  $|t' - t''| \leq k$ ) and thus have to use  $h(n)/n \rightarrow 0$  to prove  $P(E_n) \rightarrow 0$ . Now (3.7) and thus (3.6) follows.

To prove the remainder of the theorem, introduce  $Y_{ni}^k = (\dots, a(k)z(\tau_{ni}), \dots, a(-k)z(\tau_{ni}), 0, \dots)$ , put  $N_n^k = N_n^i$ , and let  $\eta_n^k = (N_n^k, Y_{n1}^k, Y_{n2}^k, \dots)$ . Since the function that maps  $\zeta_n$  into  $\eta_n^k$  is continuous, (3.5) implies that  $\eta_n^k \xrightarrow{d} \eta^k$ , where  $\eta^k$  has the distribution that is obtained from (3.2) by putting  $a(\lambda) = 0$  for  $|\lambda| > k$ . Furthermore it is immediate from Lemma 2.1 that  $\eta^k \xrightarrow{d} \eta$ . Thus, by Lemma 2.2,  $\eta_n^k \xrightarrow{d} \eta$  follows if we prove that (2.1) holds. The atoms of  $N_n^k$  are  $\tau_{n1}/n, \tau_{n2}/n, \dots$  and the atoms of  $N_n$  are  $t_{n1}/n, t_{n2}/n, \dots$  and thus, since it follows from (3.6) that  $P(|\tau_{ni}/n - t_{ni}/n| > \epsilon) \rightarrow 0$ , as  $n \rightarrow \infty \forall \epsilon > 0$ , the hypothesis of Lemma 2.4 is satisfied so (2.1) holds for  $i=0$ . Next, by definition  $\delta_j(Y_{ni}, Y_{ni}^k) \leq |Y_{ni}(j) - Y_{ni}^k(j)| = |X(j + \tau_{ni}) - a(-j)Z(\tau_{ni})| n^{-1/\alpha} \leq |X(j + \tau_{ni}) - X_k(j + \tau_{ni})| n^{-1/\alpha} + |X_k(j + \tau_{ni}) - a(-j)Z(\tau_{ni})| n^{-1/\alpha}$  if  $j \leq k$ . Hence

$$\begin{aligned} \{\delta_j(Y_{ni}, Y_{ni}^k) > 2\epsilon\} &\subset \{|X_k(j + \tau_{ni}) - a(-j)Z(\tau_{ni})| > n^{1/\alpha} \epsilon, \\ &\quad \tau_{ni} < nN - j - k\} \cup \{\tau_{ni} \geq nN - j - k\} \cup F_n \\ &\subset \{\tau_{ni} \geq nN - j - k\} \cup E_n \cup F_n, \end{aligned}$$

and as in the proof of (3.7) we obtain

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(\delta_j(Y_{ni}, Y_{ni}^k) > 2\epsilon) = 0, \quad \forall \epsilon > 0, \quad j \geq 1$$

i.e. that (2.2) holds for  $i \geq 1$ . Since this implies that also (2.1) holds for  $i \geq 1$ , it completes the proof that  $\eta_n \xrightarrow{d} \eta$ .

Finally, let  $\tau_{ni}'$  be the first time when  $\{X(t); t \in [t_{ni}', t_{ni}'+h(n)]\}$  assumes its maximum and suppose that for  $\lambda_0$  satisfying  $a(\lambda_0) = A$  we have  $\min_{\lambda \neq \lambda_0} (a(\lambda_0) - a(\lambda)) = 2\epsilon > 0$ . To verify the claim of the remark we show that it is then possible to replace  $\{\tau_{ni}'\}$  by  $\{\tau_{ni}' - \lambda_0\}$  in the statement of the lemma. To do this, it is sufficient to prove

$$(3.9) \quad P(\tau_{ni}' - \lambda_0 \neq \tau_{ni}') \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Let  $\tau_{ni}^k$  be defined from  $X_k(t) = \sum_{\lambda=-k+t}^{k+t} a(\lambda-t)z(\lambda)$  in the same way as  $\tau_{ni}'$  is defined from  $X(t)$ . Now  $\{\tau_{ni}' - \lambda_0 \neq \tau_{ni}'\} \subset \{\tau_{ni}' \neq \tau_{ni}^k, \tau_{ni}^k - \lambda_0 = \tau_{ni}'\} \cup \{\tau_{ni}^k - \lambda_0 \neq \tau_{ni}'\}$  and for  $k \geq |\lambda_0|$  we have  $\{\tau_{ni}' \neq \tau_{ni}^k, \tau_{ni}^k - \lambda_0 = \tau_{ni}'\} \subset D_n \cup E_n' \cup F_n \cup \{\tau_{ni}' > nN - k - \lambda_0 - h(n)\}$  where  $E_n'$  is defined in the same way as  $E_n$  except that  $|t' - t''| \leq 2k+1$  is replaced by  $|t' - t''| \leq h(n) + 2k$ . Furthermore  $\{\tau_{ni}^k - \lambda_0 \neq \tau_{ni}'\} \subset E_n' \cup \{\tau_{ni}' > nN - k - \lambda_0 - h(n)\}$  and thus (3.9) follows as in the proof of (3.8).  $\square$

The general result follows rather easily from Lemma 3.1. Recall the notation  $A = \max_{\lambda} a^+(\lambda)$ , put  $a = \max_{\lambda} a^-(\lambda)$  and set  $\mu' = c_{\alpha} A^{\alpha} (1+\beta) x^{-\alpha}$ ,  $\mu'' = c_{\alpha} a^{\alpha} (1-\beta) x^{-\alpha}$ . Further let  $Z'$  and  $Z''$  be independent with distribution functions  $F_1(z) = 1 - x^{\alpha} A^{-\alpha} (1+\beta)^{-1} z^{-\alpha}$ ,  $z \geq x^{-1} A (1+\beta)^{1/\alpha}$  and  $F_2(z) = 1 - x^{\alpha} a^{-\alpha} (1-\beta)^{-1} z^{-\alpha}$ ,  $z \geq x^{-1} a (1-\beta)$  respectively and let  $Y_i = (\dots a(1)Z', a(0)Z', a(-1)Z', \dots)$  with probability  $\mu' / (\mu' + \mu'')$  and  $Y_i = (\dots -a(1)Z'', -a(0)Z'', -a(-1)Z'', \dots)$  otherwise. Then the limiting distribution is that of the marked point process

(3.10)  $(N, Y_1, Y_2, \dots)$  where the components are independent,  $N$  is a Poisson process with intensity  $\mu = \mu' + \mu''$  and where the  $Y_i$ 's have the distribution given above.

THEOREM 3.2. Let  $\{a(\lambda)\}_{-\infty}^{\infty}$  satisfy (3.1) and let  $\{Z(\lambda)\}_{-\infty}^{\infty}$  be an independent, stable  $(1, \alpha, \beta)$  sequence. Suppose that the moving average sequence  $\{X(t)\}_{-\infty}^{\infty}$  is given by  $X(t) = \sum_{-\infty}^{\infty} a(\lambda-t)Z(\lambda)$ . Then there exist  $\{\tau_{ni}\}$  with  $\{\tau_{ni} - \tau_{ni}\}_{n=1}^{\infty}$  tight for each  $i \geq 1$ , such that  $\eta_n \xrightarrow{d} \eta$ , where  $\eta_n$  is the marked point process of separated exceedances of  $n^{1/\alpha}x$  by  $\{X(t); t \geq 0\}$  and where the distribution of  $\eta$  is given by (3.10).

PROOF. It is immediate from (1.1) that if  $X$  and  $Y$  are independent and

stable  $(1, \alpha, 1)$  with  $\alpha \neq 1$ , then  $(\frac{1+\beta}{2})^{1/\alpha}X - (\frac{1-\beta}{2})^{1/\alpha}Y$  is stable  $(1, \alpha, \beta)$ . If  $\alpha = 1$  a constant has to be added to

this representation, but since this introduces only trivial complications we assume  $\alpha \neq 1$  for the remainder of the proof. Let  $\{Z'(\lambda)\}_{-\infty}^{\infty}$  and  $\{Z''(\lambda)\}_{-\infty}^{\infty}$  be independent stable  $(1, \alpha, 1)$  sequences and put  $X'(t) = \sum a(\lambda-t)(\frac{1+\beta}{2})^{1/\alpha}Z'(\lambda)$  and  $X''(t) = \sum a(\lambda-t)(\frac{1-\beta}{2})^{1/\alpha}Z''(\lambda)$ . Then the stochastic process  $\{X'(t) - X''(t)\}_{t=-\infty}^{\infty}$  has the same distribution as  $\{X(t)\}_{t=-\infty}^{\infty}$  and since we are interested only in distributional properties, we may thus consider  $X'(t) - X''(t)$  instead of  $X(t)$ .

Let  $0 \leq \tau_{n1}^i < \tau_{n2}^i < \dots$  be the times when  $\{Z'(t); t \geq 0\}$  exceeds  $n^{1/\alpha}x - 1(\frac{1+\beta}{2})^{-1/\alpha}x$  and let  $0 \leq \tau_{n1}^{ii} < \tau_{n2}^{ii} < \dots$  be the times when  $\{Z''(t); t \geq 0\}$  exceeds  $n^{1/\alpha}x - 1(\frac{1-\beta}{2})^{-1/\alpha}x$ . Using  $\{\tau_{ni}^i\}$  and  $\{\tau_{ni}^{ii}\}$ , define marked point processes of separated exceedances of the level  $n^{1/\alpha}x$ ,  $\eta_n^i = (N_n^i, Y_{n1}^i, Y_{n2}^i, \dots)$  from  $\{X'(t)\}$  and  $\eta_n^{ii} = (N_n^{ii}, Y_{n1}^{ii}, Y_{n2}^{ii}, \dots)$  from  $\{X''(t)\}$ . Further let  $\eta^i$  and  $\eta^{ii}$  be independent and with the distributions obtained from (3.2) by replacing  $a(\lambda)$  with  $a(\lambda)(\frac{1+\beta}{2})^{1/\alpha}$  and with  $a(\lambda)(\frac{1-\beta}{2})^{1/\alpha}$  respectively.

From Lemma 3.1 we have  $\eta'_n \xrightarrow{d} \eta'$  and  $\eta''_n \xrightarrow{d} \eta''$ , and since  $\eta'_n$  and  $\eta''_n$  are independent,  $(\eta'_n, \eta''_n) \xrightarrow{d} (\eta', \eta'')$  (using the product metric on  $S \times S$ ). Let  $N_n^0 = N_n' + N_n''$ , let  $0 \leq \tau_{n1} \leq \tau_{n2} \leq \dots$  be the atoms of  $N_n^0$ , and if  $\tau_{ni} = \tau'_{nk}$  for some  $k$  put  $Y_{ni}^0 = Y'_{nk}$ , otherwise put  $Y_{ni}^0 = Y''_{nk}$  for the  $k$  that satisfies  $\tau_{ni} = \tau''_{nk}$ . Then the function that maps  $(\eta'_n, \eta''_n)$  into  $\eta_n^0$  is continuous except on the set where  $N_n'$  and  $N_n''$  have common atoms. Since this set has  $(\eta', \eta'')$  probability zero it follows that  $\eta_n^0 \xrightarrow{d} \eta$ , where the distribution of  $\eta$  is given by (3.10). Finally, using the independence of  $\{X'_n(t)\}$  and  $\{X''_n(t)\}$  and similar (but easier) arguments as in the proof of Lemma 3.1, it follows that  $P(d(\eta_n, \eta_n^0) > \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$ , for all  $\epsilon > 0$ , and the theorem is proven.  $\square$

Theorem 3.2 gives a rather complete description of the asymptotic behaviour of extremes of linear stable processes, but it is somewhat complicated, and we will spend the rest of this section on some (simpler) corollaries to it.

The point process  $N_n$  gives the separated exceedances of  $xn^{1/\alpha}$ , but also ordinary exceedances are interesting. For Borel sets  $B \in \mathcal{R}^+$  put  $E_n(B) = \#\{t/n \in B; X(t) > xn^{1/\alpha}\}$ , let  $v^+(z) = \#\{\lambda; za(\lambda) > x\}$  and  $v^-(z) = \#\{\lambda; -za(\lambda) > x\}$ , and let the point process  $E$  have the following distribution: atoms occur according to a Poisson process with intensity  $\mu = \mu' + \mu''$ , the multiplicities of different atoms are independent and with the distribution of  $v$ , where  $v = v^+(Z')$  with probability  $\mu'/(\mu' + \mu'')$  and  $v = v^-(Z'')$  otherwise. ( $\mu', \mu''$  and the distributions of  $Z'$  and  $Z''$  are given on p. 27.)

COROLLARY 3.3. Suppose that  $\{X(t)\}_{t=-\infty}^{\infty}$  satisfies the assumptions of Theorem 3.2. Then  $E_n \xrightarrow{d} E$ , where  $E_n$  is the point process of exceedances of  $xn^{1/\alpha}$  and where the distribution of  $E$  is given above.

PROOF. We only sketch the proof. Let  $E_n^k$  have the same atoms as  $N_n$ , but with the multiplicity of the atom at  $t_{ni}$  equal to  $\#\{t \in [t_{ni}, t_{ni}+k]; X(t) > xn^{1/\alpha}\}$ . According to the theorem  $\eta_n \xrightarrow{d} \eta$  and with probability one  $\eta$  is of the form  $(\nu, y_1, y_2, \dots)$  where  $\nu$  is a locally finite measure and where  $y_i \in \mathbb{R}^{\infty}$  is of the form  $y_i(t) = z_i a(-t)$ ,  $t=0, \pm 1, \dots$ . However, for vectors of this form, the function that maps  $\eta_n$  into  $E_n^k$  is continuous except if  $z_i a(-t) = x$  for some  $i \geq 1$ . As the set of such vectors has  $\eta$  probability zero, it follows that  $E_n^k$  converges in distribution to some point process, say  $E^k$ . It is easily seen that  $E^k \xrightarrow{d} E$  as  $k \rightarrow \infty$ , and the proof can be finished, using similar methods as in the proof of Lemma 3.1, by approximating  $E_n$  by  $E_n^k$ .  $\square$

COROLLARY 3.4. Suppose that  $\{X(t)\}_{t=-\infty}^{\infty}$  satisfies the hypothesis of Theorem 3.2 and that the Borel set  $B \subset \mathbb{R}^+$  has boundary with Lebesgue measure zero ( $|\partial B|=0$ ). Then

$$P(E_n(B)=k) \rightarrow \sum_{j=0}^k \frac{(\mu|B|)^j}{j!} e^{-\mu|B|} \Pr\left(\sum_{i=1}^j v_i = k\right)$$

as  $n \rightarrow \infty$ , where the  $v_i$ 's are independent and with the same distributions as  $\nu$ . Similarly, if  $B_1, \dots, B_\ell \subset \mathbb{R}^+$  are disjoint and have boundaries with Lebesgue measure zero, then  $P(E_n(B_1)=k_1, \dots, E_n(B_\ell)=k_\ell)$  tends to the product of the corresponding probabilities.

Also the joint limiting distribution of heights and locations of the exceedances can be obtained from Theorem 3.2, but since the limiting distributions are complicated we only give the simplest result.

COROLLARY 3.5. Suppose that  $\{X(t)\}_{t \geq 0}^{\infty}$  satisfies the hypothesis of Theorem 3.2 and let  $M_n = \max_{1 \leq t \leq n} X(t)$ . Then

$$P(M_n/n^{1/\alpha} \leq x) \rightarrow \exp\{-c_{\alpha} [A^{\alpha}(1+\beta) + a^{\alpha}(1-\beta)] x^{-\alpha}\}$$

as  $n \rightarrow \infty$ .

PROOF. Obviously  $P(M_n/n^{1/\alpha} \leq x) = P(E_n([0,1])=0)$ , and the latter probability converges to  $\exp\{-c_{\alpha} [A^{\alpha}(1+\beta) + a^{\alpha}(1-\beta)] x^{-\alpha}\}$  by Corollary 3.4.  $\square$

In Theorem 3.2 it is assumed that the independent variables have a stable distribution. However, as was noted in the proof of Lemma 3.1, the essential property of the stable distribution is that the tails decrease polynomially. Thus results similar to those above hold for moving averages as soon as the tails of the distribution of the independent variables decrease as negative powers of  $x$ , e.g. if the independent variables belong to the domain of normal attraction of a stable law with exponent  $\alpha < 2$ .

#### 4. MOVING AVERAGES OF STABLE PROCESSES IN CONTINUOUS TIME

Consider a stochastic process  $\{Z(\lambda); \lambda \in \mathbb{R}\}$  that has stationary independent increments with  $Z(0)=0$  and  $Z(1)$  stable  $(1, \alpha, \beta)$ . In the sequel we will assume that  $\alpha \neq 1$ . The usual way of obtaining an integral  $\int a(\lambda) dZ(\lambda)$  is to first define it for step functions of the form  $a(\lambda) = \sum_{i=1}^k a_i I_{(b_i, c_i]}(\lambda)$  with  $-\infty < b_1 < c_1 \leq b_2 < \dots < c_k < \infty$  by putting

$\int a(\lambda) dZ(\lambda) = \sum_{i=1}^k a_i (Z(c_i) - Z(b_i))$ . Then, as is easily seen from (1.1)  $\int a(\lambda) dZ(\lambda)$  is stable with index  $\alpha$ , scale parameter  $\{\int |a(\lambda)|^\alpha d\lambda\}^{1/\alpha}$ , and symmetry parameter  $\{\int (a^+(\lambda)^\alpha - a^-(\lambda)^\alpha) d\lambda\} / \int |a(\lambda)|^\alpha d\lambda$ . Next, if  $a(\lambda)$  is (Lebesgue) measurable and satisfies

$$(4.1) \quad \int |a(\lambda)|^\alpha d\lambda < \infty \quad (0 < \alpha < 1 \text{ or } 1 < \alpha < 2),$$

we can find a sequence  $\{a_n(\lambda)\}_{n=1}^\infty$  of step functions with  $\int |a(\lambda) - a_n(\lambda)|^\alpha d\lambda \rightarrow 0$  as  $n \rightarrow \infty$ . Then, for  $I_n = \int a_n(\lambda) dZ(\lambda)$ , the scale parameter of  $I_n - I_m$  is  $\{\int |a_n(\lambda) - a_m(\lambda)|^\alpha d\lambda\}^{1/\alpha}$  which tends to zero as  $\min(m, n) \rightarrow \infty$ . Hence  $\{I_n\}_{n=1}^\infty$  is a Cauchy sequence in the sense of convergence in probability and there is a random variable  $I$  with  $I_n \xrightarrow{P} I$ . The integral is then defined (uniquely a.s.) by  $\int a(\lambda) dZ(\lambda) = I$ .

The object of study is moving averages, i.e. processes of the form  $X(t) = \int a(\lambda - t) dZ(\lambda)$  with  $a(\lambda)$  satisfying (4.1). We always assume that a separable version has been chosen. Our approach is to approximate  $a(\lambda)$  by step functions  $a_k(\lambda) = \sum a_i I(i2^{-k} < \lambda \leq (i+1)2^{-k})$  and thus to approximate  $X(t)$  by  $X_k(t) = \sum a_i \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$ . The necessary estimates are given by the following two lemmas.

**LEMMA 4.1.** Suppose  $X(t) = \sum a_i \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$ , where  $\sum |a_i|^\alpha < \infty$  and where  $\{Z(\lambda); \lambda \in \mathbb{R}\}$  has stationary independent increments with  $Z(0) = 0$  and  $Z(1)$  stable  $(1, \alpha, 1)$ , and put  $X_k = \sup_{0 \leq t \leq 1} |X((\ell+t)2^{-k}) - X(\ell 2^{-k})|$ . If  $0 < \alpha < 1$  then, for some constant  $k_\alpha$ ,

$$(4.2) \quad P(\max_{0 \leq \ell \leq 2^k N - 1} X_\ell > x) \leq K_\alpha \sum_i A_i^\alpha N x^{-\alpha}$$

where  $A_i = \max_{0 \leq j \leq 2^k} |a_{i2^k+j}|$ . If  $1 < \alpha < 2$  then

$$(4.3) \quad P(\max_{0 \leq \ell \leq 2^k N - 1} X_\ell > x) \leq K_\alpha \sum_i |a_i|^\alpha N \{x^{-\alpha} + (2^{-k+\alpha} \sum_i |a_i|^\alpha)^{(2-\alpha)/\alpha} x^{-2}\}$$

for  $x > C_\alpha (2^{-k} \sum_i |a_i|^\alpha)^{1/\alpha}$ , where  $C_\alpha$  is a constant.

PROOF. We have

$$(4.4) \quad \begin{aligned} X((\ell+t)2^{-k}) - X(\ell 2^{-k}) &= \sum_i a_i \{Z((i+1+\ell+t)2^{-k}) - Z((i+\ell+t)2^{-k})\} \\ &\quad - \sum_i a_i \{Z((i+1+\ell)2^{-k}) - Z((i+\ell)2^{-k})\} \\ &= \sum (a_{i-1} - a_i) \{Z((i+\ell+t)2^{-k}) - Z((i+\ell)2^{-k})\}. \end{aligned}$$

Suppose  $0 < \alpha < 1$ . Then  $\{Z(\lambda)\}$  has nondecreasing sample paths and hence for  $0 \leq t \leq 1$ ,

$$\begin{aligned} |X((\ell+t)2^{-k}) - X(\ell 2^{-k})| &= \left| \sum_{i=-\infty}^{\infty} \sum_{j=1}^{2^k} (a_{i2^k+t+j-1} - a_{i2^k+t+j}) \right. \\ &\quad \left. \{Z(i+(j+\ell+t)2^{-k}) - Z(i+(j+\ell)2^{-k})\} \right| \\ &\leq 2 \sum_{i=-\infty}^{\infty} A_i \sum_{j=1}^{2^k} \{Z(i+(j+\ell+t)2^{-k}) - Z(i+(j+\ell)2^{-k})\} \\ &\leq 2 \sum_{i=-\infty}^{\infty} A_i \sum_{j=1}^{2^k} \{Z(i+(j+\ell+1)2^{-k}) - Z(i+(j+\ell)2^{-k})\} \\ &= 2 \sum A_i \{Z(i+1+(\ell+1)2^{-k}) - Z(i+\ell 2^{-k})\}. \end{aligned}$$

It follows that

$$\begin{aligned} \max_{0 \leq \ell \leq 2^k - 1} X_\ell &\leq 2 \sum A_i \{Z(i+2) - Z(i)\} \\ &= 2 \sum (A_i + A_{i-1}) \{Z(i+1) - Z(i)\}. \end{aligned}$$

The sequence  $\{z(i+1)-z(i)\}_{i=-\infty}^{\infty}$  is independent and stable  $(1, \alpha, 1)$  and thus, by (3.4),

$$\begin{aligned} P\left(\max_{0 \leq \ell \leq 2^k - 1} X_\ell > x\right) &\leq 2^\alpha k_\alpha \sum (A_i + A_{i+1})^\alpha x^{-\alpha} \\ &\leq 2 \cdot 2^\alpha k_\alpha \sum A_i^\alpha x^{-\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} P\left(\max_{0 \leq \ell \leq N2^k - 1} X_\ell > x\right) &\leq NP\left(\max_{0 \leq \ell \leq 2^k - 1} X_\ell > x\right) \\ &\leq 2 \cdot 2^\alpha k_\alpha \sum A_i^\alpha Nx^{-\alpha} \end{aligned}$$

and (4.2) holds with  $K_\alpha = 2 \cdot 2^\alpha k_\alpha$ .

Next suppose that  $1 < \alpha < 2$ . Put  $Y(t) = X(\tau 2^{-k}) - X(0)$  and set  $b_i = a_i - a_{i-1}$ . It follows from (4.4) that  $\{Y(t); 0 \leq t \leq 1\}$  has stationary independent increments with  $Y(0) = 0$  and  $Y(1)$  stable with index  $\alpha$ , scale parameter  $\gamma = (2^{-k} \sum |b_i|^\alpha)^{1/\alpha}$  and symmetry parameter  $\beta = \sum \{(b_i^+)^{\alpha} - (b_i^-)^{\alpha}\} / \sum |b_i|^\alpha$ . In analogy with the proof of Theorem 3.2, we represent  $Y(t)$  as  $Y^+(t) - Y^-(t)$ , where  $Y^+(0) = Y^-(0) = 0$  and the processes  $\{Y^+(t)\}$  and  $\{Y^-(t)\}$  are independent and have stationary independent increments with  $Y^+(1)$  stable  $\left[\gamma((1+\beta)/2)^{1/\alpha}, \alpha, 1\right]$  and  $Y^-(1)$  stable  $\left[\gamma((1-\beta)/2)^{1/\alpha}, \alpha, 1\right]$ . Further,  $Y^+(t) = Y_0^+(t) + Y_1^+(t)$  where  $Y_1^+(t)$  is the sum of jumps of  $Y^+(t)$  of size  $\geq 1$  in  $[0, t]$  and  $Y^-(t) = Y_0^-(t) + Y_1^-(t)$  where  $Y_1^-(t)$  is the sum of jumps of  $Y^-(t)$  of size  $\geq 1$  in  $[0, t]$ . Let  $\psi(u, x) = (e^{iux} - 1 - iux) |x|^{-1-\alpha}$ . Using the Lévy representation of the characteristic function of  $Y^+(t)$  as  $\phi_t^+(u) = \exp\{t \alpha_\alpha \gamma^\alpha (1+\beta) \int_0^\infty \psi(u, x) dx\}$  it follows (see Lévy (1954), Breiman (1968)) that  $\{Y_0^+(t)\}$  and  $\{Y_1^+(t)\}$  are independent and that  $Y_0^+(t)$  has the characteristic function  $\exp\{t \alpha_\alpha \gamma^\alpha (1+\beta) (\int_0^1 \psi(u, x) dx - iu \int_1^\infty x^{-\alpha} dx)\}$ . Similarly  $\{Y_0^-(t)\}$  and  $\{Y_1^-(t)\}$  are independent and  $Y_0^-(t)$  has the

characteristic function  $\exp\{t\alpha_\alpha \gamma(1-\beta) (\int_0^1 \psi(u,x) dx + iu \int_1^\infty x^\alpha dx)\}$ .

Hence

$$\begin{aligned} X_0 &= \sup_{0 \leq t \leq 1} |Y(t)| \leq \sup_{0 \leq t \leq 1} |Y_0^+(t)| + \sup_{0 \leq t \leq 1} |Y_1^+(t)| + \sup_{0 \leq t \leq 1} |Y_0^-(t)| + \sup_{0 \leq t \leq 1} |Y_1^-(t)| \\ &= \sup_{0 \leq t \leq 1} |Y_0^+(t)| + Y_1^+(1) + \sup_{0 \leq t \leq 1} |Y_0^-(t)| + Y_1^-(1) \\ &\leq 2 \sup_{0 \leq t \leq 1} |Y_0^+(t)| + Y_1^+(1) + 2 \sup_{0 \leq t \leq 1} |Y_0^-(t)| + Y_1^-(1). \end{aligned}$$

The process  $\{Y_0^+(t); 0 \leq t \leq 1\}$  has moments of all orders (Feller (1971), p. 570) and, if  $m_\alpha$  and  $v_\alpha$  are the mean and variance in the distribution

with characteristic function  $\exp\{\alpha c_\alpha (\int_0^1 \psi(u,x) dx - iu \int_1^\infty x^{-\alpha} dx)\}$ , then

$$E(Y_0^+(t)) = t\gamma(1+\beta)^{1/\alpha} m_\alpha \quad \text{and} \quad v(Y_0^+(t)) = t\gamma^2(1+\beta)^{2/\alpha} v_\alpha.$$

For  $x > 16\gamma|m_\alpha|$  Kolmogorov's inequality gives

$$\begin{aligned} P(2 \sup_{0 \leq t \leq 1} |Y_0^+(t)| > x/4) &\leq P(2 \sup_{0 \leq t \leq 1} |Y_0^+(t) - t\gamma(1+\beta)^{1/\alpha} m_\alpha| > x/4 - \gamma(1+\beta)^{1/\alpha} |m_\alpha|) \\ &\leq P(\sup_{0 \leq t \leq 1} |Y_0^+(t) - t\gamma(1+\beta)^{1/\alpha} m_\alpha| > x/16) \\ &\leq \gamma^2 v_\alpha^2 16^2 x^{-2}. \end{aligned}$$

Similarly, for  $x > 16\gamma|m_\alpha|$ ,

$$P(2 \sup_{0 \leq t \leq 1} |Y_0^-(t)| > x/4) \leq \gamma^2 v_\alpha^2 16^2 x^{-2}.$$

By (3.4) we have  $P(Y_1^+(1) > x/4) \leq k_\alpha \gamma^\alpha (1+\beta) 4^\alpha x^{-\alpha}$  and  $P(Y_1^-(1) > x/4) \leq k_\alpha \gamma^\alpha (1-\beta) 4^\alpha x^{-\alpha}$ . Thus, for  $x > 16m_\alpha \gamma$ ,

$$\begin{aligned} P(X_0 > x) &\leq P(2 \sup_{0 \leq t \leq 1} |Y_0^+(t)| > x/4) + P(Y_1^+(1) > x/4) + P(2 \sup_{0 \leq t \leq 1} |Y_0^-(t)| > x/4) + P(Y_1^-(1) > x/4) \\ &\leq k_\alpha^2 \gamma^{2-\alpha} (x^{-\alpha} + \gamma^{2-\alpha} x^{-2}) \end{aligned}$$

if  $K_\alpha = 2^\alpha \max(4.16^2 v_\alpha, k_\alpha 4^\alpha)$ , so since  $\gamma^\alpha = 2^{-k} \sum |b_i|^\alpha$  and  $b_i = a_{i-1} - a_i$  we have

$$\begin{aligned} P(\max_{0 \leq \ell \leq 12^{k-1}} X_\ell > x) &\leq 2^k NP(X_0 > x) \\ &\leq K_\alpha 2^{-\alpha k} \gamma^\alpha (x^{-\alpha} + \gamma^{2-\alpha} x^{-2}) \\ &= K_\alpha 2^{-\alpha} \sum |b_i|^\alpha N(x^{-\alpha} + (2^{-k} \sum |b_i|^\alpha)^{\frac{2-\alpha}{\alpha}} x^{-2}) \\ &\leq K_\alpha \sum |a_i|^\alpha N(x^{-\alpha} + (2^{-k+\alpha} \sum |a_i|^\alpha)^{\frac{2-\alpha}{\alpha}} x^{-2}), \end{aligned}$$

which proves (4.3). □

In order to apply Lemma 4.1 to  $X(t) = \int a(\lambda-t) dZ(\lambda)$  some conditions are needed. Let  $B_{ki} = \sup_{\lambda 2^k \in (i, i+1]} a(\lambda)$ ,  $b_{ki} = \inf_{\lambda 2^k \in (i, i+1]} a(\lambda)$  and put

$a_{ki} = a(i2^{-k})$ . One possibility is to require

$$(4.5) \quad a(\lambda) \text{ is uniformly continuous, } \sum_{i=-\infty}^{\infty} B_{ii}^\alpha < \infty \text{ and } 0 < \alpha < 1.$$

The second part of this condition is of course equivalent to  $\sum_{i=-\infty}^{\infty} B_{ki}^\alpha < \infty$ , for all  $k \geq 1$ . Another possibility is to require

$$(4.6) \quad a(\lambda) \text{ is uniformly continuous, } \sum_{i=-\infty}^{\infty} |a_{ii}|^\alpha < \infty, \text{ there exist } \delta > 0 \text{ and } K \text{ such that } 2^{k\delta} \sum |a_{ki} - a_{k-1, [i/2]}|^\alpha \leq K, \text{ and } 1 < \alpha < 2.$$

Obviously, this condition implies that  $\sum |a_{ki}|^\alpha < \infty$  for all  $k \geq 1$ . The latter part of the condition perhaps needs some motivation. Suppose that  $a(\lambda)$  is continuously differentiable, except possibly at the points

$\{i2^{-k+1}\}_{i=-\infty}^{\infty}$ , and put  $f_{ki} = \sup_{\lambda 2^k \in (i-1, i]} |a'(\lambda)|$ . Then  $|a_{ki} - a_{k-1, [i/2]}| \leq f_{ki} 2^{-k}$  and hence  $\sum |a_{ki} - a_{k-1, [i/2]}|^\alpha \leq 2^{-k(\alpha-1)} \sum f_{ki}^\alpha 2^{-k}$ . Thus the latter part of (4.6) holds with  $\delta = \alpha - 1$  if e.g.  $\sum f_{ki} 2^{-k}$  converges as  $k \rightarrow \infty$ , and to require that this holds is rather close to requiring  $\int |a'(\lambda)|^\alpha d\lambda < \infty$ .

**LEMMA 4.2.** Suppose that  $X(t) = \int a(\lambda - t) dZ(\lambda)$ , where  $a(\lambda)$  is non-negative and satisfies (4.1) and  $\{Z(\lambda); \lambda \in \mathbb{R}\}$  is as in Lemma 4.1. Furthermore put  $X_k(t) = \sum a_{ki} \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$  (the sum converges, by (4.5) or by (4.6)). If  $a(\lambda)$  satisfies (4.5) then, for some constant  $K'_\alpha$

$$(4.7) \quad P\left(\sup_{0 \leq t \leq N} |X(t) - X_k(t)| > x\right) \leq K'_\alpha \sum_{i=-\infty}^{\infty} A_{ki}^\alpha N x^{-\alpha}.$$

where  $A_{ki} = \max_{0 \leq \ell \leq 2^k} |B_{k, i2^k + j} - B_{k, i2^k + j}|$ . If  $a(\lambda)$  satisfies (4.6) then

$$(4.8) \quad P\left(\sup_{0 \leq t \leq N} |X(t) - X_k(t)| > x\right) \leq K'_\alpha 2^{-k\delta/\alpha} N(x^{-\alpha} + x^{-2}),$$

for  $k$  large enough to make  $2^k > C_{\alpha\delta} x^{-\alpha}$ , where  $C_{\alpha\delta}$  is a constant.

**PROOF.** Suppose  $0 < \alpha < 1$  and let  $\bar{X}_k(t) = \sum B_{ki} \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$ ,  $\underline{X}_k(t) = \sum b_{ki} \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$ . Since  $Z(\lambda)$  has nondecreasing sample paths,  $D_k(t) = \bar{X}_k(t) - \underline{X}_k(t) \geq |X(t) - X_k(t)|$ . Using the same methods as in the first part of Lemma 4.1 it is easily seen that  $P(\sup_{0 \leq t \leq N} D_k(t) > x) \leq K_\alpha \sum_{i=-\infty}^{\infty} A_{ki} x^{-\alpha}$  and thus (4.7) follows with  $K'_\alpha = K_\alpha$ .

Now consider  $1 < \alpha < 2$ . In this case let  $D_k(t) = X_k(t) - X_{k-1}(t)$  and put  $d_{ki} = a_{ki} - a_{k-1, [i/2]}$ , making  $D_k(t) = \sum d_{ki} \{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$ . Further let  $D_\ell = \sup_{0 \leq t \leq 1} |D_k((\ell+t)2^{-k}) - D_k(\ell 2^{-k})|$  and use (3.4) and Lemma 4.1 to obtain

$$\begin{aligned}
P\left(\sup_{0 \leq t \leq N} D_k(t) > x\right) &\leq P\left(\max_{0 \leq \ell \leq 2^{nN}-1} |D_k(\ell 2^{-k})| > x/2\right) + P\left(\max_{0 \leq \ell \leq 2^{nN}-1} D_\ell > x/2\right) \\
&\leq k_\alpha N \sum |d_{ki}|^\alpha x^{-\alpha} + k_\alpha N \sum |d_{ki}|^\alpha \{x^{-\alpha} + (2^{-k+\alpha} \sum |d_{ki}|^\alpha)^{(2-\alpha)/\alpha} x^{-2}\}
\end{aligned}$$

for  $x > C_\alpha (2^{-k} \sum |d_{ki}|^\alpha)^{1/\alpha}$ . Hence by (4.6)

$$(4.9) \quad P\left(\sup_{0 \leq t \leq N} D_k(t) > x\right) \leq KN \{(k_\alpha + k_\alpha) 2^{-k\delta} x^{-\alpha} + k_\alpha K^{(2-\alpha)/\alpha} 2^{2-\alpha} 2^{-k(2+2\delta-\alpha)/\alpha} x^{-2}\},$$

for  $x > C_\alpha (K 2^{-k(1+\delta)})^{1/\alpha}$ . Let  $x_i = 2^{-(i-1)\delta/(2\alpha)} (1 - 2^{-\delta/(2\alpha)}) x$ , so that  $\sum_{i=1}^{\infty} x_i = x$ ,  $\sum_{i=1}^{\infty} 2^{-(k+i)\delta} x_i^{-\alpha} \leq \text{constant} \times 2^{-k\delta} x^{-\alpha}$ , and  $\sum_{i=1}^{\infty} 2^{-(k+i)(2+2\delta-\alpha)/\alpha} x_i^{-2} \leq \text{constant} \times 2^{-k(2+2\delta-\alpha)/\alpha} x^{-2}$ . Since  $X_k(t) \xrightarrow{P} X(t)$  we have

$$(4.10) \quad P\left(\sup_{0 \leq t \leq N} |X(t) - X_k(t)| > x\right) \leq \sum_{i=1}^{\infty} P\left(\sup_{0 \leq t \leq N} D_{k+i}(t) > x_i\right),$$

and if  $2^k > (1 - 2^{-\delta/(2\alpha)})^{-1} C_\alpha K x^{-\alpha}$  then  $x_i > C_\alpha (K 2^{(k+i)(i+\delta)})^{1/\alpha}$ , for  $i \geq 1$ , and (4.8) follows from (4.9) and (4.10).  $\square$

The conditions used above imply that  $X(t)$  has continuous sample paths, and although we do not need this result for the sequel, it is interesting in its own right.

**THEOREM 4.3.** Let  $\{Z(\lambda): \lambda \in \mathbb{R}\}$  have stationary independent increments, which are stable with index  $\alpha$ . Further suppose that  $a(\lambda)$  satisfies (4.1) and that both  $a^+(\lambda)$  and  $a^-(\lambda)$  satisfy either (4.5) or (4.6). Then the moving average  $X(t) = \int a(\lambda-t) dZ(\lambda)$  has continuous sample paths.

**PROOF.** Obviously it is no restriction to assume that  $Z(0)=0$  and  $Z(1)$  is stable  $(1, \alpha, \beta)$ , and since  $\int a(\lambda-t) dZ(\lambda) = \int a^+(\lambda-t) dZ(\lambda) + \int a^-(\lambda-t) dZ(\lambda)$ ,

it is enough to show that each of the terms is continuous, i.e. we may also assume  $a(\lambda) \geq 0$ .

Let  $\{Z'(\lambda); \lambda \in \mathbb{R}\}$  and  $\{Z''(\lambda); \lambda \in \mathbb{R}\}$  be independent and have stationary independent increments with  $Z'(0) = Z''(0) = 0$  and  $Z'(1), Z''(1)$  stable  $(1, \alpha, 1)$ . Then  $\int a(\lambda-t) dZ(\lambda)$  has the same distribution as  $(\frac{1+\beta}{2})^{1/\alpha} \int a(\lambda-t) dZ'(\lambda) - (\frac{1-\beta}{2})^{1/\alpha} \int a(\lambda-t) dZ''(\lambda)$ , and thus we may further assume  $\beta=1$ .

The proof proceeds by approximating  $X(t)$  by  $V_k(t) = \int a_k(\lambda-t) dZ(\lambda)$ , where  $a_k(t)$  is defined by the requirement that  $a_k(t)=0$ ,  $|t| \geq k'$ , for  $k' = k'(k)$  to be specified later, that  $a_k(\ell 2^{-k}) = a(\ell 2^{-k})$ ,  $\ell = 0, \pm 1, \dots, \pm k' 2^k - 1$ , and that  $a_k(t)$  is linear between these points. Using the definition of the integral as a limit of sums and Abelian summation, it is seen that ("partial integration")

$$\begin{aligned} \int a_k(\lambda-t) dZ(\lambda) &= \int a_k(\lambda) dZ(\lambda+t) \\ &= - \int a_k'(\lambda) Z(\lambda+t) d\lambda \\ &= - \sum_{i=-k' 2^k}^{k' 2^k - 1} \left\{ \frac{a((i+1) 2^{-k}) - a(i 2^{-k})}{2^{-k}} \right\} \int_{i 2^{-k}}^{(i+1) 2^{-k}} Z(\lambda+t) d\lambda, \text{ (here} \\ &\quad a(k') = a(-k') = 0) \end{aligned}$$

where the integrals are defined as limits in probability of sums. However,  $Z(\lambda) \in D(-\infty, \infty)$  (see e.g. Breiman (1968), p. 306), and is thus locally Riemann integrable and hence  $\int_{i 2^{-k}}^{(i+1) 2^{-k}} Z(\lambda+t) d\lambda$  is a.s. a Riemann integral and is thus a.s. continuous in  $t$ , and it follows that also

$V_k(t) = \int a_k(\lambda-t) dZ(\lambda)$  is continuous in  $t$  a.s.

Hence, if we prove e.g.

$$(4.11) \quad \sup_{0 \leq t \leq 1} |X(t) - V_k(t)| \xrightarrow{P} 0,$$

then the desired result follows, since there is then a sequence  $\{k_n\}$  of integers with  $P(\sup_{0 \leq t \leq 1} |X(t) - V_{k_n}(t)| \rightarrow 0 \text{ as } n \rightarrow \infty) = 1$ , i.e.  $X(t)$  is a.s. a uniform (in  $[0,1]$ ) limit of continuous functions and is thus continuous in  $[0,1]$  and hence, by stationarity, in all of  $R$ . Now, let  $X_k(t)$  be as in Lemma 4.2 and put  $X'_k(t) = \sum_{|i2^k| < k'} a_{ki} \{z((i+1)2^{-k}+t) - z(i2^{-k}+t)\}$ . We have

$$(4.12) \quad P\left(\sup_{0 \leq t \leq 1} |X(t) - V_k(t)| > x\right) \leq P\left(\sup_{0 \leq t \leq 1} |X(t) - X_k(t)| > x/3\right) \\ + P\left(\sup_{0 \leq t \leq 1} |X_k(t) - X'_k(t)| > x/3\right) + P\left(\sup_{0 \leq t \leq 1} |X'_k(t) - V_k(t)| > x/3\right).$$

Thus, if  $0 < \alpha < 1$ , Lemmas 4.1 and 4.2 give that

$$(4.13) \quad P\left(\sup_{0 \leq t \leq 1} |X(t) - V_k(t)| > x\right) \leq K'_\alpha \left\{ \sum_{i=-\infty}^{\infty} A_{ki}^\alpha + \sum_{|i| \geq k'} B_{li}^\alpha + \sum_{i=-k'+1}^{k'-1} A_{ki}^\alpha \right\} 3^\alpha x^{-\alpha},$$

Choosing e.g.  $k'(k) \equiv k$  it follows from (4.5) and the dominated convergence theorem that the righthand side of (4.13) tends to zero as  $k \rightarrow \infty$ , and thus (4.11) holds for  $0 < \alpha < 1$ .

It is no loss of generality to assume  $\delta \leq \alpha$  in (4.6), and then it can be seen that, regardless of the value of  $k'$ ,  $a_k(\lambda)$  satisfies (4.6) with  $K$  not depending on  $k$  and with the same  $\delta$  as  $a(\lambda)$ , and thus if  $1 < \alpha < 2$  it follows from Lemma 4.2 that the first and the third terms of (4.12) are bounded by  $K'_\alpha 2^{-k\delta/\alpha} 3^\alpha x^{-\alpha}$  for large  $k$ . Furthermore, by (3.4) the second term is bounded by  $K_\alpha \sum_{|i2^k| > k'} |a_{ki}|^\alpha 3^\alpha x^{-\alpha}$ , and since  $\sum |a_{ki}|^\alpha < \infty$  by (4.6),  $k'$  can be chosen large enough to make  $\sum_{|i2^k| > k'} |a_{ki}|^\alpha \rightarrow 0$  as  $k \rightarrow \infty$ , and it follows that (4.11) holds also for  $1 < \alpha < 2$ .  $\square$

### 5. EXTREMES IN CONTINUOUS TIME

Let  $\{X(t); t \in \mathbb{R}\}$  be a moving average, and in analogy with Section 3 define recursively  $t_{T1} = \inf\{t \geq h(T); X(t) > xT^{1/\alpha}\}$ ,  $t_{Ti} = \inf\{t \geq t_{T,i-1} + h(T); X(t) > xT^{1/\alpha}\}$  for  $i \geq 2$ . For a given sequence  $\{\tau_{Ti}\}_{i=1}^{\infty}$  put  $Y_{Ti}(t) = X(t + \tau_{Ti})/T^{1/\alpha}$ , let  $N_T(B) = \#\{t_{Ti}, k \in B\}$  for Borel sets  $B \subset \mathbb{R}^+$  and consider the marked point process  $\eta_T = (N_T, Y_{T1}, Y_{T2}, \dots)$  of separated exceedances of  $xT^{1/\alpha}$ . Furthermore, put  $A = \sup_{\lambda \in \mathbb{R}} a^+(\lambda)$ , let  $\mu$  and  $Z$  be as defined on p. 17, and let  $Y_i$  have the distribution of the random variable  $\{Za(-t); t \in \mathbb{R}\}$  in  $D(-\infty, \infty)$ . If  $Z(\lambda)$  is completely asymmetric the limiting distribution will be that of

$$(5.1) \quad (N, Y_1, Y_2, \dots) \text{ where the components are independent, } N \text{ is a Poisson process with intensity } \mu, \text{ and where the } Y_i \text{'s have the distribution given above.}$$

LEMMA 5.1. Suppose that  $\{Z(\lambda); \lambda \in \mathbb{R}\}$  has stationary independent increments, with  $Z(0)=0$  and  $Z(1)$  stable  $(1, \alpha, 1)$ . Further suppose that  $a(\lambda)$  satisfies (4.1), that both  $a^+(\lambda)$  and  $a^-(\lambda)$  satisfy either (4.5) or (4.6), and that  $A > 0$ . Then there exists  $\{\tau_{Ti}\}$  such that  $\{t_{Ti} - \tau_{Ti}; T \geq 1\}$  is tight for each  $i \geq 0$  and such that the marked point process  $\eta_T$  of separated exceedances of  $xT^{1/\alpha}$  by  $X(t) = \int a(\lambda - t) dZ(\lambda)$  converges in distribution to  $\eta$ , where the distribution of  $\eta$  is given by (5.1).

PROOF. Without loss of generality we assume that  $a(0)=A$ . First suppose that  $0 < \alpha < 1$ . Recall the definitions of  $X_k(t)$  and  $a_{ki}$  from Lemma 4.2 and put  $\bar{X}_k(t) = X_k(2^{-k} \lfloor 2^k t \rfloor)$  so that  $\bar{X}_k(i2^{-k}) = X_k(i2^{-k})$  and  $\bar{X}_k(t)$

is constant for  $t \in [i2^{-k}, (i+1)2^{-k}]$ . We have

$$(5.2) \quad P\left(\sup_{0 \leq t \leq T} |X(t) - \bar{X}_k(t)| > T^{1/\alpha} x\right) \leq P\left(\sup_{0 \leq t \leq T} |X(t) - X_k(t)| > T^{1/\alpha} x/2\right) +$$

$$P\left(\sup_{0 \leq t \leq T} |X_k(t) - \bar{X}_k(t)| > T^{1/\alpha} x/2\right) \\ \leq 2K_\alpha \sum_{i=-\infty}^{\infty} A_{ki}^\alpha (T+1) T^{-1} x^{-\alpha} 2^\alpha + \\ K_\alpha \sum_{i=-\infty}^{\infty} A_{ki}^\alpha (T+1) T^{-1} x^{-\alpha} 2^\alpha$$

by Lemma (4.2) applied to both  $a^+(\lambda)$  and  $a^-(\lambda)$  and by Lemma (4.1).

From (4.5) it follows that  $\sum_{i=-\infty}^{\infty} A_{ki}^\alpha \rightarrow 0$  when  $k \rightarrow \infty$  and hence

$$(5.3) \quad \lim_{k \rightarrow \infty} \limsup_{T \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} |X(t) - \bar{X}_k(t)| > T^{1/\alpha} x\right) = 0.$$

Now, let  $\zeta_T^k = (N_T^k, Y_{T1}^k, Y_{T2}^k, \dots)$  be the marked point process of  $h(T)$ -separated upcrossings of  $xT^{1/\alpha}$  by the discrete process  $\{\bar{X}_k(i2^{-k})\}_{i=-\infty}^{\infty}$ , where  $Y_{Ti}^k(t) = \bar{X}_k(t2^{-k} + \tau_{Ti}^k)$ ,  $t=0, \pm 1, \dots$ , with  $\tau_{Ti}^k$  the time of the  $i$ 'th exceedance of  $xT^{1/\alpha}/A$  by the sequence  $\{Z(i2^{-k}) - Z((i-1)2^{-k})\}_{i=0}^{\infty}$ . According to Lemma 3.1  $\zeta_T^k$  converges in distribution, to  $\zeta^k$  say, as  $T \rightarrow \infty$ . Furthermore, if  $\eta_T^k$  is the marked point process of upcrossings of  $xT^{1/\alpha}$  by the continuous-time process  $\{\bar{X}_k(t); t \in \mathbb{R}\}$ , with the marks centered at the  $\tau_{Ti}^k$ 's, then the function  $f: \mathbb{N} \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \dots \rightarrow \mathbb{N} \times \mathbb{D}(-\infty, \infty) \times \mathbb{D}(-\infty, \infty) \times \dots$  that maps  $\zeta_T^k$  into  $\eta_T^k$  is continuous and hence  $\eta_T^k \xrightarrow{d} f(\zeta^k) = \eta^k$ , say. It is easily seen that the distribution of  $\eta^k$  is obtained from (5.1) by replacing  $a(\lambda)$  with  $a_k(\lambda) = \sum_{i=-k2^k}^{k2^k-1} a_{ki}^\alpha I(i2^{-k} < \lambda \leq (i+1)2^{-k})$ . Thus  $\eta^k = (N^k, Y_1^k, Y_2^k, \dots)$  has independent components, and if  $\eta = (N, Y_1, Y_2, \dots)$  has the distribution given by (5.1) then  $N^k$  has the same distribution as

$N$  and  $Y_i^k \xrightarrow{d} Y_i$ , since  $\sup_{\lambda \in \mathbb{R}} |a(\lambda) - a_k(\lambda)| \rightarrow 0$  by (4.5). By Lemma 2.1 it follows that  $\eta^k \xrightarrow{d} \eta$  as  $n \rightarrow \infty$ .

The appropriate centering for the  $i$ 'th mark,  $\tau_{Ti}$ , of  $\eta_T$  is the time of the  $i$ 'th jump larger than  $T^{1/\alpha}x/A$  in the process  $\{z(t); t \geq 0\}$ . To show this we first prove

$$(5.4) \quad \lim_{T \rightarrow \infty} P(|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}) = 0.$$

To do this, let  $\epsilon \in (0, x/(2A))$  and write  $Z(t) = Z^1(t) + Z^2(t) + Z^3(t) + Z^4(t)$ , where  $Z^1(t)$  is the sum of jumps by  $Z(t)$  in  $[0, t]$  of size larger than  $T^{1/\alpha}(x/A + \epsilon)$ , where  $Z^2(t)$  is the sum of jumps of size belonging to  $T^{1/\alpha}(x/A - \epsilon, x/A + \epsilon]$ , and where  $Z^3(t)$  is the sum of jumps of size belonging to  $T^{1/\alpha}(\epsilon/n, x/A - \epsilon]$ , ( $n > 0$ ). Further, for  $\ell = 1, 2, 3$ , let  $E^\ell$  be the point process which has its atoms at the times of jumps of  $Z^\ell(t)$ . We recall that, putting  $Z_j = Z(j2^{-k}) - Z((j-1)2^{-k})$ ,  $\tau_{Ti}^k$  is equal to  $2^{-k}$  times the location of the  $i$ 'th exceedance of  $T^{1/\alpha}x/A$  by the sequence  $\{Z_j\}_{j=1}^\infty$ . Put  $Z_j^\ell = Z^\ell(j2^{-k}) - Z^\ell((j-1)2^{-k})$  so that  $Z_j = \sum_{\ell=1}^4 Z_j^\ell$ , let  $N$  be a positive number and denote the event that  $E^1([0, TN]) < i$  by  $A_T$ , the event that  $E^2([0, TN]) > 0$  by  $B_T$ , the event that  $E^2(2^{-k}(j-1, j]) > 1$ , for some  $j \in [1, 2^k NT]$ , by  $C_T$ , and the event that  $E^3(2^{-k}(j-1, j]) > 1$ , for some  $j \in [1, 2^k NT]$ , by  $D_T$ . If  $|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}$  and  $A_T^* \cap B_T^*$  occurs, then at least one of the following three events must happen: either  $Z_j > T^{1/\alpha}x/A$  and  $E^1(2^{-k}(j-1, j]) = 0$ , for some  $j \in [1, 2^k NT]$ , or  $Z_j \leq T^{1/\alpha}x/A$  and  $E^1(2^{-k}(j-1, j]) > 0$ , for some  $j \in [1, 2^k NT]$ , or else  $C_T$  occurs. Moreover, if  $B_T^* \cap C_T^* \cap D_T^*$  occurs, then the first two events both imply that  $E_T = \{|Z_j^4| > T^{1/\alpha}\epsilon, \text{ some } j \in [1, 2^k NT]\}$  happens. Thus we have proved

$$\{|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}\} \subset A_T \cup B_T \cup C_T \cup D_T \cup E_T.$$

Let  $\mu_1 = c_\alpha \alpha^{-1} (x/A+\epsilon)^{-\alpha}$ ,  $\mu_2 = c_\alpha \alpha^{-1} \{(x/A-\epsilon)^{-\alpha} - (x/A+\epsilon)^{-\alpha}\}$  and  $\mu_3 = c_\alpha \alpha^{-1} \{(\epsilon/n)^{-\alpha} - (x/A-\epsilon)^{-\alpha}\}$ . Then, for  $\ell=1,2,3$ ,  $E^\ell$  is a Poisson process with intensity  $\mu_\ell/T$  (see e.g. Breiman (1968)) and thus

$$P(A_T) = P\{E^1([0, TN]) < i\} = \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!}$$

and

$$P(B_T) = P\{E^2([0, TN]) \geq 1\} = 1 - e^{-N\mu_2}.$$

Furthermore,

$$\begin{aligned} P(C_T) &\leq 2^k \text{NTP}\{E^1([0, 2^{-k}]) > 1\} \\ &= 2^k \text{NT} \left[ 1 - e^{-2^{-k} \mu_1 T^{-1}} - 2^{-k} \mu_1 T^{-1} e^{-2^{-k} \mu_1 T^{-1}} \right] \\ &\rightarrow 0, \end{aligned}$$

as  $T \rightarrow \infty$ , and similarly  $P(D_T) \rightarrow 0$ . Further, by differentiating the Lévy representation of the characteristic function of  $Z_1^4$  (c.f. the proof of Lemma 4.1) it is seen that  $E\{(Z_1^4)^2\} \leq K 2^{-k} (\epsilon/n)^{2-\alpha} T^{2/\alpha-1}$ , for some constant  $K$ , and then Chebychev's inequality gives

$$\begin{aligned} P(E_T) &\leq 2^k \text{NTP}\{|z_1^4| > T^{1/\alpha} \epsilon\} \\ &\leq 2^k \text{NTE}\{|z_1^4|^2\} T^{-2/\alpha} \epsilon^{-2} \\ &\leq K N \epsilon^{-\alpha} n^{2-\alpha}, \end{aligned}$$

as  $T \rightarrow \infty$ . Hence

$$\limsup_{T \rightarrow \infty} P(|\tau_{T_i} - \tau_{T_i}^k| > 2^{-k}) \leq \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!} + 1 - e^{-N\mu_2} + K N \epsilon^{-\alpha} n^{2-\alpha},$$

and inserting the values of  $\mu_1$  and  $\mu_2$  and letting first  $n \rightarrow \infty$ , then  $\epsilon \rightarrow 0$ , and then  $T \rightarrow \infty$ , this proves (5.4).

Now we are in a position to show that  $\{\tau_{T_i} - \tau_{T_i} : T \geq 1\}$  is tight, or equivalently to prove

$$(5.5) \quad \lim_{y \rightarrow \infty} \limsup_{T \rightarrow \infty} P(|t_{Ti} - \tau_{Ti}| > y) = 0.$$

Let  $\{t_{Ti}^k\}$  be the  $h(T)$ -separated upcrossings of  $T^{1/\alpha}(x-A\epsilon)$  by  $\bar{X}_k(t)$  and let  $\{\tilde{t}_{Ti}^k\}$  be the  $h(T)$ -separated upcrossings of  $T^{1/\alpha}(x+A\epsilon)$  by  $\bar{X}_k(t)$ .

Further, let  $\mathcal{I}_{Ti}^k$  be the location of the  $i$ 'th exceedance of  $T^{1/\alpha}(x/A-\epsilon)$  and  $\tilde{\tau}_{Ti}^k$  the location of the  $i$ 'th exceedance of  $T^{1/\alpha}(x/A+\epsilon)$  by the discrete process  $\{Z(j2^{-k}) - Z((j-1)2^{-k})\}_{j=1}^{\infty}$ , write  $F_T$  for the event that

$\sup_{0 \leq t \leq NT} |X(t) - X_k(t)| > T^{1/\alpha}A\epsilon$  and, changing the notation slightly, let  $A_T = \{\tilde{t}_{Ti}^k < NT - y\}$ . On the event  $A_T^* \cap F_T^*$  we have  $t_{Ti}^k \leq \tau_{Ti}^k \leq \tilde{t}_{Ti}^k$  and thus  $\{|t_{Ti} - \tau_{Ti}| > y\} \subset \{\tilde{t}_{Ti}^k - \tau_{Ti}^k > y\} \cup \{\tilde{\tau}_{Ti}^k - t_{Ti}^k > y\} \cup A_T \cup F_T$ . Let  $G_T$  be the event that  $z_j \in T^{1/\alpha}(x/A-\epsilon, x/A+\epsilon]$  for some  $j \in [1, 2^k NT]$ . If  $A_T^* \cap G_T^* \cap \{|\tilde{\tau}_{Ti}^k - \tau_{Ti}^k| \leq y\}$  occurs, then  $\mathcal{I}_{Ti}^k = \tau_{Ti}^k = \tilde{\tau}_{Ti}^k$  and thus

$$\begin{aligned} \{|t_{Ti} - \tau_{Ti}| > y\} &\subset \{\tilde{t}_{Ti}^k - \tau_{Ti}^k > y - 2^{-k}\} \cup \{\tau_{Ti}^k - t_{Ti}^k > y - 2^{-k}\} \cup \{|\tau_{Ti}^k - \tilde{\tau}_{Ti}^k| > 2^{-k}\} \cup A_T \cup F_T \\ &\subset \{\tilde{\tau}_{Ti}^k - \tilde{t}_{Ti}^k > y - 2^{-k}\} \cup \{\mathcal{I}_{Ti}^k - t_{Ti}^k > y - 2^{-k}\} \cup \{|\tau_{Ti}^k - \tilde{\tau}_{Ti}^k| > 2^{-k}\} \cup A_T \cup F_T \\ &\quad \cup G_T. \end{aligned}$$

Here  $P(A_T) \rightarrow \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!}$ ,  $P(|\tau_{Ti}^k - \tilde{\tau}_{Ti}^k| > 2^{-k}) \rightarrow 0$  as  $T \rightarrow \infty$ , and  $P(G_T) \leq 2^k NT P\{z_1 \in T^{1/\alpha}(x/A-\epsilon, x/A+\epsilon]\} \sim 2 N c_{\alpha} \{(x/A-\epsilon)^{-\alpha} - (x/A+\epsilon)^{-\alpha}\}$ . Hence

$$\begin{aligned} \lim_{y \rightarrow \infty} \limsup_{T \rightarrow \infty} P(|t_{Ti} - \tau_{Ti}| > y) &\leq \lim_{y \rightarrow \infty} \limsup_{T \rightarrow \infty} \{P(\tilde{t}_{Ti}^k - \tau_{Ti}^k > y - 2^{-k}) + P(\mathcal{I}_{Ti}^k - t_{Ti}^k > y - 2^{-k})\} \\ &\quad + \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!} + \lim_{y \rightarrow \infty} \limsup_{T \rightarrow \infty} P(F_T) \\ &\quad + 2 N c_{\alpha} \{(x/A-\epsilon)^{-\alpha} - (x/A+\epsilon)^{-\alpha}\}, \end{aligned}$$

and the first term is zero by Lemma 4.1, the third term tends to zero as  $k \rightarrow \infty$  by (5.3) and the remaining two terms tend to zero as first  $\epsilon \rightarrow 0$  and then  $N \rightarrow \infty$ , and thus (5.5) follows.

To complete the proof that  $\eta_T \xrightarrow{d} \eta$  it is enough to show that the conditions of Lemmas 2.3 and 2.4 are satisfied. However, the atoms of  $N_T$  are  $t_{T1}/T, t_{T2}/T, \dots$  and the atoms of  $N_T^k$  are  $t_{T1}^k/T, t_{T2}^k/T, \dots$ , and  $(t_{Ti}^k - t_{Ti}^k)/T \xrightarrow{P} 0$  follows from (5.4) and the facts that  $\{t_{Ti} - \tau_{Ti}; T \geq 1\}$  and  $\{t_{Ti}^k - \tau_{Ti}^k; T \geq 1\}$  are tight, so the hypothesis of Lemma 2.4 is satisfied. Further, let  $\epsilon_T^k = \tau_{Ti}^k - t_{Ti}^k$ . Then, by (5.4),  $\lim_{k \rightarrow \infty} \limsup_{T \rightarrow \infty} P(|\epsilon_T^k| > x) = 0$  for  $x > 0$ . Since  $Y_{Ti}^k(t) = \frac{X_k(t + \tau_{Ti}^k)}{T^{1/\alpha}}$  we have  $\{ \sup_{-l \leq t \leq l} |Y_{Ti}^k(t) - Y_{Ti}^k(t + \epsilon_T^k)| > x \} = \{ \sup_{-l \leq t \leq l} |X(t + \tau_{Ti}^k) - X_k(t + \tau_{Ti}^k)| > T^{1/\alpha} x \} \subset \{ \sup_{-l \leq t \leq NT+l} |X(t) - X_k(t)| > T^{1/\alpha} x \} \cup A_T$  and from (5.3) it then follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \limsup_{T \rightarrow \infty} P\left( \sup_{-l \leq t \leq l} |Y_{Ti}^k(t) - Y_{Ti}^k(t + \epsilon_{Tk}^k)| > x \right) &\leq \lim_{k \rightarrow \infty} \limsup_{T \rightarrow \infty} P(A_T) \\ &= \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!}, \end{aligned}$$

and since  $N$  is arbitrary it follows that (2.3) holds. Further,

$\lim_{i \rightarrow \infty} \limsup_{T \rightarrow \infty} P\left( \sup_{-l < t < l} |Y_{ni}(t)| > u \right) = 0$  follows easily from Lemma 4.2, and thus the hypothesis of Lemma 2.3 is satisfied. This concludes the proof of the lemma for the case  $0 < \alpha < 1$ .

If instead  $1 < \alpha < 2$  we have to use the second parts of Lemma 4.1 and 4.2 instead of the first ones to prove (5.3), but apart from that, the lemma follows in precisely the same way as above also in this case.  $\square$

Since the restriction that  $Z(\lambda)$  is completely asymmetric can be removed in precisely the same way as Theorem 3.2 is obtained from Lemma 3.1, we omit the details of this proof and only state the result.

Recall the notation  $A = \sup_{\lambda \in \mathbb{R}} a(\lambda)$ , put  $a = \sup_{\lambda \in \mathbb{R}} a^-(\lambda)$ , let  $\mu' = c_\alpha A^\alpha (1+\beta)x^{-\alpha}$ ,  $\mu'' = c_\alpha a^\alpha (1-\beta)x^{-\alpha}$ , let  $Z'$  and  $Z''$  have the distributions

given on p. 27 and let  $Y_i(t) = Z^+ a(-t)$ ,  $t \in R$ , with probability  $\mu' / (\mu' + \mu'')$  and  $Y_i(t) = -Z^- a(-t)$ ,  $t \in R$ , otherwise. Then the limiting distribution of  $\eta_T$  is that of the marked point process

$$(5.6) \quad (N, Y_1, Y_2, \dots) \text{ where the components are independent, } N \text{ is a Poisson process with intensity } \mu = \mu_1 + \mu_2 \text{ and the } Y_i \text{'s have the distribution given above.}$$

THEOREM 5.2. Let  $a(\lambda)$  satisfy (4.1) and let both  $a^+(\lambda)$  and  $a^-(\lambda)$  satisfy either (4.5) or else (4.6). Further let  $\{Z(\lambda); \lambda \in R\}$  have stationary independent increments with  $Z(0)=0$  and  $Z(1)$  stable  $(1, \alpha, \beta)$  and let  $X(t) = \int a(\lambda-t) dZ(\lambda)$ . Then there exist  $\{\tau_{Ti}; i=1, 2, \dots, T \geq 1\}$  with  $\{t_{Ti} - \tau_{Ti}; T \geq 1\}$  tight for each  $i \geq 1$  such that  $\eta_T \xrightarrow{d} \eta$  as  $T \rightarrow \infty$ , where  $\eta_T$  is the marked point process of separated exceedances of  $T^{1/\alpha} x$  by  $\{X(t); t \geq 0\}$  and where the distribution of  $\eta$  is given by (5.6).

Similarly as for the discrete time case, various corollaries concerning the behavior of extremes can be deduced from Theorem 5.2. Here we only give the very simplest result, concerning  $M_T = \sup_{0 \leq t \leq T} X(t)$ .

COROLLARY 5.3. Suppose that  $X(t)$  satisfies the hypothesis of Theorem 5.2. Then

$$P(M_T/T^{1/\alpha} \leq x) \rightarrow \exp\{c_\alpha [A^\alpha(1+\beta) + a^\alpha(1-\beta)] x^{-\alpha}\}$$

as  $T \rightarrow \infty$ .

We have not treated the case  $\alpha=1$  above. However, using methods rather similar to those for  $1 < \alpha < 2$ , conditions for the result of Theorem 5.2 to hold can be obtained also for  $\alpha=1$ .

Finally we note that it is easy to see that all of the limit theorems of this paper are mixing in the sense of Rényi (the first result in this direction is proved in [11]). Hence they can be extended to cases where the level is random, and possibly depending on the process  $X(t)$ .

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#### REFERENCES

- [1] Bauer, H.: *Probability Theory*. Berlin, De Gruyter 1972.
- [2] Bergström, H.: Theorie der stabilen Verteilungsfunktionen. *Archiv der Mathematik*, 4, 380-391 (1953).
- [3] Billingsley, P.: *Convergence of Probability Measures*. New York, Wiley 1968.
- [4] Brieman, L.: *Probability*. Reading, Mass., Addison-Wesley 1968.
- [5] Feller, W.: *An Introduction to Probability Theory and Its Applications*, Vol. II. New York, Wiley 1971.
- [6] Kallenberg, O.: *Random Measures*. Berlin, Akademische Verlag 1975.
- [7] Leadbetter, M.R.: Weak convergence of high level exceedances by a stationary sequence. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 34, 11-15 (1976).
- [8] Leadbetter, M.R., Lindgren, G., de Maré, J., Rootzén, H.: Lectures on extreme value theory. To appear.
- [9] Lévy, P.: *Theorie de l'addition des Variables Aleatoires*. Paris, Gauthier-Villars, second edition 1954.
- [10] Lindvall, T.: Weak convergence of probability measures and random functions in the function space  $D[0, \infty)$ . *J. Appl. Prob.* 10, 109-121 (1973).
- [11] Rootzén, H.: Some properties of convergence in distribution of sums and maxima of dependent random variables. Report 1974:7, Dept. of Math. Statist., Univ. of Lund.