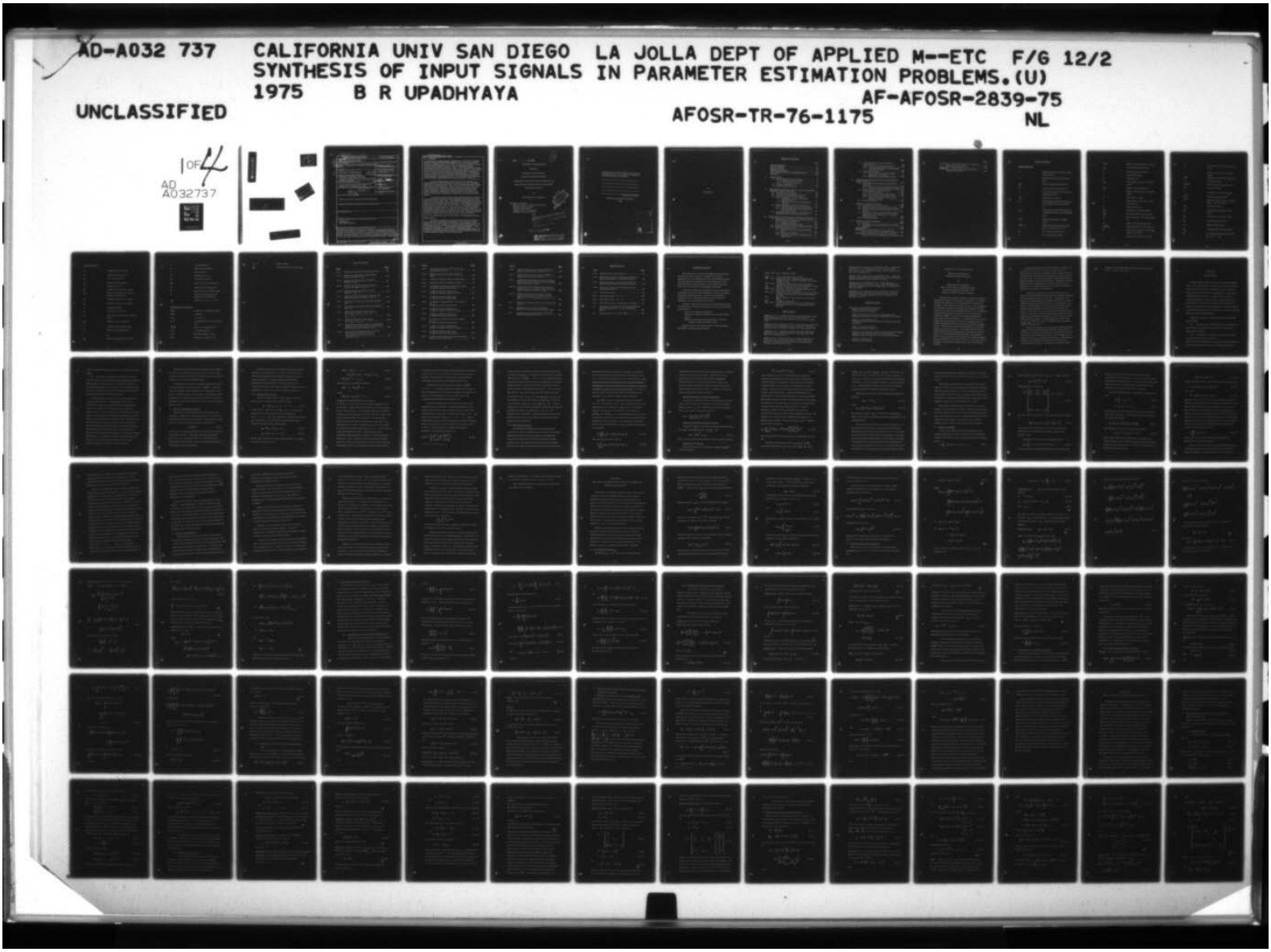
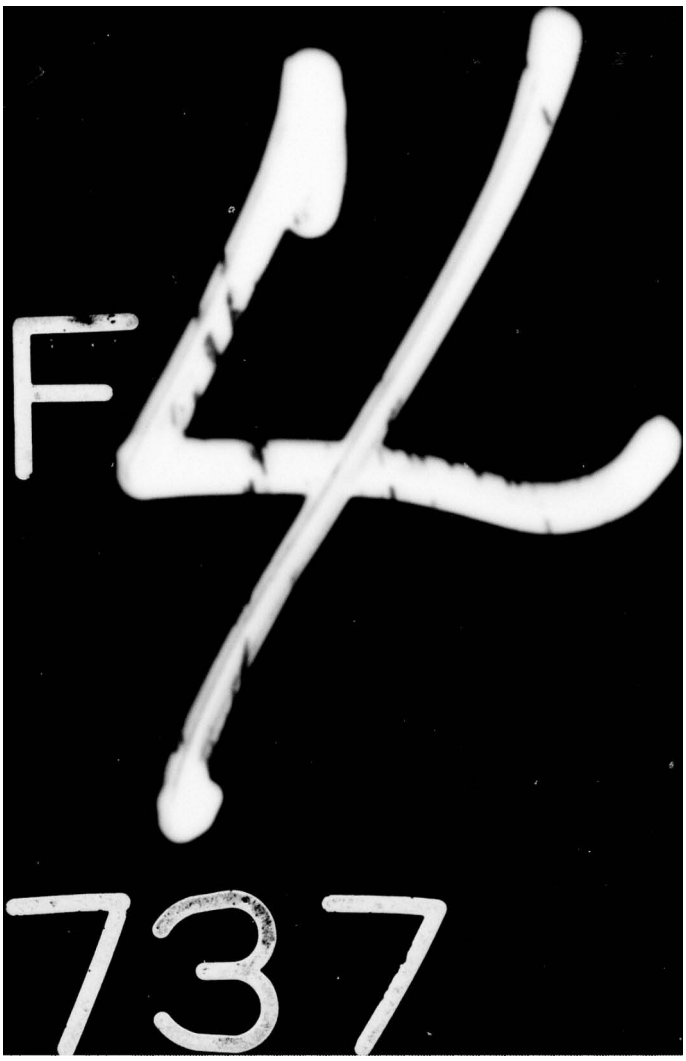
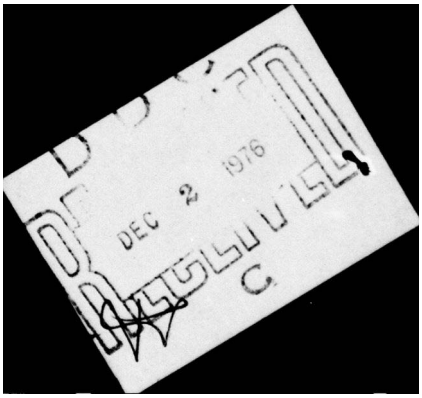
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is represented in a Baysian framework, where the a posteriori probability $P(\underline{\theta} \mid \underline{Z}^N)$ contains the information supplied by the observations about the parameters. The optimality criterion for the synthesis problem is defined in terms of the statistical distance measure between distributions characterized by distinct parameter values. In this dissertation we define this quantity in terms of the pairwise Bhattacharyya distance. The distance measure has the desirable property of ordering the Bayes' probability of error as a function of two experimental design programs. The statistical distance measures have been successfully used in studying taxonomy and characterization of racial distributions. It is shown that such an approach is also feasible in applications to more general dynamic situation.

Several useful properties of the B-distance are derived. Its relationship to the Fisher information is obtained, where the differential metric has an interpretation in terms of the sensitivity of the observations with respect to the parameters. A lower bound on the mean square error is derived using the relationship between the distance measure and the mutual information. Application to dynamic systems is made via the adaptive filtering formulation where the parameters have a finite set of discrete values and the estimation is performed in a parallel processing scheme.

An alternative approach to the input synthesis problem is obtained by generating these signals from linear stochastic processes of the auto-regressive moving average type. This approach to the selection of random inputs gives the additional degrees of freedom of choosing the parameters of the input process, maximizing the cost function. The asymptotic property of the input correlation matrix is studied using the asymptotic eigenvalue distribution of Toeplitz matrices. When the variance of the white noise generating the linear inputs is constrained, it is shown that among the class of all linear inputs that input for which the optimality criterion is maximum also has the largest value for the intergral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{uu}(\lambda)\, d\lambda$ where $f_{uu}(\lambda) = S_{uu}(\lambda)/\sigma_\nu^2$ with $S_{uu}(\lambda)$ the spectrum of corresponding stationary linear process.

The effect of feedback on the input sequence has been studied in terms of the open-loop feedback formulation. At each stage k, of the dynamic process, future inputs are selected based on the past and present observation program, and the first component of this subsequence, $\underline{U}_{k,N'}$ is selected as the input u_k. The feedback is introduced when an additional observation is incorporated in computing the future inputs. It is shown here, that the feedback at stage k, can be represented as the sum of open-loop input at stage k and a correction term.

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UNIVERSITY OF CALIFORNIA

San Diego

Synthesis of Input Signals in Parameter Estimation Problems

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering Sciences

by

Belle Raghavendra Upadhyaya

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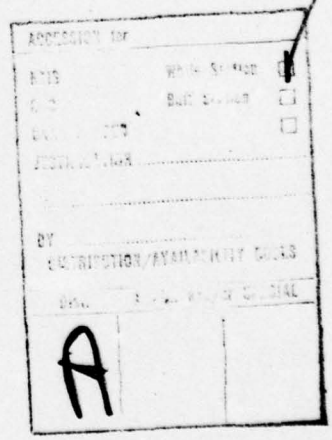
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Committee Chairman

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1975



To

My Parents

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LIST OF SYMBOLS

ROMAN SYMBOLS

A	Matrix defining autoregressive inputs; system matrix
B	Matrix defining steady state Kalman gain
$B_{\alpha\gamma}$	Pairwise Bhattacharyya distance
$\bar{B}$	Average Bhattacharyya distance
$\underline{b}$	Vector defining the linear term of the optimality criterion
Det	Determinant of a matrix
$\underline{e}_i$	N-dimensional normalized eigenvector corresponding to the ith eigenvalue λ_i
$\underline{e}_m$	N-dimensional normalized eigenvector corresponding to the largest eigenvalue λ_{max}
$f(x)$	Function generating the Toeplitz matrix
$G(\underline{m}, \Sigma)$	Multivariate Gaussian density with mean $\underline{m}$ and covariance Σ
H	$s \times n$ measurement matrix
$\underline{h}$	$n \times 1$ measurement matrix

$I(x;y)$	Mutual information between x and y
$J_{\alpha\gamma}$	Pairwise divergence measure
$\bar{J}$	Average divergence measure
k	Discrete time index
L	Likelihood ratio
M_{ij}	i-jth element of Fisher information matrix
$M_{\alpha\gamma}$	Pairwise Matusita distance
m	Number of discrete sets of parameter values
N	Total number of stages
n	System order in single-input single-output dynamics
O	Observability matrix
OLF	Acronym for open-loop feedback
$P(\underline{\theta})$	Prior distribution of parameter $\underline{\theta}$
P_i	Prior probability of the ith parameter set
p	Order of autoregressive process
$p(\underline{Z}^N \mid \underline{\theta})$	Conditional density of $\underline{Z}^N$ given $\underline{\theta}$
Q	Plant noise covariance matrix of $\underline{w}_k$; quadratic cost matrix
q	Order of moving average process

R	Measurement noise covariance matrix of $\underline{v}_k$
r	Order of input gain vector in SISO system
SISO	Acronym for single-input single-output
T	Transpose operator
$T[f(x)]$	Toeplitz matrix generated by $f(x)$
Tr	Trace of a matrix
t	Parameter in the function $\rho_{\alpha\gamma}(t)$
$\underline{U}^{N-1}$	N-dimensional input vector
$\underline{U}^{k,N}$	Open-loop feedback inputs computed at stage k for (N-k) stages
$u_{k\|k}$	OLF input at stage k
$\underline{v}_k$	s-dimensional measurement noise vector
$\underline{w}_k$	n-dimensional plant noise vector
$\underline{x}_k$	n-dimensional state vector
y_k	Output of SISO system
$\underline{z}_k$	s-dimensional measurement vector
$\underline{Z}^K$	s k dimensional measurement vector $(\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_k)$

GREEK SYMBOLS

α_i	Autoregressive parameters
$\underline{\beta}$	n-dimensional gain vector
γ_k	k th input correlation
ϵ_k	Sensitivity index at stage k
Θ	Compact set in R^p
$\underline{\theta}$	p-dimensional system parameters
Λ_N	Matrix representing the correlation function of ARMA inputs
Λ_1	Correlation matrix of AR inputs
Λ_2	Correlation matrix of MA inputs
λ	Lagrange multiplier
λ_i	ith eigenvalue of a matrix
ν_k	White noise generating linear inputs at stage k
$\underline{\nu}_k$	Innovations process at stage k
Ξ	Matrix of moving average inputs
ξ_i	ith moving average parameter
π	Pi
$\rho_{\alpha\gamma}$	Pairwise Bhattacharyya coefficient

Σ	Covariance matrix
Y	Total energy constraint
υ	Input constraint
Φ	System matrix, $n \times n$
φ_i	ith system parameter
$\hat{\underline{X}}_0$	Mean value vector of initial state
Ω	Sample space of observations
$\mathcal{B}$	Input gain matrix of SISO system
θ	A priori information about the parameters and initial state
$\mathcal{I}_{Nk}$	Index set $\{k, k+1, \ldots, N-1\}$

MATHEMATICAL SYMBOLS

$A \underline{\Delta} B$	A equals B by definition; denotes
$A \rightarrow B$	A implies B
$A \supset B$	A contains B
$A \subset B$	A is contained in B ; A is a subset of B
$\{x \mid P\}$	Set of x's having property P
$x \in A$	x is an element of A
(a, b)	Open interval $\{t \mid a < t < b\}$
$[a, b]$	Closed interval $\{t \mid a \leq t \leq b\}$

$\langle \cdot, \cdot \rangle$	Scalar product
$\boxed{X}$	End of definition, theorem, proof

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Synthesis of Input Signals in
Parameter Estimation Problems

by

Belle Raghavendra Upadhyaya
Doctor of Philosophy in Engineering Sciences
University of California, San Diego, 1975
Professor Harold W. Sorenson, Chairman

Design of input signals to enhance the estimation of unknown parameters in discrete time dynamics is considered. The system identification problem can be considered as the initial phase of a stochastic control problem of a dynamic system. In such a situation it is desirable to estimate the parameters as rapidly as possible without disturbing the normal operation of the process. The problem is represented in a Baysian framework where the a posteriori probability $P(\underline{\theta}|\underline{Z}^N)$ contains the information supplied by the observations about the parameters. The optimality criterion for the synthesis problem is defined in terms of the statistical distance measure between distributions characterized by distinct parameter values. In this dissertation we define this quantity in terms of the pairwise Bhattacharyya distance. The distance measure has the desirable property of ordering the Bayes' probability of error as a function of two experimental design programs. The statistical distance measures have been successfully used in studying taxonomy and characterization of racial distributions. It is shown that such an approach is also feasible in applications to more general dynamic situation.

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CHAPTER I

INTRODUCTION

A large number of physical systems require the determination of their structural parameters. Some of these require the estimation of the parameters during normal operations while for the others a separate test can be carried out to accomplish this. A wide range of systems belong to this category - chemical industry processes, aerospace applications, nuclear reactors, biological systems, etc.. These systems are identified by exciting them with suitable input signals and observing the resulting response of the system. The inputs can be deliberately chosen in many instances so as to extract maximum information from the physical system.

The type of response of a system may belong to the following three categories:

(i) If the input U is merely observed (not controlled in any way), what kind of information is conveyed by the observations Z under normal operation?

(ii) If the input U is changed in some specific way what change is induced in the behavior of Z ?

For the first case we need to observe the data without interfering with the normal operation of the system. The second question can be

answered by deliberately changing the input by a proper experimental design.

(iii) A third class of problems is that of driving a system from an initial state to a designated target and the system requires the determination of unknown parameters. This combined problem of control and identification is called "dual control" (see Fel'dbaum [26], pp. 319-330). Here the control action is required to divide the total energy optimally between estimation and control.

In this dissertation we address ourselves to the solution of the question posed by (ii) above. Even though no control aspect is included in the analysis, the system identification problem can be treated as an initial phase of a stochastic control problem for a dynamic system. In such a situation it is desirable that the parameters be determined as rapidly as possible without disturbing the system so as to cause a deterioration of its normal operation. This phase can be used to identify the parameters up to a desirable accuracy and the complete learning will eventually take place during the control phase. This separation of the identification and control differs from the definition of dual control where the identification and control take place instantaneously. The solution to a dual control problem can be very expensive and since the control policy depends heavily on the parameter learning, the total error could be very large.

The necessity of a fast and efficient identification phase requires the selection of an appropriate input sequence where the total input energy is constrained and such that the optimally criterion uses all the available a priori information about the system.

The problem is represented in a Bayesian framework where the a posteriori distribution of the parameters, $P(\underline{\theta}|\underline{Z}^k)$, contains the information supplied by the observations about the parameters, and there is a sufficient statistic for the problem. In the following sections the approach to the input synthesis problem is outlined, showing how the application of discriminant functions of statistics can be extended to dynamic systems.

1.1 The Input Signal Selection Problem

The problem of experimental design was considered by many statisticians in application to regression problems. Since there is no controllable objective in general, application to input problem has been very few. The statistical regression model is described by:

$$y = \underline{f}(\xi)^T \underline{\theta} + v \qquad (1.1\text{-}1)$$

where $\underline{\theta}$ are the regression parameters and $\underline{f}(\xi)$ is a function of the experimental variable ξ. The major studies have been those of Kiefer [46], Kiefer and Wolfowitz [47], Elfving [24], Karlin and Studden [44] and Fedorov [25]. These are primarily directed towards analyzing linear regression problems.

In communication systems the optimum signals are those that minimize the probability of error. Generally referred to as the sphere packing problem they are solved by geometrical means by several investigators - Landau and Slepian [51], Stutt [74], Balakrishnan [7] and Weber [82]. In the present work we are concerned with design of input signals as applied to dynamic systems represented by a state space model or an input-output model.

1.1.1 Statement of the Problem

Consider a sequence of observations $\{\underline{z}_k\}$ of a stochastic process which is parameterized by $\underline{\theta} \in R^q$ such that

$$\underline{z}_k = \underline{y}_k(\underline{\theta}, u_k, k) + \underline{v}_k, \quad k = 1, 2, \ldots \qquad (1.1\text{-}2)$$

Let $\underline{\theta} \in \Theta \subset R^q$ where Θ is a compact set in R. The unknown parameters $\underline{\theta}$ are to be estimated with the knowledge of $\{\underline{z}_k, k = 1, 2, \ldots, N\}$. More specifically, consider that the signal $\underline{y}_k = H\underline{x}_k$ where $\underline{x}_k$ is the output of a linear dynamic system. Consider the following system representation:

$$\underline{x}_k = \Phi \underline{x}_{k-1} + \underline{\beta} u_{k-1} + \underline{w}_{k-1} \qquad (1.1\text{-}3)$$

$$\underline{z}_k = H\underline{x}_k + \underline{v}_k, \quad k = 1, 2, \ldots \qquad (1.1\text{-}4)$$

$\Phi(nxn)$, $\beta(nx1)$ and $H(sxn)$ are time invariant matrices. $\underline{w}_k$ and $\underline{v}_k$ are white noise sequences with

$$E[\underline{w}_k] = E[\underline{v}_k] = 0 \ , \ \forall \ k \qquad (1.1\text{-}5)$$

$$E[\underline{w}_j \underline{w}_k^T] = Q \delta_{jk} \ , \ E[\underline{v}_j \underline{v}_k^J] = R \delta_{jk} \ .$$

$$E[\underline{w}_j \underline{v}_k^T] = 0 \ , \ \forall_{j,k} \ . \qquad (1.1\text{-}6)$$

Initial state $\underline{x}_o$ is a random variable with

$$E[\underline{x}_o] = \underline{0} \ , \ E[\underline{x}_o \underline{x}_o^T] = P_o \ . \qquad (1.1\text{-}7)$$

Further

$$E[\underline{x}_o \underline{w}_k^T] = E[\underline{x}_o \underline{v}_k^T] = 0 \ , \ \forall_k \ . \qquad (1.1\text{-}8)$$

In general the structural parameters are taken as $\underline{\theta} = (\Phi, \underline{\beta}, H, Q, R)$. We assume that the noise statistics Q and R are known. Further the measurement matrix H is also assumed to have a known structure. The unknown parameters are taken to be the dynamic system parameters Φ and $\underline{\beta}$. Thus $\underline{\theta} = (\Phi, \underline{\beta})$. With a prior distribution $P(\underline{\theta})$ let the system be in the canonical form such that the pair $(\Phi, \underline{\beta})$ has a maximum of 2n parameters. Let the eigenvalues of matrix Φ be within the unit circle (Φ is stable). It is required to choose the inputs $\{u_k, k = 0, 1, \ldots N-1\}$ such that a prescribed optionality criterion representing the learning performance of the parameters is maximized. The presence of the exciting force $\{u_k\}$ is necessary so that the observation program retains the maximum information about the stable dynamic matrix Φ. In the absence of this, as time progresses the information about the

Φ matrix in the observations will be reduced and the observations will be dominated by the noise variable, thus diminishing the parameter learning. We require the system to be stable, controllable and observable; these conditions are discussed in Section (3-1).

We pointed out that the a posteriori distribution $P(\underline{\theta}|\underline{Z}^k)$, $k = 1, 2, \ldots$, is a sufficient statistic for the parameter learning. A direct maximization of this quantity is not feasible. Consequently, we look for a criterion which has an ordering with respect to the a posteriori probability generated by Bayes' rule.

It is also necessary that the cost function use all the a priori information about the parameters. An alternate approach is to use adaptive filtering theory. In practical filters the parameters $\underline{\theta}$ are restricted to have a finite number of known values $\underline{\theta}_i$, $i = 1, 2, \ldots, m$, with a priori distribution $P(\underline{\theta}_i) = P_i$, $i = 1, 2, \ldots, m$. Then the adaptive filter consists of m filters operating in parallel such that the output of the processor is a linear combination of the output of these m filters. The learning device calculates the a posteriori probabilities recursively using Bayes' rule

$$P(\underline{\theta}_i|\underline{Z}^k) = \frac{p(\underline{z}_k|\underline{Z}^{k-1}, \underline{\theta}_i)\, P(\underline{\theta}_i|\underline{Z}^{k-1})}{\sum_{j=1}^{m} p(\underline{z}_k|\underline{Z}^{k-1}, \underline{\theta}_j)\, P(\underline{\theta}_j|} \qquad (1.1\text{-}8)$$

$i = 1, 2, \ldots, m$, $k = 1, 2, \ldots$.

In the framework of hypotheses testing the m discretizations can be considered as constituting m-hypotheses with which one can associate the joint densities $p(\underline{Z}^N|_i\underline{\theta})$, $i = 1, 2, \ldots, m$. A suitable criterion of distinguishing these distributions can be formulated as a measure of distance between the pairwise distributions. We would like to have for these distance measures the following property. For a given pair $(\underline{\theta}_\alpha , \underline{\theta}_\gamma) \epsilon \Theta$ if $d_{\alpha\gamma}(U_\eta)$ and $d_{\alpha\gamma}(U_\zeta)$ are the distance measures for two input designs U_η and U_ζ , and if $d_{\alpha\gamma}(U_\eta) > d_{\alpha\gamma}(U_\zeta)$ then the probability of errors (in the sense of minimum Bayes' risk) will have the ordering $P_e(\underline{\theta}_\alpha , \underline{\theta}_\gamma , U_\eta) < P_e(\underline{\theta}_\alpha , \underline{\theta}_\gamma , U_\zeta)$. By Bayes' minimum risk criterion when the hypothesis with the greatest posterior probability is selected, the probability of error P_e is also minimized (Van, Trees [81]). In the following a brief history of the application of these measures in statistics and the extension to the present context are discussed.

1.1.2 The Proposed Approach

In 1967 Gagliardi [29] proposed that by considering the parameters as members of a finite set, the criterion for the optimal input can be defined as that which maximizes the probability of correctly determining the true parameter value from a multiple hypotheses test. He showed that this probability is maximized by those input sets whose corresponding output has the largest perimeter

when plotted in the output vector space. However, in many cases either directly maximizing the probability of identifying the correct set, or minimizing the probability of error is often impossible. This may be because an analytical expression for the error probability is too difficult to find, or even if an expression can be derived, it may be too difficult for analytical or numerical optimization. Therefore, we look for a signal selection criterion that is weaker than the error probability, but is related to it and is easier to implement.

Let $P(\underline{\theta})$ be the prior probability distribution of $\underline{\theta} \in \Theta \subset R$. Where Θ is a compact set in R^q. Then a suitable criterion can be expressed in terms of the distance $d(\underline{\theta}_\alpha, \underline{\theta}_\gamma)$ in Θ. In binary hypotheses testing these are the distributions of observations $p(\underline{Z}^N|\underline{\theta}_\alpha)$ and $p(\underline{Z}^N|\underline{\theta}_\gamma)$. By making this distance large, we would be able to distinguish between the two distributions, making the error probability as small as possible. In an adaptive filtering problem where we have multiple hypotheses we can define the average distance as the criterion.

$$\bar{d} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} P(\underline{\theta} = {}_i\underline{\theta})\, P(\underline{\theta} = {}_j\underline{\theta})\, d({}_i\underline{\theta}, {}_j\underline{\theta}) \qquad (1.1\text{-}9)$$

The continuous analog of the above is:

$$\bar{d} = \int_{\Theta \chi \Theta} d(\underline{\theta}_\alpha, \underline{\theta}_\gamma)\, p(\underline{\theta}_\alpha)\, p(\underline{\theta}_\gamma)\, d\underline{\theta}_\alpha\, d\underline{\theta}_\gamma \qquad (1.1\text{-}10)$$

In statistical literature the discriminant functions such as - Pearson's coefficient of racial likeness (Rao [66, p 356]), Fisher's linear discriminant function (Anderson [2, p 134]) and Mahalanobis D^2-statistic [58] were employed in studying race mixtures and taxonomy (orderly classification of plants and animals according to their presumed natural relationships). For purposes of comparison the above quantities are defined as follows:

Pearson's Coefficient of Racial Likeness (CRL)

If n_{1i} and n_{2i} denote the number of observations on which the means m_{i1} and m_{i2} of the ith character for the first and second groups are based, s_i is the standard deviation of the ith character and p is the number of characters used, the n

$$CRL = \frac{1}{p} \sum_{i=1}^{p} \frac{n_{1i}\, n_{2i}}{n_{1i}+n_{2i}} \left[\frac{m_{i1}-m_{i2}}{s_i}\right]^2 \qquad (1.1\text{-}11)$$

Fisher's Linear Discriminant Function (LDF)

If $p_1(\underline{x})$ and $p_2(\underline{x})$ are two multivariate Gaussian densities such that $p_1(\underline{x}) = G(\underline{\mu}_1, \sum)$ and $p_2(\underline{x}) = G(\underline{\mu}_2, \sum)$, then

$$LDF = \underline{x}^T \sum^{-1} (\underline{\mu}_1 - \underline{\mu}_2) \qquad (1.1\text{-}12)$$

This is a linear function of the components of the observation vector $\underline{x}$.

Mahalanobis D^2 - Statistic

For two multivariate Gaussian densities $p_1(\underline{x})$ and $p_2(\underline{x})$ D^2 - statistic is given by

$$D^2 = (\underline{\mu}_1 - \underline{\mu}_2)^T \dot{\sum}^{-1} (\underline{\mu}_1 - \underline{\mu}_2) \qquad (1.1\text{-}13)$$

Only recently the interest in application of distance measures in engineering has been focused towards the problem of signal design. Grettenberg [34] used the divergence measure introduced by Kullback and Leibler [49], for signal selection in communication and radar systems. Divergence has been used by several others in application to pattern recognition problems - Kadota and Shepp [40], Tou and Heydorn [75] Fu, Min and Li [28] and Caprihan and de Figueiredo [20]. Mosca [64] considered the problem of selecting optimal probing signals for identifying time-varying linear M-ary channels by maximizing the average divergence under the energy constraint. In a recent application to adaptive filtering, Smith [71] used the average divergence to compute test signals in linear multistage processes. The adaptive filter is similar to that of Magill [57] and consists of m Kalman filters in parallel. For a given pair $(\theta_\alpha, \theta_\gamma) \epsilon \Theta$. The pair-wise divergence is given by

$$J_{\alpha\gamma} = \int_{Z^N \epsilon \Omega} \left[p(\underline{Z}^N | \underline{\theta}_\alpha) - p(\underline{Z}^N | \underline{\theta}_\gamma) \right] \ln \left[\frac{p(\underline{Z}^N | \underline{\theta}_\gamma)}{p(\underline{Z}^N | \underline{\theta}_\gamma} \right] d\underline{Z}^N \qquad (1.1\text{-}14)$$

Several properties of this measure are given in Kullback [48, pp. 12-28].

Bradt and Karlin [16] showed that if $J_{\alpha\gamma}(U_\eta)$ and $J_{\alpha\gamma}(U_\zeta)$ correspond to signals sets U_η and U_ζ and if $P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma, U_\eta) \leq$

$P_e(\underline{\theta}_{\alpha\gamma}, \underline{\theta}_{\gamma}, U_\zeta)$ then $J_{\alpha\gamma}(U_\eta) \geq J_{\alpha\gamma}(U_\zeta)$. Where P_e is the probability of error. The result holds for all prior probabilities (P_α, P_γ). Grettenberg's studies showed that the use of divergence could result in very poor signal sets compared to direct minimization of P_e. We define another distance measure which has the above ordering property with respect to the probability of error.

This measure, generally called the Bhattacharyya distance, is defined by

$$B_{\alpha\gamma} = -\ln \rho_{\alpha\gamma} \tag{1.1-15}$$

where

$$\rho_{\alpha\gamma} = \int \{p(\underline{Z}^N|\underline{\theta}_\alpha)\, p(\underline{Z}^N|\underline{\theta}_\gamma)\}^{\frac{1}{2}}\, d\underline{Z}^N \tag{1.1-16}$$

The ordering property of $B_{\alpha\gamma}$ as well as that of the divergence follows from a theorem due to Hardy, Littlewood and Polýa [35]. Also see Blackwell [14].

The divergence and B-distance are not equal in general. For dynamic systems where they are unequal the results obtained by applying these discriminant functions will differ. There is no general result relating the performance of these two measures. It was shown by Grettenberg [34] and Kailath [41] that for the problem of selecting communication links, the result obtained from the B-distance was better than that obtained by the divergence, coinciding with the result obtained by the direct minimization of the error probability. This is

also the case in input selection problems in linear dynamic systems with nonzero dynamic noise, as illustrated in Chapter 3. (See example 3.5.1).

The first part of the dissertation is primarily concerned with applying the distance measures to synthesize input signals in discrete-time linear stochastic systems. The learning performance with the inputs obtained from the B-distance, divergence and nonoptional inputs are compared. Numerical results show that the optimal inputs can be adequately represented by low-order difference equations.

The synthesis of random inputs is considered as an alternate approach to the signal selection problem. These inputs are generated by linear stochastic processes of the autoregressive moving average type, and the method gives rise to a reduced optimization in the space of input process parameters. The characterization of optimal stochastic inputs is studied in terms of their spectral densities.

1.2 Review of Literature

In this section a brief history of the development of input selection problem is discussed. In 1960 Levin [54] considered the estimation of the impulse response function of a discrete time model given by:

$$z_k = \sum_{l=0}^{p} \alpha_i \, g_i(k) + w_k \, , \quad k = 0, 1, \ldots, N \qquad (1.2\text{-}1)$$

The least squares estimate of $\underline{\alpha} = (\alpha_o, \alpha_1, \ldots, \alpha_p)^T$ is given by

$$\underline{\alpha}_{LS} = \left[G\,G^T\right]^{-1} G\,\underline{Z}^N \tag{1.2-2}$$

where the matrix G is given by

$$G^T = \begin{bmatrix} g_o(o) & g_1(o) \ldots . & g_p(o) \\ g_o(1) & g_1(1) \ldots . & g_1(1) \\ \vdots & & \\ g_o(N) & g_1(N) \ldots . & g_p(N) \end{bmatrix} \quad (N+1) \times (p+1) \tag{1.2-3}$$

For white noise w_k with variance σ_w^2 the error covariance matrix is

$$E\left[(\underline{\alpha} - \underline{\alpha}_{LS})(\underline{\alpha} - \underline{\alpha}_{LS})^T\right] = \sigma_w^2 \left[G\,G^T\right]^{-1} \tag{1.2-4}$$

It was shown by Levin that the trace of the covariance matrix is minimized by using a white noise sequence as inputs (i.e. the quantities $g_i(k)$).

Levadi [53] used a continuous time model of the form

$$x(t) = \int_0^T w(f, \tau, \beta)\, u(\tau)\, d\tau + n(t), \quad t \in [o, T] \tag{1.2-5}$$

$\underline{\beta}$ are the parameters of the weight in function. He showed that the

optimal, energy-constrained inputs are obtained by solving a Fredholm integral equation of the second kind. Arimoto and Kimura [5] considered the system given by (1.1-9) and derived results similar to those of Levin by the maximization of the approximate mutual information when the inputs are amplitude constrained.

Aoki and Staley [3] consider a scalar difference equation of the form

$$x_k = \sum_{i=1}^{n} \alpha_i x_{k-i} + u_k \qquad (1.2\text{-}6)$$

$$z_k = x_k + v_k \,, \quad k = 1, 2, \ldots . \qquad (1.2\text{-}7)$$

and maximize the trace of the Fisher information matrix to design energy constrained inputs. The Fisher information is defined by a $q \times q$ matrix with elements

$$M_{ij} = \left[E \frac{\partial}{\partial \theta_i} \ln p(\underline{Z}^N | \underline{\theta}) \frac{\partial}{\partial \theta_j} \ln p(\underline{Z}^N | \underline{\theta}) \right] \qquad (1.2\text{-}8)$$

The approximation to the trace is made in the sense that

$$\text{trace } M^{-1} \to (\text{trace } M)^{-1} \text{ as } N \to \infty \qquad (1.2\text{-}9)$$

Nahi and Wallis [65] give an optimal control formulation of the synthesis problem and use the Cramér-Rao lower bound to determine optimal energy constrained inputs. The Cramér-Rao lower bound on the estimation error is given by

$$E\left[(\underline{\theta} - \underline{\theta})\,(\underline{\theta} - \underline{\theta})^T\right] \geq M^{-1} \tag{1.2-10}$$

The nonlinear deterministic plant of the following form has been used.

$$\dot{x} = f(x,(t),u(t),t,a),\ z(o) = x_o, t\epsilon[o,T] \tag{1.2-11}$$

where $a\epsilon R^1$

$$z(t) = g[x(t),u(t),t,a] + v(t), t\epsilon[0,T] \tag{1.2-12}$$

Dynamics similar to the above are also used by Kalaba and Spingarn [42] who maximized the sensitivity of the dynamic function with respect to the parameter subject to an energy constraint. The solution to a two point boundary value problem is obtained via quasilinerazation. Goodwin, Murdoch and Payne [30] employed the trace of the inverse of Fisher information matrix to obtain input sequences for a single-input single-output system having amplitude constraints. A vector form of this case was the topic of a dissertation by Lopez-Toledo [56] who considered feedback laws of the form

$$u_k = \sum_{i=1}^{\ell} g_i(k)\, y_k + h_k \tag{1.2-13}$$

where $g_i(k)$ and h_k are the quantities to be determined by optimization of the sensitivity function.

In all of the above mentioned works [3,30,42,56,65] the authors used basically the same criterion for the synthesis problem and (except for [30])did not compare the performance of these signals

with nonoptimal inputs. Absent also is a discussion of the validity and approximations involved in the above procedures. It is very important to realize that the cost criterion is a function of the unknown parameters to be identified, and holds only for large N.

Recently Mehra [62, 63] used some concepts from experimental designs in linear repression models (see, e.g. Fedorov [25) to design optimal inputs in linear systems. Approximation to the regression form is carried out by a Taylor series expansion of the system transfer function about an a priori mean $\underline{\theta}_o$. Applying the results of repression design techniques Mehra showed that under steady-state conditions, an input consisting of a finite number of frequencies can be found such that it has the same information matrix as any other stationary input of equal power. The optimal inputs are obtained by repeating the experiment through the same length of time for each run, until the estimates of consecutive runs converge. Mehra's approach, though by far the most novel method for the solution of this class of problems, requires a good justification of the approximation involved in the Taylor series expansion. The approximation includes only terms up to the first order. It is quite likely that the successive estimates of the parameters are such that the difference $\Delta\theta_i = \theta_i - \hat{\theta}_i$, $i = 1, 2, \ldots,$ may be very large and the approximation may be very poor. This could easily give rise to a divergence in both the input sequence and the resulting estimation,

and a solution may not exist. In off-line applications this may not prove to be a hindrance, but in systems where the identification has to be carried out during the process itself, more analysis is required in implementing this method. In spite of these drawbacks, the method of [62, 63] constitutes a distinct departure from the previous approaches and has broadened the scope of input selection for applications in physical systems.

Some authors have also experimented with signals of special character. Cumming [22] applied a synchronous random telegraph wave and varied its bandwidth to match the frequency response of the system. Since the frequency response of the system is assumed to be known, these signals performed better than the pseudo-random binary perturbations discussed by Briggs et al, [18]. Box and Jenkins ([15], pp. 416-420) used a first order autoregressive input in application to a simple system. Litman and Huggins [55] used growing exponentials in identifying transfer function models.

1.3 Outline of The Dissertation

The deterministic signals derived from the maximization of B-distance and divergence measures is of the open-loop type. Only a priori information about the system is used in synthesizing the input and no observation information is associated. The solution to the constrained problem will be simplified if the cost function has a numerically tractable form so that the solution is globally optimum.

It was shown in [80] that the distance measures for linear systems is indeed quadratic in the inputs and the constrained maximization has a global optimum.

The effect of feedback is studied in terms of the open-loop feedback synthesis. Here at each stage k the inputs for the future steps are computed based on the past observations and the first element of this subsequence is taken as u_k. The feedback is introduced when a subsequent measurement is taken and the procedure is repeated.

Synthesis of random inputs is considered as an alternative approach to the signal selection problem. The signals are generated using linear stochastic processes. The optimization problem consists of choosing the process parameters by the maximization of sensitivity of the observations with respect to the parameters.

In Chapters 2 and 3 we present the discussion of distance measures and synthesis of open-loop inputs. Design and characterization of stochastic signals is given in Chapter 4. Chapter 5 presents a study of open-loop feedback signals.

In Chapter 2 the B-distance is defined in terms of the likelihood ratio. Some properties of this distance measure are given in Section 2.2. Sections 2.3 and 2.4 establish the B-distance as an optimality criterion for signal selection. The ordering property with respect to

the error probability is derived. Upper bounds on the mutual information in terms of the divergence and B-distance are derived in Section 2.5 and a lower bound on the mean-square error is obtained using the rate-distortion theory. The geometric interpretation of $\rho_{\alpha\gamma}$ is used to relate the distance measure to the Fisher information through the differential metric.

Synthesis of open-loop inputs in linear discrete-time systems is considered in Chapter 3. A description of the identifiability of parameters is summarized in Section 3.1. The B-distance is derived for the state-space model. This is shown to be the sum of three terms - a quadrative term in $\underline{U}^{N-1}$, a linear term in $\underline{U}^{N-1}$ and a constant term. For the energy constrained case the inputs are shown to be the eigenvector of a symmetric positive definite matrix corresponding to its largest eigenvalue. A sufficient condition for the accuracy of learning is derived as the nonnegativity of a quadratic form of the inputs. Numerical results show that the optimal inputs derived above can be approximated by a low order difference equation. Two distinct numerical examples illustrating the open-loop synthesis are detailed in Section 3.5.

Chapter 4 is concerned with the synthesis of stochastic inputs generated from linear models of the mixed autoregressive moving average type. The use of these inputs provide us with additional degrees of freedom (in selecting the optimal sequence) in terms of the

parameters of the process. Thus these inputs can be considered as being generated from a white noise sequence, using a linear filter. Section 4.1 gives a brief discussion of various linear models.

In Section 4.2 properties of these signals are studied in terms of the spectral density of the inputs. The system is considered in the form of a single-input single-output dynamics and the sensitivity of the output with respect to the parameters is used as the optimality criterion. By approximating the input correlation matrix by appropriate Toeplitz forms and using the results on asymptotic eigenvalue distribution of Toeplitz matrices, the optimum value of the cost function is derived. When the variance σ_ν^2 of the white noise is fixed, at the optimum cost the integral

$$\frac{1}{2\pi\sigma_\nu^2}\int_{-\pi}^{\pi} S_{uu}(\lambda)\, d\lambda$$

also attains its maximum value. Examples are presented in Section 4.3 studying the characteristics of these inputs.

Chapter 5 deals with the synthesis of feedback inputs which are obtained using the open-loop feedback technique. At each stage k of the process, future inputs are selected based on the past and present observation program, and the first component of this subsequen $\underline{U}_{k,N}$ is selected as the input u_k. The feedback is introduced when an additional observation is incorporated in computing the future inputs. The feedback input is shown to be equal to the sum of open-

loop and a correction term. For scalar systems with unknown gain the open-loop and feedback signals are shown to coincide.

Some concluding remarks and suggestions for future work in this field are given in Chapter 6.

CHAPTER II

THE DISTANCE MEASURE AND OPTIMALITY CRITERION FOR INPUT SIGNAL SYNTHESIS

In this chapter the optimality criterion for input signal selection is obtained in terms of the statistical distance measure between distributions characterizing distinct pairs of parameter sets. In particular the distance function is defined in terms of the Bhattacharyya measure (B-distance) and its properties are studied. In Section 2.1 the B-distance is defined and some of the related quantities are stated. Section 2.2 presents a discussion of the properties of this measure. Some of these results are useful in studying the asymptotic behavior of the B-distance. The important ordering property with respect to the probability of error is presented in Section 2.3.

The discussions of Section 2.4 show that the statistical distance measures are closely related to Shannon's definition of mutual information [69] and thus provide an alternative to using the latter. The distance measures are much easier to compute than the mutual information. In Section 2.5 the geometric interpretation of the B-distance is employed in relating this to the Fisher information.

2.1 The Bhattacharyya Distance:

Let $\underline{Z}^N \triangleq \{z_1, z_2, \ldots, z_n\}^T$ be a set of N observations of a

stochastic process parameterized by a p-dimensional parameter $\underline{\theta} \in \Theta \subset R^p$ where Θ is a compact sub set in R^p. Let us assume that the joint probability density of $\underline{Z}^N$ exists and is given by $p(\underline{Z}^N|\underline{\theta})$. For a given pair $(\underline{\theta}_\alpha, \underline{\theta}_\gamma)$ in Θ define the likelihood ratio

$$L = \frac{p(\underline{Z}^N|\underline{\theta}_\alpha)}{p(\underline{Z}^N|\underline{\theta}_\gamma)} \tag{2.1-1}$$

Define the quantity $\rho_{\alpha\gamma}(t)$ as (Chernoff [21], Zacks [84, p 212])

$$\rho_{\alpha\gamma}(t) = \int_\Omega e^{t\ln L} \; p(\underline{Z}^N|\underline{\theta}_\gamma)\, d\underline{Z}^N, \quad t \in (0,1) \tag{2.1-2}$$

where Ω is the sample space of $\underline{Z}^N$. The quantity $\rho_{\alpha\gamma}$ for $t=\frac{1}{2}$ is defined at the Blattacharyya coefficient [13,41],

$$\rho_{\alpha\gamma}(\tfrac{1}{2}) = \int_\Omega \left[p(\underline{Z}^N|\underline{\theta}_\alpha)\, p(\underline{Z}^N|\underline{\theta}_\gamma)\right]^{\frac{1}{2}} d\underline{Z}^N \tag{2.1-3}$$

The above integral also appears in a non-statistical context in a paper by Hellinger [33]. Ideally we like to define

$$\rho_{\alpha\gamma}(t^*) = \inf_{t \in (0,1)} \rho_{\alpha\gamma}(t) \tag{2.1-4}$$

Such a quantity is analytically hard to work with and it is clear that any advantage gained by (2.1-4) is overshadowed by the numerical

complexity involved in implementing $\rho_{\alpha\gamma}(t^*)$. Moreover, it is shown in App. A that $t^* = \frac{1}{2}$ for Gaussian densities with equal covariances. The Bhattacharyya distance is defined as

$$B_{\alpha\gamma} = -\ln \rho_{\alpha\gamma} \tag{2.1-5}$$

This quantity has also been used by Feller [27] to study the limit theorems of some of random variables. Let

$$F_{x(\mu)} = P\{x \leq \mu\} \tag{2.1-6}$$

and

$$\rho(t) = \int_{\infty}^{\infty} e^{t\xi}\, dF(\xi) \tag{2.1-7}$$

Denoting the new random variable by $\bar{x}$ the distribution of $\bar{x}$ is given by

$$F_{\bar{x}(\mu)} = \frac{\int_{\infty}^{\mu} e^{t\xi}\, dF(\xi)}{\rho(t)} \tag{2.1-8}$$

It can be shown that $F_{\bar{x}(\mu)}$ is a distribution function and the random variable $\bar{x}$ has mean and variance given by

$$E[\bar{x}] = \frac{1}{\rho(t)} \int_{\infty}^{\infty} \xi\, e^{t\xi}\, dF(\xi) = \frac{d}{dt} \ln \rho(t) \quad . \tag{2.1-9}$$

and

$$\mathrm{Var}(\bar{x}) = \frac{d^2}{dt^2} [\ln \rho(t)] \quad . \tag{2.1-10}$$

Some discussion with regard to performance bounds is also given in Van Tres [81, pp 116-133].

A distance function closely related to the B-distance has been studied by Matusita [59] and is a particular case of the measure defined by Jeffreys [39, p 174].

$$M_{\alpha\gamma}(m) = \int_{\Omega} \left| \{p(\underline{Z}^N|\underline{\theta}_\alpha)\}^{\frac{1}{m}} - \{p(\underline{Z}^N|\theta_\gamma)\}^{\frac{1}{m}} \right|^m d\underline{Z}^N \qquad (2.1\text{-}11)$$

Now the Matinta measure is defined as

$$[M_{\alpha\gamma}(\tfrac{1}{2})]^{\frac{1}{2}} = D_{\alpha\gamma} = \left[\int \left\{ \left[p(\underline{Z}^N|\underline{\theta}_\alpha) \right]^{\frac{1}{2}} - \left[p(\underline{Z}^N|\theta_\gamma)^{\frac{1}{2}} \right\}^2 d\underline{Z}^N \right]^{\frac{1}{2}} \qquad (2.1\text{-}12)$$

Simplifying the above we have

$$D_{\alpha\gamma} = \sqrt{2} \left(1 - \rho_{\alpha\gamma}(\tfrac{1}{2})\right)^{\frac{1}{2}} \qquad (2.1\text{-}13)$$

Appendix B gives the Bhattacharyya distances for multivariate Gaussian, Gamma, Poisson and multinominal distributions.

2.2 Properties of Bhattacharyya Measure

In this section we give several properties of the B-distance.

Proposition 2.2.1 If x and y are two independent random variables then

$$B_{\alpha\gamma}(x, y) = B_{\alpha\gamma}(x) + B_{\alpha\gamma}(y) \qquad (2.2\text{-}1)$$

[X]

Proof:

$$\rho_{\alpha\gamma}(x, y) = \int_x \int_y \left\{ p(x, y|\theta_\alpha)\, p(x, y|\theta_\gamma) \right\}^{\frac{1}{2}} dxdy$$

$$= \int_x \int_y \left\{ p(x|\theta_\alpha)\, p(y|\theta_\alpha)\, p(x|\theta_\gamma)\, p(y|\theta_\gamma) \right\}^{\frac{1}{2}} dxdy$$

$$= \int_x \left\{ p(x|\theta_\alpha)\, p(x|\theta_\gamma) \right\}^{\frac{1}{2}} dx \int_y \left\{ p(y|\theta_\alpha)\, p(y|\theta_\gamma) \right\}^{\frac{1}{2}} dy$$

or $\rho_{\alpha\gamma}(x, y) = \rho_{\alpha\gamma}(x)\ \ \rho_{\alpha\gamma}(y)$.

now $B_{\alpha\gamma}(x, y) = -\ln \rho_{\alpha\gamma}(x, y)$

$$= -\ln \rho_{\alpha\gamma}(x) - \ln \rho_{\alpha\gamma}(y)$$

$$= B_{\alpha\gamma}(x) + B_{\alpha\gamma}(y)$$

[X]

Then in general if the random variables $x_1, x_2, \ldots, x_n$ are independent then

$$B_{\alpha\gamma}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} B_{\alpha\gamma}(x_i) \qquad (2.2\text{-}2)$$

Proposition 2.2-2 $B_{\alpha\gamma}$ has the following metric related properties

(i) $0 < B_{\alpha\gamma} < \infty$ (2.2-3a)

(ii) $B_{\alpha\gamma}$ is symmetric, $B_{\alpha\gamma} = B_{\gamma\alpha}$ (2.2-3b)

(III) $B_{\alpha\alpha} = 0$ (2.2-3c)

[X]

$B_{\alpha\gamma}$ does not satisfy the triangular inequality for a metric. Let us refer to $B_{\alpha\gamma}$ as the pseudometric. The transformation obtained via the Matusita measure, $D_{\alpha\gamma}$, indeed satisfies the trinagle inequality.

Proposition 2.2.3 $D_{\alpha\gamma} + D_{\gamma\delta} \geq D_{\alpha\delta}$ (2.2-4)

[X]

Proof: We use Schwartz inequality to derive this

$$D_{\alpha\delta} = \left\{ \int \left[\left\{ p(\underline{Z}^N | \underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N | \underline{\theta}_\delta) \right\}^{\frac{1}{2}} \right]^2 d\underline{Z}^N \right\}$$

$$= \left\{ \int \left[\left(\left\{ p(\underline{Z}^N | \underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N | \underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) - \left(\left\{ p(\underline{Z}^N | \underline{\theta}_\delta) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N | \underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) \right]^2 d\underline{Z}^N \right\}^{\frac{1}{2}}$$

Now we can write the right hand side as follows

$$D^2_{\alpha\delta} = \left\{ \left[\int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 d\underline{Z}^N \right]^{\frac{1}{2}} \right.$$

$$\left. + \left[\int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\delta) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 d\underline{Z}^N \right]^{\frac{1}{2}} \right\}^2$$

$$-2 \left[\left\{ \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 d\underline{Z}^N \right\}^{\frac{1}{2}} \left\{ \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\delta) \right\}^{\frac{1}{2}} \right.\right.\right.$$

$$\left.\left.\left. - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 \right\} \left\{ \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\delta) \right\}^{\frac{1}{2}} \right.\right.$$

$$\left.\left. - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) d\underline{Z}^N \right] \qquad (2.2\text{-}5)$$

From Couchy - Schwartz inequality

$$\left| \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\delta) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right) d\underline{Z}^N \right|$$

$$\leq \left\{ \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\alpha) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 d\underline{Z}^N \right\}^{\frac{1}{2}}$$

$$\cdot \left\{ \int \left(\left\{ p(\underline{Z}^N|\underline{\theta}_\delta) \right\}^{\frac{1}{2}} - \left\{ p(\underline{Z}^N|\underline{\theta}_\gamma) \right\}^{\frac{1}{2}} \right)^2 d\underline{Z}^N \right\}^{\frac{1}{2}}$$

The second term in square brackets in (2.2-5) is nonnegative. Hence it follows from (2.2-5) that

$$D_{\alpha\delta} \leq D_{\alpha\gamma} + D_{\gamma\delta}$$

[X]

By defining

$$\mu(t) = \ln \int p(\underline{Z}^N|\underline{\theta}_\alpha)^t \; p(\underline{Z}^N|\underline{\theta}_\gamma)^{1-t} \; d\underline{Z}^N \,, \quad t \in [0, 1] \qquad (2.2\text{-}6)$$

We will show that $\mu(t)$ is a convex function of t. From this it is possible to study $\mu(t)$ as a function of t.

Proposition 2.2-4 $\mu(t)$, $0 \le t \le 1$ is a convex function of t.

Proof: Let $p_\alpha = p(\underline{Z}^N|\underline{\theta}_\alpha)$, $p_\gamma = p(\underline{Z}^N|\underline{\theta}_\gamma)$.

$$\frac{d\mu}{dt} = \frac{\int p_\alpha^t\, p_\gamma^{1-t}\, (\log p_\alpha - \log p_\gamma)\, d\underline{Z}^N}{\int_{-\infty}^{\infty} p_\alpha^t\, p_\gamma^{1-t}\, d\underline{Z}^N}$$

$$= \frac{\int_{-\infty}^{\infty} p_\alpha^t\, p_\gamma^{1-t}\, \log \frac{p_\alpha}{p_\gamma}\, d\underline{Z}^N}{o(t)}$$

$$\frac{d^2\mu}{dt^2} = \frac{1}{o(t)^2}\left[\int p_\alpha^t\, p_\gamma^{1-t}\, \ln \int p_\alpha^t\, p_\gamma^{1-t}\, \left(\ln \frac{p_\alpha}{p_\gamma}\right)^2 d\underline{Z}^N \right. \tag{2.2-6}$$

$$\left. - \left\{\int p_\alpha^t\, p_\gamma^{1-t}\, \ln \frac{p_\alpha}{p_\gamma}\, d\underline{Z}^N\right\}^2\right]$$

By the Schwartz inequality we know that

$$\left[\int fg\right]^2 \le \int f^2 \cdot \int g^2$$

Let $f = \left[p_\alpha^t\, p_\gamma^{1-t}\right]^{\frac{1}{2}}$, $g = \left[p_\alpha^t\, p_\gamma^{1-t}\right]^{\frac{1}{2}} \ln \frac{p_\alpha}{p_\gamma}$

Then clearly

$$\left[\int p_\alpha^t p_\gamma^{1-t} \ln\frac{p_\alpha}{p_\gamma}\, d\underline{Z}^N\right]^2 < \int p_\alpha^t p_\gamma^{1-t}\, d\underline{Z}^N \int p_\alpha^t p_\gamma^{1-t} \ln\left(\frac{p_\alpha}{p_\gamma}\right)^2 d\underline{Z}^N \tag{2.2-7}$$

From (2.2-6) and (2.2-7) we have

$\frac{d^2\mu}{dt^2} > 0 \Longrightarrow \mu(t)$ is a convex function of $t\epsilon[0,1]$. X

Another important property of the distance measure is its monotonic behavior with an increasing number of observations. If B_k is the pair wire distance for observations $\underline{Z}^k$, then the sequence $\{B_k\}_{k=1}^{\infty}$ is nondecreasing. Hence we have the following result.

Proposition 2.2-5 If $\{z_k\}_{k=1}^{\infty}$ is a sequence of observations of a stochastic process then

$$B_{k+1} \geq B_k \qquad \text{X}$$

Proof:

$$\rho_{\alpha\gamma,k+1} = \int \left\{ p(\underline{Z}^{k+1}|\underline{\theta}_\alpha)\, p(\underline{Z}^{k+1}|\underline{\theta}_\gamma)\right\}^{\frac{1}{2}} d\underline{Z}^{k+1}$$

$$= \int d\underline{Z}^k \int \left[p(\underline{Z}^k|\underline{\theta}_\alpha)\, p(\underline{Z}^k|\underline{\theta}_\gamma)\right]^{\frac{1}{2}}$$

$$\left[p(z_{k+1}|\underline{Z}^k, \underline{\theta}_\alpha)\, p(z_{k+1}|\underline{Z}^k, \underline{\theta}_\gamma)\right]^{\frac{1}{2}} dz_{k+1}$$

Now $$\int \left[p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\alpha)\, p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\gamma) \right]^{\frac{1}{2}} dz_{k+1}$$

$$\leq \left\{ \int p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\alpha)\, dz_{k+1} \right\}^{\frac{1}{2}} \left\{ \int p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\gamma)\, dz_{k+1} \right\}^{\frac{1}{2}}$$

$$\Longrightarrow \quad \int \left[p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\alpha)\, p(z_{k+1} | \underline{Z}^k, \underline{\theta}_\gamma) \right]^{\frac{1}{2}} dz_{k+1} \leq 1$$

Hence it follows that

$$\rho_{\alpha\gamma, k+1} \leq \int \left\{ p(\underline{Z}^k | \underline{\theta}_\alpha)\, p(\underline{Z}^k | \underline{\theta}_\gamma) \right\}^{\frac{1}{2}} d\underline{Z}^k$$

or $$\rho_{\alpha\gamma, k+1} \leq \rho_{\alpha\gamma, k}$$

But

$$B_{\alpha\gamma, k} = -\ln \rho_{\alpha\gamma, k}$$

$$B_{\alpha\gamma, k+1} \geq B_{\alpha\gamma, k} \quad .$$

[X]

This result is useful in studying the asymptotic relationship between the distance function and the probability of error.

2.3 Ordering Property of the B-distance

We would like for a distance measure $d_{\alpha\gamma}$, which is defined as the statistical measure of the affinity of probability measures of the observations characterized by $\underline{\theta}_\alpha$ and $\underline{\theta}_\gamma$, to have the following property: For two experimental designs U_η and U_ζ if $d_{\alpha\gamma}(U_\eta) > d_\alpha(U_\zeta)$, then $P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\eta) < P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\zeta)$ where P_e is the Bayes' probability of error for specified prior probabilities. In such a situation we say that the experiment U_η is more informative than U_ζ (symbolically $U_\eta > U_\zeta$). The input signal U appears in the distance function via the system dynamics and measurement process. The ordering of the B-distance with respect to the probability of error can be proved using a theorem of Hardy, Littlewood and Polya [35]. Results similar to that of these authors has also been obtained by Blackwell [14]. We will use a lemma due to Bradt and Karlin [16] to express the probability of error in an integral form.

2.3.1 Probability of Error for the Multi-Hypotheses Case

Define the total probability of error as the probability of error incurred in identifying the correct values of the system parameters. Let $P(\underline{\theta} = i\ \theta) = P_i$, $i=1, 2, \ldots, m$ be the set of prior probabilities. Let the cost of each course of action be C_{ij}. The first subscript signifies that the ith hypothesis is chosen and the second signifies that the jth hypothesis is true. Denoting the region of the observation space Ω in which we choose $\underline{\theta}_i$ as Ω_i, the total risk is

given by

$$R = \sum_{i=1}^{m} \sum_{j=1}^{m} P_j C_{ij} \int_{\Omega_i} p(\underline{Z}^N | \underline{\theta}_j) \, d\underline{Z}^N \tag{2.3-1}$$

If $C_{ii} = 0 \ \forall \ i \neq j$ then the resulting expression represents the total probability of error. Thus the probability of error P_e is given by

$$P_e = \sum_{i=1}^{m} \sum_{j=i}^{m} P_j \int_{\Omega_i} p(\underline{Z}^N | \underline{\theta}_j) \, d\underline{Z}^N \tag{2.3-2}$$

Bayes' likelihood ratio decision rule applied to the sample $\underline{Z}^N$ requires that we rank the parameter sets $\underline{\theta}_i$, $i=1,2,\ldots,m$ by the following rule. For all $i \neq j$ the set $\underline{\theta}_i$ is favored to the set $\underline{\theta}_j$ if

$$\frac{p(\underline{Z}^N | \underline{\theta}_i)}{p(\underline{Z}^N | \underline{\theta}_j)} > \Lambda_{ij} = \frac{P_j}{P_i} \tag{2.3-3}$$

The maximal element of this ordering is the class to which $\underline{Z}^N$ is assigned. Let

$$S_{ij} = \left\{ \underline{Z}^N \left| \frac{p(\underline{Z}^N | \underline{\theta}_i)}{p(\underline{Z}^N | \underline{\theta}_j)} > \frac{P_j}{P_i} \right. \right\} \tag{2.3-4}$$

Then the probability of error incurred when the decision rule is Bayes' and $\underline{Z}^N$ belongs to hypotheses k is

$$P_{ek} = P\left\{ \bigcup_{\substack{i=1 \\ i \neq k}}^{m} S_{ik} \,\middle|\, \underline{\theta} = {}_{k}\underline{\theta} \right\} \leq \sum_{\substack{i=1 \\ i \neq k}}^{m} P\left\{ S_{ik} \,\middle|\, \underline{\theta} = {}_{k}\underline{\theta} \right\} \qquad (2.3\text{-}5)$$

Hence the total probability of error is

$$P_e = \sum_{k=1}^{m} P_k P_{ek} \qquad (2.3\text{-}6)$$

An upper bound for the P_e can be obtained in terms of the distance function. Using (2.3-5)

$$P_e \leq \sum_{i=1}^{m} P_i \sum_{\substack{j=1 \\ j \neq i}}^{m} P\left\{ S_{ji} \,\middle|\, \underline{\theta} = {}_{i}\underline{\theta} \right\}$$

$$= \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \left[P_i P\left\{ S_{ji} \,\middle|\, \underline{\theta} = {}_{i}\underline{\theta} \right\} + P_j P\left\{ S_{ij} \,\middle|\, \underline{\theta} = {}_{j}\underline{\theta} \right\} \right] \qquad (2.3\text{-}7)$$

Let $P_e(k, j) = P_i P\left\{ S_{ji} \,\middle|\, \underline{\theta} = {}_{i}\underline{\theta} \right\} + P_j P\left\{ S_{ij} \,\middle|\, \underline{\theta} = {}_{j}\underline{\theta} \right\}$ (2.3-8)

Now $P_e(i, j) = \int_{\Omega} \min \left\{ P_i \, p(\underline{Z}^N \,|\, {}_{i}\underline{\theta}) \, , \; P_j \, p(\underline{Z}^N \,|\, {}_{j}\underline{\theta}) \right\} d\underline{Z}^N$ (2.3-9)

Since

$$\min (x_i, x_j) \leq x_i^{\alpha} x_j^{\beta} \, , \quad x_i \geq 0 \, , \quad \alpha + \beta = 1 \qquad (2.3\text{-}10)$$

we have

$$P_e(i,j) \leq \int_{\Omega} \left\{ P_i \, p(\underline{Z}^N | \underline{\theta}_i) \right\}^t \left\{ P_j \, p(\underline{Z}^N | \underline{\theta}_j) \right\}^{1-t} d\underline{Z}^N$$

or

$$P_e \leq \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} P_i^t P_j^{1-t} \int p(\underline{Z}^N | \underline{\theta}_i)^t \, p(\underline{Z}^N | \underline{\theta}_j)^{1-t} \, d\underline{Z}^N \quad (2.3\text{-}11)$$

The last integral is the moment generating function $\rho(t)$

$$P_e \leq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P_i^t P_j^{1-t} \, \rho_{ij}(t) \quad (2.3\text{-}12)$$

The tightest bound is obtained for which the right hand side of (2.3-10) is minimized with respect to t. This gives

$$P_e \leq \inf_{t \in (0,1)} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P_i^t P_j^{1-t} \, \rho_{ij}(t) \quad (2.3\text{-}13)$$

For the class of upper bounds for which $t = \frac{1}{2}$ we have [50]

$$P_e \leq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \left\{ P_i P_j \right\}^{\frac{1}{2}} \rho_{ij}(\tfrac{1}{2}) \quad (2.3\text{-}14)$$

The upper bounds obtained here do not assume any form of distribution for the $\underline{Z}^N$.

2.3.2 The Ordering Property of the Distance Measure

For two experimental designs U_η and U_ζ, if $B_{\alpha\gamma}(U_\eta) \geq B_{\alpha\gamma}(U_\zeta)$ implies $P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\eta) \leq P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\zeta)$ then such a property of the distance measure is referred to as the ordering property. A necessary condition for the probability ordering can be derived using a theorem of Hardy, Littlewood and Polŷa [35]. Result similar to that of the above authors has been obtained by Blackwell [14].

The following lemma of Bradt and Karlin [16] will be used in deriving the ordering property of the B-distance.

Lemma 2.3-1 A necessary and sufficient condition that $P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\eta) \leq P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\zeta)$ is that

$$\int \min\left\{ \frac{p(\underline{Z}^N; U_\eta | \underline{\theta}_\alpha)}{p(\underline{Z}^N; U_\eta | \underline{\theta}_\gamma)} - \xi, 0 \right\} p(\underline{Z}^N; U_\eta | \underline{\theta}_\gamma)\, d\underline{Z}^N$$

$$\leq \int \min\left\{ \frac{p(\underline{Z}^N; U_\zeta | \underline{\theta}_\alpha)}{p(\underline{Z}^N; U_\zeta | \underline{\theta}_\gamma)} - \xi, 0 \right\} p(\underline{Z}^N; U_\zeta | \underline{\theta}_\gamma)\, d\underline{Z}^N \qquad (2.3\text{-}15)$$

where $\xi = P_\gamma / P_\alpha$.

[X]

This result expresses the error probability as a function of the likelihood ratio

$$L = p(\underline{Z}^N | \underline{\theta}_\alpha) / P(\underline{Z}^N | \underline{\theta}_\gamma) \qquad (2.3\text{-}16)$$

The following theorem of Hardy, Littlewood and Polỳa [35] can be specialized to obtain Blackwell's theorem as follows.

Theorem 2.3-1 Suppose $p(x)$ and $q(x)$ are given such that

$$\int_R p(x)\,dx = \int_R q(x)\,dx \tag{2.3-17}$$

Then a necessary and sufficient condition that

$$\int_R \phi\left\{g(x)\right\} q(x)\,dx \le \int_R \phi\left\{f(x)\right\} p(x)\,dx \tag{2.3-18}$$

for every $\phi(\cdot)$, convex and and continuous in a closed interval including all values of f and g, is that

$$\int_R \max\left\{g(x)-y,\ 0\right\} q(x)dx \le \int_R \max\left\{f(x)-y,\ 0\right\} p(x)dx \tag{2.3-19}$$

for all y.

⊠

Using the lemma (2.3-1) and theorem (2.3-1) we have the following theorem relating the probability of error to the ordering of (2.3-18).

Theorem 2.3-2 If U_η and U_ζ are two experimental designs,

$$P_e(\underline{\theta}_\alpha, \underline{\theta}_{\gamma i}\ U_\zeta) \le P_e(\underline{\theta}_\alpha, \underline{\theta}_{\gamma i}\ U_\zeta) \tag{2.3-20}$$

for all prior probabilities (P_α, P_γ), if and only if

$$E\left[\phi(L_\eta)|\underline{\theta}_\gamma\right] \leq E\left[\phi(L_\zeta)|\underline{\theta}_\gamma\right] \tag{2.3-21}$$

for all continuous concave functions $\phi(L)$

[X]

The necessary condition for the ordering of error probability in terms of the B-distance is given by the following theorem.

Theorem 2.3-3 If $P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\eta) \leq P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\zeta)$ for all prior probabilities (P_α, P_γ) then

$$B_{\alpha\gamma}(U_\eta) \geq B_{\alpha\gamma}(U_\zeta) \tag{2.3-22}$$

[X]

Proof: By writing $\rho_{\alpha\gamma}$ as

$$\rho_{\alpha\gamma} = \int \sqrt{\frac{p(\underline{Z}^N|\underline{\theta}_\alpha)}{p(\underline{Z}^N|\underline{\theta}_\gamma)}} \cdot p(\underline{Z}^N|\underline{\theta}_\gamma)\, d\underline{Z}^N$$

$$= E\left[\sqrt{L}\,|\underline{\theta}_\alpha\right] \tag{2.3-23}$$

We notice that (2.3-21) is of the form $\phi(L) - \sqrt{L}$, a continuous concave function of L. Thus it is clear that whenever

$P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\eta) \leq P_e(\underline{\theta}_\alpha, \underline{\theta}_\gamma; U_\zeta)$ we have

$$\rho_{\alpha\gamma}(U_\eta) \leq \rho_{\alpha\gamma}(U_\zeta) \tag{2.3-24}$$

By definition $B_{\alpha\gamma} = -\ln \rho_{\alpha\gamma}$ and (2.3-22) follows.

[X]

Theorem (2.3-3) gives a necessary condition for the optimality of error probability. Sufficiency of the ordering property is true only when the conditions of theorem (2.3-2) are satisfied. The performance of B-distance and divergence will be studied for the case when the prior probabilities are all equal.

For the special case of independently and identically distributed observations it was shown by Chenoff [21] that the B-distance is exponentially optimum, that is, $\rho_{\alpha\gamma}$ essentially represents the probability of error -

$$P_e \rightarrow e^{-B_{\alpha\gamma}} \quad \text{for large } N \tag{2.3-25}$$

Generalization of this result to nonidentical distributions can be obtained using a limit theorem due to Feller [27].

Since the Bayes' probability of error is minimized by choosing the parameter set with the largest a posteriori probability $P(\underline{\theta}|\underline{Z}^N)$, whenever the condition of theorem (2.3-3) is satisfied, the optimal inputs obtained by maximizing the B-distance will also result in the highest a posteriori probability, containing maximum information about the parameters.

The generalization of the above theorem to the case when

when the parameter space Θ contains more than two points can be obtained using the average distance. Each pairwise error probability can be used to uniformly approximate the corresponding Bhattacharyya coefficient $\rho_{\alpha\gamma}, \alpha \neq \gamma$, through the minimum function. Thus with each probability pair we can associate the corresponding pairwise distance. Since the total probability is the sum of all such measures, the distance function is also the sum of all possible pairs of distances. The following theorem generalizes this result.

Theorem 2.3-4 When the parameter space Θ contains more than two points, the total probability of error P_e is given by (2.3-2) and whenever $P_e(U_\eta) \leq P_e(U_\zeta)$ then $\overline{B}(U_\eta) \geq \overline{B}(U_\zeta)$. ☒

The average distance $\overline{B}$ is given by

$$\overline{B} = \int_{\Theta \times \Theta} \int B_{\alpha\gamma} \, dP(\underline{\theta}_\alpha) \, dP(\underline{\theta}_\gamma) \tag{2.3-26}$$

When the parameters take on a finite number of discrete values

$$\overline{B} = \sum_{i=1}^{m} \sum_{j=1}^{m} P_i P_j B_{ij} \tag{2.3-27}$$

2.4 Mutual Information and Distance Measures

It is an interesting fact to note that the Bhattacharyya distance and the divergence (defined in Chapter 1) are closely related to Shannon's mutual information [68, 69]. The average mutual information $I(\underline{Z}^N; \underline{\theta})$

represents the mean amount of information that knowledge of the value assumed by $\underline{Z}^N$ supplies about the value of the parameter $\underline{\theta}$. In communication systems these two quantities can be treated as input $\tilde{X}$ and output $\tilde{Y}$ of a given channel and the capacity of the channel is expressed as

$$C = \max I(\tilde{X};\tilde{Y}) \tag{2.4-1}$$

where the maximum is taken over all the possible choices of the input distributions.

The distance function, though not defined in a similar manner, analogously represents the affinity of two distributions characterizing the parameter sets. Thus larger the distance measure, greater is the knowledge of correct parameter sets provided by the observations. Hence it is not surprising that there exists a certain relationship between these two information measures and it is shown in this section that the measures - divergence and B-distance bound the mutual information. This also leads to a lower bound on the mean-square error through the rate distortion theory.

2.4.1 Upper Bound on the Mutual Information

Definition: The average mutual information $I(\underline{Z}^N;\underline{\theta})$ is given by

$$I(\underline{Z}^N;\underline{\theta}) = \int_{\underline{Z}^N}\int_{\underline{\theta}} p(\underline{Z}^N,\underline{\theta}) \ln\left[\frac{p(\underline{Z}^N,\underline{\theta})}{p(\underline{Z}^N)\,p(\underline{\theta})}\right] d\underline{Z}^N\, d\underline{\theta} \tag{2.4-2}$$

This can also be written as

$$I(\underline{Z}^N; \underline{\theta}) = H(\underline{\theta}) - H(\underline{\theta} \mid \underline{Z}^N) \tag{2.4-3}$$

where $H(\underline{\theta})$ is the entropy of $\underline{\theta}$

$$H(\underline{\theta}) = -\int_{\theta} p(\underline{\theta}) \ln p(\underline{\theta})\, d\underline{\theta} \tag{2.4-4}$$

and $H(\underline{\theta} \mid \underline{Z}^N)$ is the conditional entropy

$$H(\underline{\theta} \mid \underline{Z}^N) = -\int_{\underline{Z}^N} \int_{\underline{\theta}} p(\underline{Z}^N, \underline{\theta}) \ln p(\underline{\theta} \mid \underline{Z}^N)\, d\underline{Z}^N\, d\underline{\theta} \tag{2.4-5}$$

"In words (2.5-3) expresses the intuitively plausible fact that the average information supplied about $\underline{\theta}$ by specification of $\{\underline{z}_k\}_1^N$ equals the average a priori uncertainty in $\underline{\theta}$ minus the average uncertainty in $\underline{\theta}$ that still remains after $\{\underline{z}_k\}_1^N$ is specified." The following theorem provides the upper bound on the mutual information in terms of the distance function.

Theorem 2.4-1 The upper bound on the mutual information is given by

$$I(\underline{Z}^N; \underline{\theta}) \leq \frac{1}{2} \overline{J} \tag{2.4-6}$$

For the Gaussian case with equal covariance functions $\overline{J} = 8\overline{B}$ and

$$I(\underline{Z}^N; \theta) \leq 4\overline{B} \tag{2.4-7}$$

Here

$$\overline{J} = E_{\alpha\gamma}\left[J_{\alpha\gamma}\right] \tag{2.4-8}$$

and

$$J_{\alpha\gamma} = \int_{\underline{Z}^N} \left[p(\underline{Z}^N|\underline{\theta}_\alpha) - p(\underline{Z}^N|\underline{\theta}_\gamma) \right] \ln \frac{p(\underline{Z}^N|\underline{\theta}_\alpha)}{p(\underline{Z}^N|\underline{\theta}_\gamma)} \, d\underline{Z}^N \qquad (2.4\text{-}9)$$

Proof: Using the definition (2.4 -2) we can write

$$I(\underline{Z}^N; \underline{\theta}) = -\int_{\underline{Z}^N} p(\underline{Z}^N) \ln p(\underline{Z}^N) \, d\underline{Z}^N$$

$$+ \int_{\underline{Z}^N} \int_{\underline{\theta}} p(\underline{Z}^N, \underline{\theta}) \ln p(\underline{Z}^N|\theta) \, d\underline{Z}^N \, d\underline{\theta}$$

$$= I_1 + I_2 \qquad (2.4\text{-}10)$$

Consider the first integral I_1.

$$\int p(\underline{Z}^N) \ln p(\underline{Z}^N) \, d\underline{Z}^N = \int \left\{ \int_{\underline{\theta}} p(\underline{Z}^N|\underline{\theta}) \, p(\underline{\theta}) \, d\underline{\theta} \right\}$$

$$\ln \left\{ \int_{\underline{\theta}} p(\underline{Z}^N|\underline{\theta}) \, p(\underline{\theta}) \, d\underline{\theta} \right\} \, d\underline{Z}^N$$

Using the inequality [36, p 150, theorem 204]

$$\ln \int_{\underline{\theta}} p(\underline{Z}^N|\underline{\theta}) \, p(\underline{\theta}) \, d\underline{\theta} \geq \int_{\underline{\theta}} \ln \left[p(\underline{Z}^N|\underline{\theta}) \right] \, p(\underline{\theta}) \, d\underline{\theta} \qquad (2.4\text{-}11)$$

we obtain

$$- I_1 \geq \int_{\underline{Z}^N} \left\{ \int_{\underline{\theta}_\alpha} \int_{\underline{\theta}_\gamma} \left[\ln p(\underline{Z}^N | \underline{\theta}) \right] p(\underline{Z}^N | \underline{\theta}_\alpha)\, p(\underline{\theta}_\alpha)\, p(\underline{\theta}_\gamma)\, d\underline{\theta}_\alpha\, d\underline{\theta}_\gamma \right\} d\underline{Z}^N \tag{2.4-12}$$

Add and subtract

$$\frac{1}{2} \left[\int_{\underline{Z}^N} \left\{ \int_{\underline{\theta}_\alpha} \int_{\underline{\theta}_\gamma} \left[p(\underline{Z}^N | \underline{\theta}_\alpha) \ln p(\underline{Z}^N | \underline{\theta}_\alpha) + p(\underline{Z}^N | \underline{\theta}_\gamma) \ln p(\underline{Z}^N | \underline{\theta}_\gamma) \right] \right.$$

$$\left. p(\underline{\theta}_\alpha)\, p(\underline{\theta}_\gamma)\, d\underline{\theta}_\alpha\, d\underline{\theta}_\gamma \right\} d\underline{Z}^N \Bigg]$$

on the right hand side of (2.4 -11). Rearranging terms and using the definition (2.4 -8) it follows

$$- I_1 \geq -\frac{1}{2} \int_{\underline{\theta}_\alpha} \int_{\underline{\theta}_\gamma} J_{\alpha\gamma}\, p(\underline{\theta}_\alpha)\, p(\underline{\theta}_\gamma)\, d\underline{\theta}_\alpha\, d\underline{\theta}_\gamma$$

$$+ \int_{\underline{Z}^N} \int_{\underline{\theta}} p(\underline{Z}^N, \underline{\theta}) \ln p(\underline{Z}^N | \underline{\theta})\, d\underline{Z}^N\, d\underline{\theta}$$

or

$$I_1 \leq \frac{1}{2} \bar{J} - I_2 \tag{2.4-13}$$

From (2.4 -9) and (2.4 -12) we get

$$I(\underline{Z}^N; \underline{\theta}) \leq \frac{1}{2} \bar{J} \tag{2.4-14}$$

For Gaussian signals with equal covariances [see Chapter 3] we have $\bar{J} = 8\bar{B}$ and

$$I(\underline{Z}^N; \underline{\theta}) \leq 4\bar{B} \qquad (2.4\text{-}15)$$

[X]

When the parameters have a discrete distribution the average divergence takes the form

$$\frac{1}{2}\bar{J} = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} P_i P_j J_{ij} \qquad (2.4\text{-}16)$$

Evaluation of the distance measures is much easier than the evaluation of the mutual information for multiple observations ($N \geq 2$). The above relationships indicate that maximization of the distance function will also maximize the mutual information. The results of (2.4-14) and (2.4-15) represent the distance functions as alternatives to using the mutual information. These results are general and are independent of the underlying distributions of the observation process.

2.4.2 The Rate Distortion Lower Bound on the Mean Square Error

The Cramēr-Rao bound is a widely used lower bound on the mean square error and obtained in terms of the Fisher information. For unbiased estimates of $\theta \in R^1$, the C-R bound is given by [81, p 66]

$$E\left[(\theta - \hat{\theta})^2\right] \geq \left\{E\left[\left\{\frac{\partial}{\partial \theta} \ln p(\underline{Z}^N, \theta)\right\}^2\right]\right\}^{-1} \qquad (2.4\text{-}17)$$

Lower bounds are also derived using Shannon's rate distortion theory [85]. In this section a lower bound on the estimation error is derived by using the upper bound on the mutual information. $I(\underline{Z}^N; \theta)$ derived in Section (2.4 -1).

Let the estimate of $\theta \epsilon R^1$ corresponding to the observation $\underline{Z}^N$ be denoted by $\theta = \mu(\underline{Z}^N)$. Let $p(\underline{Z}^N | \theta)$ denote the transition density of $\underline{Z}^N$ for a given θ. The distortion in the estimate of θ is

$$d(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2 \tag{2.4-18}$$

and the average squared distortion is

$$= \iint_{\theta\,\hat{\theta}} d(\theta, \hat{\theta})\, p(\theta, \hat{\theta})\, d\theta\, d\hat{\theta} \tag{2.4-19}$$

Define the set $\Gamma_\theta(D)$ as

$$\Gamma_\theta(D) = \left\{ p(\epsilon | \hat{\theta}) \;\middle|\; E_{\theta, \hat{\theta}}\left[d(\theta, \hat{\theta}) \right] \leq D \right\} \tag{2.4-20}$$

Then the Shannon's rate distortion function is given by [70, 68 , pp. 141-155].

$$R_\theta(D) = \inf_{p(\theta | \hat{\theta}) \epsilon \Gamma_\theta(D)} I(\theta; \hat{\theta}) \tag{2.4-21}$$

where

$$I(\theta;\hat{\theta}) = \int_{\theta}\int_{\hat{\theta}} p(\theta,\hat{\theta}) \ln \frac{p(\theta,\hat{\theta})}{p(\theta)\,p(\hat{\theta})} \, d\theta \, d\hat{\theta} \qquad (2.4\text{-}22)$$

Thus $R_\theta(D)$ represents the minimum amount of information to be conveyed to the estimator in order to attain the required accuracy D. A lower bound for the rate distortion $R_\theta(D)$ can be obtained when D is equal to the mean square fidelity criterion ϵ^2. Following the development in Berger [12, pp. 92-102] it can be shown that

$$R_\theta(\epsilon^2) \geq H(\theta) - \frac{1}{2} \ln (2\pi e \epsilon^2) \qquad (2.4\text{-}23)$$

From the definition of $R_\theta(\epsilon^2)$ we have

$$R_\theta(\epsilon^2) \leq I(\theta;\hat{\theta}) \leq I(\theta;\underline{Z}^N) \qquad (2.4\text{-}24)$$

The last inequality follows from the fact that any information contained in $\hat{\theta}$ is also contained in $\underline{Z}^N$. Noting that $I(\theta;\underline{Z}^N) = I(\underline{Z}^N;\theta)$ we get

$$I(\underline{Z}^N;\theta) \geq H(\theta) - \frac{1}{2} \ln (2\pi e \epsilon^2) \qquad (2.4\text{-}25)$$

From this the lower bound is

$$E\left[(\theta-\hat{\theta})^2\right] \geq \frac{1}{2\pi e} \exp\left[2\,H(\theta) - I(\underline{Z}^N;\theta)\right] \qquad (2.4\text{-}26)$$

The main result is obtained in the following theorem.

Theorem 2.4-2 Let $p(\theta)$ represent the prior distribution of θ and $\overline{J}$ be the average divergence measure given by (2.5-8). Then

$$E\left[(\theta - \hat{\theta})^2\right] \geq \frac{1}{2\pi e} \exp\left[2\,H(\theta) - \bar{J}\right] \qquad (2.4\text{-}27)$$

Proof: The result follows from (2.5-26) and the inequality

$I(\underline{Z}^N; \theta) \leq \frac{1}{2}\,\bar{J}$ X

Remarks

1. When the parameter set is discrete with equal prior probabilities, $P_i = \frac{1}{m}$ $\forall$ i, the lower bound becomes

$$E\left[(\theta - \hat{\theta})^2\right] \geq \frac{m^2}{2\pi e} \exp\left[-\bar{J}\right] \qquad (2.4\text{-}28)$$

2. For Gaussian random variables with equal covariance functions $\bar{J} = 8\,\bar{B}$ and we have

$$E\left[(\theta - \hat{\theta})^2\right] \geq \frac{1}{2\pi e} \exp\left[2\,H(\theta) - 8\,\bar{B}\right] \qquad (2.4\text{-}29)$$

The appearance of the distance measure in the lower bound is intuitively appealing for its use as a criterion for signal selection. It has been observed that there is a region of input signal-to-noise ratio in which the lower bound of (2.4-26) performs better than the Cramĕr-Rao bound [80]. Even though the lower bounds derived here cannot be considered as strong competitors for the existing bounds, the relationship between the mutual information and the statistical distance functions is quite useful in applications to signal design and feature extraction problems.

2.5 Geometric Interpretation of the B-distance and its Relationship to the Fisher Information

Bhattacharyya proposed that the numbers $\left\{p(\underline{Z}^N|\underline{\theta}_\alpha)\right\}^{\frac{1}{2}}$ and $\left\{p(\underline{Z}^N|\underline{\theta}_\gamma)\right\}^{\frac{1}{2}}$ can be regarded as the direction cosines of two vectors in the space of $\underline{Z}^N$. Alternately we can consider them as defining two points on a unit hypersphere with the angle between them given by

$$\cos \Delta = \int_{\Omega} \left\{p(\underline{Z}^N|\underline{\theta}_\alpha)\, p(\underline{Z}^N|\underline{\theta}_\gamma)\right\}^{\frac{1}{2}} d\underline{Z}^N = \rho_{\alpha\gamma} \qquad (2.5\text{-}1)$$

The angle between them must clearly be between 0 and $\frac{\pi}{2}$.

The above definition becomes clear by considering two multinominal populations characterized by the population probability $(\pi_1, \pi_2, \ldots, \pi_n)$ and $(\pi_1', \pi_2', \ldots, \pi_n')$. Then as $\sum_{i=1}^{n} \pi_i = 1$ and $\sum_{i=1}^{n} \pi_i' = 1$, $(\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_n})$ and $(\sqrt{\pi_1'}, \sqrt{\pi_2'}, \ldots, \sqrt{\pi_n'})$ can be considered as the direction cosines of two straight lines in n-dimensional space referred to a system of orthogonal coordinate axes. These lines may be called population lines. The angle between the lines may be called the angle of divergence; any monotonically increasing single-valued function of this angle may be used as a measure of divergence. Then if Δ is the angle between the lines we have

$$\cos \Delta = \sum_{i=1}^{n} \left\{ \pi_i \, \pi_i \right\}^{\frac{1}{2}} \qquad (2.5\text{-}2)$$

When $\Delta = \frac{\pi}{2}$ the separation between the populations is maximum and when $\Delta = 0$, the two populations essentially coincide, as the affinity between them becomes maximum.

The Fisher information matrix defined in Chapter 1 gives us the sensitivity of the observations with respect to the parameters $\underline{\theta} \in R^p$, when $\underline{\theta}$ has a continuous distribution. The Fisher information can be related to the B-distance via the quadratic differential metric. The Fisher information is defined by the elements of the pxp matrix as

$$M_{ij} = E\left[\frac{\partial}{\partial \theta_i} \ln p(\underline{Z}^N | \underline{\theta}) \; \frac{\partial}{\partial \theta_j} \ln p(\underline{Z}^N | \underline{\theta}) \right] \qquad (2.5\text{-}3)$$

Consider a perturbation $\Delta \underline{\theta}$ of $\underline{\theta} \in R^p$. If $\|\delta s\|$ is the arcual length along the hypersphere of $\{\underline{Z}^N\}$ between the points with direction cosines $p(\underline{Z}^N | \underline{\theta})^{\frac{1}{2}}$ and $p(\underline{Z}^N | \underline{\theta} + \Delta \underline{\theta})^{\frac{1}{2}}$ then we can write

$$\rho_{\underline{\theta}, \theta + \Delta \underline{\theta}} = \cos \|\delta s\| = \int_{\Omega} \left\{ p(\underline{Z}^N | \underline{\theta}) \, p(\underline{Z}^N | \underline{\theta} + \Delta \underline{\theta}) \right\}^{\frac{1}{2}} d\underline{Z}^N \qquad (2.5\text{-}4)$$

Let us state the following regularity conditions on the density function $p = p(\underline{Z}^N | \underline{\theta})$.

(i) The partial derivatives $\frac{\partial}{\partial \theta_i} \ln p$ and $\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln p$ exist $\forall \; i, j = 1, 2, \ldots, p$.

(ii)

$$\left|\frac{\partial p}{\partial \theta_i}\right| < F(\underline{Z}^N) \quad , \quad \left|\frac{\partial^2 p}{\partial \theta_i \partial \theta_j}\right| < G(\underline{Z}^N) \tag{2.5-5}$$

$\forall$ i, j = 1, 2, ..., p where $F(\underline{Z}^N)$ and $G(\underline{Z}^N)$ are intergrals over Ω .

iii)

$$\int_{\Omega} \frac{\partial p}{\partial \theta_i} \, d\underline{Z}^N = 0 \quad , \int_{\Omega} \frac{\partial^2 p}{\partial \theta_i \partial \theta_j} \, d\underline{Z}^N = 0 , \quad \forall \, i, j = 1, 2, ..., p \tag{2.5-6}$$

Expanding $\left\{p(\underline{Z}^N|\underline{\theta} + \Delta\underline{\theta}\right\}^{\frac{1}{2}}$ by Taylor series about $\underline{\theta}$

$$\left\{p(\underline{Z}^N|\underline{\theta} + \Delta\underline{\theta})\right\}^{\frac{1}{2}} = \left[p(\underline{Z}^N|\underline{\theta})\right]^{\frac{1}{2}} + \frac{1}{2}\sum_{i=1}^{p} p^{-\frac{1}{2}} \frac{\partial p}{\partial \theta_i} \Delta\theta_i$$

$$+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left[-\frac{1}{4}\frac{1}{p}\frac{\partial p}{\partial \theta_i}\frac{\partial p}{\partial \theta_j} + \frac{\partial^2 p}{\partial \theta_i \partial \theta_j}\right] + \ldots \tag{2.5-7}$$

Equation (2.5-4) becomes

$$\cos ||\delta s|| = \int_{\Omega}\left\{p(\underline{Z}^N|\underline{\theta}) + \frac{1}{2}\sum_{i=1}^{p}\frac{\partial p}{\partial \theta_i}\Delta\theta_i\right.$$

$$\left. + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left[-\frac{1}{4}\frac{1}{p}\frac{\partial p}{\partial \theta_i}\frac{\partial p}{\partial \theta_j} + \frac{\partial^2 p}{\partial \theta_i \partial \theta_j}\right]\Delta\theta_i \Delta\theta_j \, d\underline{Z}^N + \ldots\right. \tag{2.5-8}$$

Using the regularity conditions we obtain

$$\cos \|\delta s\| = 1 - \frac{1}{8} \sum_{i=1}^{p} \sum_{j=1}^{p} \left[\int_{\Omega} \frac{\partial}{\partial \theta_i} \ln p(\underline{Z}^N | \underline{\theta}) \frac{\partial}{\partial \theta_j} \ln p(\underline{Z}^N | \underline{\theta}) \cdot \right.$$

$$\left. p(\underline{Z}^N | \underline{\theta}) \, d\underline{Z}^N \right] \cdot \Delta\theta_i \Delta\theta_j + \ldots \qquad (2.5\text{-}9)$$

Rewriting (2.5-9)

$$1 - \cos \|\delta s\| \approx \frac{1}{8} \sum_{i=1}^{p} \sum_{j=1}^{p} M_{ij} \Delta\theta_i \Delta\theta_j \qquad (2.5\text{-}10)$$

Now

$$1 - \cos \|\delta s\| \approx 2 \sin^2 \frac{\|\delta s\|}{2} \approx 2 \cdot \frac{\|\delta s\|^2}{4} \qquad (2.5\text{-}11)$$

From (2.5-10) and (2.5-11)

$$\|\delta s\|^2 = \frac{1}{4} \sum_{i-1}^{p} \sum_{j=1}^{p} M_{ij} \Delta\theta_i \Delta\theta_j \qquad (2.5\text{-}12)$$

where M_{ij} are the elements of Fisher information matrix defined by (2.6-3). By definition

$$\rho \, \underline{\theta}, \underline{\theta} + \Delta\underline{\theta} = \cos \|\delta s\| \approx 1 - \frac{\|\delta s\|^2}{2} \qquad (2.5\text{-}13)$$

and

$$B_{\underline{\theta},\,\underline{\theta}+\Delta\underline{\theta}} = -\ln \rho_{\underline{\theta},\,\underline{\theta}+\Delta\underline{\theta}}$$

$$= -\ln\left[1-\frac{\|\delta s\|^2}{2}\right] \qquad (2.5\text{-}14)$$

For small $\|\delta s\|$ we have

$$-\ln\left[1-\frac{\|\delta s\|^2}{2}\right] \simeq \frac{\|\delta s\|^2}{2}$$

Thus

$$B_{\underline{\theta},\,\underline{\theta}+\Delta\underline{\theta}} = \frac{\|\delta s\|^2}{2} \approx \frac{1}{8}\sum_{i=1}^{p}\sum_{j=1}^{p} M_{ij}\,\Delta\theta_i\,\Delta\theta_j \qquad (2.5\text{-}15)$$

This indicates that in order to distinguish between unlike parameter sets increasing the differential metric is equivalent to increasing the sensitivities of the observations with respect to infinitesimal changes in the values of the parameters. Thus a close relationship between the distance measure and the Fisher information exists, the distance function representing the difference in the distributions for neighboring parameters. The differential metric suggests that this quantity can itself be used as a criterion function for signal synthesis. However, there is no guarantee that $\Delta\theta_i$ are small enough to obtain a meaningful approximation and it may be that these approximations are very bad from stage to stage in the estimation of $\underline{\theta}$. Expansion of the transfer function matrix in the frequency domain has been used by Mehra [62] in approximating a state space model to a regression model.

The purpose here is only to show the connection between different information measures, rather than to use it for problem solutions.

Summarizing the discussions in this Chapter, the Bhattacharyya distance is defined and its ordering with respect to the probability of error P_e is obtained as a necessary condition for the optimality of P_e. Thus the input signals obtained this way will result in the maximum a posterori probability $P(\underline{\theta}|\underline{Z}^N)$ representing the parameter learning. It is also shown that the distance measures are closely related to the mutual information and the maximization of distance functions will also maximize the information rate. It is much easier to implement the distance functions numerically in designing the input signals. It is further shown that the differential metric $B_{\underline{\theta}, \underline{\theta}+\Delta\underline{\theta}}$ represents the sensitivity of the observations with respect to the parameters, in terms of the Fisher information. Thus the distance measure, in a way reflects the sensitivity index to be maximized to choose the optimal signals. In the next chapter we apply the distance functions and make a comparison of their performance in parameter learning.

CHAPTER III

SIGNAL SELECTION IN LINEAR DYNAMICAL SYSTEMS

The problem of designing optimal inputs in linear systems is described in this chapter. The distance measure is derived for the general linear Gaussian case. The optimization problem is solved when the inputs are constrained to have total energy less than or equal to a given value. For this case the optimal inputs are given by the eigenvector corresponding to the largest eigenvalue of a symmetric positive definite matrix. When the number of stages N is large, a numerical method is derived to obtain the optimal signal vector. Comparison of the performances using the B-distance and the divergence measures is made as a function of the dynamic noise covariance. Optimal input sequences are obtained with these two criteria and the numerical results show that for increasing value of the noise covariance, the learning with B-distance is better than with the divergence. This comparison is made for the case of equal prior probabilities of the parameter sets. Two examples are presented illustrating various aspects of the signal synthesis problem.

The statement of the problem and a brief discussion of the identifiability of parameters in linear discrete-time systems is given in Section (3.1). The distance measure is derived in Section (3.2) and the optimization problem is discussed in Section (3.3). A sufficient

condition is obtained for the asymptotic accuracy of learning. This is similar to the condition of persistent excitation of the input sequence given by Åström and Bohlin [6]. The open-loop inputs can be characterized as satisfying a linear difference equation of low order. This is discussed in Section (3.4). Two numerical examples are presented in Section (3.5).

3.1 Statement of the Problem for the General Discrete Time Stochastic Linear System

The signal selection problem for the linear Gaussian system is presented with a brief discussion of parameter identifiability in discrete time systems.

3.1.1 Problem Statement

Consider the following representation of the dynamic system

$$\underline{x}_k = \Phi \underline{x}_{k-1} + \underline{\beta} u_{k-1} + \underline{w}_{k-1} \qquad (3.1\text{-}1)$$

$$\underline{z}_k = H\underline{x}_k + \underline{v}_k \quad , \quad k=1,2,\ldots \qquad (3.1\text{-}2)$$

$\Phi(n \times n)$, $H(s \times n)$ and $\underline{\beta}(n \times 1)$ are time invariant matrices. $\underline{w}_k$ and $\underline{v}_k$ are Gaussian white noise sequences

$$\underline{w}_k \sim G(0, Q) \qquad (3.1\text{-}3)$$

$$\underline{v}_k \sim G(0, R) \qquad (3.1\text{-}4)$$

$$E\left[\underline{w}_j \underline{v}_k^T\right] = 0 \quad , \quad \forall j,k \qquad (3.1\text{-}5)$$

The initial state $\underline{x}_0$ has a Guassian distribution with

$$\underline{x}_0 \sim G(\underline{x}_0 , P_0) \tag{3.1-6}$$

It is further assumed that $\underline{x}_0$ is independent of $\{\underline{w}_k , k=0, 1\ldots\}$ and $\{\underline{v}_k , k = 1, 2, \ldots.\}$ so that

$$E\left[\underline{x}_0 \underline{w}_k^T\right] = 0 , \quad k = 0, 1, 2, \ldots \tag{3.1-7}$$

$$E\left[\underline{x}_0 \underline{v}_k^T\right] = 0 , \quad k = 1, 2, \ldots \tag{3.1-8}$$

The above system representation is popularly known as the state space representation (e.g. Sorenson [73, p 111]). The unknown parameters $\underline{\theta}$ contained in the set $(\Phi , \underline{\beta})$. Let Φ be a stable matrix. Let $\theta \in \Theta \subset R^q$ where Θ is a compact set in R^q. Let $P(\underline{\theta})$ denote the joint a priori distribution of $\underline{\theta}$. When $\underline{\theta}$ has a discrete distribution such that $\underline{\theta}$ belongs to a finite set of discrete values, let

$$P(\underline{\theta} = \underline{\theta}_i) = P_i , \quad i = 1, 2, \ldots, m \tag{3.1-9}$$

$$P_i \geq 0 \text{ and } \sum_{i=1}^{m} P_i = 1 \tag{3.1-10}$$

Define the following notations

$$\underline{Z}^k \underline{\Delta} (\underline{z}_1 , \underline{z}_2 , \ldots , \underline{z}_k)^T \in R^{mk} \tag{3.1-11}$$

$$\underline{U}^k \underline{\Delta} (u_0 , u_1 , \ldots , u_k)^T \in R^{k+1} \tag{3.1-12}$$

Let the vector $\underline{U}^{N-1} \in R^N$ of inputs be constrained to the set υ

where υ is a compact set in R^N. Two types of constraints belong to this category - the energy constraint

$$\upsilon_E = \left\{ \underline{U}^{N-1} \mid \| \underline{U}^{N-1} \|^2 \leq Y \right\} \tag{3.1-13}$$

and the magnitude constraint

$$\upsilon_M = \left\{ \underline{U}^{N-1} \mid a_k \leq u_k \leq b_k \, , \; k = 0, 1, \ldots, N-1 \right\} \tag{3.1-14}$$

The optimal input design is defined by the class of admissible inputs that maximize the distance measure derived from the system described by (3.1-1) - (3.1-10). No probability distribution is associated with these inputs and they belong to the class of open-loop deterministic inputs.

The multistate structure of (3.1-1) and (3.1-2) in the cononical form reduces to the scalar single-input single-output (SISO) form which we will discuss in Section (3.1-2). We also make use of this representation in Chapter 4 where the synthesis of random inputs will be discussed.

3.1.2 Identifiability of Parameters

In this section we define the concepts of identifiability and state conditions under which a linear system is identifiable. These results require that the system be completely controllable and observable. First we define these terms for a deterministic system. These concepts were first introduced by Kalman [43] in the

mathematical representation of dynamic systems.

Consider the following deterministic system

$$\underline{x}_k = \Phi \underline{x}_{k-1} + \Gamma \underline{u}_{k-1} \qquad (3.1\text{-}15)$$

(D)

$$\underline{z}_k = H\underline{x}_k \qquad , \; k = 1, 2, \ldots \qquad (3.1\text{-}16)$$

Definition 3.1.1 [73, p 118] The dynamical system (D) is said to be completely controllable on an interval $[t_0, t_n]$ if, for a given t_0 and t_n, each initial state $\underline{x}_0$ can be transferred to any terminal state $\underline{x}_n$ using some control policy $\{\underline{u}\}$ over the interval. ☒

The following theorem gives the mathematical condition for controllability of (D).

Theorem 3.1.1 [60, p 56] The system (D) is completely controllable if and only if the nxnr matrix

$$C = \left[\Gamma, \Phi\Gamma, \Phi^2\Gamma, \ldots \Phi^{n-1}\Gamma\right] \qquad (3.1\text{-}17)$$

has rank n.

Definition 3.1.2 [73, p 130] The system (D) is said to be completely observable on an interval $[t_0, t_n]$ if for given t_0 and t_n every initial state $\underline{x}_0$ can be determined from knowledge of $\underline{z}_k$ $(k = 1, 2, \ldots, n)$. ☒

Theorem 3.1.2 [60, p 48] The system (D) is completely observable if and only if the nxmn matrix

$$O = \left[H^T, \Phi^T H^T, \ldots (\Phi^T)^{n-1} H^T \right] \tag{3.1-18}$$

has rank n .

[X]

Now let us define the identifiability of $\underline{\theta}$ and the conditions for identifiability when the system is observed with noise present.

Definition 3.1.3 [76] Let $\hat{\underline{\theta}}_k = \hat{\underline{\theta}}(\underline{Z}^k)$ where $\left\{ \underline{z}_k \right\}_{k=1}^{\infty}$ is a sequence of observations parameterized by $\underline{\theta}$. A sequence of estimates is said to be consistent in probability, if for any δ , ϵ arbitrarily small, there exists an $N(\epsilon, \delta) > \infty$ such that for $n > N(\epsilon, \delta)$

$$P_r \left\{ d(\hat{\underline{\theta}}_n, \underline{\theta}) > \delta \right\} < \epsilon \tag{3.1-19}$$

where d is a metric defined in $\Theta \subset R^q$.

[X]

Definition 3.1.4 [76] The parameter $\underline{\theta}$ is said to be identifiable if there exists a sequence of estimates $\left\{ \hat{\underline{\theta}}_n \right\}_{n=1}^{\infty}$ which is consistent in probability, viz.

$$\hat{\underline{\theta}}_n \xrightarrow{P} \underline{\theta} \tag{3.1-20}$$

[X]

Consider the system described in (3.1-1) and (3.1-2) without the control variables u_k .

$$\underline{x}_k = \Phi \underline{x}_{k-1} + \underline{w}_{k-1} \tag{3.1-21}$$

(S)

$$\underline{z}_k = H\underline{x}_k + \underline{v}_k \tag{3.1-22}$$

The steady state Kalman filter representation of (S) is the following.

$$\hat{\underline{x}}_{k+1|k} = \Phi \hat{\underline{x}}_{k|k-1} + B\underline{v}_k \tag{3.1-23}$$

$$\underline{z}_k = H\hat{\underline{x}}_{k|k-1} + \underline{v}_k \tag{3.1-24}$$

The steady state gain and covariances are given by

$$B = P H^T \left[HPH^T + R\right]^{-1} \tag{3.1-25}$$

and

$$P = \Phi\left[I - BH\right]P\Phi^T + Q \tag{3.1-26}$$

$\underline{\nu}_k$ is called the innovations sequence.

$$\underline{\nu}_k = \underline{z}_k - H\hat{\underline{x}}_{k|k-1} \tag{3.1-27}$$

Under steady state the innovation sequence $\underline{\nu}_k$ is a stationary white Gaussian sequence [61]. The original problem of estimating $\underline{\theta} = (\Phi, H, R_0, Q_o)$ can be reformulated as (where R_0 and Q_0 are the steady state values of R_k and Q_k) that of estimating the parameters (Φ, H, B). The following theorem can now be stated:

Theorem 3.1.3 [76] Let the system (S) satisfy the following conditions:

(i) The eigenvalues of Φ are within the unit circle.

(ii) (Φ, H) is an observable pair.

(iii) (Φ, B) is a controllable pair viz. the matrix

$$\left[B, \Phi B, \ldots \Phi^{n-1} B\right] \tag{3.1-28}$$

has rank n.

Then the linear system (S) is identifiable up to its equivalent steady-state Kalman filter representation.

[X]

"Regardless of the identification algorithm if two systems of the form (S) have the steady-state Kalman filter with the same impulse response and the same innovations covariance, then they cannot be resolved using stead-state measurements. For a given system (S) there may be many steady-state Kalman filters with identical impulse response and innovations covariance. Thus, without additional structural constraint the model is in general not identifiable." In the steady-state form (3.1-23) - (3.1-24) if the matrices (Φ, H, B) are in a unique form (that is, the unknown parameters (Φ, H, B) must be invariant under state coordinate transformation) then the steady-state Kalman filter is identifiable. Some canonical forms for multivariable systems are given by Weinert and Anton [83]. The conditions of controllability and

and observability will make the parameters of the system matrix Φ not all independent. In this case the system can be reduced to a unique canonical form as follows.

Theorem 3.1.4 [52, p 86] Given a dynamic system

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \underline{\beta}\, \underline{u}_k \tag{3.1-29}$$

$$z_k = \underline{h}^T \underline{x}_k \tag{3.1-30}$$

where the apir (Φ, h) is observable, $(\Phi, \underline{\beta}, \underline{h})$ can be transformed to the canonical form (Φ^*, β^*, h^*) by using the observability matrix 0 where Φ^*, β^*, h^* and O are defined by

$$\Phi^* = \left[\begin{array}{c|c} \begin{matrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} & I_{n-1} \\ \hline \Phi_1 & \Phi_2 \cdots \cdots \quad n \end{array}\right] \tag{3.1-31}$$

$$\underline{\beta}^* = \left[\beta_1, \beta_2, \ldots, \beta_n\right]^T \tag{3.1-32}$$

$$h^* = \left[1 \;\; 0 \ldots . \;\; 0\right] \tag{3.1-33}$$

and

$$O = \left[h^T,\; h^T\Phi, \ldots . \; h^T\Phi^{n-1}\right] \tag{3.1-34}$$

[X]

The canonical structure assumes that all the unknown parameters in the Φ matrix are identifable. The canonical form of theorem

(3.1-4) is also equivalent to a difference equation representation as given by the following theorem.

Theorem 3.1.5 [52, p 90] Given a difference equation

$$z_k = \sum_{i=1}^{n} \varphi_i z_{k-i} + \sum_{i=1}^{n} \beta_i u_{k-i} \tag{3.1-35}$$

it can be reduced to the standard canonical form $(\varphi^*, \underline{h}^*, \underline{d}^*)$ given by

$$\underline{x}_k = \varphi^* \underline{x}_{k-1} + \underline{d}^* u_{k-1} \tag{3.1-36}$$

$$z_k = \underline{h}^{*T} \underline{x}_k \tag{3.1-37}$$

$$\underline{d}^* = \begin{bmatrix} 1 & & & \\ -\varphi_1 & 1 & & \\ -\varphi_2 & -\varphi_1 & \ddots & 0 \\ \vdots & & \ddots & \\ -\varphi_{n-1} & -\varphi_{n-2} & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{3.1-38}$$

Reduction to canonical forms assumes that the system matrix φ is uniquely represented by n parameters which are identifiable. The form (3.1-35) is also referred to as the single-input single-output (SISO) model. In time series analysis these are referred to as the autoregressive moving average (ARMA) models. We shall consider

these representations in Chapter 4.

This concludes our discussion of system representation and identifiability. For more thorough discussion the reader is referred to [52, 61, 76, 83].

3.2 Distance Measure For Linear Gaussian Systems

Consider the linear system described by (3.1-1) and (3.1-2). For this Gauss-Markov model the joint conditional density function of the observations $\underline{Z}^N$ is a Gaussian density function $p(\underline{Z}^N|\underline{\theta})$. The mean and covariances are given by

$$\hat{\underline{Z}}^N = E\left[\underline{Z}^N|\underline{\theta}\right] \tag{3.2-1}$$

$$\sum_N = E\left[(\underline{Z}^N - \hat{\underline{Z}}^N)(\underline{Z}^N - \hat{\underline{Z}}^N)^T|\underline{\theta}\right] \tag{3.2-2}$$

For a given pair $(\underline{\theta}_\alpha, \theta_\gamma)$ in Θ the Bhattacharyya distance is given by

$$B_{\alpha\gamma} = \frac{1}{8}(\hat{\underline{Z}}_\alpha^N - \hat{\underline{Z}}_\gamma^N)^T\left[\sum_{\alpha\gamma}^{-1}\right](\hat{\underline{Z}}_\alpha^N - \hat{\underline{Z}}_\gamma^N)$$

$$+\frac{1}{2}\ln\left\{\frac{\det\sum_{\alpha\gamma}}{\left[\det\sum_{\alpha N}\det\sum_{\gamma N}\right]^{\frac{1}{2}}}\right\} \tag{3.2-3}$$

where

$$\sum_{\alpha\gamma} = \frac{1}{2}\left\{\sum_{\alpha N} + \sum_{\gamma N}\right\} \tag{3.2-4}$$

The covarance matrix $\sum_N$ is not a function of the deterministic open-loop inputs and hence the second term in (3.2-3) is independent of the inputs $\underline{U}^{N-1}$. Neglecting this term we have

$$B_{\alpha\gamma} = \frac{1}{8}\,(\hat{\underline{Z}}^N_\alpha - \hat{\underline{Z}}^N_\gamma)^T \sum_{\alpha\gamma}^{-1} (\hat{\underline{Z}}^N_\alpha - \hat{\underline{Z}}^N_\gamma) \tag{3.2-5}$$

When the covariances are equal (this happens when the dynamic noise is zero and the initial state is non random) such that $\sum_{\alpha N} = \sum_{\gamma N} = \sum$, the B-distance has the simple form

$$B_{\alpha\gamma} = \frac{1}{8}\,(\hat{\underline{Z}}^N_\alpha - \hat{\underline{Z}}^N_\gamma)^T \sum^{-1} (\hat{\underline{Z}}^N_\alpha - \hat{\underline{Z}}^N_\gamma) \tag{3.2-6}$$

$\sum$ consists of the observation noise covariance R. The following two theorems give the expressions for the B-distance.

Theorem 3.2.1 For the system described by (3.1-1) - (3.1-8) the joint Guassian probability density $p(\underline{Z}^N|\underline{\theta})$ is given by the mean and covariance functions as follows.

$$\hat{\underline{Z}}^N = E\left[\hat{\underline{Z}}^N|\underline{\theta}\right] = (\hat{\underline{z}}_1, \hat{\underline{z}}_2, \ldots, \hat{\underline{z}}_N)^T \tag{3.2-7}$$

where

$$\hat{\underline{z}}_k = H\left(\Phi^k \hat{\underline{x}}_0 + \sum_{i=1}^{k} \Phi^{k-i} \underline{\beta}\, u_{i-1}\right) \tag{3.2-8}$$

and

$$\sum_{N,k\ell} = \left\{ E\left[(\underline{z}_k - \hat{\underline{z}}_k)\,(\underline{z}_\ell - \hat{\underline{z}}_\ell)^T \,|\, \underline{\theta} \right] \right\} \tag{3.2-9}$$

The elements of the covariance matrix $\sum_N$ are given by

$$E\left[(\underline{z}_k - \hat{\underline{z}}_k)\,(\underline{z}_k - \hat{\underline{z}}_k)^T \,|\, \underline{\theta} \right] = H P_k H^T + R \tag{3.2-10}$$

$$k = 1, 2, \ldots, N$$

$$E\left[(\underline{z}_k - \hat{\underline{z}}_k)\,(\underline{z}_\ell - \hat{\underline{z}}_\ell)^T \,|\, \underline{\theta} \right] = H P_k \Phi^{\ell-k} H^T \tag{3.2-11}$$

$$k = 1, 2, \ldots, N-1$$

$$\ell = k+1, \ldots, N$$

with

$$P_k = E\left[(\underline{x}_k - \hat{\underline{x}}_k)\,(\underline{x}_k - \hat{\underline{x}}_k)^T \,|\, \theta \right] \tag{3.2-12}$$

generated recursively by

$$P_k = \Phi P_{k-1} \Phi^T + Q \quad , \; k = 1, 2, \ldots, N \tag{3.2-13}$$

with initial condition P_0.

[X]

Proof: Since $\underline{x}_0$ and $\underline{w}_k$ are Gaussian random variables and $\underline{x}_k$ is generated by the Gauss-Markov model of (3.1-1) $(\underline{x}_1, \ldots \underline{x}_N)$ are jointly Gaussian and $\underline{z}_k$ being the sum of Gaussian random variables $H\underline{x}_k$ and $\underline{v}_k$ it follows that $\underline{Z}^N$ is jointly Gaussian.

Now

$$\hat{\underline{Z}}^N = E\left[\underline{Z}^N \middle| \underline{\theta}\right] = \left[\hat{\underline{z}}_1, \hat{\underline{z}}_2, \ldots, \hat{\underline{z}}_N\right]^T \quad \text{and}$$

$$\hat{\underline{z}}_k = H\, E\left[\underline{x}_k \middle| \underline{\theta}\right] = H\left(\Phi^k \hat{\underline{x}}_0 + \sum_{i=1}^{k} \Phi^{k-i} \underline{\beta}\, u_{i-1}\right) \tag{3.2-14}$$

Further

$$E\left[(\underline{z}_k - \hat{\underline{z}}_k)(\underline{z}_k - \hat{\underline{z}}_k)^T \middle| \underline{\theta}\right]$$

$$= E\left\{(H\underline{x}_k + \underline{v}_k - H\hat{\underline{x}}_k)(H\underline{x}_k + \underline{v}_k - H\hat{\underline{x}}_k)^T \middle| \underline{\theta}\right\}$$

$$= H\, E\left[(\underline{x}_k - \hat{\underline{x}}_k)(\underline{x}_k - \hat{\underline{x}}_k)^T \middle| \theta\right] + R_k \tag{3.2-15}$$

Let $P_k = E\left[(\underline{x}_k - \hat{\underline{x}}_k)(\underline{x}_k - \hat{\underline{x}}_k)^T \middle| \underline{\theta}\right]$

Then (3.2-15) becomes

$$E\left[(\underline{z}_k - \hat{\underline{z}}_k)(\underline{z}_i - \hat{\underline{z}}_k) \middle| \theta\right] = H P_k H^T + R \quad , \; k=1, 2, \ldots, N \tag{3.2-16}$$

Because of the Markov nature of the system equation P_k can be written recursively as

$$P_k = \Phi P_{k-1} \Phi^T + Q \, , \; k=1, 2, \ldots, N \tag{3.2-17}$$

with initial condition $P_0 = E\left[(\underline{x}_0 - \hat{\underline{x}}_0)(\underline{x}_0 - \hat{\underline{x}}_0)^T\right]$. To compute off-diagonal elements of $\sum_N$, consider

$$E\left[(\underline{z}_k - \hat{\underline{z}}_k)(\underline{z}_\ell - \hat{\underline{z}}_\ell)^T|\theta\right]$$

$$= E\left\{(H\underline{x}_k + \underline{v}_k - H\hat{\underline{x}}_k)(H\underline{x}_\ell + \underline{v}_\ell - H\hat{\underline{x}}_\ell)^T|\underline{\theta}\right\} \qquad (3.2\text{-}18)$$

$$k < \ell$$

$\underline{x}_\ell$ is written in terms of $\underline{x}_k$ as follows

$$\underline{x}_\ell = \Phi^{\ell-k}\underline{x}_k + \sum_{i=1}^{\ell-k}\Phi^{\ell-k-i}\underline{\beta}\,u_{k+i-1} + \sum_{i=1}^{\ell-k}\Phi^{\ell-k-i}\,\underline{w}_{k+i-1}$$

$$\underline{x}_\ell - \hat{\underline{x}}_\ell = \Phi^{\ell-k}(\underline{x}_k - \hat{\underline{x}}_k) + \sum_{i=1}^{\ell-k}\Phi^{\ell-k-i}\,\underline{w}_{k+i-1}$$

Hence

$$E\,(\underline{z}_k - \hat{\underline{z}}_k)(\underline{z}_\ell - \hat{\underline{z}}_\ell)^T|\underline{\theta} = H\,E\left[(\underline{x}_k - \hat{\underline{x}}_k)(\underline{x}_k - \hat{\underline{x}}_k)^T|\underline{\theta}\right]\Phi^{\ell-k}H^T$$

$$= H P_k \Phi^{\ell-k} H^T \qquad (3.2\text{-}19)$$

$$k=1,2,\ldots,N-1$$

$$\ell=k+1,\ldots,N$$

□

Theorem 3.2.2 The pairwise distance measure between the densities $p(\underline{Z}^N|\underline{\theta}_\alpha)$ and $p(\underline{Z}^N|\underline{\theta}_\gamma)$ is given by

$$B^N_{\alpha\gamma} = \frac{1}{8}\underline{X}_0^T\sum_{\alpha\gamma}^{-1}\underline{X}_0 + \frac{1}{8}\underline{U}^{N-1T}M^T_{\alpha\gamma}\sum_{\alpha\gamma}^{-1}M_{\alpha\gamma}\underline{U}^{N-1}$$

$$+ \frac{1}{4}\underline{X}_0^T\sum_{\alpha\gamma}^{-1}M_{\alpha\gamma}\underline{U}^{N-1} \qquad (3.2\text{-}20)$$

where

$$\underline{x}_0^T = \left[(H_\alpha \underline{\Phi}_\alpha - H_\gamma \Phi_\gamma)\,\hat{\underline{x}}_0\,,\ (H_\alpha \Phi_\alpha^2 - H_\gamma \Phi_\gamma^2)\,\hat{\underline{x}}_0\,,\ \ldots\right.$$

$$\left.(H_\alpha \Phi_\alpha^N - H_\gamma \Phi_\gamma^N)\,\hat{\underline{x}}_0\right] \qquad (3.2\text{-}21)$$

$$\sum_{\alpha\gamma} = \frac{1}{2}\left(\sum_{\alpha N} + \sum_{\gamma N}\right) \text{ is } (Nn \times Nn) \text{ matrix} \qquad (3.2\text{-}22)$$

$$M_{\alpha\gamma} = M_\alpha - M_\gamma \qquad (3.2\text{-}23)$$

$$M_\alpha = \begin{bmatrix} H_\alpha \underline{\beta}_\gamma & & & \\ H_\alpha \underline{\Phi}_\alpha \underline{\beta}_\alpha & H_\alpha \underline{\beta}_\alpha & & \\ \vdots & & \ddots & \\ H_\alpha \Phi_\alpha^{N-1} \beta_\alpha & H_\alpha \Phi_\alpha^{N-2} \beta_\alpha & \cdots & H_\alpha \underline{\beta}_\alpha \end{bmatrix} \text{ is } (Nn \times N) \text{ matrix}$$

(3.2-24)

[X]

Proof: From equation (3.2-5) we have

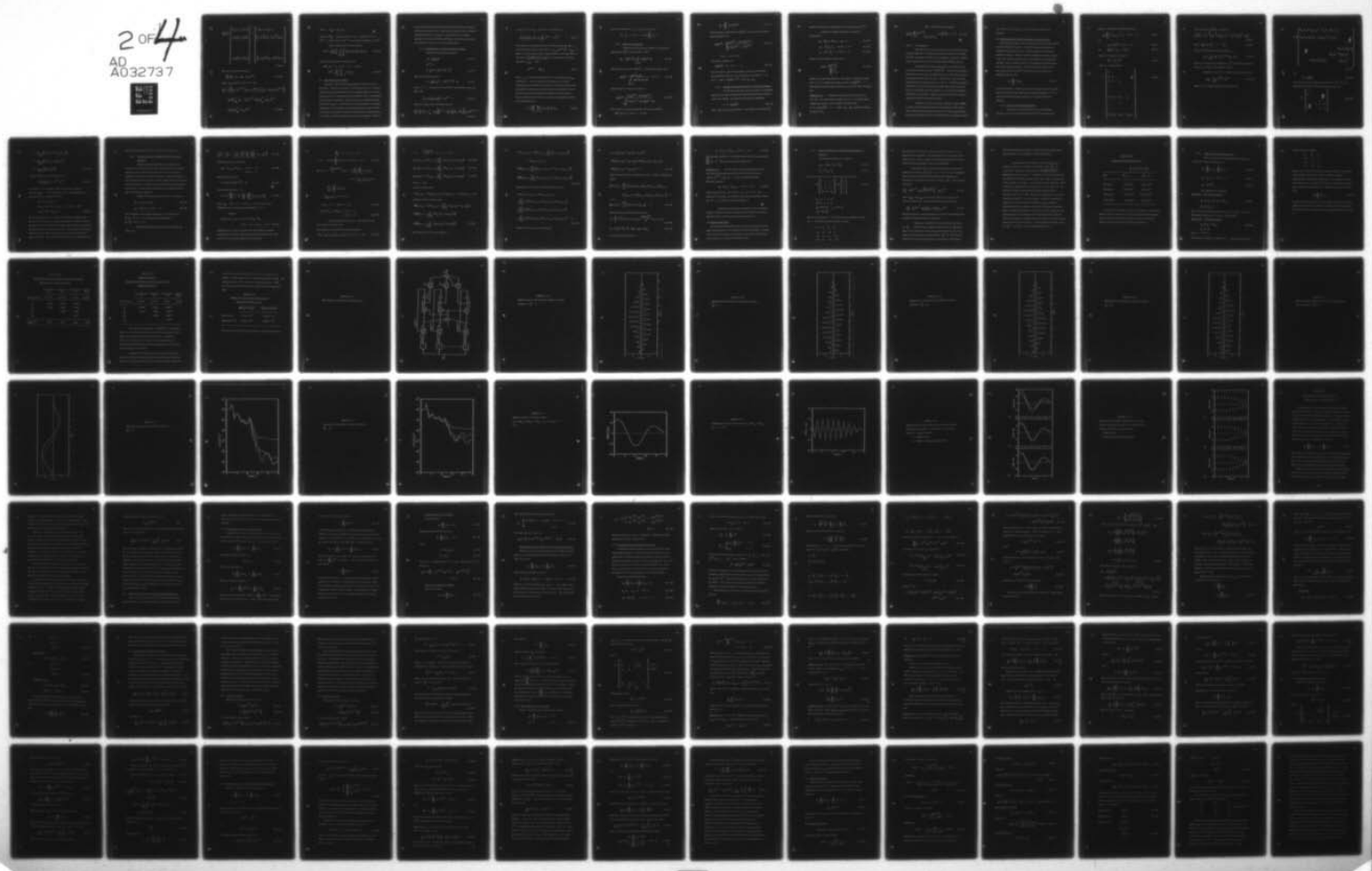
$$B_{\alpha\gamma}^N = \frac{1}{8}\left(\hat{\underline{Z}}_\alpha^N - \hat{\underline{Z}}_\gamma^N\right)^T \sum_{\alpha\gamma}^{-1} \left(\hat{\underline{Z}}_\alpha^N - \hat{\underline{Z}}_\gamma^N\right) \qquad (3.2\text{-}25)$$

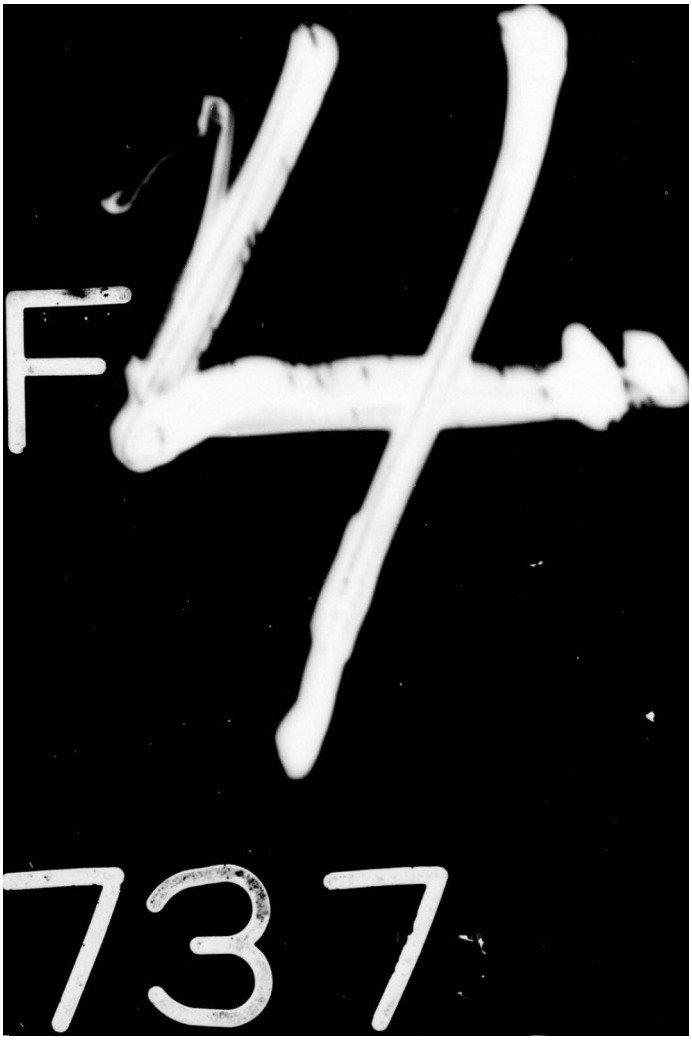
where

$$\sum_{\alpha\gamma} = \frac{1}{2}\left(\sum_{\alpha N} + \sum_{\gamma N}\right)$$

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$$\hat{\underline{Z}}_{\alpha}^{N} - \hat{\underline{Z}}_{\gamma}^{N} = \begin{bmatrix} (H_{\alpha}\Phi_{\alpha} - H_{\gamma}\Phi_{\gamma})\hat{\underline{x}}_0 \\ (H_{\alpha}\Phi_{\alpha}^{2} - H_{\gamma}\Phi_{\gamma}^{2})\hat{\underline{x}}_0 \\ \vdots \\ (H_{\alpha}\Phi_{\alpha}^{N} - H_{\gamma}\Phi_{\gamma}^{N})\hat{\underline{x}}_0 \end{bmatrix} + \begin{bmatrix} H_{\alpha}\underline{\beta}_{\alpha} u_0 - H_{\gamma}\underline{\beta}_{\gamma} u_0 \\ \sum_{i-1}^{2} (H_{\alpha}\Phi_{\alpha}^{2-i}\underline{\beta}_{\alpha} - H_{\gamma}\Phi_{\gamma}^{2-i}\underline{\beta}_{\gamma})u_{i-1} \\ \vdots \\ \sum_{i=1}^{N} (H_{\alpha}\Phi_{\alpha}^{N-i}\underline{\beta}_{\alpha} - H_{\gamma}\Phi_{\gamma}^{N-i}\underline{\beta}_{\gamma})u_{i-1} \end{bmatrix} \tag{3.2-26}$$

This can be written concisely as

$$\hat{\underline{Z}}_{\alpha}^{N} - \hat{\underline{Z}}_{\gamma}^{N} = \underline{X}_0 + (M_{\alpha} - M_{\gamma})\,\underline{U}^{N-1} \tag{3.2-27}$$

where M_{α} is given by (3.2-24). Thus

$$P_{\alpha\gamma}^{N} = \frac{1}{8}\left[\left\{\underline{X}_0 + \underline{U}^{N-1\,T}(M_{\alpha} - M_{\gamma})^{T}\right\}\sum_{\alpha\gamma}^{-1}\left\{\underline{X}_0 + (M_{\alpha} - M_{\gamma})\,\underline{U}^{N-1}\right\}\right]$$

$$= \frac{1}{8}\,\underline{X}_0^{T}\sum_{\alpha\gamma}^{-1}\underline{X}_0 + \frac{1}{8}\,\underline{U}^{N-1\,T} M_{\alpha\gamma}^{T}\sum_{\alpha\gamma}^{-1} M_{\alpha\gamma}\,\underline{U}^{N-1}$$

$$+ \frac{1}{4}\,\underline{X}_0^{T}\sum_{\alpha\gamma}^{-1} M_{\alpha\gamma}\,\underline{U}^{N-1} \tag{3.2-28}$$

where $M_{\alpha\gamma} = M_\alpha - M_\gamma$. [X]

Note that $B^N_{\alpha\gamma}$ consists of three terms - a quadratic term in $\underline{U}^{N-1}$, a linear term in $\underline{U}^{N-1}$ and a constant independent of $\underline{U}^{N-1}$.

The average B-distance then becomes

$$\bar{B}^N = E\left[B^N_{\alpha\gamma}\right] = \int_{\Theta} \int_{\Theta} B^N_{\alpha\gamma}\, p(\theta_\alpha)\, p(\theta_\gamma)\, d\theta_\alpha\, d\theta_\gamma \qquad (3.2\text{-}29)$$

When $\underline{\theta}$ has a discrete distribution with

$P(\underline{\theta} = {}_i\underline{\theta}) = P_i$, $i=1, 2, \ldots, m$ we have

$$\bar{B}^N = 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P_i P_j B_{ij} \qquad (3.2\text{-}30)$$

3.3 The Optimization Problem

The form of the optimal inputs depends on the constraint on the inputs. When the constraint is of the energy type, the constraint region is an N-dimensional ball, and the optimal inputs are given by the eigenvector of a symmetric positive definite matrix corresponding to the maximum eigenvalue. When the inputs are amplitude limited the optimal sequence is controlled by a switching function, the inputs saturating at their maximum values giving bang-bang type of signal. Using Bayes' estimation scheme it is shown that for a given accuracy of learning, a certain quadratic form must be nonnegative. This is a

function of the total input energy and the matrices that are functions of discrete parameter sets. This result is similar to that of Åström and Bohlin [6] where the inputs are required to be persistently exciting. The result of Åström and Bohlin, unlike the one derived in this section, is a function only of the input signals.

3.3.1 Optimal Inputs for Energy Constrained Case

Consider the optimization problem

$$\max \; E_{\underline{\theta}} \left[B_{\alpha\gamma} \right] \tag{3.3-1}$$

$$\underline{U}^{N-1} \in \upsilon$$

$$\upsilon = \left\{ \underline{U}^{N-1} \mid \| U^{N-1} \|^2 \leq Y \right\} \tag{3.3-2}$$

form the Lagrangian function

$$L = E_{\underline{\theta}} \left[B_{\alpha\gamma} \right] - \lambda \left(\| U^{N-1} \|^2 - Y \right) \tag{3.3-3}$$

where $\lambda > 0$. The necessary condition for optimization is given by [19, p 70] .

$$\frac{1}{2} \nabla_u \left\{ E_{\underline{\theta}} \left[B_{\alpha\gamma} \right] \right\} - \lambda \underline{U}^{N-1} = 0 \tag{3.3-4}$$

Using (3.2-20) the above condition becomes

$$\frac{1}{8} \left\{ E_{\underline{\theta}} \left[M_{\alpha\gamma}^T \; \Sigma_{\alpha\gamma}^{-1} \; M_{\alpha\gamma} \right] \right\} \underline{U}^{N-1} + \frac{1}{8} \left\{ E_{\underline{\theta}} \left[M_{\alpha\gamma}^T \; \Sigma_{\alpha\gamma}^{-1} \right] \right\} \underline{X}_0 - \lambda \underline{U}^{N-1} = 0 \tag{3.3-5}$$

Let $\hat{\underline{x}}_0 = \underline{0}$ then $\underline{x}_0 = \underline{0}$ and (3.3-5) simplifies to

$$\frac{1}{8}\left\{E_\theta\left[M_{\alpha\gamma}^T\ \Sigma_{\alpha\gamma}^{-1}\ M_{\alpha\gamma}\right]\right\}\ \underline{U}^{N-1} = \lambda\ \underline{U}^{N-1} \qquad (3.3\text{-}6)$$

The maximum of the quadratic form is achieved when the RHS in (3.3-6) has its maximum value which is $\lambda_{max.} \left\| U^{N-1} \right\|^2 = \lambda_{max.}\ Y$. Hence it is clear that the optimal input sequence is the eigenvector of the matrix $\frac{1}{8}\left\{E_\theta\left[M_{\alpha\gamma}^T \Sigma_{\alpha\gamma}^{-1} M_{\alpha\gamma}\right]\right\}$ corresponding to its largest eigenvalue $\lambda_{max.}$. Thus

$$\underline{U}^{N-1^*} = \sqrt{Y}\ \underline{e}_m \qquad . \qquad (3.3\text{-}7)$$

where $\underline{e}_m$ is the normalized eigenvector corresponding to $\lambda_{max.}$. For large N, direct evaluation of the eigenvectors is time consuming and in Appendix C we present a numerical method based on gradient calculations to solve the optimization problem (3.3-1). In actual numerical calculations the parameter set is assumed discrete taking on finite number of values and the quadratic matrix Q is calculated using

$$Q = \frac{1}{4}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} P_i\ P_j\ M_{ij}^T\ \Sigma_{ij}^{-1}\ M_{ij} \qquad (3.3\text{-}8)$$

The prior probabilities P_i are all assumed same.

$$P_i = \frac{1}{m} \ , \ i = 1, 2, \ldots, m \text{ and } \sum_{i-1}^{m} P_i = 1 \ .$$

3.3.2 Parameter Estimation

The minimum mean-square estimate of $\underline{\theta}$ based on observations $\left\{\underline{Z}^k\right\}$ is given by [72, p 227]

$$\underline{\theta}_k = E\left[\underline{\theta} \mid \underline{Z}^k\right] = \int_{\theta \in \Theta} \underline{\theta}\ p(\underline{\theta} \mid \underline{Z}^k)\ d\underline{\theta} \qquad (3.3\text{-}9)$$

The a posteriori density $p(\underline{\theta} \mid \underline{Z}^k)$ is given by the Bayes' rule

$$p(\underline{\theta} \mid \underline{Z}^k) = \frac{p(\underline{Z}^k \mid \underline{\theta})\, p(\underline{\theta})}{\int_{\theta \in \Theta} p(\underline{Z}^k \mid \underline{\theta})\ p(\underline{\theta})\ d\underline{\theta}} \ , \ k=1, 2, \ldots . \qquad (3.3\text{-}10)$$

Recursively (3.3-10) can be written as

$$p(\underline{\theta} \mid \underline{Z}^k) = \frac{p(\underline{z}_k \mid \underline{Z}^{k-1}, \underline{\theta})\ p(\underline{\theta} \mid \underline{Z}^{k-1})}{\int_{\theta \in \Theta} p(\underline{z}_k \mid \underline{Z}^{k-1}, \underline{\theta})\ p(\underline{\theta} \mid \underline{Z}^{k-1})\ d\underline{\theta}} \qquad (3.3\text{-}11)$$

When $\underline{\theta}$ has a discrete distribution with prior probability

$P(\underline{\theta} = {}_i\underline{\theta}) = P_i$, i=1, 2, ..., m then

$$\underline{\theta}_k = \sum_{i=1}^{m} {}_i\underline{\theta}\, P({}_i\underline{\theta} \,|\, \underline{Z}^k) \tag{3.3-12}$$

The a posteriori probabilities $P({}_i\theta \,|\, \underline{Z}^k)$ are recursively updated using the Bayes' rule.

$$P({}_i\underline{\theta} \,|\, \underline{Z}^k) = \frac{p(\underline{z}_k \,|\, \underline{Z}^{k-1}, {}_i\theta)\, P({}_i\theta \,|\, \underline{Z}^{k-1})}{\sum_{j=1}^{m} p(\underline{z}_k \,|\, \underline{Z}^{k-1}, {}_j\theta \,|\, \underline{Z}^{k-1})} \tag{3.3-13}$$

$$i=1, 2, \ldots, m; \quad k=1, 2, \ldots$$

The initial conditions are

$$P({}_i\underline{\theta} \,|\, \underline{Z}^0) = P_i, \quad i=1, 2, \ldots, m \tag{3.3-14}$$

Braverman [17, p 29] showed that the a posteriori probability corresponding to the true set of parameters converges to 1 as $N \to \infty$. That is $P({}_i\underline{\theta} \,|\, \underline{Z}^k) \xrightarrow{a.s.} 1$ when ${}_i\underline{\theta} = \underline{\theta}_0$.

3.3.3 Accuracy of Learning-Condition on the Input Sequence

For the linear Gaussian system we obtain a relationship between the accuracy of parameter learning and the input signal-to-noise ratio (ISNR). The accuracy of learning is defined as

$$\epsilon = 1 - P(\underline{\theta}_0 \,|\, \underline{Z}^N) \tag{3.3-15}$$

where $\underline{\theta}_0$ is the true parameter. The condition is derived as the

positive definiteness of a quadratic form involving inputs $\underline{U}^{N-1}$.

Consider the following dynamics with scalar observations.

$$\underline{x}_k = \Phi \underline{x}_{k-1} + \underline{\beta}\, u_{k-1} \ , \ \underline{x}_0 = \underline{a} \tag{3.3-16}$$

$$z_k = \underline{h}^T \underline{x}_k + v_k \ , \ k=1,2,\ldots, N \tag{3.3-17}$$

$$v_k \sim G(0, \sigma^2_{v_k}) \ , \ k=1,2,\ldots, N \tag{3.3-18}$$

Define the input signal-to-noise ratio as

$$\text{ISNR} = \frac{\| \underline{U}^{N-1} \|^2}{\sum_{k=1}^{N} \sigma^2_{v_k}} \tag{3.3-19}$$

Suppose we require the following: For a given N (large but finite) and $\epsilon > 0$ (ϵ sufficiently small) we require $P(\underline{\theta}_0 | \underline{Z}^N) > 1 - \epsilon$. The following theorem shows that there is a value of ISNR satisfying this.

Theorem 3.3.1 Consider the system described by (3.3-16) - (3.3-18). For a given finite intergal N and $\epsilon > 0$ (ϵ sufficiently small) there exists a value of ISNR such that for all $n \geq N$, $P(\underline{\theta}_0 | \underline{Z}^N) > 1 - \epsilon$, where $\underline{\theta}_0 = \iota \underline{\theta}$ is the true value of the parameters.

This is governed by the inequality

$$\frac{1}{2(m-1)} \sum_{i \neq \ell}^{m} \frac{\| \underline{U}^{N-1} \|^2}{M_{\ell i}^T R_N^{-1} M_{\ell i}} > \ln \left[\frac{m-1}{\epsilon} \right], \quad m \geq 2 \qquad (3.3\text{-}20)$$

[X]

Proof: See Appendix D

The required accuracy in learning can be achieved by using a proper ISNR above a certain threshold as dictated by the inequality (3.3-20). The above condition can be treated as a sufficient condition for the estimation error to be less than a given fidelity criterion.

Assuming that the observation noise covariance is fixed, for a given total input energy $Y = \| \underline{U}^{N-1} \|^2$, it may not be possible to attain this accuracy since the ISNR given by (3.3-19) need not satisfy the conditions of the above theorem. The theorem requires that the input energy be variable for a given ϵ and N. Conditions analogous to the above are given by Åström and Bohlin [6] and Aoki and Staley [3], to be satisfied by an input sequence for the maximum likelihood estimators to be efficient. These are stated exclusively in terms of sample correlation of the input sequence.

In Chapter 4 where we consider stochastic inputs we need to impose certain mild restrictions on the spectrum of the signals. These requirements primarily characterize the input signals and by the positive definiteness of the corresponding input correlation matrix,

the condition of "persistently exciting" inputs is automatically satisfied.

3.4 Characterization of Open-loop Inputs for Linear Systems

The open-loop signals derived in Section (3.3) can be approximated by linear difference equations. When the unknown parameters are contained in the gain vector $\underline{\beta}$, the inputs satisfy a second order difference equation. When the unknown parameters are also contained in the state transition matrix such a representation is not explicitly possible. Here the optimal inputs can be adequately represented by a low order difference equation. The numerical fit is obtained by a least squares determination of the parameters of the difference equation having the form

$$u_k = \sum_{i=1}^{s} \xi_i u_{k-i} \qquad (3.4\text{-}1)$$

It is shown that for a simple linear regression type system the optimal input corresponds to the eigenvector of a banded Toeplitz matrix. In this case the input sequence naturally obeys a linear difference equation.

3.4.1 The Linear Regression Problem

The linear regression model, because of its special structure, gives rise to closed form solution for the optimal input

problem. Consider the following dynamics:

$$z_k = \sum_{i=1}^{p} \theta_i u_{k-i+1} + v_k \ , \quad k=1,2,\ldots, N \tag{3.4.2}$$

let

$$v_k \sim G(0, \sigma^2), \quad k=1,2,\ldots, N \tag{3.4-3}$$

let $$\underline{\theta} \triangleq (\theta_1 , \theta_2 , \ldots , \theta_p)^T \tag{3.4-4}$$

Then for N stages we can write

$$\underline{Z}^N = A\,\underline{\theta} + \underline{V}^N \tag{3.4-5}$$

where A is Nxp matrix.

$$A = \begin{bmatrix} u_1 & & & & 0 \\ u_2 & u_1 & & & \\ u_3 & u_2 & u_1 & & \\ \cdot & & \ddots & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \\ u_p & u_{p-1} & u_{p-2} & \cdots & u_1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ u_N & u_{N-1} & u_{N-2} & \cdots & u_{N-p+1} \end{bmatrix} \tag{3.4-6}$$

The joint density $p(\underline{Z}^N|\underline{\theta})$ is Gaussian

$$p(\underline{Z}^N|\underline{\theta}) = (2\pi)^{-\frac{N}{2}} |R_N|^{-\frac{1}{2}} \exp\left[-\frac{1}{2} (\underline{Z}^N - A\underline{\theta})^T R_N^{-1} (\underline{Z}^N - A\underline{\theta})\right] \tag{3.4-7}$$

where $R_N \underline{\Delta} \text{diag}\,(\sigma_v^2, \sigma_v^2, \ldots, \sigma_v^2)$ (3.4-8)

The pairwise distance measure is given by

$$B_{\alpha\gamma} = \frac{1}{8\sigma_v^2} (\underline{\theta}_\alpha - \underline{\theta}_\gamma)^T A^T A (\underline{\theta}_\alpha - \underline{\theta}_\gamma) \tag{3.4-9}$$

This can be rewritten as a quadratic in $\underline{U}^N$ such that the quadratic matrix is a function of the parameter values.

Consider the case $p=2$. Then (3.4-9) can be written as

$$B_{\alpha\gamma} = \frac{1}{8\sigma_v^2} \underline{U}^{N^T} Q' \underline{U}^{N-1} \tag{3.4-10}$$

where Q' is a tridiagonal matrix having the form

$$\Omega' \triangleq \begin{bmatrix} (\theta_{1\alpha}-\theta_{1\gamma})^2+(\theta_{2\alpha}-\theta_{2\gamma})^2 & (\theta_{1\alpha}-\theta_{1\gamma})(\theta_{2\alpha}-\theta_{2\gamma}) & & & \\ (\theta_{1\alpha}-\theta_{1\gamma})(\theta_{1\alpha}-\theta_{2\gamma}) & (\theta_{1\alpha}-\theta_{1\gamma})^2+(\theta_{2\alpha}-\theta_{2\gamma})^2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & (\theta_{1\alpha}-\theta_{1\gamma})^2+(\theta_{2\alpha}-\theta_{2\gamma})^2 & (\theta_{1\alpha}-\theta_{1\gamma})(\theta_{2\alpha}-\theta_{2\gamma}) \\ & & & (\theta_{1\alpha}-\theta_{1\gamma})(\theta_{2\alpha}-\theta_{2\gamma}) & (\theta_{1\alpha}-\theta_{1\gamma})^2 \end{bmatrix} \tag{3.4-11}$$

Let

$$\Omega = E_\theta\left[\Omega'\right] \tag{3.4-12}$$

where the averaging is made over the space $\Theta \times \Theta$. Ω is a tridiagonal form and is a function of the moments of θ .

$$\Omega' = \begin{bmatrix} a & b & & & & \\ b & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & b & & \\ & & b & a & b & \\ & & & b & a' \end{bmatrix} \tag{3.4-13}$$

$$a = E_{\theta_\alpha \theta_\gamma}\left[(\theta_{1\alpha}-\theta_{1\gamma})^2 + (\theta_{2\alpha}-\theta_{2\gamma})^2\right]$$

$$b = E_{\theta_\alpha \theta_\gamma}\left[(\theta_{1\alpha}-\theta_{1\gamma})(\theta_{2\alpha}-\theta_{2\gamma})\right]$$

$$a' = E_{\theta_\alpha \theta_\gamma}\left[(\theta_{1\alpha}-\theta_{1\gamma})^2\right] \qquad (3.4\text{-}14)$$

Finally the distance measure has the form

$$8\sigma_v^2 E_\theta\left[B_{\alpha\gamma}\right] = \underline{U}^{N^T} Q \underline{U}^N \qquad (3.4\text{-}15)$$

The matrix Q is an almost toeplitz form except for the last element $q_{NN} = a'$. The eigenvector of Q corresponding to any eigenvalue can be computed as follows:

$$(a - \lambda) u_1 + b u_2 = 0$$

$$b u_{i-1} + (a - \lambda) u_i + b u_{i+1} = 0 \quad i=2, 3, \ldots, N-1$$

$$b u_{N-1} + (a' - \lambda) u_N = 0 \qquad (3.4\text{-}16)$$

We see that except for $i-1$ and N the remaining components are generated by the same order difference equation. When the number of parameters $p > 2$, the order of the difference equation increases and the approximation becomes less accurate. This special case, however, illustrates that a difference equation representation of the optimal inputs is feasible. The next section is devoted to the representation of

optimal signals in dynamic systems described in Section 3.3.

3.4.2 Generating Function for Optimal Signals in Linear Dynamics

Discrete optimal control theory is applied to the characterization of open-loop inputs to show that when the unknown parameters are contained in the gain vector the input sequence satisfies a second order difference equation. The generalization to the case when the state transition matrix contains unknown parameters does not yield such a result because of the difficulty in manipulating the equations. In this situation the signals can be approximated by low order difference equations.

Consider the following scalar discrete time system:

$$x_k = \varphi x_{k-1} + \beta u_{k-1} \tag{3.4-17}$$

$$z_k = h x_k + v_k , \quad k=1,2,\ldots, N \tag{3.4-18}$$

Let us consider a discrete distribution for $\theta = \beta$ taking values $\beta = \beta_i$, $i=1,2,\ldots, m$ with equal prior probabilities $P_i = \frac{1}{m}$, $i=1,2,\ldots, m$.

The following optimization problem is stated for the above case:

$$\max_{\underline{U}^{N-1}} \bar{B} = \max_{\underline{U}^{N-1}} \frac{h^2}{2m^2\sigma_v^2} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \left(\sum_{k=1}^{N} (x_{ik} - x_{jk})^2 \right) \qquad (3.4\text{-}19)$$

subject to the linear constraints

$$x_{ik} = \varphi\, x_{i,k-1} + \beta_i\, u_{k-1}\,, \quad i=1,2,\ldots,m \qquad (3.4\text{-}20)$$

$$k=1,2,\ldots,N$$

and the input constraint

$$\upsilon = \left\{ \underline{U}^{N-1} \mid \|\underline{U}^{N-1}\|^2 \le \Upsilon \right\} \qquad (3.4\text{-}21)$$

[X]

Define the Hamiltonian H as

$$H = \bar{B} - \lambda \left(\sum_{i=1}^{N-1} u_i^2 - \Upsilon \right) + \sum_{i=1}^{m} \left(\sum_{k=0}^{N-1} p_{ik}\, f_{ik}(x_k, u_k) \right) \qquad (3.4\text{-}22)$$

where p_{ik}, $i=1,2,\ldots,m$; $k=0,1,\ldots,N-1$ are the costate multipliers and λ is the Lagrange multiplier, $\lambda > 0$.

Further

$$f_{ik}(x_k, u_k) = x_{i,k+1} - x_{ik} = (\varphi - 1)\, x_{ik} + \beta_i u_k$$

$$i=1,2,\ldots,m \ ; \ k=0,1,\ldots,N-1 \qquad (3.4\text{-}23)$$

Following Canon, Cullum and Polak [19, pp 75-84] the necessary conditions for the discrete optimization problem defined by (3.4-19)-(3.4-21) are given by the following set of equations:

$$\frac{\partial H}{\partial u_k} = 0 \ , \ k=0, 1, \ldots, N-1$$

$$\Longrightarrow \qquad -2\lambda u_k + \sum_{i=1}^{m} p_{ik} \beta_i = 0 \ , \quad k=0, 1, \ldots, N-1 \qquad (3.4\text{-}24)$$

$$p_{ik} - p_{i,k+1} = \frac{\partial f_{ik}}{\partial x}(x_k, u_k)\, p_{i,k+1} + \frac{h^2}{m^2\sigma_v^2} \sum_{j=i+1}^{m} (x_{ik} - x_{jk})$$

for $i \leq m-1$

$$- \frac{h^2}{m\sigma_v^2} \sum_{j=1}^{i-1} (x_{jk} - x_{ik})$$

for $i \geq 2$

$$i = 1, 2, \ldots, m \ ; \ k=0, 1, \ldots, N-1 \qquad (3.4\text{-}25)$$

$$x_{ik} = \Phi x_{i,k-1} + \beta_i u_{k-1} \ , \quad i=1, 2, \ldots, m$$

$$k-1, 2, \ldots, N \qquad (3.4\text{-}26)$$

Let us consider the case $m=3$. The result for any $m > 3$ follows from this case.

Then equation (3.4-24) and (3.4-25) take the form

$$-2\lambda u_k + p_{1k} \beta_1 + p_{2k} \beta_2 + p_{3k} \beta_3 = 0 \ , \quad k=0, 1, \ldots, N-1 \qquad (3.4\text{-}27)$$

with $\dfrac{\partial f_{ik}(x_k, u_k)}{\partial k} = \Phi - 1 = \alpha$, we have

$$p_{1k}-p_{1,k+1} = \alpha p_{1,k+1} + \frac{h^2}{m^2\sigma_v^2}\left|(x_{1k}-x_{2k}) + (x_{1k}-x_{3k})\right| \qquad (3.4\text{-}28a)$$

$$p_{2k}-p_{2,k+1} = \alpha p_{2,k+1} + \frac{h^2}{m^2\sigma_v^2}\left|(x_{2k}-x_{3k}) - (x_{1k}-x_{2k})\right| \qquad (3.4\text{-}28b)$$

$$p_{3k}-p_{3,k+1} = \alpha p_{3,k+1} - \frac{h^2}{m^2\sigma_v^2}\left|(x_{1k}-x_{3k}) + (x_{2k}-x_{3k})\right| \qquad (3.4\text{-}28c)$$

$k=0, 1, \ldots, N-1$

From (3.4-27) we have

$$-2\lambda(u_k-u_{k+1}) + \beta_1(p_{1k}-p_{1,k+1}) + \beta_2(p_{2k}-p_{2,k+1}) + \beta_3(p_{3k}-p_{3,k+1}) = 0$$

Using (3.4-28) in the above gives

$$-2\lambda\left((u_k-u_{k+1}) + \beta_1\left(\alpha p_{1,k+1} + \frac{h^2}{m^2\sigma_v^2}\left|(x_{1k}-x_{2k}) + (x_{1k}-x_{3k})\right|\right)\right.$$

$$+ \beta_2\left(\alpha p_{2,k+1} + \frac{h^2}{m^2\sigma_v^2}\left|(x_{2k}-x_{3k}) - (x_{1k}-x_{2k})\right|\right)$$

$$\left. + \beta_3\left(\alpha p_{3,k+1} - \frac{h^2}{m^2\sigma_v^2}\left|(x_{1k}-x_{3k}) + (x_{2k}-x_{3k})\right|\right)\right) \qquad (3.4\text{-}29)$$

Also, putting $k=k+1$ in (3.4-29) gives

$$-2\lambda(u_{k+1}-u_{k+2}) + \beta_1\left(\alpha p_{1,k+2} + \frac{h^2}{m^2\sigma_v^2}\left[(x_{1,k+1}-x_{2,k+1})\right]\right)$$

$$+ (x_{1,k+1}-x_{3,k+1})$$

$$+\beta_2\left(\alpha p_{2,k+2} + \frac{h^2}{m^2\sigma_v^2}\left[(x_{2,k+1}-x_{3,k+1}) - (x_{1,k+1}-x_{2,k+1})\right]\right)$$

$$+\beta_3\left(\alpha p_{3,k+2} - \frac{h^2}{m^2\sigma_v^2}\left[(x_{1,k+1}-x_{3,k+1}) + (x_{2,k+1}-x_{3,k+1})\right]\right)$$

(3.4-30)

Subtracting (3.4-30) from (3.4-29) and using (3.4-27)

$$-2\lambda(u_k - 2u_{k+1} + u_{k+2}) + 2\alpha\lambda(u_{k+1}-u_{k+2})$$

$$+ \frac{h^2}{m^2\sigma_v^2}\beta_1\left[2(x_{1k}-x_{1,k+1})-(x_{2k}-x_{2,k+1})-(x_{3k}-x_{3,k+1})\right]$$

$$+ \frac{h^2}{m^2\sigma_v^2}\beta_2\left[2(x_{2k}-x_{2,k+1})-(x_{3k}-x_{3,k+1})-(x_{1k}-x_{1,k+1})\right]$$

$$+ \frac{h^2}{m^2\sigma_v^2}\beta_3\left[2(x_{3k}-x_{3,k+1})-(x_{1k}-x_{1,k+1})-(x_{2k}-x_{2,k+1})\right] = 0$$

(3.4-31)

Using (3.4-26) this can be simplified as

$$A_1 + u_k \left|(\beta_1-\beta_2)^2 + (\beta_2-\beta_3)^2 + (\beta_3-\beta_1)^2\right|$$

$$+ \alpha\beta_1 \left[(x_{1k}-x_{2k}) + (x_{1k}-x_{3k})\right] + \alpha\beta_2 \left[(x_{2k}-x_{3k}) + (x_{2k}-x_{1k})\right]$$

$$+ \alpha\beta_3 \left[(x_{3k}-x_{1k}) + (x_{3k}-x_{2k})\right] = 0 \qquad (3.4\text{-}32)$$

The terms in (3.4-32) are related to (3.4-29). Further simplification yields

$$\frac{1}{\alpha}(A_1+A_2) + \frac{m^2\sigma_v^2}{h^2}\left[\beta_1 p_{1k}+\beta_2 p_{2k}+\beta_3 p_{3k} - (\beta_1 p_{1,k+1}+\beta_2 p_{2,k+1}\right.$$

$$\left. +\beta_3 p_{3,k+1}) - \alpha(\beta_1 p_{1,k+1}+\beta_2 p_{2,k+1}+\beta_3 p_{3,k+1})\right] = 0$$

$$\Longrightarrow$$

$$\frac{1}{\alpha}(A_1+A_2) + \frac{m^2\sigma_v^2}{h^2}\left[2\lambda\,(u_k-(1+\alpha)u_{k+1})\right] = 0 \qquad (3.4\text{-}33)$$

The last identity is obtained from (3.4-27).

$$A_1 = \frac{2\lambda}{h^2}\, m^2\sigma_v^2\,(u_k-2u_{k+1}+u_{k+2}) + \frac{2\alpha\lambda\, m^2\sigma_v^2}{h^2}\,(u_{k+1}-u_{k+2}) \qquad (3.4\text{-}34)$$

$$A_2 = 2u_k(\beta_1^2+\beta_2^2+\beta_3^2-\beta_1\beta_2-\beta_2\beta_3-\beta_3\beta_1) \qquad (3.4\text{-}35)$$

So (3.4-33) can be written as

$$u_k = \xi_1 u_{k-1} + \xi_2 u_{k-2}, \quad k=1,2,\ldots, N-1 \qquad (3.4\text{-}36)$$

The Lagrange multiplier λ is finally obtained when the inputs satisfy $\sum_{k=0}^{N-1} u_k^2 = Y$. Thus we have the following theorem.

Theorem 3.4.1 For the scalar system described by (3.4-17)-(3.4-18) the optimization problem of (3.4-19)-(3.4-21) has the solution $\underline{U}^{N-1^*} = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$ satisfying the linear difference equation.

$$u_k = \xi_1 u_{k-1} + \xi_2 u_{k-2}, \quad k=1,2,\ldots, N-1 \qquad (3.4\text{-}37)$$

with inertial condition $u_0 \neq 0$ and $u_{-\ell} = 0$ for $\ell > 0$. The parameters ξ_1 and ξ_2 are functions of the discrete parameter values β_i, $i=1,\ldots, m$ and the Lagrange multiplier λ.

[X]

In Section (3.6) we present examples where the optimal input for multistate systems is approximated by difference equations upto forth order using least square fit to determine $\underline{\xi}$.

3.5 Numerical Examples

The examples discussed here bring out the salient features of the signal design problem. We consider two different examples. A fourth order state space dynamics and a single-input single-output system. Two examples are considered in the latter case.

3.5.1 *Fourth Order System with Unknown Parameters in Φ matrix*

The following dynamics is considered.

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \underline{\beta}\, u_k + \underline{\Gamma} w_k \qquad (3.5\text{-}1)$$

$$z_k = \underline{h}^T \underline{x}_k + v_k \qquad (3.5\text{-}2)$$

The matrixes above have the following numerical values.

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.66 & .78 & -.18 & 1 \end{bmatrix}, \quad \underline{\beta} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad (3.5\text{-}3)$$

$$\underline{\Gamma} = (1 \quad 1 \quad 2 \quad 1)^T$$

$$\underline{h}^T = [1 \quad 0 \quad 0 \quad 0]$$

$$\underline{x}_o \sim G(0, 10I) \quad , \quad v_k \sim G(0, .25)$$

$$w_k \sim G(0, \sigma_w^2)$$

The number of stages $N = 50$ and the discrete parameter values $m = 3$. The following discretizations are used

φ_1	φ_2	φ_3	φ_4
-.66	.78	-.18	1.
-.75	.85	-.3	1.19
-.24	.26	.18	.45

The solution to the estimation problem defined here consists of m Kalman filters in parallel. The parallel processing scheme is shown in fig. (3.5-1). This is similar to the parallel computation techniques used by several investigators (see e.g. Alspach [1]).

The optimal inputs are obtained for different values of the plant noise covariance σ_w^2 ; $\sigma_w^2 = 0, .1, .25$ are used. Comparison of the estimates is made using the Bhattacharyya distance and divergence, as a function of the noise covariance σ_w^2. The B-distance is given by

$$B^N_{\alpha\gamma} = \frac{1}{8} \underline{U}^{N-1^T} M_{\alpha\gamma} \left\{ \frac{\Sigma_{\alpha N} + \Sigma_{\gamma N}}{2} \right\}^{-1} M_{\alpha\gamma} \underline{U}^{N-1} \qquad (3.5\text{-}4)$$

where $M_{\alpha\gamma}$ and $\Sigma_{\alpha N}$ are as in (3.2-28); (3.2-24) and (3.2-9)-(3.2-13) respectively. The pairwise divergence has the form

$$J^N_{\alpha\gamma} = \frac{1}{2} \underline{U}^{N-1^T} M^T_{\alpha\gamma} \left(\Sigma^{-1}_{\alpha N} + \Sigma^{-1}_{\gamma N} \right) M_{\alpha\gamma} \underline{U}^{N-1} \qquad (3.5\text{-}5)$$

The optimal energy constrained inputs are obtained numerically using a gradient technique outlined in Appendix C.

Figures (3.6-2a) and (3.6-2b) show the optimal inputs for $\sigma_w^2 = .1$ with B-distance and divergence criterion respectively. Figures (3.6-3a) and (3.6-3b) are the optimal inputs for $\sigma_w^2 = 0.25$. Figure (3.6-4) shows the optimal inputs for $\sigma_w^2 = 0$. The presence of plant noise has a marked influence in changing the signal shape.

The learning with optimal inputs is compared with nonoptimal pulse inputs because of its resemblance to the optimal inputs.

The convergence of estimates of $\underline{\theta} = (\varphi_1, \omega_2, \varphi_3, \varphi_4)$ is compared in figs. (3.5-5) and (3.5-6) for $\sigma_w^2 = .1$ and $\sigma_w^2 = .25$ respecitvely. Stagewise normalized error $\| \underline{\theta} - \hat{\underline{\theta}}_k \|^2 / \| \underline{\theta} - \hat{\underline{\theta}}_0 \|^2$ is plotted against the time step k. Comparison of the learning with the two signals as a function of the noise covariance σ_w^2 is of importance in this example. For $\sigma_w^2 = .1$ the results are almost identical. When the dynamic signal power is increased, that is, when $\sigma_w^2 = .25$, the B-distance yields an improved performance compared to the divergence (see fig. (3.5-6)). This is true at least for the case when the prior probabilities P_i are all equal. A similar conclusion was obtained in an entirely different problem of choosing optimum communication links in Gaussian processes with unequal covariance [41]. The pulse input exhibits less effective learning compared to the optimal signals, thus showing that the modulation of the signals as obtained by approriate syntheses has amarked influence on the learning time and the accuracy of learning. The normalized estimation errors for $\sigma_w^2 = .1$ and $\sigma_w^2 = .25$ are listed in table (3.5-1).

Table (3.5-1)

Comparison of Normalized Errors

$$\|\underline{\theta} - \hat{\underline{\theta}}_N\|^2 / \|\underline{\theta} - \hat{\underline{\theta}}_0\|^2$$

Input	$\sigma_w^2 = .1$	$\sigma_w^2 = .25$
B-distance	$.512 \times 10^{-5}$	1.953×10^{-3}
Divergence	1.012×10^{-5}	2.914×10^{-3}
Pulse Input	$.917 \times 10^{-3}$	10.679×10^{-3}
White noise	$.654 \times 10^{-3}$	8.89×10^{-3}

For Q=.25 the B-distance criterion has 60% of the input energy between stages 21 and 40, while the divergence has 43% of the input energy. Thus the B-distance criterion has larger portion of the input energy during the second half of the input history.

3.5-2 Single-Input Single-Output Systems

Here we consider the following discrete form of the scalar input-output system given by

$$x_k = \sum_{i=1}^{m} \varphi_i x_{k-i} + \sum_{i=1}^{r} \beta_i u_{k-i} \qquad (3.5-6)$$

$$z_k = x_k + v_k \ , \quad k=1,2,\ldots, N \qquad (3.5-7)$$

$$v_k \sim G(0, \sigma_v^2) \qquad (3.5-8)$$

Two examples are considered.

Example (i) Second Order System:

$$x_k = \varphi_1 x_{k-1} + \varphi_2 x_{k-2} + \beta u_{k-1} \qquad (3.5-9)$$

$$\underline{\theta} = (\varphi_1, \varphi_2, \beta)$$

The actual values of the parameters are $\varphi_1 = 1.5$, $\varphi_2 = -.7$, $\beta = 1.$. This system has poles at $.75 \pm j.37$ in the Z-plane.

Example (ii) First Order System:

$$x_k = \varphi x_{k-1} + \beta u_{k-1} \qquad (3.5-10)$$

$$\underline{\theta} = (\varphi, \beta)$$

with $\varphi = -.7$, $\beta = 1.$.

Total number of stages $N = 20$ and $m = 3$. The discrete parameter

values are taken as follows:

φ_1	φ_2	β
1.5	-.7	1.
1.1	-.4	.6
1.6	-.8	1.1

Figures (3.5-7) and (3.5-8) show the optimal inputs for second order and first order systems respectively. Difference equation approximation of the optimal input sequences is obtained for each of the above two cases. Figures (3.5-9) and (3.5-10) show respectively these results for the two dynamics. The difference equation has the form

$$u_k = \sum_{i=1}^{s} \xi_i u_{k-i} \qquad (3.5\text{-}11)$$

Second, third and fourth order fits are displayed in the above figures. Tables (3.5-2) and (3.5-3) show the comparison of the cost functions for each of the above systems and the corresponding parameter values $\underline{\xi}$.

Table (3.5-2)

Second Order System-Comparison of Difference Equation Approximation to the Optimal Inputs

	Second Order Fit	Third Order Fit	Fourth Order Fit	Optimal Input
Cost Function	72.3134	78.1179	78.7619	80.2267
ξ_1	1.54994	1.32858	1.29982	
ξ_2	- .76833	- .31829	- .34751	
ξ_3		- .29294	- .16464	
ξ_4			- .0986	
$P(\underline{\theta}_0 \mid \underline{Z}^{20})$	.8903	.9723	.9815	.9928

Table (3.5-3)

First Order System

Comparison of Difference Equation Approximation

to the Optimal Inputs

	Second Order Fit	Third Order Fit	Fourth Order Fit	Optimal Input
Cost Function	19.2525	19.6215	19.863	20.1219
ξ_1	- 1.27082	- 1.21546	- 1.18881	
ξ_2	- .30199	- .05506	- .06018	
ξ_3		.19808	.02459	
ξ_4			- .14764	

The a posteriori probabilities $P(\underline{\theta}_0|\underline{Z}^{20})$ are compared in Table (3.6-2) for the second order system with optimal input and the inputs from second, third and fourth order fits. A significant difference is seen from the second order to the third order fit. Thereafter the improvement in learning is gradual as is the increase in the respective cost functions.

A comparison of the learning performances with optimal inputs and nonoptimal exponential inputs of equal energy, exhibits the superior learning with optimal inputs. Growing exponential inputs are

used by Litman and Huggins [55] in identifying transfer function models. In this example both the transition parameters (φ_1, φ_2) and the gain parameter β are treated as unknown quantities. Table (3.5-6) compares the learnings with optimal input and exponential input.

Table (3.5-4)

Comparison of the Learning Performances

for the Second Order System

	$\|\underline{\Phi}-\hat{\underline{\Phi}}_N\|^2 / \|\underline{\Phi}-\hat{\underline{\Phi}}_0\|^2$	$\|\beta-\hat{\beta}_N\|^2 / \|\beta-\hat{\beta}_0\|^2$
Optimal Input	5.7546×10^{-4}	5.3843×10^{-4}
Exponential Input	4.8278×10^{-2}	4.8449×10^{-2}

The superior learning with optimal inputs is clearly demonstrated.

FIGURE 3.5-1

Block diagram of parallel processing scheme.

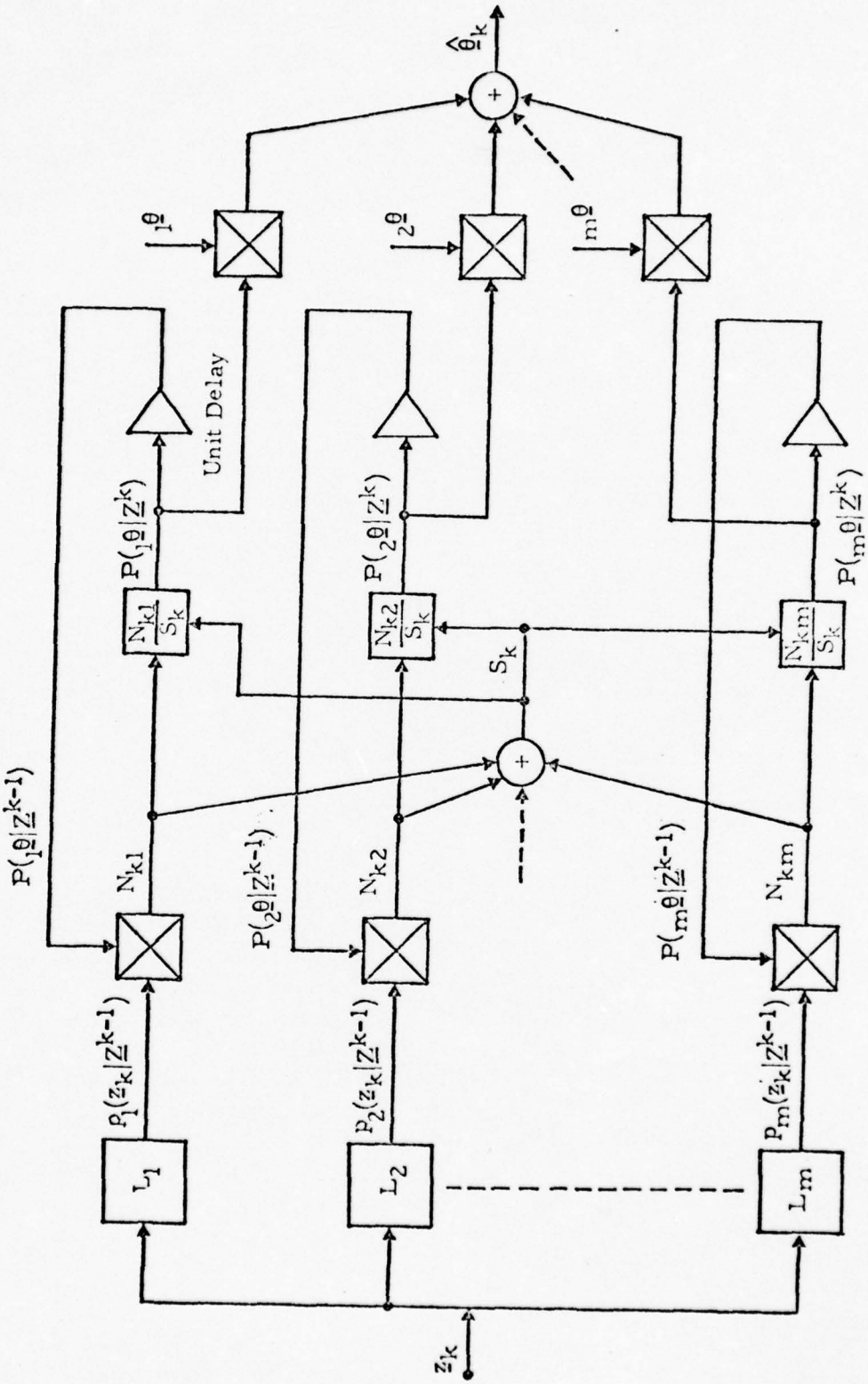

$\hat{\theta}_k$
$_1\theta$
$_2\theta$
$_m\theta$
Unit Delay
$P(_1\theta|Z^k)$
$P(_2\theta|Z^k)$
$P(_m\theta|Z^k)$
$\frac{N_{k1}}{S_k}$
$\frac{N_{k2}}{S_k}$
$\frac{N_{km}}{S_k}$
S_k
$P(_1\theta|Z^{k-1})$
$P(_2\theta|Z^{k-1})$
$P(_m\theta|Z^{k-1})$
N_{k1}
N_{k2}
N_{km}
$p_1(z_k|Z^{k-1})$
$p_2(z_k|Z^{k-1})$
$p_m(z_k|Z^{k-1})$
L_1
L_2
L_m
z_k

FIGURE 3.5-2a

Optimal input with Bhattacharyya distance criterion;

Example 1, $\sigma_w^2 = .1$.

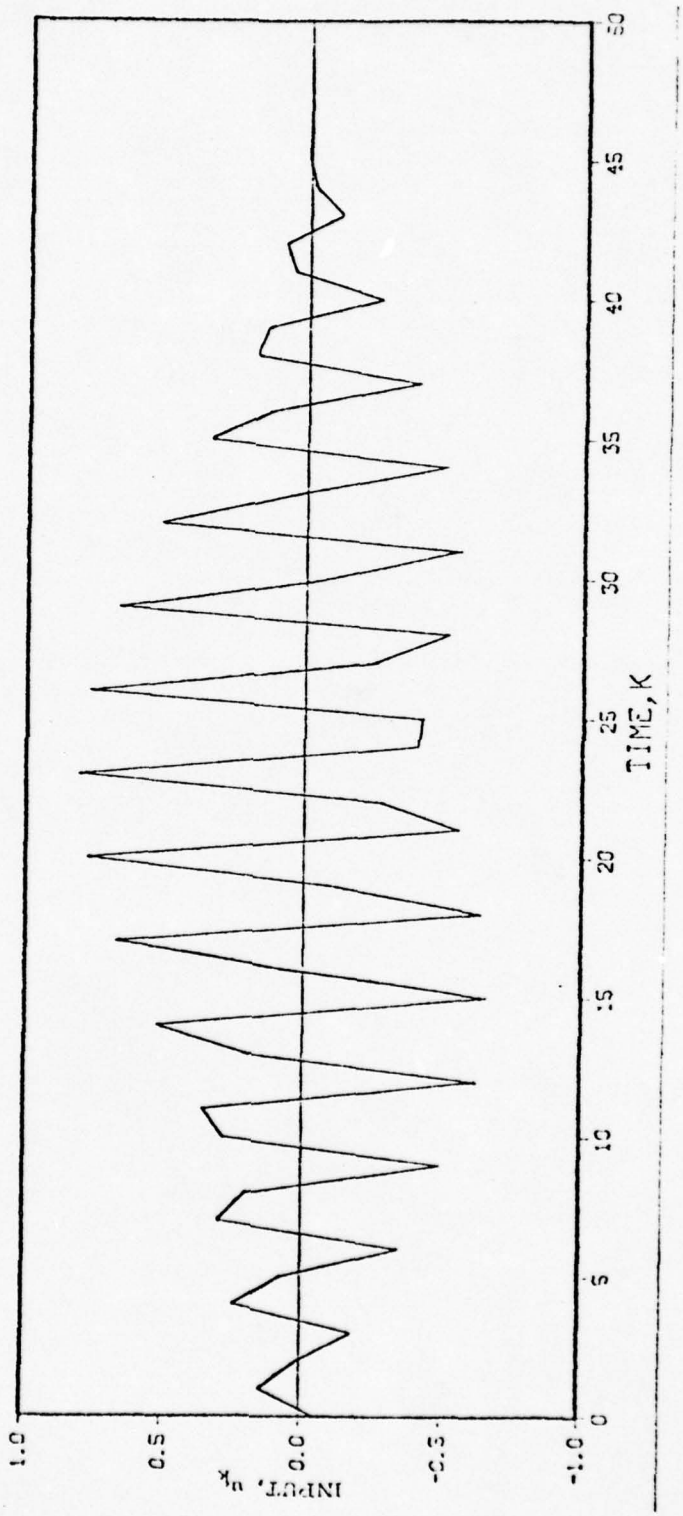

INPUT, u_k
1.0
0.5
0.0
-0.5
-1.0
0
5
10
15
20
25
30
35
40
45
50
TIME, K

FIGURE 3.5-2b

Optimal input with divergence criterion; Example 1,

$\sigma_w^2 = .1$.

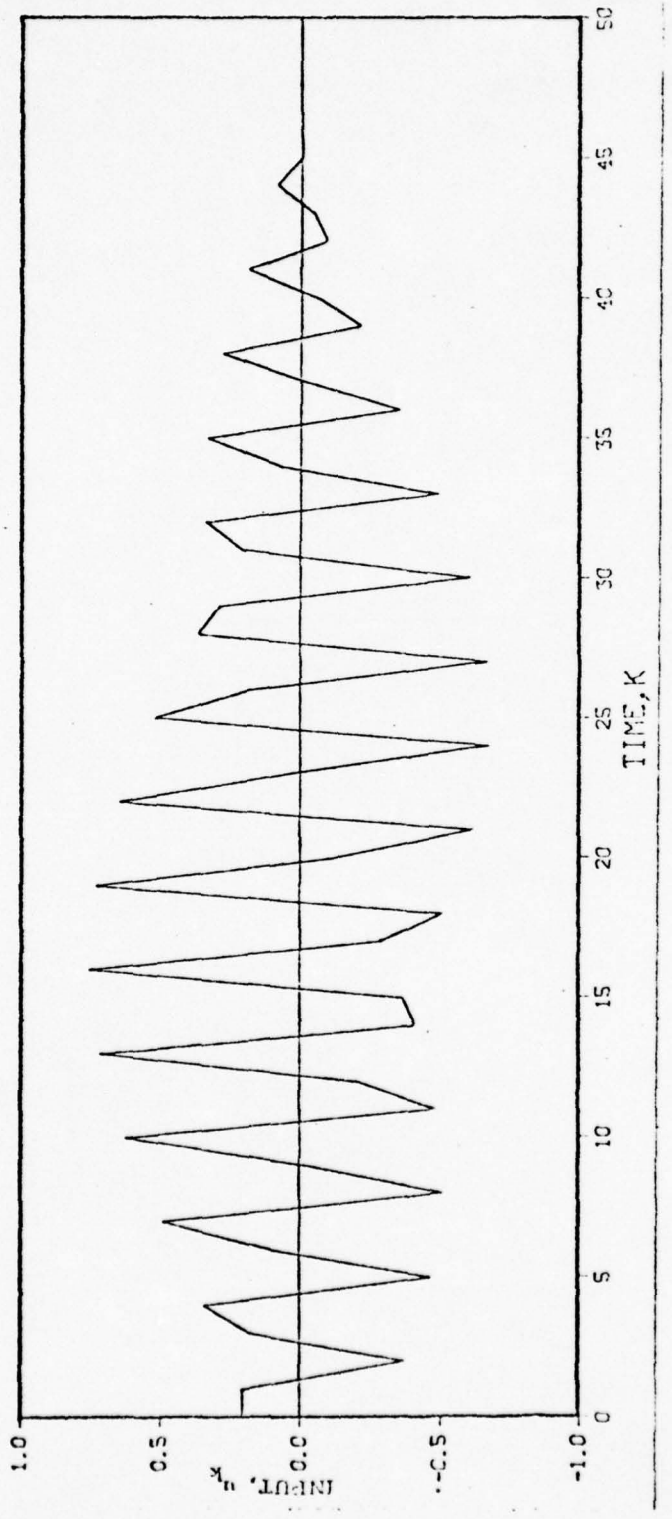

1.0
0.5
0.0
-0.5
-1.0
INPUT, u_k
0
5
10
15
20
25
30
35
40
45
50
TIME, K

FIGURE 3.5-3a

Optimal input with Bhattacharyya distance criterion;

Example 1, $\sigma_w^2 = .25$.

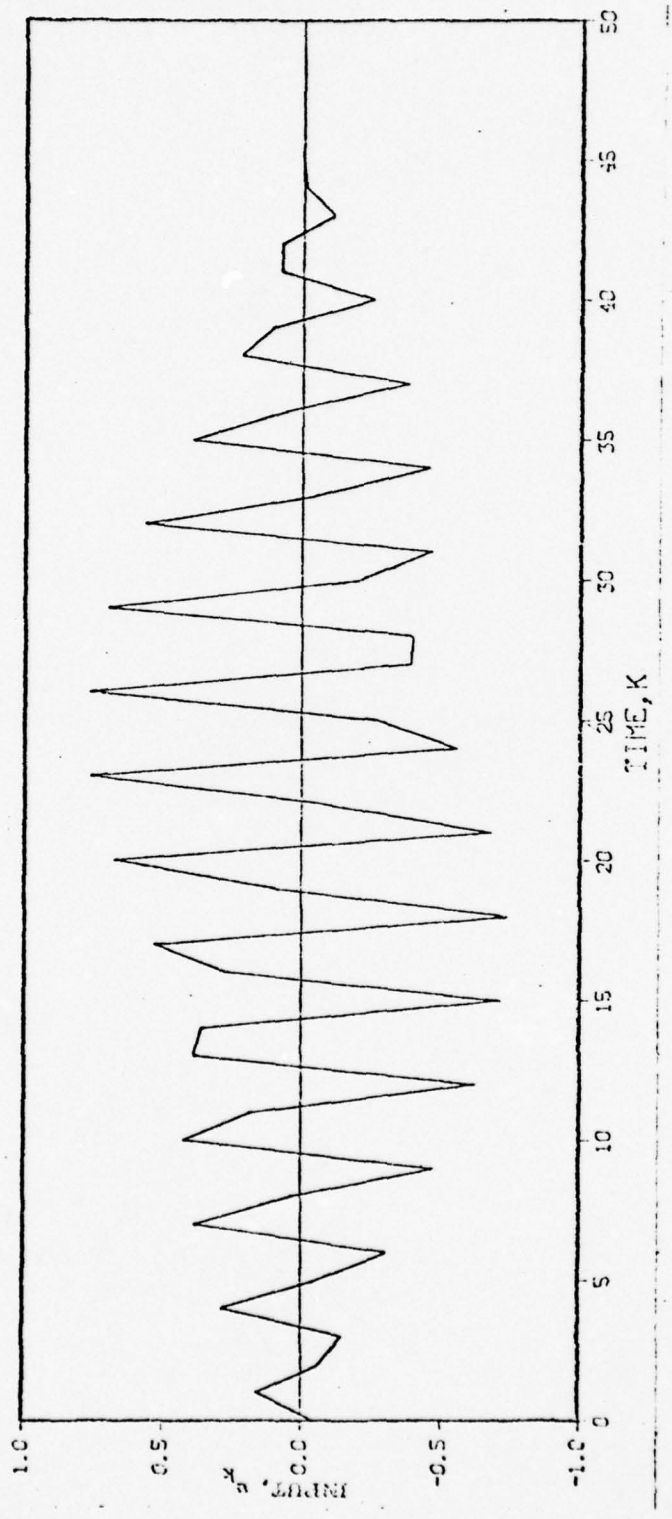

INPUT, u_k
1.0
0.5
0.0
-0.5
-1.0
0
5
10
15
20
25
30
35
40
45
50
TIME, K

FIGURE 3.5-3b

Optimal input with divergence criterion; Example 1,

$\sigma_w^2 = .25$.

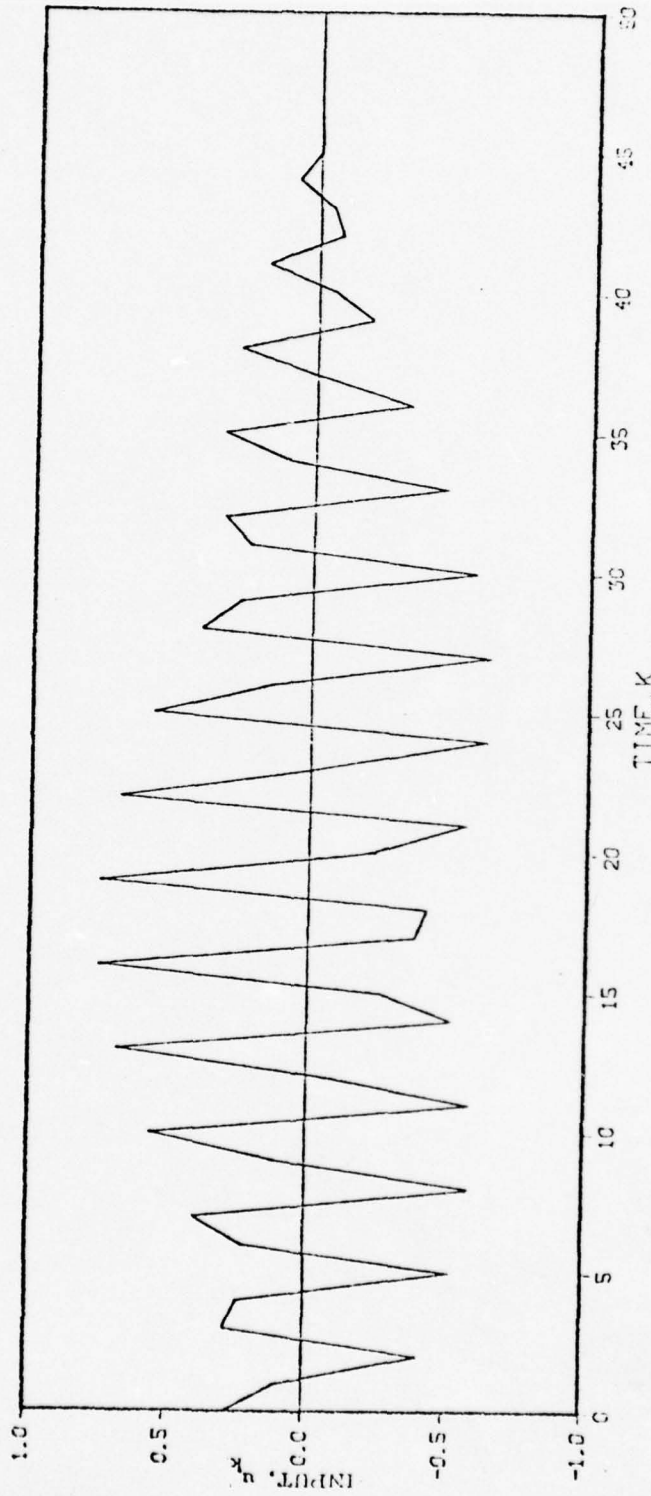

1.0
0.5
0.0
-0.5
-1.0
INPUT, u_k
0
5
10
15
20
25
30
35
40
45
50
TIME, K

FIGURE 3.5-4

Optimal input with Bhattacharyya distance and divergence criteria; Example 1, $\sigma_w^2 = 0$.

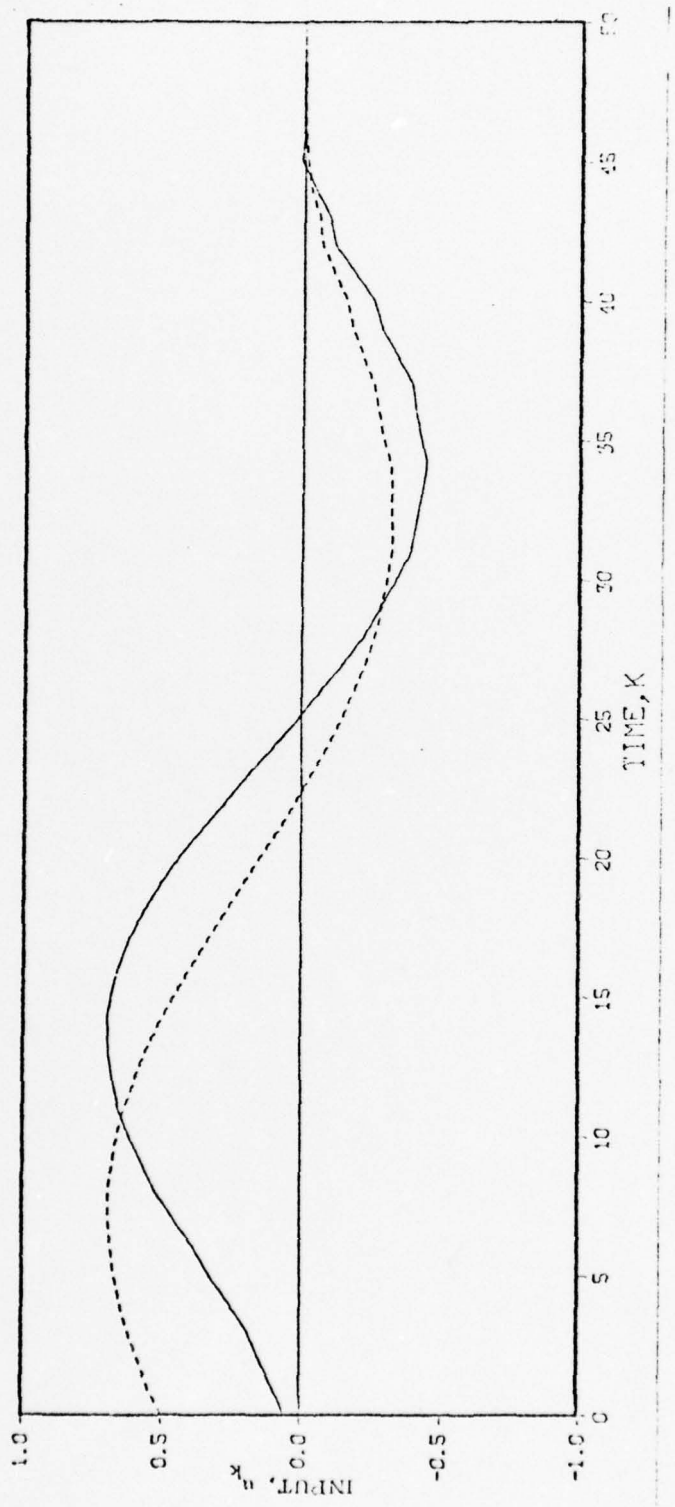

1.0
0.5
0.0
-0.5
-1.0
INPUT, u_k
0
5
10
15
20
25
30
35
40
45
TIME, K

FIGURE 3.5-5

Convergence of parameter estimates; Example 1, $\sigma_w^2 = .1$.

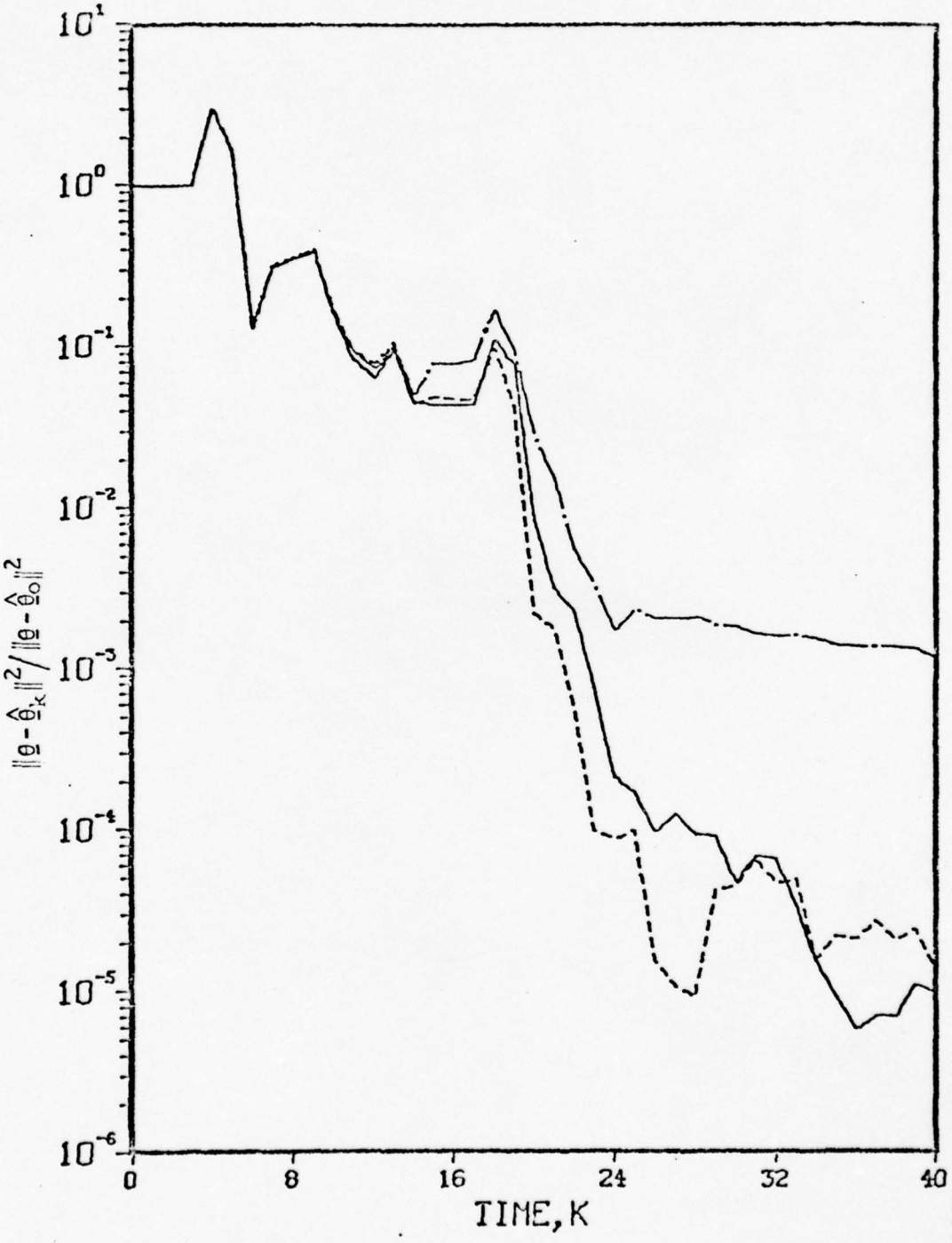

10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\|\underline{\theta}-\hat{\underline{\theta}}_K\|^2/\|\underline{\theta}-\hat{\underline{\theta}}_0\|^2$
0
8
16
24
32
40
TIME, K

FIGURE 3.5-6

Convergence of parameter estimates; Example 1,

$\sigma_w^2 = .25$.

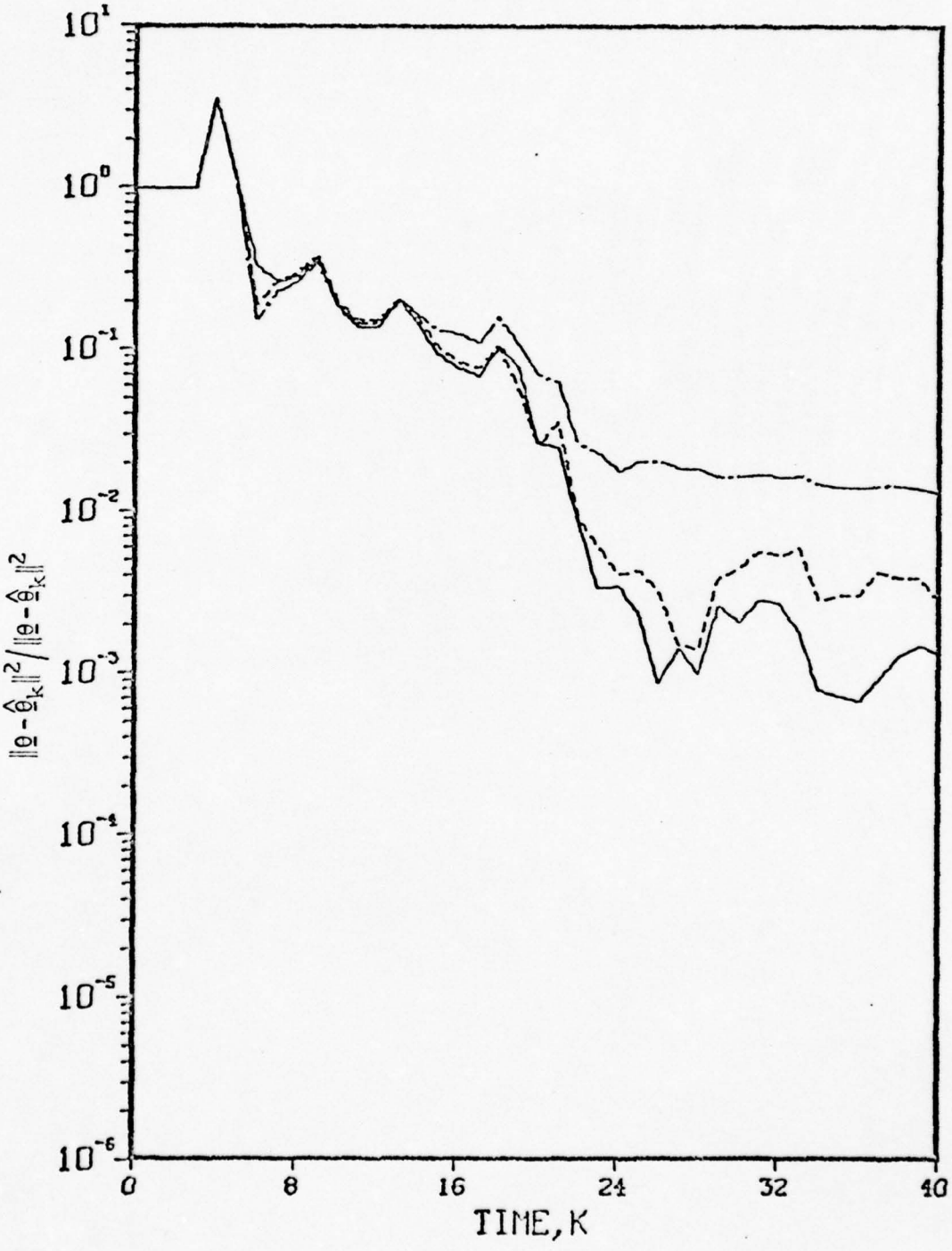

10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\|\underline{\theta} - \hat{\underline{\theta}}_k\|^2 / \|\underline{\theta} - \hat{\underline{\theta}}_k\|^2$
0
8
16
24
32
40
TIME, K

FIGURE 3.5-7

Optimal input for second order system

$X_k = \varphi_1 X_{k=1} + \varphi_2 X_{k-2} + \beta U_{k-1}$; $\varphi_1 = 1.5$, $\varphi_2 = .7$,

$\beta = 1$.

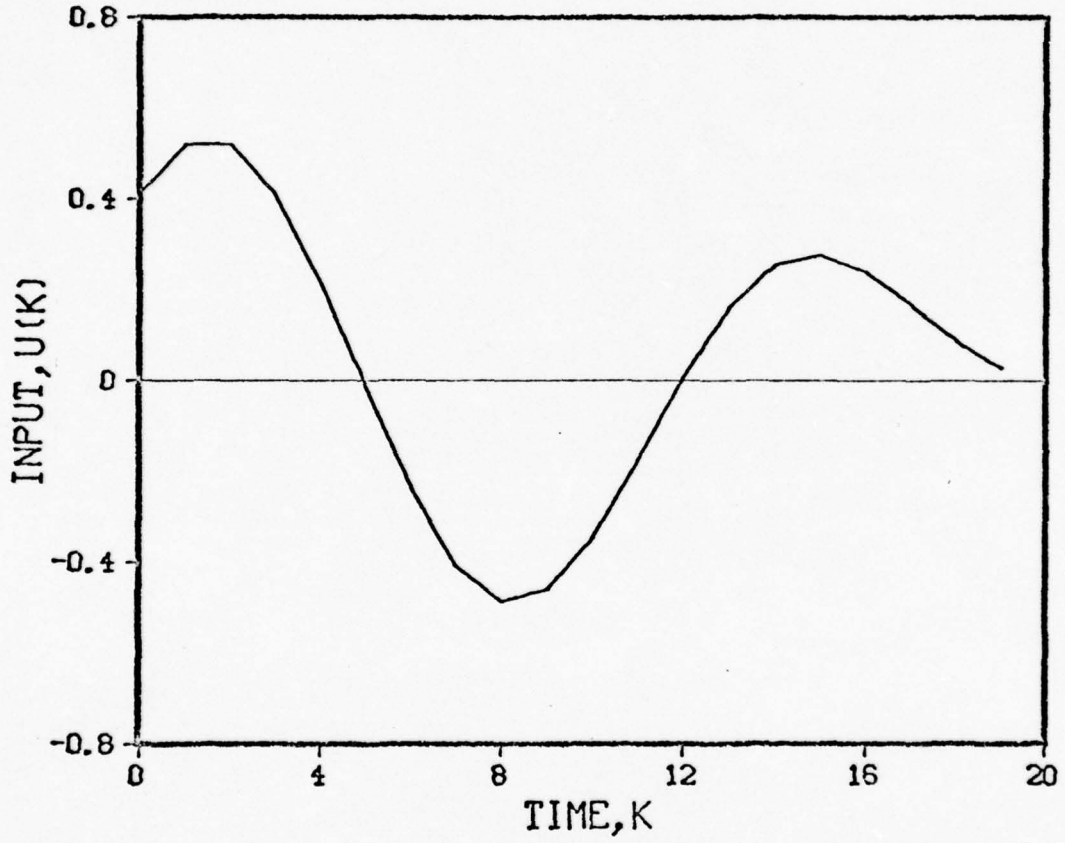

0.8
0.4
0
-0.4
-0.8
INPUT,U(K)
0
4
8
12
16
20
TIME,K

FIGURE 3.5-8

Optimal input for first order system $X_k = \varphi X_{k-1} + \beta U_{k-1}$;

$\varphi = -.7, \quad \beta = 1.$

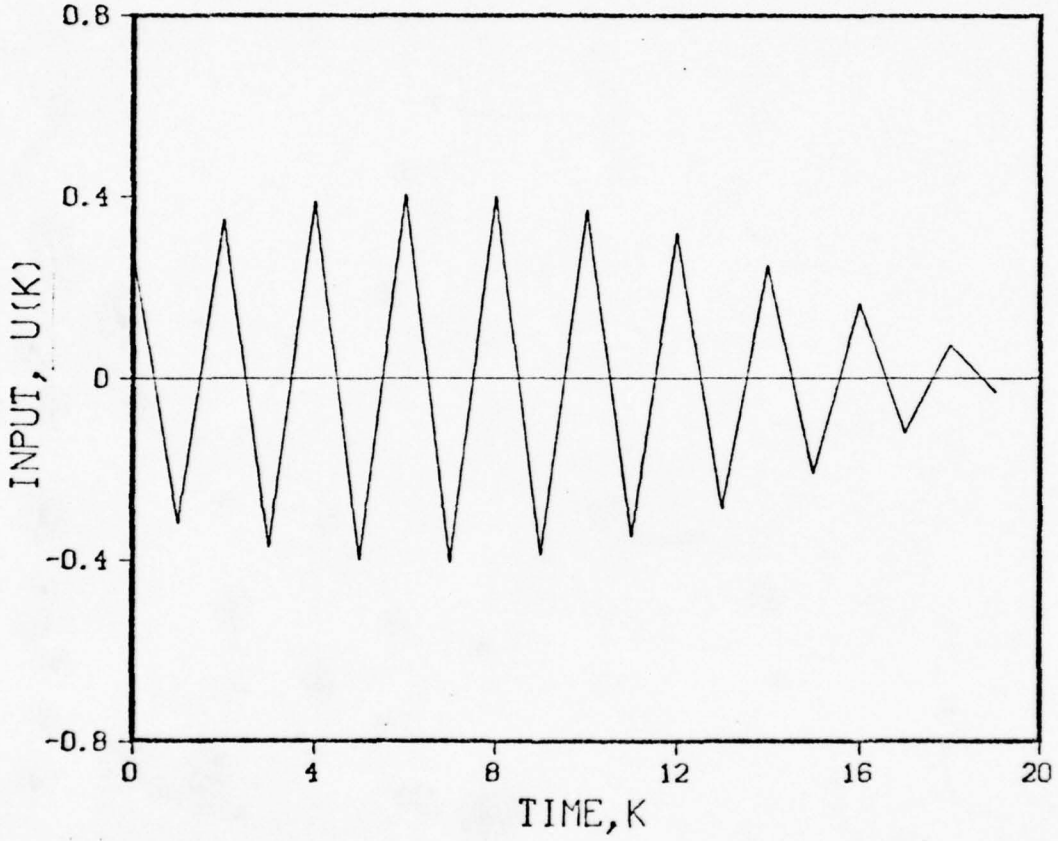

0.8
0.4
0
-0.4
-0.8
INPUT, U(K)
0
4
8
12
16
20
TIME, K

FIGURE 3.5-9

Approximation of optimal inputs for second order system by difference equations: a) second order fit; b) third order fit; c) fourth order fit.

——— *Optimal input*

------ Difference equation approximation

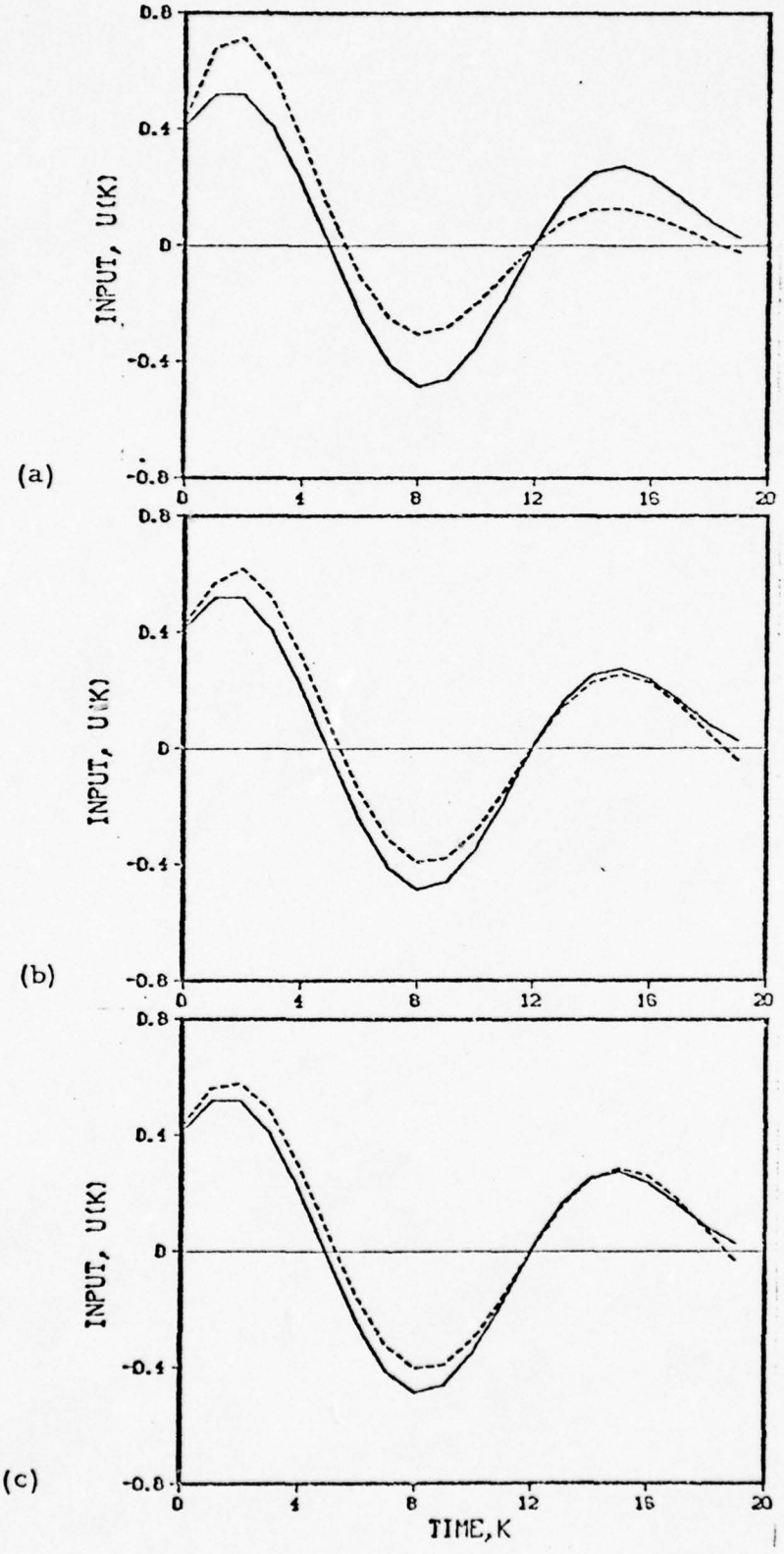
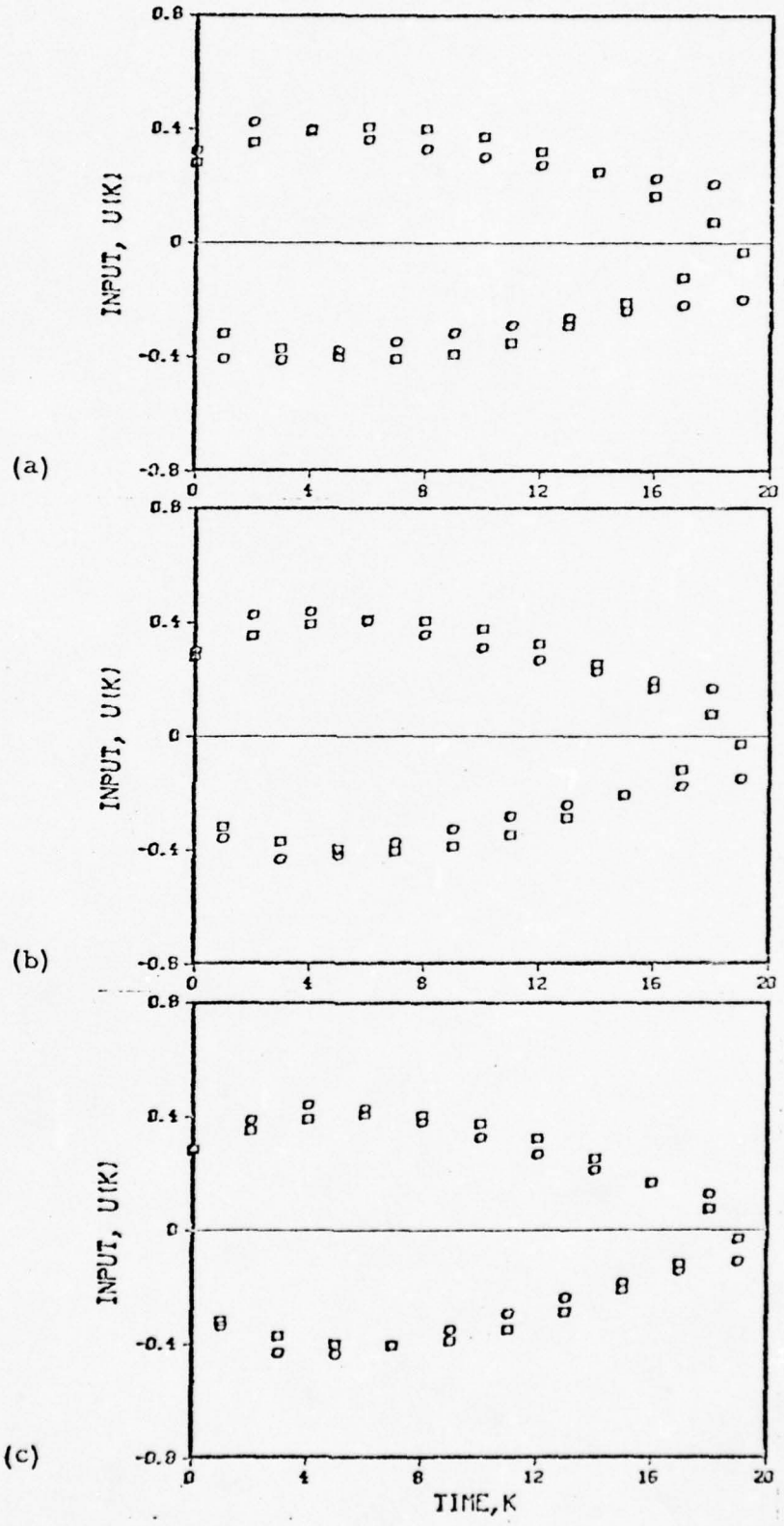

0.8
0.4
0
-0.4
-0.8
INPUT, U(K)
0
4
8
12
16
20
(a)
(b)
(c)
TIME, K

FIGURE 3.5-10

Approximation of optimal inputs for the first order system by difference equations: a) second order fit; b) third order fit; c) fourth order fit.

□ Optimal input

O Difference equation approximation



(a)

(b)

(c)

CHAPTER IV

SYNTHESIS OF STOCHASTIC INPUTS GENERATED BY LINEAR MODELS

In this chapter an alternate approach to the signal design problem is presented. Here linear stochastic processes are used to generate inputs for identification problems. Such signals can be treated as linearly filtered white noise. A degenerate case of this class of signals is a pure white noise sequence. Considerable improvement in learning can be achieved by using inputs that are correlated. The stochastic signals in this chapter are generated from the linear stochastic processes of the mixed autoregressive moving average type (ARMA(p,q)) given by

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \qquad (4.1)$$

where $\{\nu_k\}$ is a white noise sequence. The signals thus generated are independent of the observation noise and the resulting problem gives rise to a reduced optimization in the space of parameters $(\underline{\alpha}, \underline{\xi})$ ensuring stationarity and invertibility of the processes.

Choice of random inputs has been studied by Box and Jenkins [15, pp. 416-420]. Here a specific example is considered and a first order autoregressive model is used to generate the input

sequence. Cumming [22] used a synchronous random telegraph signal in studying the bandwidth effects on the identification accuracy. Simulation studies showed that an input signal bandwidth of about one half that of the system gave the best results.

Here we are interested in generating random inputs from linear processes and comparing the performance of these input signals. Instead of merely choosing the variance of the white noise, the technique developed in this chapter gives us the additional degrees of freedom of choosing the input parameters via the optimality criterion. For each type of system various inputs are tried and the one that gives the maximum cost is selected as the optimum one. No bandwidth considerations are included in the analysis.

We use the sensitivity index of the observations as the cost function. The quadratic nature of this function is exploited in deriving the property of optimal signals. It is also shown that these sensitivities are related to the Bhattacharyya distance.

In section (4.1) a brief discussion of the stationary processes is given and the optimality criterion for the single-input single-output system is derived. In section (4.2) the properties of these inputs as applied to linear systems are studied via the spectral density, $s_{uu}(\lambda)$ of these signals. The input sequence u_k, $k = 0, 1, 2, \ldots$ is asymptotically stationary under certain constraints on $(\underline{\alpha}, \underline{\xi})$.

By writing the correlation matrix in the form

$$C_{uu} = \sigma_\nu^2 \, MM^T \tag{4.2}$$

and using the results on asymptotic eigenvalue distribution of Toeplitz matrices it is shown that

$$\lim_{N\to\infty} \frac{1}{N} \, \text{Trace}[MM^T] = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \tag{4.3}$$

With constraints on σ_ν^2 it is established that at the optimum value of the cost function the corresponding effective spectral area given by the above relationship also has the largest value. In section (4.3) examples are presented comparing the performances of stochastic inputs obtained by several different processes. Numerical results show that the inputs obtained from the autoregressive and ARMA(1, 1) processes give better learning characteristics than the moving average process. It is also observed that if the parameter space is taken to be small surrounding the actual parameter values then the power of the input process is concentrated in the region where the gain of the system is high.

4.1 Input Signals Generated by Linear Stochastic Processes

Linear stochastic models are briefly discussed giving their salient features. The system dynamics is presented in terms of

single-input single-output (SISO) form. The cost function is calculated using the sensitivity index which is quadratic in the input variables.

4.1.1 Description of Linear Stochastic Models

A general linear stochastic model supposes that a time series is generated by a linear aggregation of random noise. Such a process can be represented by the autoregressive moving average model

$$u_k = \sum_{i=1}^{p} \alpha_1 u_{k-i} + \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \qquad (4.1\text{-}1)$$

Using the backward shift operator

$$Bu_k = u_{k-1} \qquad (4.1\text{-}2)$$

(4.1-1) can be written as

$$(1 - \sum_{i=1}^{p} \alpha_i B^i) u_k = (1 - \sum_{i=1}^{q} \xi_i B^i) \nu_k \qquad (4.1\text{-}3)$$

This can be further written as

$$u_k = (1 - \sum_{i=1}^{p} \alpha_i B^i)^{-1} (1 - \sum_{i=1}^{q} \xi_i B^i) \nu_k \qquad (4.1\text{-}4)$$

The convergence of the series $\psi(B) = (1 - \sum_{i=1}^{p} \alpha_i B^i)^{-1}$ ensures that the process has a finite variance. This is ensured by the condition

on $\underline{\alpha}$ that the roots of the equation

$$1 - \sum_{i=1}^{p} \alpha_i B^i = 0 \qquad (4.1\text{-}5)$$

must be outside the unit circle. Thus when the noise process is stationary, the above condition ensures that the process u_k is also stationary. The condition of invertibility is concerned with recovering ν_k from the present and past happenings. Writing (4.1-3) as

$$\nu_k = (1 - \sum_{i=1}^{q} \xi_i B^i)^{-1} (1 - \sum_{i=1}^{p} \alpha_i B^i) u_k \qquad (4.1\text{-}6)$$

the linear process is invertible if the infinite series expansion of $\pi(B) = (1 - \sum_{i=1}^{q} \xi_i B^i)^{-1}$ converges. This requires that the roots of the equation

$$1 - \sum_{i=1}^{q} \xi_i B^i = 0 \qquad (4.1\text{-}7)$$

lie outside the unit circle. The conditions for invertibility are independent of the conditions for stationarity of the time series. Thus we see that the conditions imposed by the roots of (4.1-5) and (4.1-7) restrict the values taken by $\underline{\alpha}$ and $\underline{\xi}$. In the following we briefly discuss these processes. For a complete discussion see Box and Jenkins [15, Ch. 3].

Autoregressive Process (AR(p))

This is given by

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k \qquad (4.1\text{-}8)$$

The autocorrelation of the process is

$$\rho_t = \sum_{i=1}^{p} \alpha_i \rho_{t-i}, \quad t > 0 \qquad (4.1\text{-}9)$$

where

$$\gamma_t = E[u_k u_{t+k}] \qquad (4.1\text{-}10)$$

$$\rho_t = \gamma_t / \gamma_0 \qquad (4.1\text{-}11)$$

We assume a sampling period of T = 1 sec. The spectrum of the process is

$$s_{uu}(f) = \sigma_\nu^2 \left[\left| 1 - \alpha_1 e^{-i2\pi f} - \alpha_2 e^{-i4\pi f} - \ldots - \alpha_p e^{-i2\pi pf} \right|^2 \right]^{-1}$$

$$|f| \leq \tfrac{1}{2} \qquad (4.1\text{-}12)$$

Moving Average Process (MA(q))

This is of the form

$$u_k = \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \qquad (4.1\text{-}13)$$

The autocorrelation function of the process is

$$\rho_t = \begin{cases} -\xi_t + \xi_1 \xi_{t+1} + \ldots + \xi_{q-k} \xi_q, & k = 1, 2, \ldots, q \\ 0 & k > q \end{cases} \qquad (4.1\text{-}14)$$

The spectrum of the process is

$$s_{uu}(f) = \sigma_\nu^2 \left| 1 - \xi_1 e^{-i2\pi f} - \xi_q e^{-i2\pi q f} \right|^2 \qquad |f| \leq \tfrac{1}{2} \qquad (4.1\text{-}15)$$

Mixed Autoregressive Moving Average Process (ARMA(p, q))

This process has the characteristics of both AR(p) and MA(q) processes and is the general form of the stationary time series. This has the form

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \qquad (4.1\text{-}16)$$

The correlation function is given by

$$\rho_t = \alpha_1 \rho_{t-1} + \alpha_2 \rho_{t-2} + \ldots + \alpha_p \rho_{t-p}, \quad t \geq q+1 \qquad (4.1\text{-}17)$$

There are q autocorrelations ρ_q, ρ_{q-1}, ..., ρ_1 whose values depend directly on the choice of the q moving average parameters $\underline{\xi}$ and on the p autoregressive parameters $\underline{\alpha}$. The spectrum of the mixed process is given by

$$s_{uu}(f) = \sigma_\nu^2 \frac{\left| 1 - \xi_1 e^{-i2\pi f} - \xi_2 e^{-i4\pi f} - \dots - \xi_q e^{-i2\pi qf} \right|^2}{\left| 1 - \alpha_1 e^{-i2\pi f} - \alpha_2 e^{-i4\pi f} - \dots - \alpha_p e^{-i2\pi qf} \right|^2}$$

$$|f| \leq \tfrac{1}{2} \qquad (4.1\text{-}18)$$

In the next section we derive the optimality criterion for systems represented by input-output dynamics.

4.1.2 Sensitivity Index and Optimality Criterion

We define the system equations in terms of the single-input single-output dynamics. This representation facilitates the easy computation of sensitivity index and comparison of the results obtained in this chapter with those of earlier investigators. (This class of linear models is also used by Box and Jenkins [15, Ch. 11] in time series analysis.) The quadratic nature of the cost function is exploited in deriving the properties of the optimal inputs.

Consider the following dynamics:

$$y_k = \sum_{i=1}^{n} \varphi_i y_{k-i} + \sum_{i=1}^{r} \beta_i u_{k-i} + w_k \qquad (4.1\text{-}19)$$

$$y_\ell = 0, \quad u_{\ell-1} = 0 \qquad \forall\, \ell \leq 0 \qquad (4.1\text{-}20)$$

$$w_k \sim G(O, \sigma_k^2), \qquad k = 1, 2, \dots, N \qquad (4.1\text{-}21)$$

$\{u_k\}$ is a zero mean random sequence uncorrelated with $\{w_k\}$

$$E\{u_k w_\ell\} = 0 \qquad \forall\, k, \ell \tag{4.1-22}$$

Define the following $N \times N$ matrices

$$\Phi_N = I_N - \sum_{i=1}^{n} \varphi_i S^i \tag{4.1-23}$$

$$\mathcal{B}_N = \begin{cases} \beta_1 I_N + \sum_{i=2}^{r} \beta_i S^{i-1}, & r \geq 2 \\ \beta_1 I_N, & r = 1 \end{cases} \tag{4.1-24}$$

S is the $N \times N$ shift matrix with elements $s_{ij} = \delta_{i-1,j}$. The vector $\underline{y}^N \triangleq (y_1, y_2, \ldots, y_N)^T$ can be written as

$$\underline{y}^N = \Phi_N^{-1} \mathcal{B}_N \underline{U}^{N-1} + \Phi_N^{-1} \underline{w}^N \tag{4.1-25}$$

For further analysis of the stochastic input problem we derive the sensitivity index of the observations with respect to the parameters $\underline{\theta} = (\underline{\varphi}, \underline{\beta}) \in R^{n+r}$ and show that this index is closely related to the B-distance. The quadratic form of the sensitivities is useful in obtaining the spectral properties of the input sequence.

The sensitivity of the output y_k with respect to $\underline{\theta} = (\underline{\varphi}, \underline{\beta})$ is given by

$$\frac{\partial y_k}{\partial \underline{\theta}} = (y_{k-1}, \ldots, y_{k-n}, u_{k-1}, \ldots, u_{k-r}) \tag{4.1-26}$$

and the squared index is given by

$$\epsilon_k = \frac{\partial y_k^T}{\partial \underline{\theta}} \frac{\partial y_k}{\partial \underline{\theta}} = \sum_{i=1}^{n} y_{k-i}^2 + \sum_{i=1}^{r} u_{k-i}^2 \qquad (4.1\text{-}27)$$

The average cost function can be written as

$$J = E\left\{ \sum_{k=1}^{N} \frac{\partial y_k^T}{\partial \underline{\theta}} \frac{\partial y_k}{\partial \underline{\theta}} \right\} \qquad (4.1\text{-}28)$$

The quantities ϵ_k, $k = 1, \ldots, N$ can be computed as a quadratic form in the input sequence $\{u_k\}_{k=0}^{N-1}$ as follows.

$$\epsilon_1 = u_0^2$$

$$\epsilon_2 = y_1^2 + u_1^2 + u_0^2$$

$$\vdots$$

$$\epsilon_r = y_{r-1}^2 + y_{r-2}^2 + \ldots + y_1^2 + u_{r-1}^2 + \ldots + u_0^2$$

$$\epsilon_{r+1} = y_r^2 + y_{r-1}^2 + \ldots + y_1^2 + u_r^2 + \ldots + u_1^2$$

$$\vdots$$

$$\epsilon_n = y_{n-1}^2 + y_{n-2}^2 + \ldots + y_1^2 + u_{n-1}^2 + u_{n-2}^2 + \ldots + u_{n-r}^2$$

$$\epsilon_{n+1} = y_n^2 + y_{n-1}^2 + \ldots + y_1^2 + u_n^2 + \ldots \quad \ldots + u_{n+1-r}^2$$

$$\vdots$$

$$\epsilon_N = y_{N-1}^2 + y_{N-2}^2 + \ldots + y_{N-n}^2 + u_{N-1}^2 + \ldots + u_{N-r}^2$$

In the above we have assumed $n \geq r$. So J can be written as

$$\sum_{k=1}^{N} \epsilon_k = \underline{y}^{N^T} W_1 \underline{y}^N + \underline{U}^{N-1^T} W_2 \underline{U}^{N-1} \tag{4.1-29}$$

The matrices W_1 and W_2 are given by

$$W_1 = \text{diag}[\underbrace{n, n, \ldots n,}_{N-n} \quad \underbrace{n-1, n-2, \ldots, 1, 0}_{n}] \tag{4.1-30}$$

$$W_2 = \text{diag}[\underbrace{r, r, \ldots, r,}_{N-r+1} \quad \underbrace{r-1, r-2, \ldots, 1}_{r-1}] \tag{4.1-31}$$

Averaging over all the random variables

$$\bar{J} = E_{y,u}[J] \tag{4.1-32}$$

Using (4.1-25)

$$J = \{\Phi_N^{-1} \mathcal{B}_N \underline{U}^{N-1} + \Phi_N^{-1} \underline{W}^N\}^T W_1 \{\Phi_N^{-1} \mathcal{B}_N \underline{U}^{N-1} + \Phi_N^{-1} \underline{W}^N\}$$
$$+ \underline{U}^{N-1^T} W_2 \underline{U}^{N-1} \tag{4.1-33}$$

$$\bar{J} = E[\underline{U}^{N-1^T} \mathcal{B}_N^T \Phi_N^{-T} W_1 \Phi_N^{-1} \mathcal{B}_N \underline{U}^{N-1}] + E[\underline{U}^{N-1^T} W_2 \underline{U}^{N-1}]$$

$$+ E[\underline{W}^{N^T} \Phi_N^{-T} W_1 \Phi_N^{-1} \underline{W}^N] \qquad (4.1\text{-}34)$$

The cross terms are zero since $\{u_k\}$ and $\{w_k\}$ are independent, zero mean, random sequences. The last term in (4.1-34) is independent of the input variables and we can write

$$\bar{J} = E_u[\underline{U}^{N-1^T} Q \underline{U}^{N-1}] \qquad (4.1\text{-}35)$$

where

$$Q = E_{\underline{\theta}}[\mathcal{B}_N^T \Phi_N^{-T} W_1 \Phi_N^{-1} \mathcal{B}_N + W_2] \qquad (4.1\text{-}36)$$

When the parameter set takes on finite number of discrete values we have

$$J_i = E\left\{\underline{U}^{N-1^T} [\mathcal{B}_{N_i}^T \Phi_{N_i}^{-T} W_1 \Phi_{N_i}^{-1} \mathcal{B}_{N_i} + W_2] \underline{U}^{N-1}\right\}$$

$$= E\{\underline{U}^{N-1^T} Q_i \underline{U}^{N-1}\} \qquad (4.1\text{-}37)$$

and the total cost function can be written as

$$\bar{J} = E\left[\sum_{i=1}^{m} J_i\right] \qquad (4.1\text{-}38)$$

Let us now derive the B-distance for this case. The pairwise distance is given by

$$B_{ij} = \ell n \left[\frac{\det\left(\frac{\Sigma_i + \Sigma_j}{2}\right)}{(\det \Sigma_i \det \Sigma_i)^{\frac{1}{2}}} \right] \qquad (4.1\text{-}39)$$

where Σ_i and Σ_j are the covariances of $\underline{y}^N$ and $\underline{y}_j^N$. Now

$$B_{ij} = \ell n \left[\prod_{k=1} \lambda_k \left(\frac{\Sigma_i + \Sigma_j}{2}\right)\right] - \left(|\Sigma_i||\Sigma_j|\right)^2$$

$$\leq \ell n \left[\prod_{k=1} \lambda_k \left(\frac{\Sigma_i + \Sigma_j}{2}\right)\right]$$

$$B_{ij} \leq \ell n \left[\frac{1}{N} \sum_{k=1}^{N} \lambda_k \left(\frac{\Sigma_i + \Sigma_j}{2}\right)\right]$$

$$= N \ell n \frac{1}{2N} \text{ Trace } (\Sigma_i + \Sigma_j) \qquad (4.1\text{-}40)$$

The covariance matrices Σ_i are given by

$$\Sigma_i = E[\underline{y}_i^N \underline{y}_i^{N^T}]$$

$$= E\left[\{\Phi_{N_i}^{-T} \mathcal{B}_{N_i} \underline{U}^{N-1} + \Phi_{N_i}^{-1} \underline{W}^N\} \{\Phi_{N_i}^{-T} \mathcal{B}_{N_i} \underline{U}^{N-1} + \Phi_{N_i}^{-1} W^N\}^T \right]$$

$$= \Phi_{N_i}^{-1} \underline{W}^N \underline{W}^{N^T} \Phi_{N_i}^{-T} + \Phi_{N_i}^{-1} \mathcal{B}_{N_i} \underline{U}^{N-1} \underline{U}^{N-1^T} \mathcal{B}_{N_i}^T \Phi_{N_i}^{-T}$$

(4.1-41)

Since the first term in (4.1-41) is not a function of $\{u_k\}_k$ we have

$$\text{Trace}[\Sigma_i + \Sigma_j] = E\left[\underline{U}^{N-1^T}\{\mathcal{B}^T_{N_i}\Phi^{-T}_{N_i}\Phi^{-1}_{N_i}\mathcal{B}_{N_i} + \mathcal{B}^T_{N_j}\Phi^{-T}_{N_j}\Phi^{-1}_{N_j}\mathcal{B}_{N_j}\}\underline{U}^{N-1}\right] \quad (4.1\text{-}42)$$

$$E[J_i + J_j] = E\left\{\underline{U}^{N-1^T}[\mathcal{B}^T_{N_i}\Phi^{-T}_{N_i}W_1\Phi^{-1}_{N_i}\mathcal{B}_{N_i} + \mathcal{B}^T_{N_j}\Phi^{-T}_{N_j}W_1\Phi^{-1}_{N_j}\mathcal{B}_{N_j} + 2W_2]\underline{U}^{N-1}\right\} \quad (4.1\text{-}43)$$

The matrices in (4.1-42) and (4.1-43) differ in the following way - the weighting matrix W_1 is a function of the order n of the system. The quadratic matrix has its diagonal elements increased by the addition of the diagonal matrix $2W_2$, where W_2 is a function of the order of the moving average part of the dynamics. Within these limits the sensitivity cost index has the interpretation given by the upper bound (4.1-40).

Assuming that the input sequences $\{u_k\}_k$ is stationary the cost function can be written as

$$\bar{J} = \sum_{k=0}^{N-1}\mu_k\gamma_k$$

$$= \gamma_0\sum_{k=0}^{N-1}\mu_k\rho_k \quad (4.1\text{-}44)$$

where $\rho_k = \gamma_k/\gamma_0$, $k = 0, 1, \ldots, N-1$ and γ_0 is the variance of the inputs. Assume that the inputs are constrained to have finite energy Y such that

$$\gamma_0 \leq Y \tag{4.1-45}$$

The μ_k are sums of diagonal elements of Q and matrix given by

$$\mu_k = \sum_{i=1}^{N-k} q_{i,\,i+k}, \quad k = 0, 1, \ldots, N-1 \tag{4.1-46}$$

The optimization problem is that of maximizing $\bar{J}$ with respect to parameters of the input process - $(\underline{\alpha}, \underline{\xi}) \in R^{p+q}$. The constraint region for $(\underline{\alpha}, \underline{\xi})$ is such that these parameters must satisfy conditions for stationarity and invertibility of the given input process. For a given total input energy γ_0 the optimization problem can be written as

$$\bar{J}^* = \sup_{(\underline{\alpha}, \underline{\xi}) \in \mathcal{B}} \sum_{k=0}^{N-1} \mu_k \rho_k \tag{4.1-47}$$

where $\mathcal{B}$ is the admissible region in R^{p+q} for the input parameters. The constraint regions for input processes with two parameters is given below:

AR(2) Inputs

$$u_k = \alpha_1 u_{k-1} + \alpha_2 u_{k-2} + \nu_k, \quad k = 0, 1, \ldots \tag{4.1-48}$$

$\mathcal{D}$:

$$\alpha_2 + \alpha_1 < 1$$

$$\alpha_2 - \alpha_1 < 1$$

$$|\alpha_2| < 1 \qquad (4.1\text{-}49)$$

MA(2) Inputs

$$u_k = \nu_k - \xi_1 \nu_{k-1} - \xi_2 \nu_{k-2} \qquad (4.1\text{-}50)$$

$\mathcal{D}$:

$$\xi_2 + \xi_1 < 1$$

$$\xi_2 - \xi_1 < 1$$

$$|\xi_2| < 1 \qquad (4.1\text{-}51)$$

ARMA(1, 1) Inputs

$$u_k = \alpha_1 u_{k-1} + \nu_k - \xi_1 \nu_{k-1} \qquad (4.1\text{-}52)$$

$\mathcal{D}$:

$$|\alpha_1| < 1, \quad |\xi_1| < 1 \qquad (4.1\text{-}53)$$

From the expression for the sensitivity index derived in (4.1-27) and (4.1-28) it is clear that when the unknown parameters are contained only in the gain vector $\underline{\beta}$, the cost function has the form

$$\overline{J} = E\left\{\sum_{k=1}^{N} \sum_{i=1}^{r} u_{k-i}^2\right\} \qquad (4.1\text{-}54)$$

When $\{u_k\}$ are stationary with finite energy $\bar{J}$ is maximized by any stationary input. In this particular case a white noise process will perform as well as any other stationary input of equal energy.

4.2 Characterization of Stochastic Inputs

The stochastic inputs are derived from the above processes by considering u_k for $k \geq 0$. These input processes are asymptotically stationary when the input process parameters satisfy the conditions discussed in section (4.1). The properties of input signals as applied to linear dynamical systems are studied by considering the spectral density of the stationary processes. Using the results on asymptotic eigenvalue distribution of Toeplitz matrices the input correlation matrix can be approximated by the corresponding spectral density of the stationary process in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \text{ Trace } \left\{ E[\underline{U}\ \underline{U}^T] \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \qquad (4.2\text{-}1)$$

Furthermore, if we fix the variance of the input white noise, σ_ν^2, by writing the correlation matrix in the form

$$C_{uu} = \sigma_\nu^2 MM^T \qquad (4.2\text{-}2)$$

it follows that

$$\lim_{N\to\infty} \frac{1}{N} \text{ Trace } [MM^T] = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \qquad (4.2\text{-}3)$$

Thus for this case at the maximum value of $\bar{J}$ the input process is such that it has the largest value for the effective spectral area in the sense of (4.2-3).

The results on Toeplitz asymptotic theorems used here have been applied by Gray [31] in deriving information rates of ARMA processes. This was the first time that such theory was applied in engineering problems. Even though the results of the following sections related to the Toeplitz matrices are not new, the application of such theory in the present context has enabled us to obtain new insight into the behavior of random input signals. Such a characterization has many attractive features for theoretical as well as numerical studies. We will use some basic theorems on asymptotic eigenvalue distribution of Toeplitz matrices credited to Grenander and Szegö [33, Ch. 5], and some results of Gray [31].

4.2.1 The Cost Function

The optimality criterion from (4.1-35) is

$$\bar{J} = E_{\underline{U}}[\underline{U}^{N-1^T} Q \underline{U}^{N-1}] \tag{4.2-4}$$

$$= \mathrm{Tr}\left[Q E_{\underline{U}}(\underline{U}^{N-1} \underline{U}^{N-1^T})\right] \tag{4.2-5}$$

$\bar{J}$ has the following upper bound

$$\mathrm{Tr}\left[Q E_{\underline{U}}(\underline{U}^{N-1} \underline{U}^{N-1^T})\right] \leq \lambda_{\max}(Q)\, \mathrm{Tr}\, E_{\underline{U}}[\underline{U}^{N-1} \underline{U}^{N-1^T}] \tag{4.2-6}$$

Thus for this case at the maximum value of $\bar{J}$ the input process is such that it has the largest value for the effective spectral area in the sense of (4.2-3).

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The optimal value of $\bar{J}$ is

$$\bar{J}^* = \lambda_{max}(Q) \cdot \sup \mathrm{Tr}\, E_u [\underline{U}^{N-1} \underline{U}^{N-1^T}] \qquad (4.2\text{-}7)$$

If the inputs are stationary with finite energy then

$$\bar{J}^* = \lambda_{max}(Q) \cdot c \qquad (4.2\text{-}8)$$

where c is a constant. We will show that when the inputs are asymptotically stationary $\bar{J}^*$ has the optimum value in the sense

$$\frac{1}{N} \bar{J}^* \to \lambda_{max}(Q) \cdot Y \text{ for large } N \qquad (4.2\text{-}9)$$

where Y is the input energy constraint. When the input noise variance σ_ν^2 is fixed we have

$$\bar{J}^* = \lambda_{max}(Q)\sigma_\nu^2 \sup \mathrm{Tr}[MM^T] \qquad (4.2\text{-}10)$$

We will show that $\bar{J}^*$ attains its optimum value asymptotically when the following relationship is satisfied

$$\frac{1}{N} \mathrm{Tr}[MM^T] \to \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \quad \text{for large } N \qquad (4.2\text{-}11)$$

Thus at the maximum value of $\bar{J}$ the corresponding input process also has the largest spectral area in the sense of (4.2-11). When the condition of stationarity is satisfied the cost function can also

be written as

$$\bar{J} = \sum_{n=0}^{N-1} \mu_n \gamma_n \qquad (4.2\text{-}12)$$

The correlations γ_n are given by

$$\gamma_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, e^{in\lambda}\, d\lambda \qquad n = 0,\ 1,\ \ldots \qquad (4.2\text{-}13)$$

Since $s_{uu}(\lambda)$ is symmetric, $\bar{J}$ can be written as

$$\bar{J} = \frac{1}{\pi} \int_{0}^{\pi} \left(\sum_{n=0}^{N-1} \mu_n \cos n\lambda \right) s_{uu}(\lambda)\, d\lambda \qquad (4.2\text{-}14)$$

This can be interpreted as the area under the product curve of $s_{uu}(\lambda)$ and $\sum_{n=0}^{N-1} \mu_n \cos n\lambda$. Since the first term in the parentheses in (4.2-10) is fixed the maximum value of $\bar{J}$ is a function of $s_{uu}(\lambda)$, the spectral density which must be shaped so as to maximize the area of the product curve $\left(\sum_{n=0}^{N-1} \mu_n \cos n\lambda \right) s_{uu}(\lambda)$. In the next few sections we discuss these aspects with reference to the finite order linear inputs.

4.2.2 The Autoregressive AR(p) Inputs

Consider the finite order autoregressive process

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k, \qquad k \geq 0$$

$$u_k = 0 \qquad , \qquad k < 0 \qquad (4.2\text{-}15)$$

$\{\nu_k\}_k$ is a zero-mean white noise sequence with finite variance σ_ν^2.

We can write (4.2-15) as

$$\nu^{N-1} = A\underline{U}^{N-1} \tag{4.2-16}$$

where

$$A = \begin{bmatrix} 1 & & & & \\ -\alpha_1 & 1 & & & \\ -\alpha_2 & -\alpha_1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -\alpha_p & -\alpha_{p-1} & \cdots & 1 & \\ 0 & -\alpha_p & -\alpha_{p-1} & & 1 \end{bmatrix} \quad N \times N \text{ matrix} \tag{4.2-17}$$

Rewriting (4.2-16) as

$$\underline{U}^{N-1} = A^{-1}\underline{\nu}^{N-1} \tag{4.2-18}$$

The correlation matrix becomes

$$C_{uu} = \sigma_\nu^2 (A^T A)^{-1} \tag{4.2-19}$$

The matrix $A^T A$ is shown in Fig. (4.2-1). Instead of using the matrix $(A^T A)^{-1}$ consider its inverse $\Lambda_N = A^T A$. The elements of matrix $A^T A$ are given by

$$\Lambda_N(i,j) = \sum_{k=0}^{N-\max(i,j)} \alpha_k \alpha_{k+|i-j|}, \qquad \alpha_0 = 1$$

$$i, j = 1, 2, \ldots, N \qquad (4.2\text{-}20)$$

Since $\alpha_k = 0$ for $k > p$, all entries more than p diagonals away from the main diagonal of Λ_N are zero. For large N all entries of Λ_N except those in the $p \times p$ submatrix in the lower right hand corner depend only on $|i-j|$. Hence the inverse of C_{uu} represented by Λ_N closely resembles a Toeplitz matrix. This is the key point to relating the trace of Λ_N^{-1} to the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda$. Define the $N \times N$ symmetric Toeplitz matrix T as

$$T_N = T\left[f(x)\right] = [t_{kj}] = [t_{|k-j|}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{i|k-j|x}\, dx \qquad (4.2\text{-}21)$$

$f(x)$ is a real-valued, continuous, bounded function on $[-\pi, \pi]$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln f(x) > -\infty \qquad (4.2\text{-}22)$$

so that $f(x) > 0$ almost everywhere. Let us refer to this class of function as L.

Definition 4.2.1 [33, p. 62]. Consider the nonnegative sequences $\{a_{k,N}\}$ and $\{b_{k,N}\}$. Assume that for each k and N

$$a_{k,N} < R, \qquad b_{k,N} < R \qquad (4.2\text{-}23)$$

where R is independent of both k and N. The sets $\{a_{k,N}\}$ and $\{b_{k,N}\}$ are said to be asymptotically equally distributed as $N \to \infty$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left[F(a_{k,N}) - F(b_{k,N}) \right] = 0 \qquad (4.2\text{-}24)$$

where $F(t)$ is an arbitrary continuous function on $[0, R]$. ⊠

Definition 4.2.2 Let L_N be an $N \times N$ Hermitian matrix with entries $L_N(i,j)$ and eigenvalues $\lambda_{k,N}$, $k = 1, 2, \ldots, N$. The strong norm of L_N is

$$\| L_N \| = \max_k |\lambda_{k,N}| \qquad (4.2\text{-}25)$$

The weak norm of L_N is

$$|L_N| = \left\{ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |L_N(i,j)|^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{1}{N} \sum_{k=1}^{N} |\lambda_{k,N}|^2 \right. \qquad (4.2\text{-}26)$$

⊠

Definition (4.2.3) We say that two sequences of Hermitian matrices $\{L_N\}$ and $\{G_N\}$ exhibit mutual approximation, denoted by $L_N \sim G_N$ if they possess the following properties.

i) $$\| L_N \|, \| G_N \|, |L_N|, |G_N| \le M < \infty \qquad (4.2\text{-}27)$$

ii) $$\lim_{N\to\infty} |L_N - G_N| = 0 \qquad (4.2\text{-}28)$$

⊠

Now we state the fundamental theorem on the asymptotic eigenvalue distribution of Teoplitz matrices (See [33, theorem 5.2, p. 64]).

Theorem 4.2.1 (Asymptotic Eigenvalue Distribution of Toeplitz Matrices).

Let $f(x)$ be a real-valued function of class L.

Denote by m and M the essential lower and upper bounds of $f(x)$ respectively, and assume that m and M are finite. Let $F(\lambda)$ be any continuous function defined in the finite interval $m \le \lambda \le M$. Let $\lambda_{k,N}$, $k = 1, 2, \ldots, N$ be the eigenvalues of the corresponding Toeplitz matrix. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(\lambda_{k,N}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left[f(x)\right] dx \qquad (4.2\text{-}29)$$

⊠

The next theorem is the basic theorem for finding the asymptotic eigenvalue distribution of Hermitian matrices approximated by Toeplitz forms. We state this theorem as given by Gray [32].

Theorem 4.2.2 Given a Toeplitz matrix L_N and a Hermitian matrix G_N such that $L_N \sim G_N$. Let $\{\alpha_{k,N}\}_{k=1}^{N}$ and $\{\beta_{k,N}\}_{k=1}^{N}$

be the eigenvalues of L_N and G_N respectively. Since L_N and G_N are bounded, there exist finite numbers m and M such that

$$m \leq \alpha_{k,N}, \quad \beta_{k,N} \leq M, \quad k = 1, 2, \ldots, N \tag{4.2-30}$$

Let $F(x)$ be an arbitrary continuous function on $[m, M]$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F[\alpha_{k,N}] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F[\beta_{k,N}] \tag{4.2-31}$$

⊠

if either of the limits exist.

Theorems (4.2.1) and (4.2.2) are applied to the matrix $A^T A$ to obtain the desired result. Let us identify the $N \times N$ matrix $A^T A$, which is Hermitian and positive definite, as G_N. Let

$$G_N = A^T A \tag{4.2-32}$$

Define the $N \times N$ Toeplitz matrix L_N as

$$L_N = T\left[f(x)\right] = [L_N(i, j) = \sum_{k=0}^{p} \alpha_k \alpha_{k+|i-j|} \tag{4.2-33}$$

with elements beyond the pth diagonal equal to zero. L_N and G_N agree element by element except in a $p \times p$ submatrix at the lower right corner. It is shown by Gray [31] that the Hermitian matrix G_N is approximated by the Toeplitz matrix L_N in the sense that

$$\lim_{N\to\infty} |L_N - G_N| = 0 \tag{4.2-34}$$

Since the matrix L_N is Toeplitz it follows from theorem (4.2.1) that the eigenvalues $\{\alpha_{k,N}\}$ of L_N are distributed asymptotically as $f(\lambda)$ where

$$f(\lambda) = \left| 1 - \sum_{k=j}^{p} \alpha_k e^{-ik\lambda} \right|^2 \tag{4.2-35}$$

Further

$$\lim_{N\to\infty} |L_N| = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\lambda)|^2 \, d\lambda \right] < \infty \tag{4.2-36}$$

since

$$0 < f(\lambda) \leq \left(\sum_{k=0}^{p} |\alpha_k| \right)^2 < \infty \tag{4.2-37}$$

Hence from (4.2-34) and (4.2-35) the conditions of theorem (4.2.2) are satisfied giving the result that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(\alpha_{k,N}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(\beta_{k,N}) \tag{4.2-38}$$

where $\{\beta_{k,N}\}_{k=1}^{N}$ are the eigenvalues of $\Lambda_N = A^T A$. Thus theorem (4.2.1) and (4.2.38) yield)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(\beta_{k,N}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left[f(\lambda) \right] d\lambda \tag{4.2-39}$$

Note that $\beta_{k,N} > 0 \quad \forall k$. and letting

$$F(\beta_{k,N}) = \beta_{k,N}^{-1} \tag{4.2-40}$$

we finally obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \beta_{k,N}^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f(\lambda)} \, d\lambda \qquad (4.2\text{-}41)$$

where

$$f(\lambda) = \left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ik\lambda} \right|^2, \quad \lambda \in [-\pi, \pi] \qquad (4.2\text{-}42)$$

But for the autoregressive process AR(p) the spectrum is given by

$$s_{uu}(\lambda) = \sigma_\nu^2 / \left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ik\lambda} \right|^2, \quad \lambda \in [-\pi, \pi] \qquad (4.2\text{-}43)$$

Hence we have

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} E[\underline{U}\,\underline{U}^T] = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \qquad (4.2\text{-}44)$$

We have the following theorem for autoregressive inputs.

Theorem 4.2.3 For the autoregressive inputs

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k$$

the trace of the input correlation matrix is related to the spectrum of the corresponding stationary process, $s_{uu}(\lambda)$ in the sense

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}[A^T A]^{-1} = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \qquad (4.2\text{-}55)$$

where the spectrum of the stationary AR(p) inputs is

$$s_{uu}(\lambda) = \sigma_\nu^2 \Big/ \Big| 1 - \sum_{k=1}^{p} \alpha_k d^{-ik\lambda} \Big|^2, \quad \lambda \in [-\pi, \pi] \tag{4.2-56}$$

⊠

For a given value of σ_ν^2, the variance of white noise generating the inputs, we have from (4.2-10) that at the maximum value of $\bar{J}$ the corresponding effective spectral area of the input spectrum also has the largest value. That is

$$\frac{1}{N} \bar{J}^* = \lambda_{max}(Q) \sup_{\underline{\alpha} \in \mathcal{B}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f(\lambda)} \, d\lambda \tag{4.2-57}$$

for large N which follows from (4.2-55).

4.2.3 The Moving Average MA(q) Inputs

This class of inputs has the form

$$u_k = \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \tag{4.2-58}$$

For N stages this can be written as

$$\underline{U}^{N-1} = \Xi \, \underline{\nu}^{N-1} \tag{4.2-59}$$

where

$$\Xi = \begin{bmatrix} 1 & & & & \\ -\xi_1 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ -\xi_q & & \ddots & \ddots & \\ 0 & -\xi_q & & -\xi_1 & 1 \end{bmatrix} \quad N \times N \text{ matrix} \tag{4.2-60}$$

The correlation matrix is given by

$$E[\underline{U}\,\underline{U}^T] = \sigma_\nu^2\, \Xi\,\Xi^T \tag{4.2-61}$$

The eigenvalues of $\Xi\Xi^T$ are the same as that of $\Xi^T\Xi$ which has the same form as A^TA with $\underline{\xi}$ substituted for $\underline{\alpha}$ up to $p = q$ in Figure (4.2-1). Thus the matrix $\Xi^T\Xi$ can be approximated by a Toeplitz matrix as shown in section (4.2-2) where

$$f(\lambda) = \left| 1 - \sum_{k=1}^{q} \xi_k e^{-ik\lambda} \right|^2, \quad \lambda \in [-\pi, \pi] \tag{4.2-62}$$

If $\{\lambda_{k,N}\}$ are the eigenvalues of $\Xi^T\Xi$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \lambda_{k,N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda)\, d\lambda \tag{4.2-63}$$

The following theorem now follows:

Theorem 4.2.4 For the finite moving average inputs

$$u_k = \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \tag{4.2-64}$$

The trace of the input correlation matrix is related to the spectrum of the corresponding stationary process, $s_{uu}(\lambda)$ in the sense

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}[\Xi\Xi^T] = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda)\, d\lambda \tag{4.2-65}$$

where the spectrum of the stationary MA(q) inputs is

$$s_{uu}(\lambda) = \sigma_\nu^2 \left| 1 - \sum_{k=1}^{q} \xi_k e^{-ik\lambda} \right|^2, \quad \lambda \in [-\pi, \pi] \tag{4.2-66}$$

□

For the moving average inputs we can obtain an estimate of the error due to the above approximation. For MA(2) process

$$\frac{1}{N} \text{ Trace}[\Xi \Xi^T] = \frac{1}{N} \quad N + (N - 1)\, \xi_1^2 + (N - 2)\, \xi_2^2 \tag{4.2-67}$$

$$= 1 + 1 - \frac{1}{N} \quad \xi_1^2 + 1 - \frac{2}{N} \quad \xi_2^2 \tag{4-2-68}$$

$$\frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_u(\lambda)\, d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 + \xi_1 e^{-1\lambda} + \xi_2 e^{-i2\lambda} \right|^2 d\lambda$$

$$= 1 + \xi_1^2 + \xi_2^2 \tag{4.2-69}$$

$$\text{Error} = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_u(\lambda)\, d\lambda - \frac{1}{N} \text{ Trace}[\quad\quad]$$

$$e = \frac{1}{N} \quad (\xi_1^2 + 2\xi_2^2) \tag{4.2-70}$$

From the constraint region for (ξ_1, ξ_2) we have an upper bound for the error e.

$$e < \frac{6}{N} \tag{4.2-71}$$

Percentage error

$$e_p = \frac{1}{N} \quad \frac{\xi_1^2 + 2\xi_2^2}{1 + \xi_1^2 + \xi_2^2} \approx \frac{1}{N} \tag{4.2-72}$$

Thus we see that the error in the approximation is like $1/N$. In practice we do not have to compute these traces for optimization procedure. For processes of order greater than two, the computation of the trace is much simpler than the computation of the spectral area.

4.2.4 The Autoregressive-Moving Average ARMA(p, q) Inputs

Consider the finite order mixed input process

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \qquad (4.2\text{-}73)$$

$$u_k = 0, \quad k < 0$$

We can write (4.2-73) using the results of AR(p) and MA(q) processes (4.2-16) and (4.2-59) as

$$A \underline{U}^{N-1} = \Xi \underline{\nu}^{N-1}$$

or

$$\underline{U}^{N-1} = A^{-1} \Xi \underline{\nu}^{N-1} \qquad (4.2\text{-}75)$$

The input correlation matrix is given by

$$E[\underline{U}\ \underline{U}^T] = \sigma_\nu^2 A^{-1} \Xi \Xi^T A^{-T} \qquad (4.2\text{-}76)$$

We are interested in showing that

$$\frac{1}{N}\,\mathrm{Tr}\left[A^{-1}\Xi\Xi^{T}A^{-T}\right] \rightarrow \frac{1}{2\pi\sigma_{\nu}^{2}}\int_{-\pi}^{\pi} s_{uu}(\lambda)\,d\lambda \qquad (4.2\text{-}77)$$

as $N\rightarrow\infty$. $s_{uu}(\lambda)$ is the spectral density of the stationary ARMA process -

$$s_{uu}(\lambda) = \sigma_{\nu}^{2}\;\frac{\left|1-\sum_{k=1}^{q}\xi_{k}\,e^{-ik\lambda}\right|^{2}}{\left|1-\sum_{k=1}^{q}\alpha_{k}\,e^{-ik\lambda}\right|^{2}}\;,\quad \lambda\in[-\pi,\pi] \qquad (4.2\text{-}78)$$

The results summarized below are obtained from Gray [31] and applied to the optimization problem of input synthesis. The implication of these results are also discussed. A complete summary of the asymptotic eigenvalue properties of Toeplitz matrices is given in Appendix E.

Eigenvalues of the product of nonsingular matrices is unaffected by a cyclic rotation of the matrices, that is

$$\lambda\left[A^{-1}\Xi\Xi^{T}A^{-T}\right] = \lambda\left[A^{-T}A^{-1}\Xi\Xi^{T}\right] \qquad (4.2\text{-}79)$$

This is clear because the above operation is like affecting a similarity transformation under which the eigenvalues are invariate. Now we can write

$$\Lambda_N = A^{-T} A^{-1} \Xi \Xi^T = (AA^T)^{-1} \Xi \Xi^T \qquad (4.2\text{-}80)$$

The matrix Λ_N is of the form

$$\Lambda_N = \Lambda_1^{-1} \Lambda_2 \qquad (4.2\text{-}81)$$

$$\Lambda_1 = AA^T, \quad \Lambda_2 = \Xi \Xi^T \qquad (4.2\text{-}82)$$

Λ_N is not a Toeplitz matrix. But we have seen that the eigenvalues of Λ_1 and Λ_2 can be approximated by the Toeplitz matrices $T\left[f(x)\right]$ and $T\left[g(x)\right]$ respectively, where

$$f(x) = \left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ikx} \right|^2, \quad x \in [-\pi, \pi] \qquad (4.2\text{-}83)$$

and

$$g(x) = \left| 1 - \sum_{k=1}^{p} \xi_k e^{-ikx} \right|^2, \quad x \in [-\pi, \pi] \qquad (4.2\text{-}84)$$

We use the following theorem of Gray [31] to show that the trace of $\Lambda_1^{-1} \Lambda_2$ approximates the desired integral.

Theorem 4.2.5 Let $f(x)$ and $g(x)$ belong to class L with $f(x) > 0$ everywhere. Then

$$\lim_{N \to \infty} \left| T\left[g(x)\right] T^{-1}\left[f(x)\right] - T\left[g(x) / f(x)\right] \right| = 0 \qquad (4.2\text{-}85)$$

and the eigenvalues of $T\left[g(x)\right] T^{-1}\left[f(x)\right]$ are asymptotically distributed as $g(x) / f(x)$, $x \in [-\pi, \pi]$. ⊠

Corollary 4.2.1 Let L_N and M_N be Hermitian matrices approximating $T_N[g]$ and $T_N[f]$ respectively. Then

$$\lim_{N\to\infty} \left| L_N M_N^{-1} - T_N[g/f] \right| = 0 \qquad (4.2\text{-}86)$$

⊠

The proof of the above results are given in Gray [31]. These results can be applied to the matrix

$$\Lambda_N = (AA^T)^{-1}(\Xi\Xi)^T = \Lambda_1^{-1}\Lambda_2 \qquad (4.2\text{-}87)$$

We know from results of sections (4.2.2) and (4.2.3) that the matrices AA^T and $\Xi\Xi^T$ can be approximated by Toeplitz matrices $T\left[f(x)\right]$ and $T\left[g(x)\right]$. Thus from the theorem (4.2.5) and corollary (4.2.1) we have

$$\lim_{N\to\infty} \left| \Lambda_1^{-1}\Lambda_2 - T[g/f] \right| = 0 \qquad (4.2\text{-}88)$$

Hence the eigenvalues of $\Lambda_1^{-1}\Lambda_2$ are distributed asymptotically as $g(x)/f(x)$, $x \in [-\pi, \pi]$. It is also shown by Gray that by a similar argument as above if $g(x) > 0$ everywhere then the eigenvalues of Λ^{-1} are distributed asymptotically as $f(x)/g(x)$, $x \in [-\pi, \pi]$. Thus if either $f(x) > 0$ or $g(x) > 0$ everywhere then corollary 4.2.1 gives the asymptotic behavior of Λ_N or Λ_N^{-1} respectively. The result for the ARMA(p, q) inputs is summarized in the following theorem.

Theorem 4.2.6 Let the ARMA(p, q) inputs be given by

$$u_k = \sum_{i=1}^{p} \alpha_i u_{k-i} + \nu_k - \sum_{i=1}^{q} \xi_i \nu_{k-i} \tag{4.2-89}$$

Let

$$f(\lambda) = \left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ik\lambda} \right|^2 \tag{4.2-90}$$

$$g(\lambda) = \left| 1 - \sum_{k=1}^{q} \xi_k e^{-ik\lambda} \right|^2, \quad \lambda \in [-\pi, \pi] \tag{4.2-91}$$

If $f(\lambda) > 0$ everywhere in $[-\pi, \pi]$ then the eigenvalues of the matrix

$$\Lambda_N = (AA^T)^{-1} \Xi \Xi^T \tag{4.2-92}$$

are distributed asymptotically as $g(\lambda)/f(\lambda)$ in the following sense

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \lambda_{k,N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(\lambda)}{f(\lambda)} \, d\lambda \tag{4.2-93}$$

The trace of the input correlation matrix is related to the spectrum of the corresponding stationary process, $s_{uu}(\lambda)$ in the sense

$$\lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}[A^{-1} \Xi \Xi^T A^{-T}] = \frac{1}{2\pi\sigma_\nu^2} \int_{-\pi}^{\pi} s_{uu}(\lambda) \, d\lambda \tag{4.2-94}$$

where the spectrum of the stationary ARMA(p, q) inputs is

$$s_{uu}(\lambda) = \sigma_\nu^2 \frac{\left| 1 - \sum_{k=1}^{q} \xi_k e^{-ik\lambda} \right|^2}{\left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ik\lambda} \right|^2}, \quad \lambda \in [-\pi, \pi] \tag{4.2-95}$$

⊠

At the optimum value $\bar{J}^*$ the spectral curve must be shaped in such a way that the cost function $\bar{J}$ is maximized and

$$\bar{J}^* = \frac{1}{\pi} \int_0^{\pi} \left(\sum_{n=0}^{N-1} \mu_n \cos n\lambda \right) s_{uu}^*(\lambda)\, d\lambda \qquad (4.2\text{-}96)$$

when σ_ν^2 is fixed the optimum $\bar{J}^*$ is obtained for which the effective spectral area of the input process is also maximized in the sense of (4.2-94). In this case

$$\lim_{N\to\infty} \frac{1}{N} \bar{J}^* = \lambda_{max}(Q)\, \sigma_\nu^2 \sup_{(\underline{\alpha}, \underline{\xi})} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(\lambda)}{f(\lambda)}\, d\lambda \qquad (4.2\text{-}97)$$

The results are studied numerically in the next section where different input sequences are compared. The assumption of $s_{uu}(\lambda) > 0$ is satisfied by all the inputs considered in this section, and is a necessary and sufficient condition for the basic Toeplitz theorem. This condition is also referred to as "persistently exciting" by Mehra [62] and Åström and Bohlin [6]. This condition makes the input correlation matrix positive definite and is a sufficient condition for the maximum likelihood estimates to be efficient [6]. Here we see that this condition is being satisfied naturally in the development of the asymptotic properties for the input processes generated by linear models discussed in this section. The conditions stated by earlier investigators is embedded in the results derived.

The circuit diagram for generating the ARMA(p, q) inputs can be interpreted as the u_k passed through an autoregressive filter and the noise ν_k passed through a moving average filter in cascade. Figure (4.2-2) shows this block diagram.

4.3 Numerical Results

In this section we will present two examples of single-input and single-output system. Comparison of the different inputs obtained by the stationary models is made. The system is represented by

$$y_k = \sum_{i=1}^{n} \varphi_i y_{k-i} + \sum_{i=1}^{r} \beta_i u_{k-i} + w_k \qquad (4.3\text{-}1)$$

$$w_k \sim G(0, \sigma^2) \qquad (4.3\text{-}2)$$

We consider first order and second order systems obtained from (4.3-1).

(i) Second Order System

$$y_k = \varphi_1 y_{k-1} + \varphi_2 y_{k-2} + \beta u_{k-1} + w_k \qquad (4.3\text{-}3)$$

The system has the transfer function

$$G(z) = \frac{\beta_2^{-1}}{1 - \varphi_1 z^{-1} - \varphi_2 z^{-2}} \qquad (4.3\text{-}4)$$

The frequency response is given by

$$|G(f)| = \frac{|\beta e^{-i2\pi f}|}{|1 - \varphi_1 e^{-i2\pi f} - \varphi_2 e^{-i4\pi f}|} , \quad |f| \leq \tfrac{1}{2}$$

Simplifying

$$|G(f)| = \frac{|\beta|}{1 + \varphi_1^2 + \varphi_2^2 - 2\varphi_1(1 - \varphi_2)\cos 2\pi f - 2\varphi_2 \cos 4\pi f)^{\frac{1}{2}}}$$

$$|f| \leq \tfrac{1}{2} \qquad (4.3\text{-}5)$$

(ii) First Order System

$$y_k = \varphi y_{k-1} + \beta u_{k-1} + w_k \qquad (4.3\text{-}6)$$

Frequency response is given by

$$|G(f)| = \frac{|\beta e^{-i2\pi f}|}{|1 - \varphi e^{-i2\pi f}|} , \quad |f| \leq \tfrac{1}{2}$$

Simplifying

$$|G(f)| = \frac{|\beta|}{1 + \varphi^2 - 2\varphi \cos 2\pi f)^{\frac{1}{2}}} , \quad |f| \leq \tfrac{1}{2} \qquad (4.3\text{-}7)$$

Only input processes up to second order are considered. The theoretical spectrum of these give processes is given below.

(i) AR(2) Process:

$$u_k = \alpha_1 u_{k-1} + \alpha_2 u_{k-2} + \nu_k \qquad (4.3\text{-}8)$$

Spectrum

$$s_u(f) = \sigma_\nu^2 \left(1 + \alpha_1^2 + \alpha_2^2 - 2\alpha_1(1 - \alpha_2)\cos 2\pi f - 2\varphi_2 \cos 4\pi f\right)^{-1}$$

$$|f| \le \tfrac{1}{2} \qquad (4.3\text{-}9)$$

(ii) MA(2) Process

$$u_k = \nu_k - \xi_1 \nu_{k-1} - \xi_2 \nu_{k-2} \qquad (4.3\text{-}10)$$

Spectrum

$$s_u(f) = \sigma_\nu^2 \left(1 + \xi_1^2 + \xi_2^2 - 2\xi_1(1 - \xi_2)\cos 2\pi f - 2\xi_2 \cos 4\pi f\right) \qquad (4.3\text{-}11)$$

(iii) ARMA(1, 1) Process

$$u_k = \alpha_1 u_{k-1} + \nu_k - \xi_1 \nu_{k-1} \qquad (4.3\text{-}12)$$

Spectrum is

$$s_u(f) = \sigma_\nu^2 \frac{1 + \xi_1^2 - 2\xi_1 \cos 2\pi f}{1 + \alpha_1^2 - 2\alpha_1 \cos 2\pi f}, \quad |f| \le \tfrac{1}{2} \qquad (4.3\text{-}13)$$

(iv) AR(1) Process

$$u_k = \alpha_1 u_{k-1} + \nu_k \qquad (4.3\text{-}14)$$

The spectrum is

$$s_u(f) = \sigma_\nu^2 (1 + \alpha_1^2 - 2\alpha_1 \cos 2\pi f)^{-1}, \quad |f| \leq \tfrac{1}{2} \qquad (4.3\text{-}15)$$

(v) MA(1) Process

$$u_k = \nu_k - \xi_1 \nu_{k-1} \qquad (4.3\text{-}16)$$

The spectrum is

$$s_u(f) = \sigma_\nu^2 (1 + \xi_1^2 - 2\xi_1 \cos 2\pi f), \quad |f| \leq \tfrac{1}{2} \qquad (4.3\text{-}17)$$

The maximization of the cost function with respect to the parameters requires these to satisfy certain constraints such that the stability and invertibility of the input processes is ensured. The constraint regions are as follows:

AR(1) Process: $$|\alpha_1| < 1 \qquad (4.3\text{-}18)$$

MA(1) Process: $$|\xi_1| < 1 \qquad (4.3\text{-}19)$$

AR(2) Process: $$\alpha_2 + \alpha_1 < 1$$

$$\alpha_2 - \alpha_1 < 1$$

$$|\alpha_2| < 1 \qquad (4.3\text{-}20)$$

MA(2) Process:

$$\xi_2 + \xi_1 < 1$$

$$\xi_2 - \xi_1 < 1$$

$$|\xi_2| < 1 \qquad (4.3\text{-}21)$$

ARMA(1, 1) Process:

$$|\alpha_1| < 1, \quad |\xi_1| < 1 \qquad (4.3\text{-}22)$$

The constraint regions for AR(2), MA(2) and ARMA(1, 1) are given in figures (4.3-1) and (4.3-2).

In the following examples the parameters have three discrete values with equal probabilities as follows:

φ_1	φ_2	β	
1.5	-.7	1.	(actual values)
1.1	-.4	.6	
1.6	-.8	1.1	

Under the assumption that the input process is stationary the optimal inputs are computed for the above five input models. The input parameters, optimal cost functions and the theoretical effective spectral areas are compared in tables (4.3-1)-(4.3-3). Table (4.3-3) gives results when the φ value for the first order system is changed from -.7 to .7. Optimization for the second order system is made by taking β as a known parameter. Figures

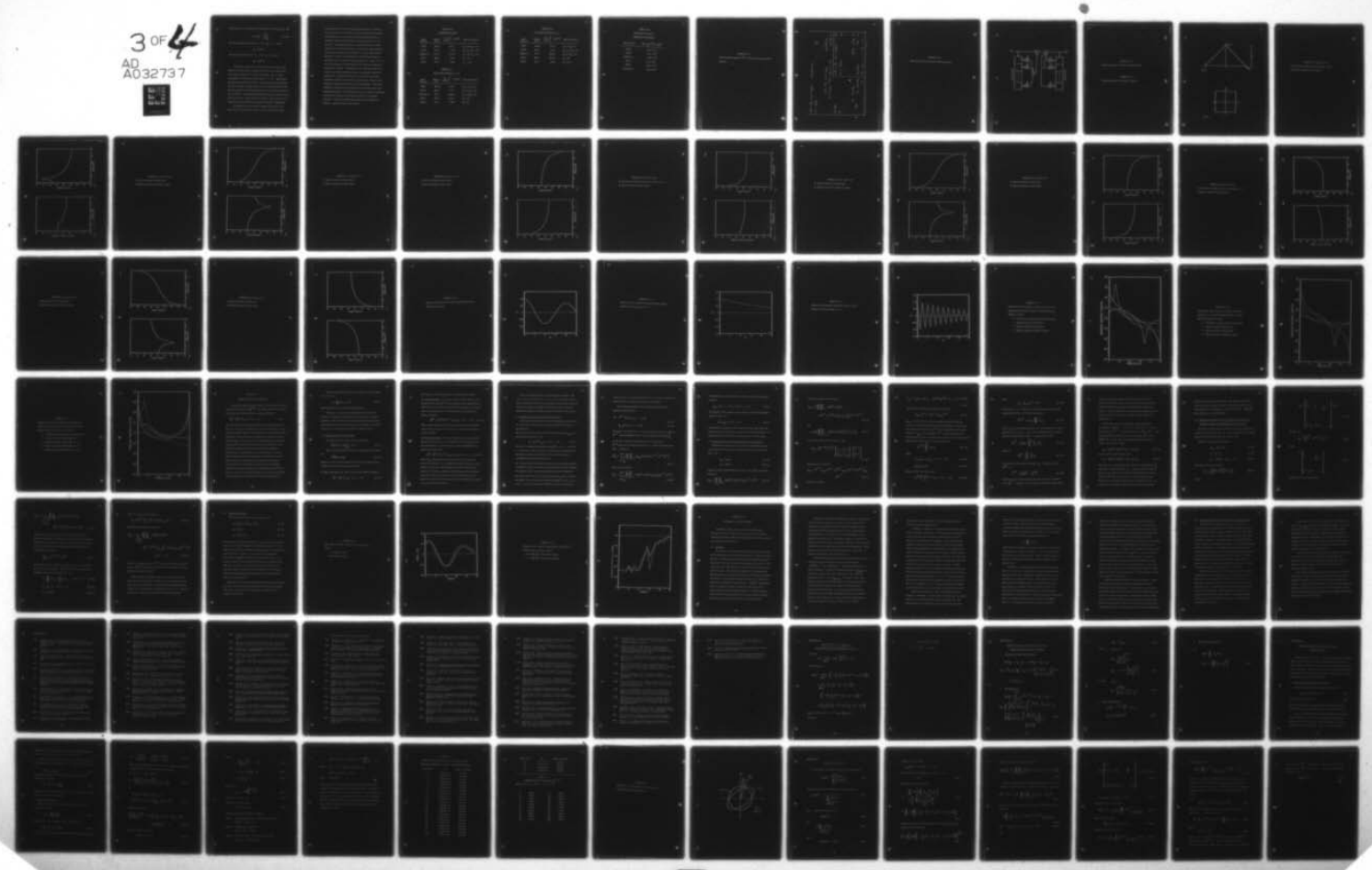
(4.3-3a)-(4.3-3f) show the frequency response of the second order system and the spectrum of input processes. Figures (4.3-4a)-(4.3-4f) are the frequency response for the first order system with $\varphi = .7$ and figures (4.3-5a)-(4.3-5f) correspond to $\varphi = -.7$. The autocorrelation of the optimal AR(2) inputs for the second order system and the ARMA(1, 1) inputs for the first order system for $\varphi = .7$ and $\varphi = -.7$ are shown in figures (4.3-6)-(4.3-8) respectively. When the autocorrelation function slowly decays exponentially we see that the spectrum of the process is dominated by low frequencies. This results in the neighboring points of the input sequence being similar and then exhibits marked trends. In contrast, when the autocorrelation function alternates in sign, the spectrum is dominated by high frequencies and the input sequence tends to oscillate rapidly.

Figure (4.3-9) shows the system response and input spectrum of AR(2), MA(2), ARMA(1, 1) processes for the second order system. It is seen here that the power of the input spectrum is concentrated at frequencies where the system gain is also high. Thus we see a matching of the frequency responses. This tuning is caused due mainly to the fact that the parameter space is restricted to a small region surrounding the actual parameter values. The same is also true for the first order system as evidenced by figure (4.3-10).

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The frequency corresponding to the peak gain is given by [15, p. 60].

$$\cos 2\pi f_0 = \frac{|\varphi_1|}{2\sqrt{-\varphi_2}} \qquad (4.3\text{-}23)$$

For the second order system, $\varphi_1 = 1.5$, $\varphi_2 = -.7$. Hence

$$f_0 = .0738 \text{ Hz}$$

For the optimal AR(2) input, $\alpha_1 = 1.8$, $\alpha_2 = -.95$, and

$$f_0 = .0628 \text{ Hz}.$$

The learning performance for the second order system with different random inputs is compared in table (4.3-4). Comparison is also made with pure white noise as the input. These results correspond to a total experimental length of $N = 20$. AR(2) inputs give the best parameter learning; the moving average signal being the least favorable. Comparison of the cost functions indicates that the autoregressive and ARMA inputs perform better than the moving average inputs. The optimization problem is easy for input processes up to second order. Beyond this the constraint region becomes very complicated and the optimization requires expensive computation.

It is pointed out earlier that the optimal input spectrum is such that there is maximum transfer from the input to the output

at frequencies where the corresponding gains match. In order to investigate if this is the result of tuning the parameter sets close to the actual values of the system parameters experiments are conducted with the number of discrete parameter sets $m = 3, 5, 10$ and 15. The parameter sets are chosen to include the stable region of the second order system. For the same input process it is seen that the learnings with different m do not show marked difference. Hence it is not possible to correlate the number of parameter sets m and the learning performance. Figure (4.3-11) shows the spectrum of AR(2) inputs and the frequency response of second order system. This indicates that as the parameter set is reduced in size the optimal input spectrum gradually matches the system gain. No general conjecture can be derived from these regarding a specific behavior of the input process. However, this suggests a sequential approach to the design problem. This can be obtained by successively reducing the parameter space after each experimental stage. As the parameter space is reduced successively the best input spectrum will emerge eventually as the optimal spectrum. Further theoretical and computational analyses are needed to explain the above observations.

Table 4.3-1

Second Order System

Input Process	Optimal Cost	$\frac{1}{2\pi\sigma_\nu^2}\int_{-\pi}^{\pi} s_{uu}(\lambda)\,d\lambda$	Input Parameters
AR(2)	3401.3	69.33	$\alpha_1 = 1.8, \alpha_2 = -.95$
MA(2)	1524.6	3.113	$\xi_1 = -1.1, \xi_2 = -.95$
ARMA(1, 1)	2146.6	12.676	$\alpha_1 = .85, \xi_1 = -.95$
AR(1)	2008.1	5.263	$\alpha_1 = .95$
MA(1)	967.7	1.903	$\xi_1 = -.95$

Table 4.3-2

First Order System, $\omega = -.7$

Input Process	Optimal Cost	$\frac{1}{2\pi\sigma_\nu^2}\int_{-\pi}^{\pi} s_{uu}(\lambda)\,d\lambda$	Input Parameters
AR(2)	500.03	8.276	$\alpha_1 = -.7, \alpha_2 = .25$
MA(2)	204.8	3.113	$\xi_1 = 1.1, \xi_2 = -.95$
ARMA(1, 1)	507.3	38.026	$\alpha_1 = -.95, \xi_1 = .95$
AR(1)	496.2	10.256	$\alpha_1 = -.95$
MA(1)	146.1	1.908	$\xi_1 = .95$

Table 4.3-3

First Order System, $\rho = .7$

Input Process	Optimal Cost	$\frac{1}{2\pi\sigma_\nu^2}\int_{-\pi}^{\pi} s_{uu}(\lambda)\,d\lambda$	Input Parameters
AR(2)	500.03	8.276	$\alpha_1 = .7, \alpha_2 = .25$
MA(2)	204.8	3.113	$\xi_1 = -1.1, \xi_2 = -.95$
ARMA(1, 1)	507.3	38.026	$\alpha_1 = .95, \alpha_2 = -.95$
AR(1)	496.2	10.256	$\alpha_1 = .95$
MA(1)	146.1	1.903	$\xi_1 = -.95$

Table 4.3-4

Second Order System -

Comparison of Learning

Input Process	$\|\underline{\varphi} - \underline{\hat{\varphi}}_N\|^2 / \|\underline{\varphi} - \underline{\hat{\varphi}}_0\|^2$
AR(2)	8.56×10^{-7}
MA(2)	3.21×10^{-4}
ARMA(1,1)	7.98×10^{-6}
AR(1)	2.9×10^{-5}
MA(1)	8.01×10^{-4}
White Noise	8.29×10^{-4}

FIGURE 4.2-1

The correlation matrix $A^T A$ of pth order autoregressive process.

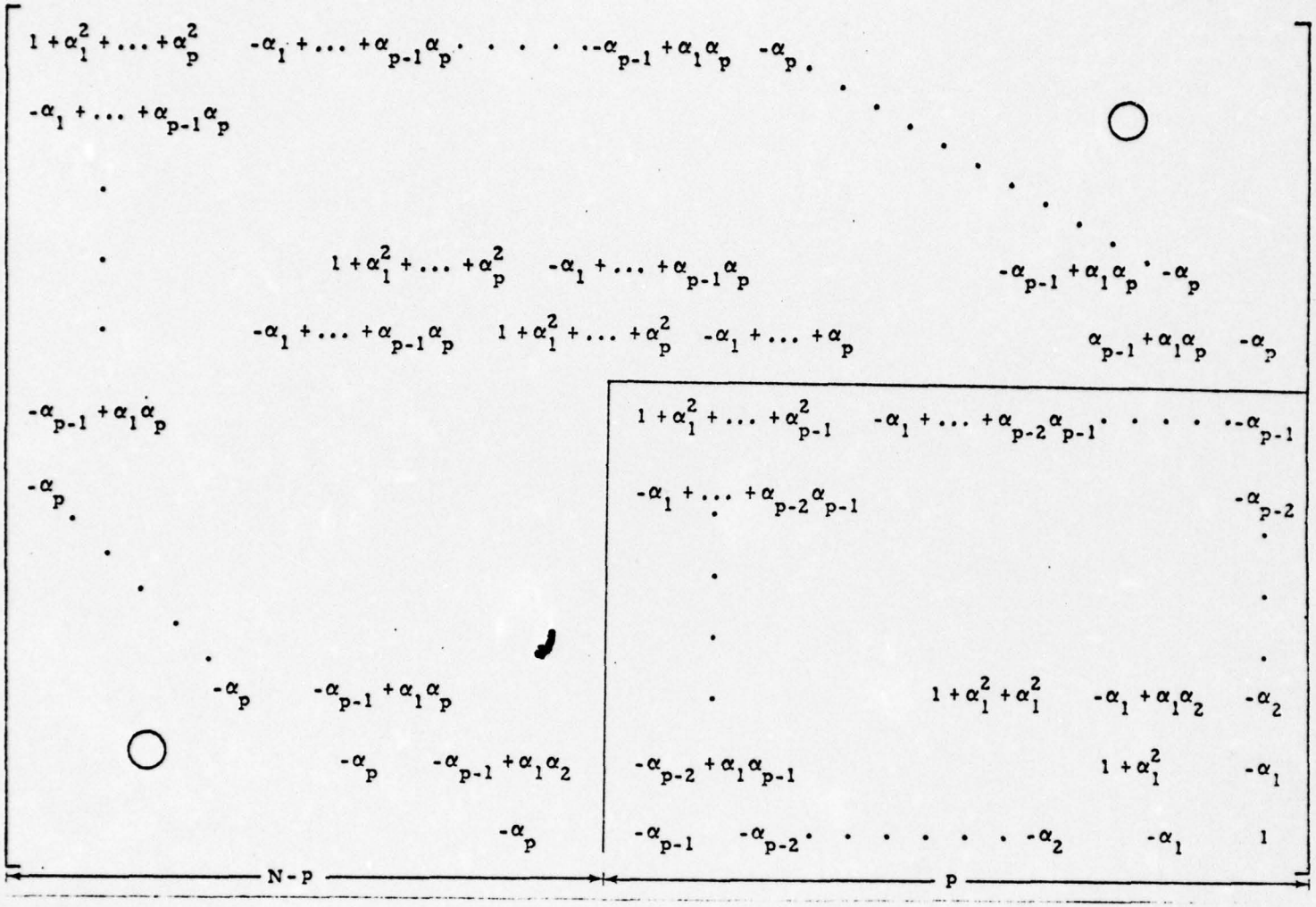

$1+\alpha_1^2+\ldots+\alpha_p^2$ $-\alpha_1+\ldots+\alpha_{p-1}\alpha_p$ $-\alpha_{p-1}+\alpha_1\alpha_p$ $-\alpha_p$
$-\alpha_1+\ldots+\alpha_{p-1}\alpha_p$
$1+\alpha_1^2+\ldots+\alpha_p^2$ $-\alpha_1+\ldots+\alpha_{p-1}\alpha_p$ $-\alpha_{p-1}+\alpha_1\alpha_p$ $-\alpha_p$
$-\alpha_1+\ldots+\alpha_{p-1}\alpha_p$ $1+\alpha_1^2+\ldots+\alpha_p^2$ $-\alpha_1+\ldots+\alpha_p$ $\alpha_{p-1}+\alpha_1\alpha_p$ $-\alpha_p$
$-\alpha_{p-1}+\alpha_1\alpha_p$
$-\alpha_p$
$-\alpha_p$ $-\alpha_{p-1}+\alpha_1\alpha_p$
$-\alpha_p$ $-\alpha_{p-1}+\alpha_1\alpha_2$
$-\alpha_p$
$1+\alpha_1^2+\ldots+\alpha_{p-1}^2$ $-\alpha_1+\ldots+\alpha_{p-2}\alpha_{p-1}$ $-\alpha_{p-1}$
$-\alpha_1+\ldots+\alpha_{p-2}\alpha_{p-1}$ $-\alpha_{p-2}$
$1+\alpha_1^2+\alpha_1^2$ $-\alpha_1+\alpha_1\alpha_2$ $-\alpha_2$
$-\alpha_{p-2}+\alpha_1\alpha_{p-1}$ $1+\alpha_1^2$ $-\alpha_1$
$-\alpha_{p-1}$ $-\alpha_{p-2}$ $-\alpha_2$ $-\alpha_1$ 1
N - p
p

FIGURE 4.2-2

Block diagram for generating ARMA(p, q) inputs.

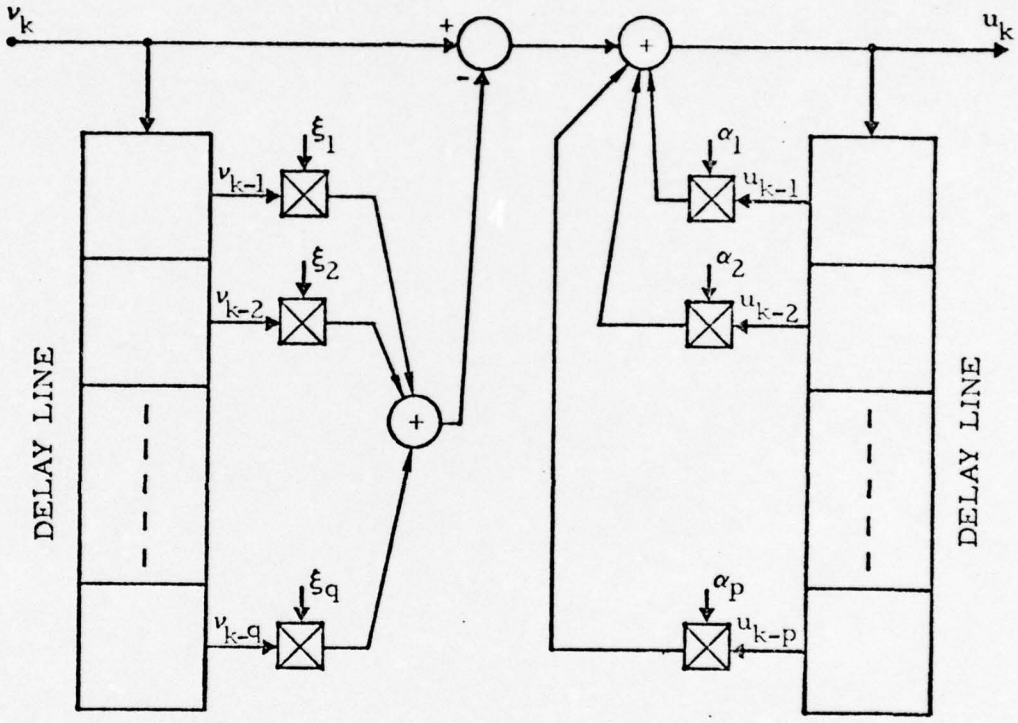
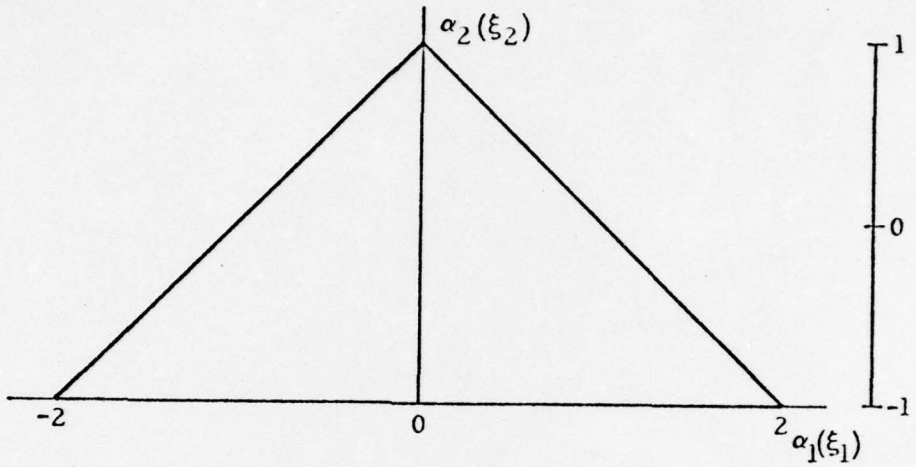

v_k
+
-
+
u_k
ξ_1
v_{k-1}
ξ_2
v_{k-2}
ξ_q
v_{k-q}
+
DELAY LINE
α_1
u_{k-1}
α_2
u_{k-2}
α_p
u_{k-p}
DELAY LINE

FIGURE 4.3-1

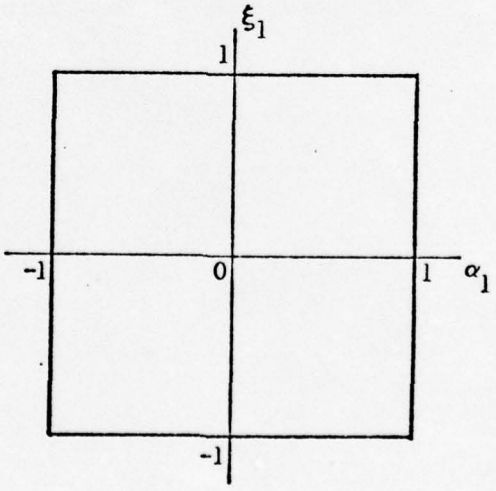
Constraint region for AR(2) and MA(2) inputs.

FIGURE 4.3-2

Constraint region for ARMA(1, 1) inputs.



(4.3-1)

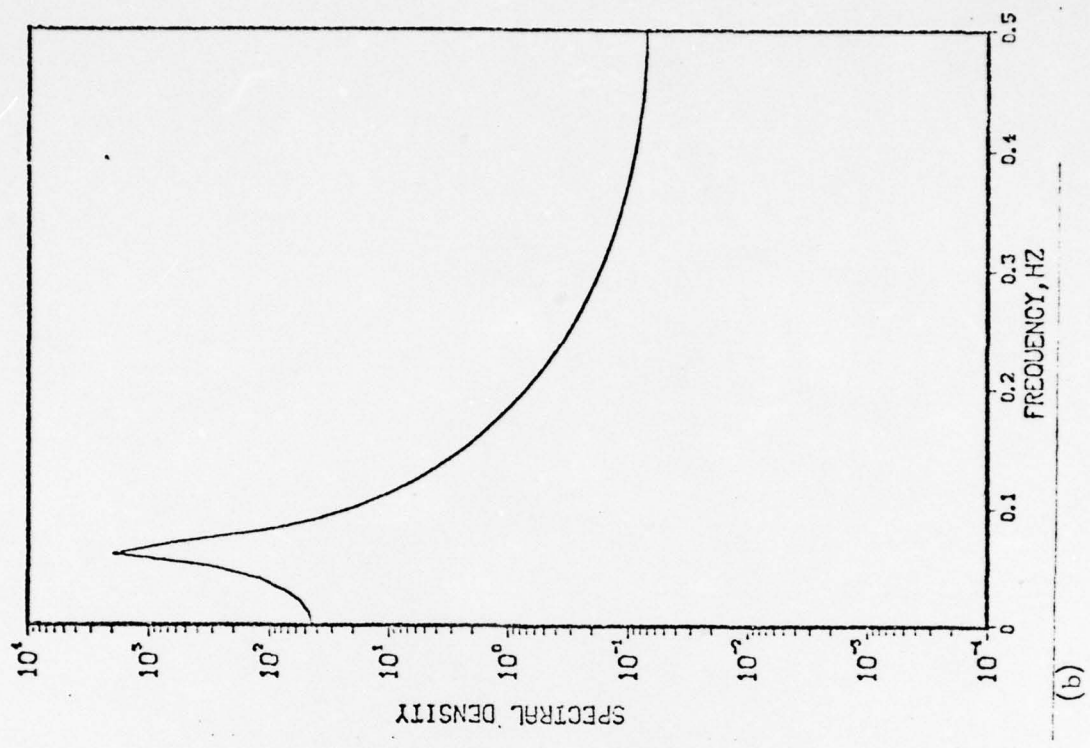


(4.3-2)

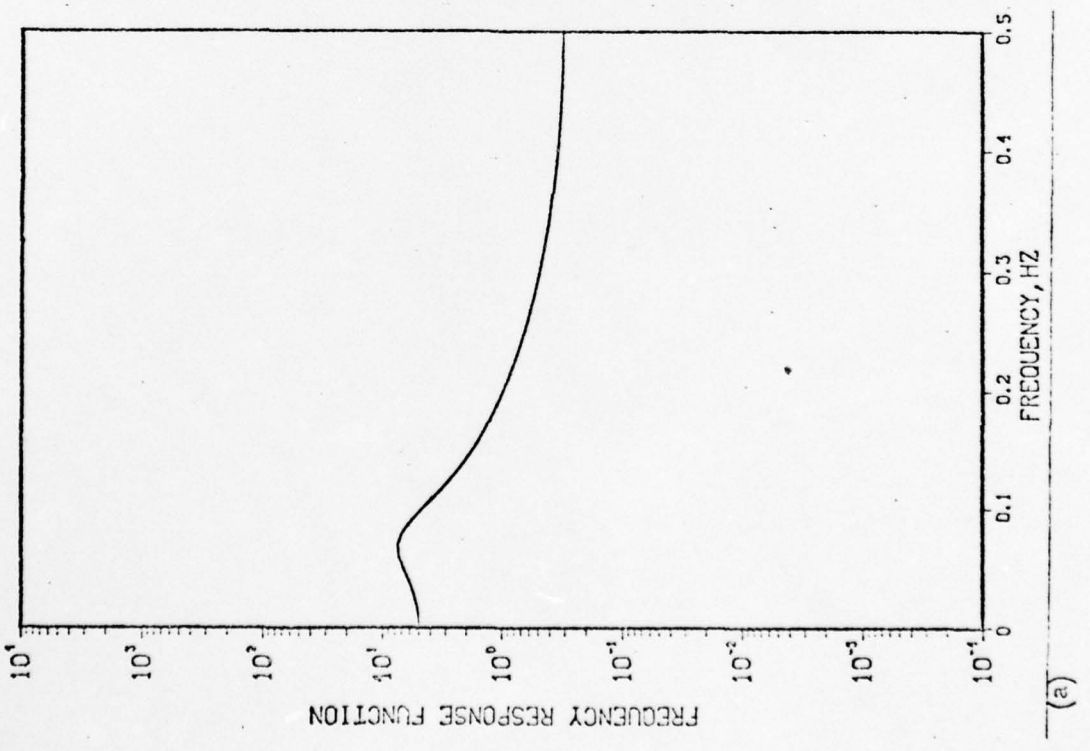
FIGURES 4.3-3a and 4.3-3b

a) Frequency response of second order system

b) Spectral density of AR(2) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(b)

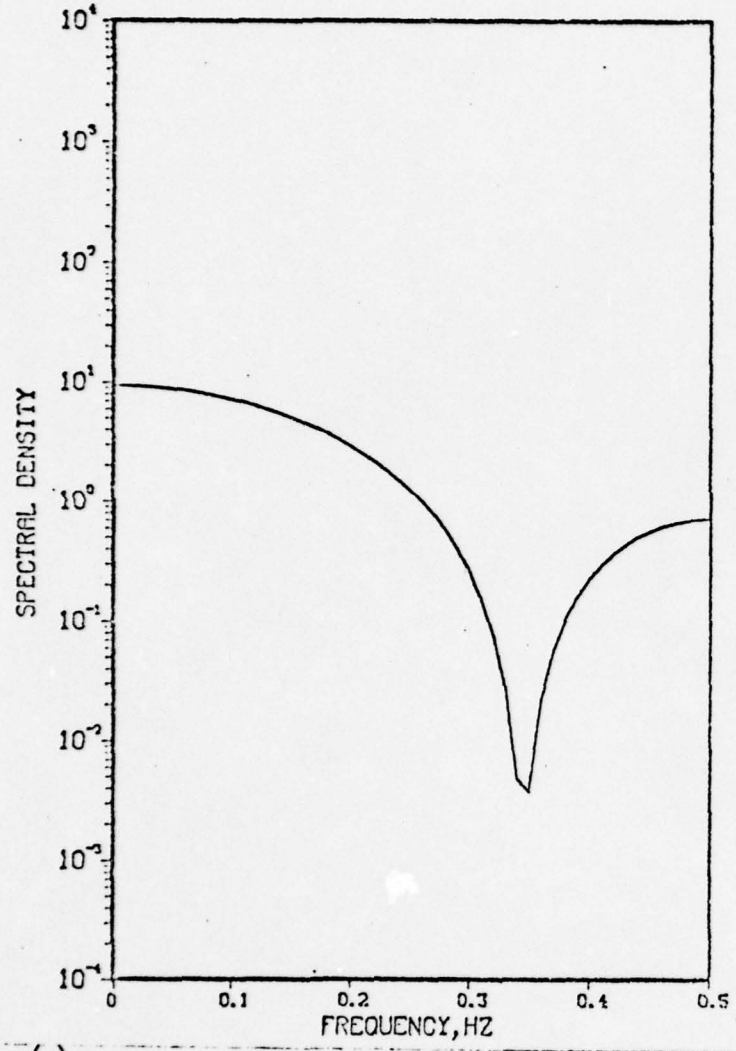

FREQUENCY RESPONSE FUNCTION
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(a)

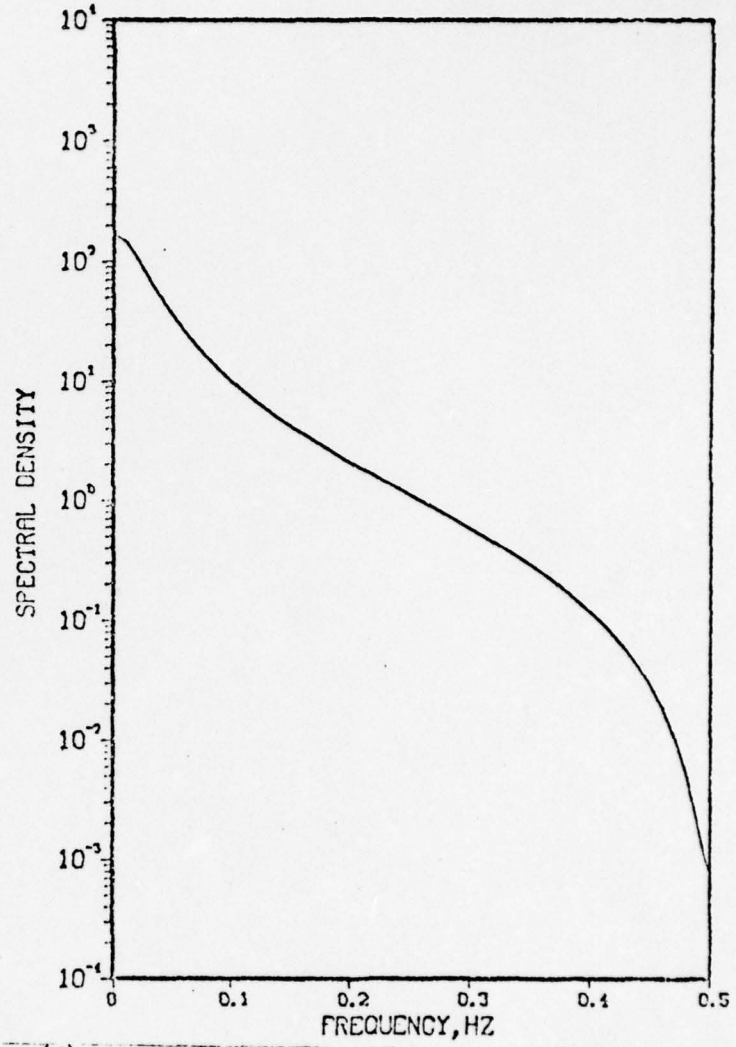
FIGURE 4.3-3c and 4.3-3d

c) Spectral density of MA(2) inputs

d) Spectral density of ARMA(1, 1) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(c)


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(d)

FIGURES 4.3-3e and 4.3-3f

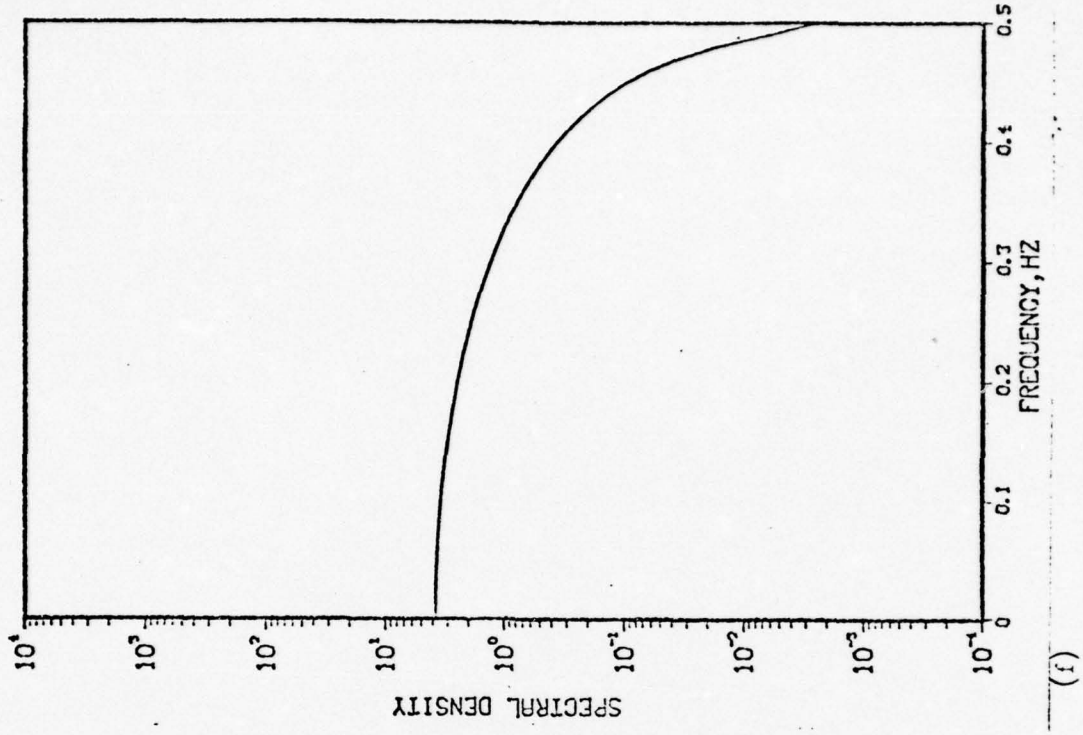
e) Spectral density of AR(1) inputs

f) Spectral density of MA(1) inputs.

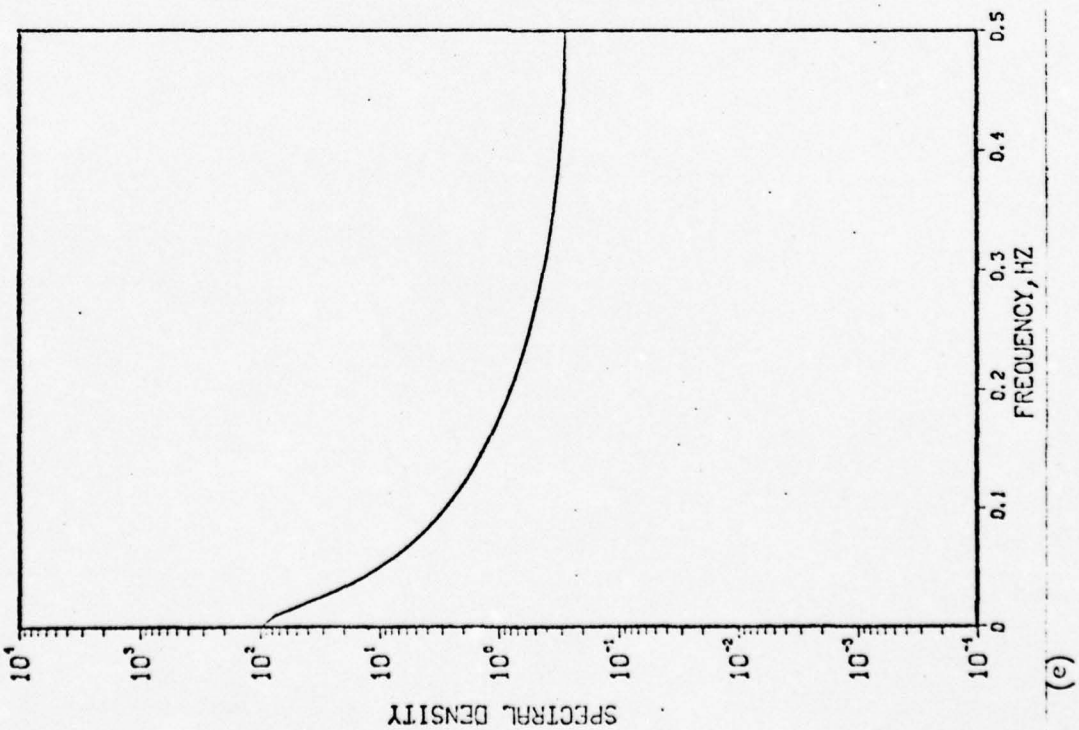
FIGURES 4.3-3e and 4.3-3f

e) Spectral density of AR(1) inputs

f) Spectral density of MA(1) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(f)

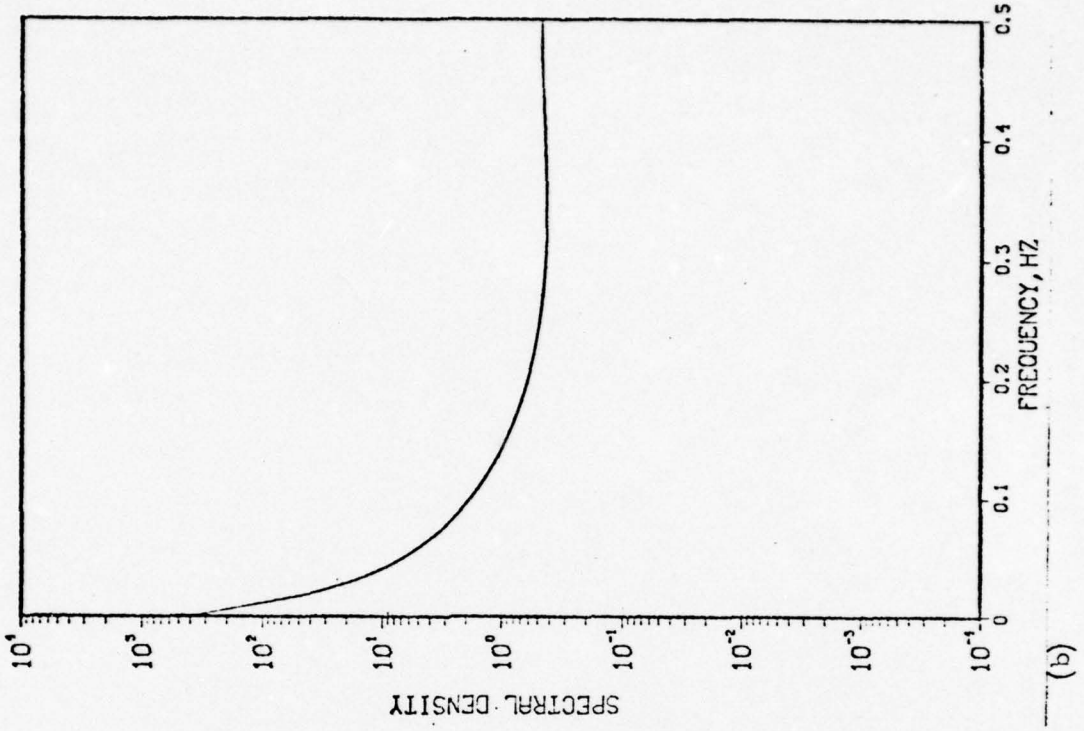

SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(e)

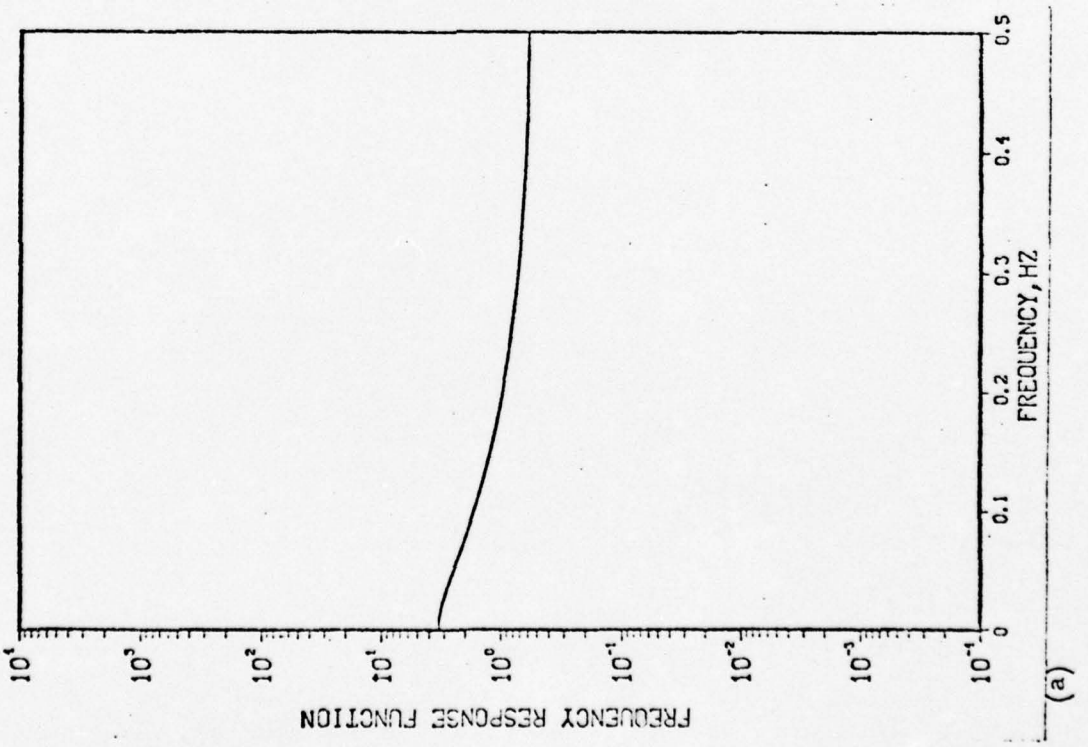
FIGURES 4.3-4a and 4.3-4b

a) Frequency response of first order system, $\varphi = .7$

b) Spectral density of AR(2) inputs.


SPECTRAL DENSITY
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(b)

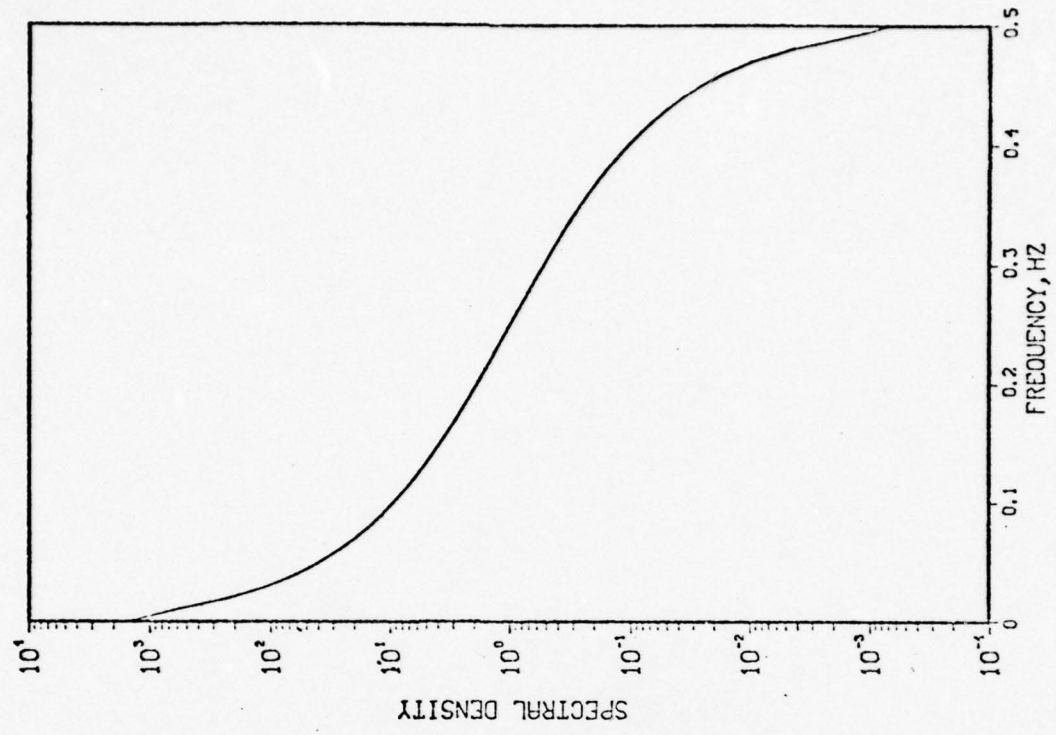

FREQUENCY RESPONSE FUNCTION
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(a)

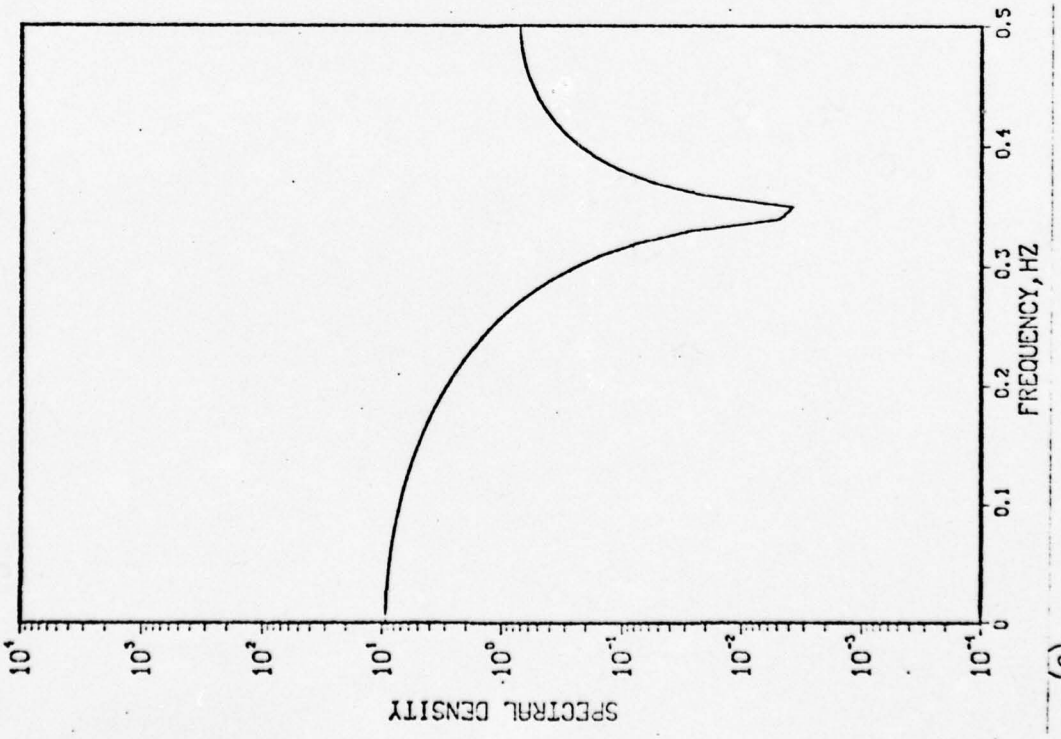
FIGURES 4.3-4c and 4.3-4d

c) Spectral density of MA(2) inputs

d) Spectral density of ARMA(1, 1) inputs.


SPECTRAL DENSITY
FREQUENCY, HZ
0
0.1
0.2
0.3
0.4
0.5
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}

(d)

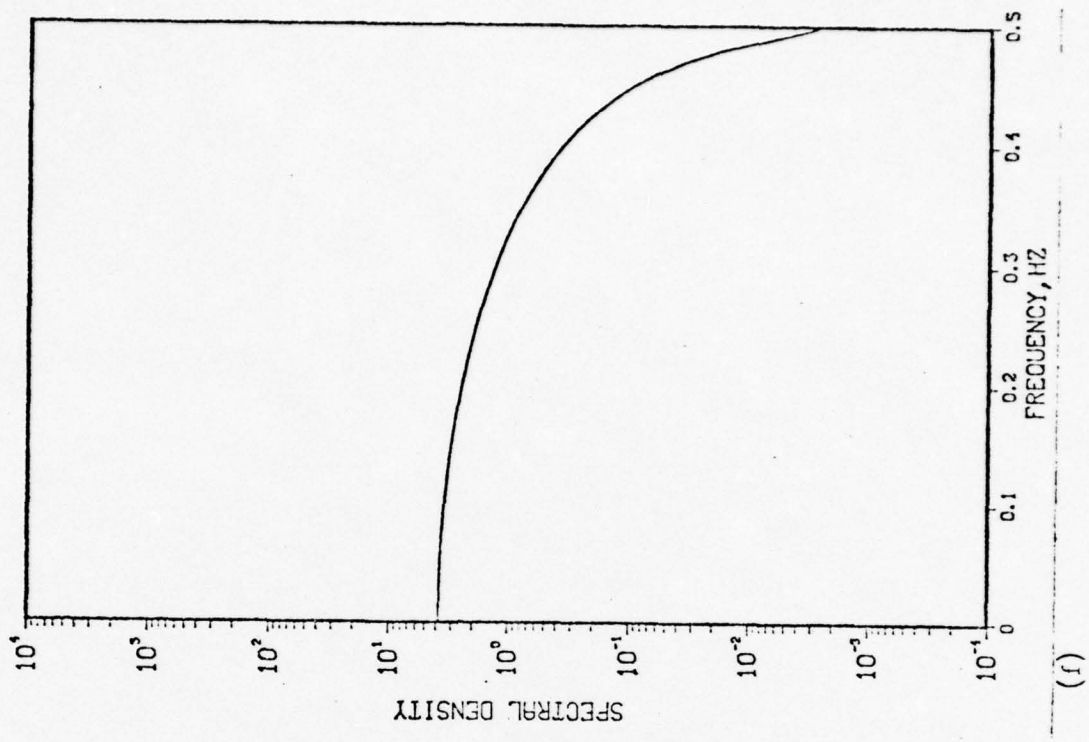

SPECTRAL DENSITY
FREQUENCY, HZ
0
0.1
0.2
0.3
0.4
0.5
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}

(c)

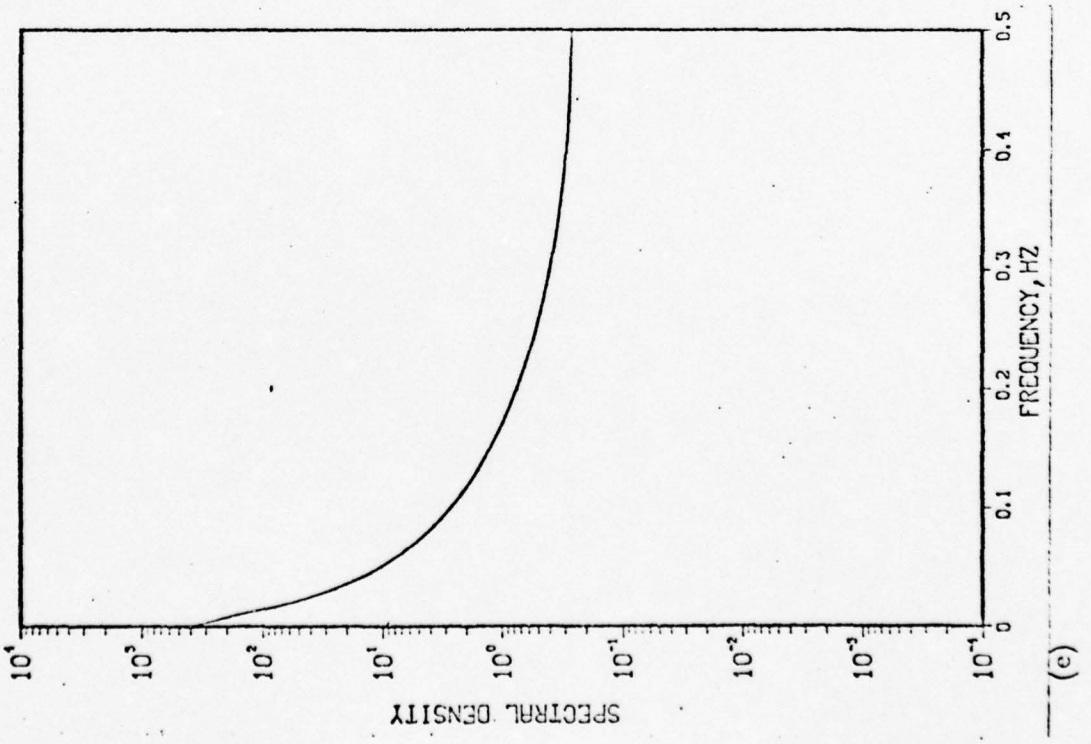
FIGURES 4.3-4e and 4.3-4f

e) Spectral density of AR(1) inputs

f) Spectral density of MA(1) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(f)

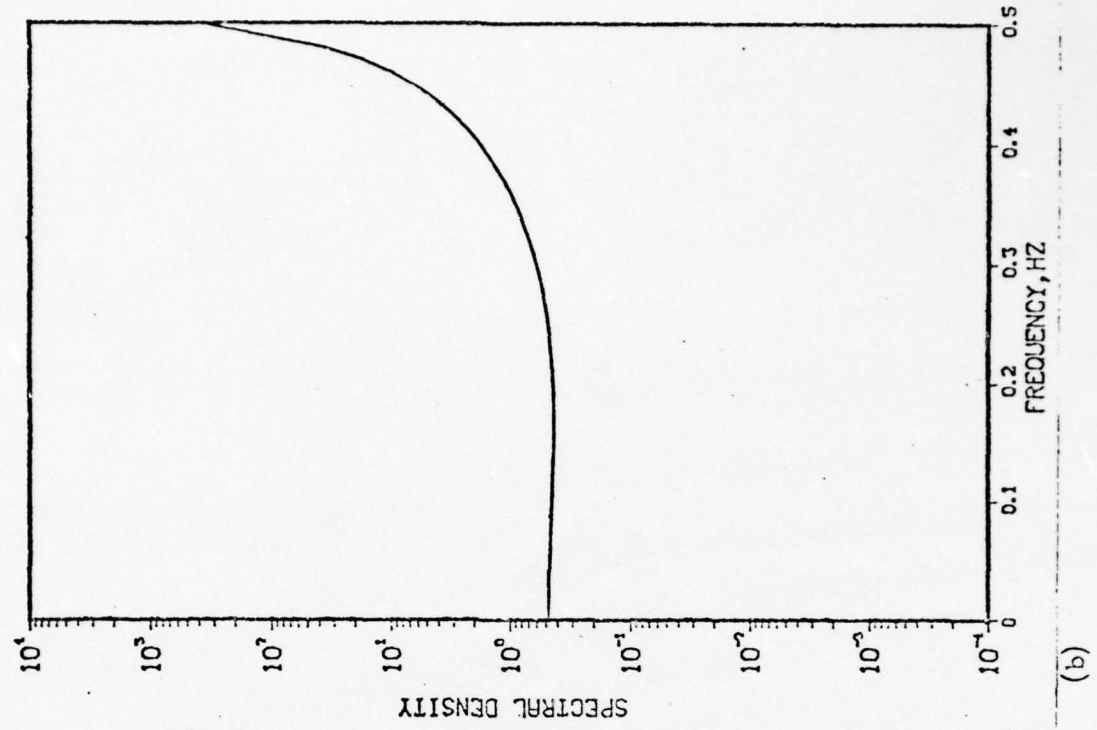

SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(e)

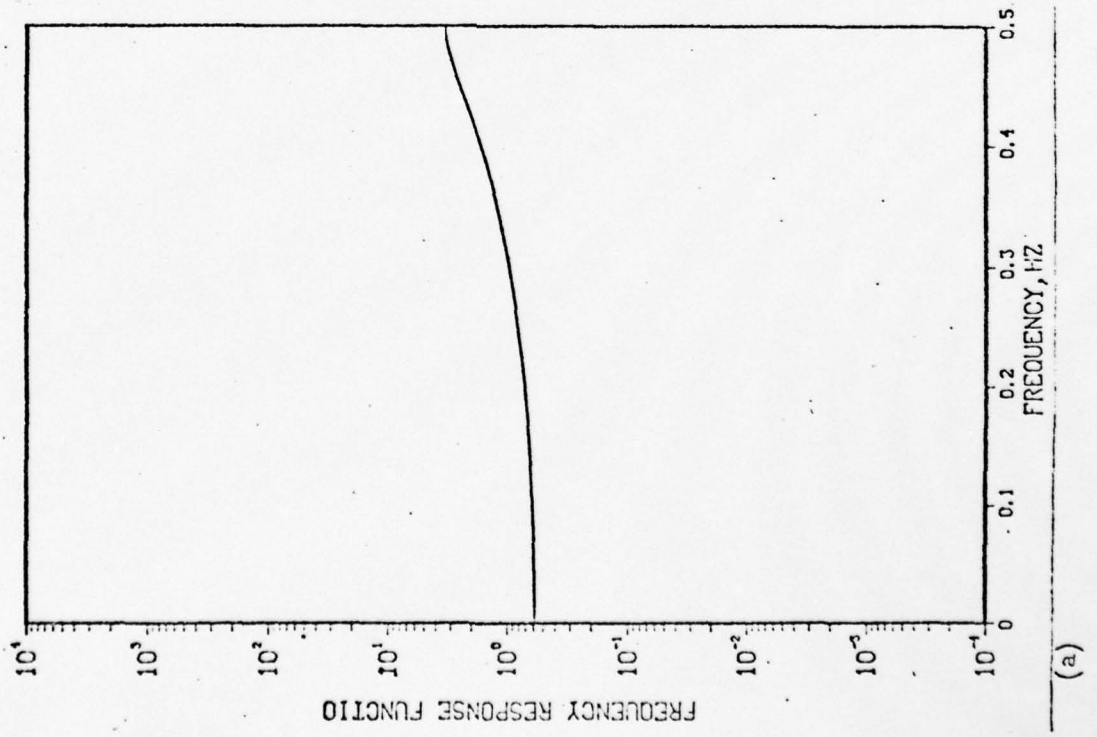
FIGURES 4.3-5a and 4.3-5b

a) Frequency response of first order system, $\varphi = -.7$

b) Spectral density of AR(2) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(b)

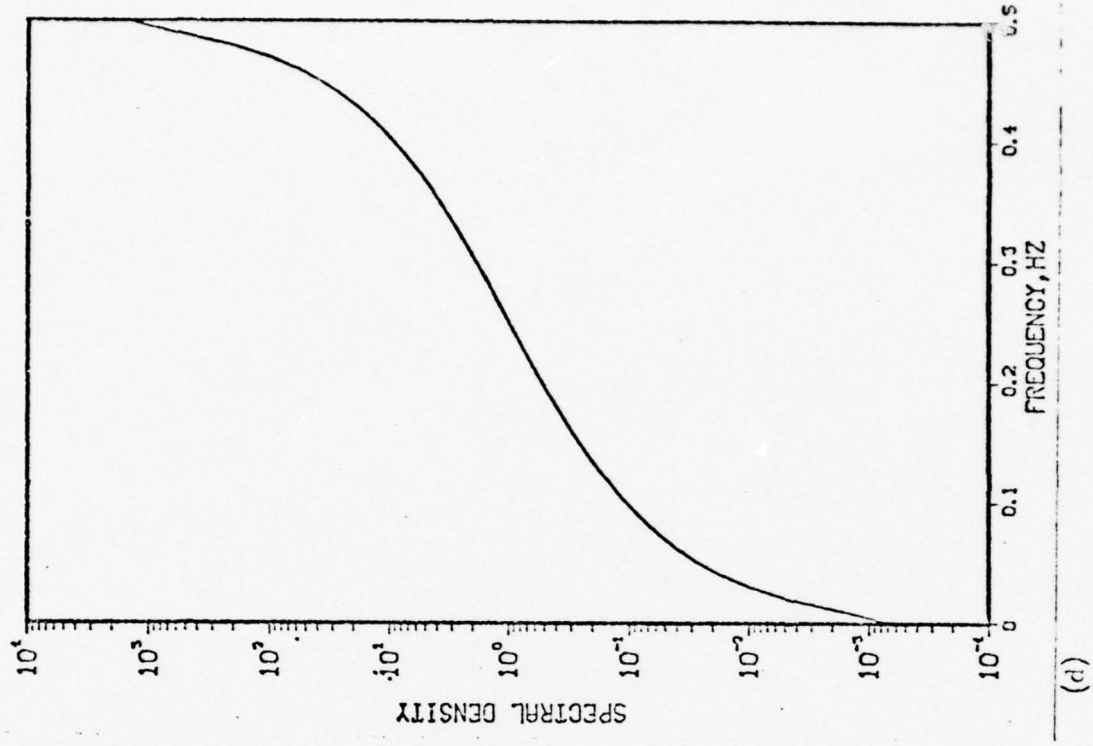

FREQUENCY RESPONSE FUNCTIO
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(a)

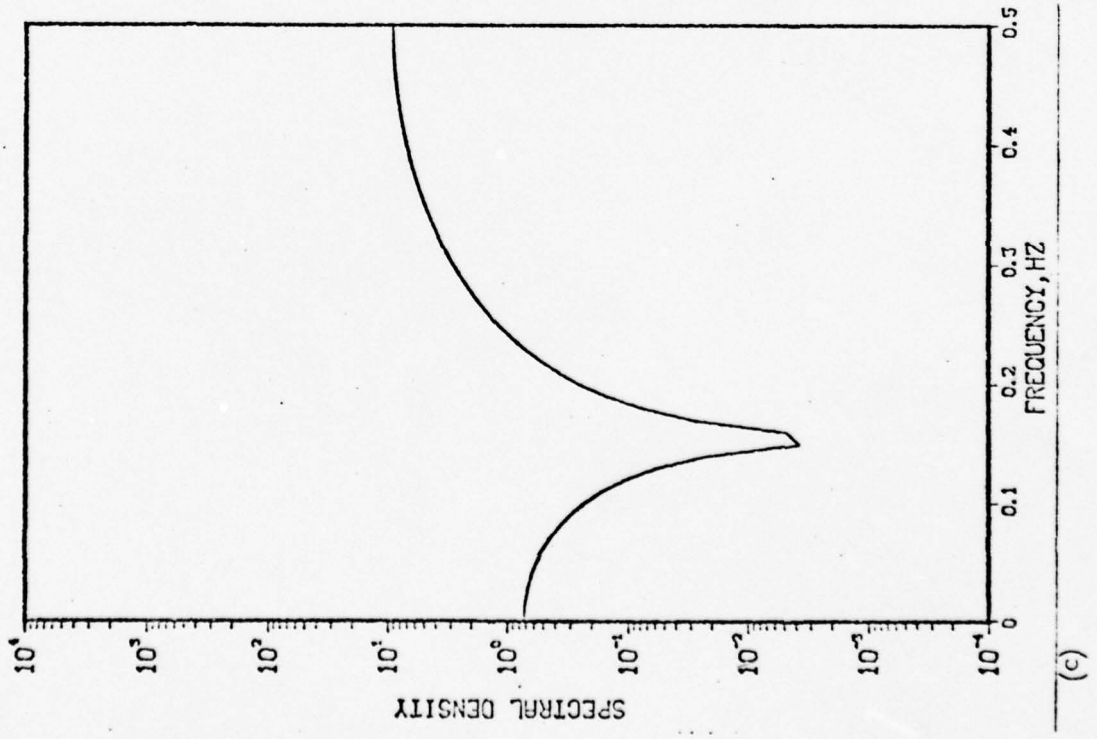
FIGURES 4.3-5c and 4.3-5d

c) Spectral density of MA(2) inputs

d) Spectral density of ARMA(1, 1) inputs.


SPECTRAL DENSITY
FREQUENCY, HZ
0
0.1
0.2
0.3
0.4
0.5
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}

(d)

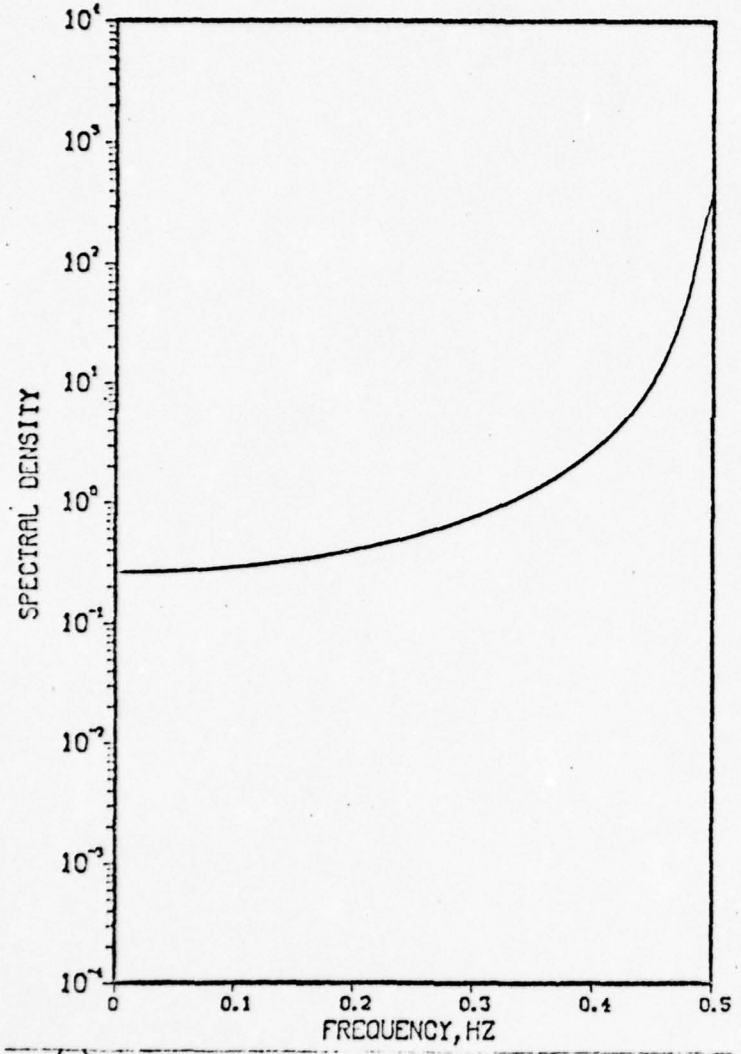

SPECTRAL DENSITY
FREQUENCY, HZ
0
0.1
0.2
0.3
0.4
0.5
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}

(c)

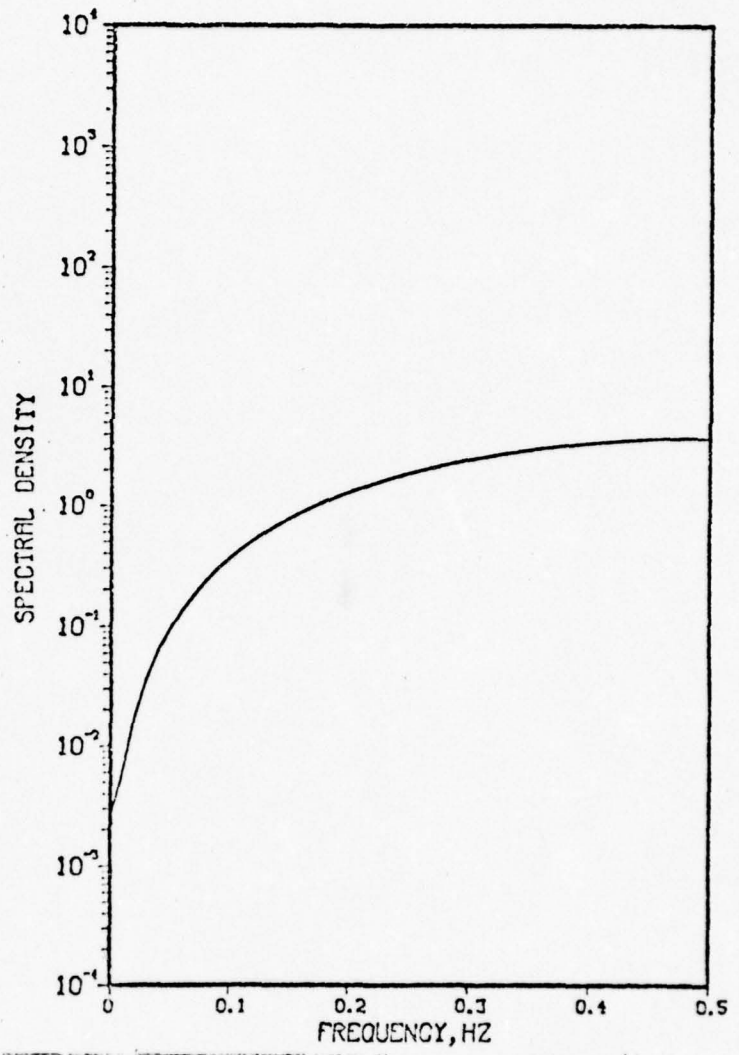
FIGURES 4.3-5e and 4.3-5f

e) Spectral density of AR(1) inputs

f) Spectral density of MA(1) inputs.


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(e)


SPECTRAL DENSITY
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

(f)

FIGURE 4.3-6

Autocorrelation function of optimal AR(2) inputs for the second order system.

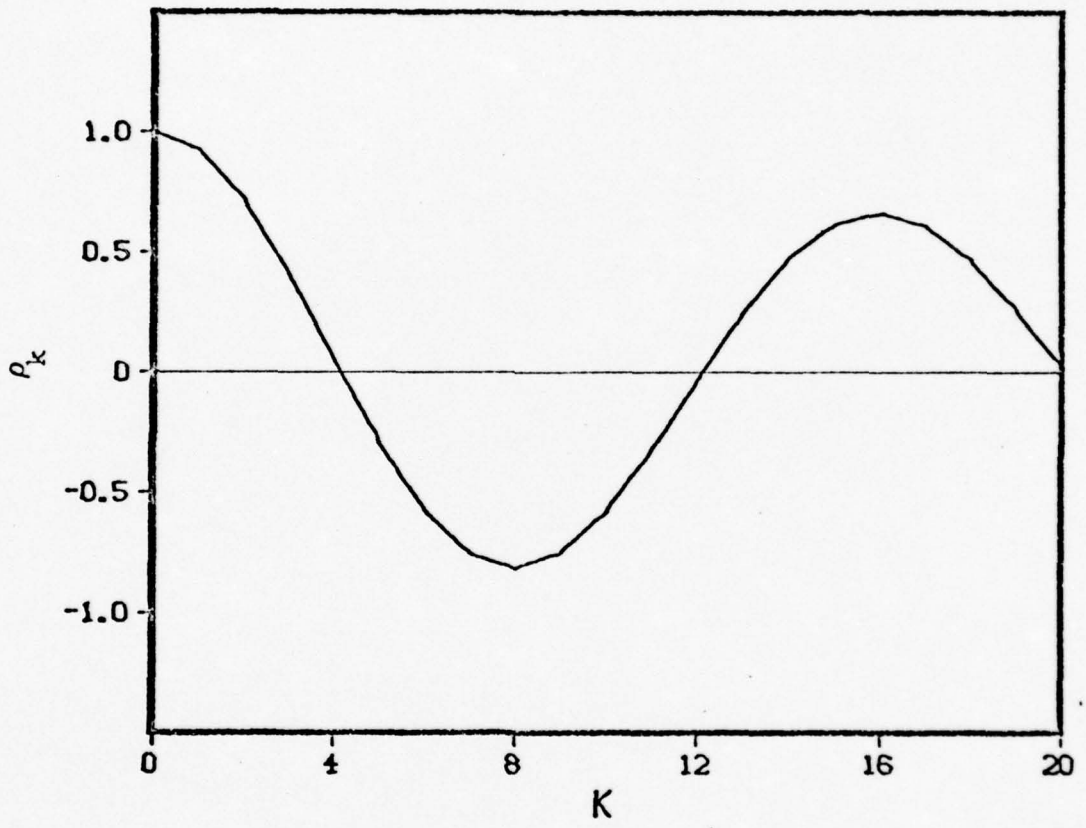

1.0
0.5
0
-0.5
-1.0
ρ_k
0
4
8
12
16
20
K

FIGURE 4.3-7

Autocorrelation function of optimal ARMA(1, 1) inputs for the first order system, $\varphi = .7$.

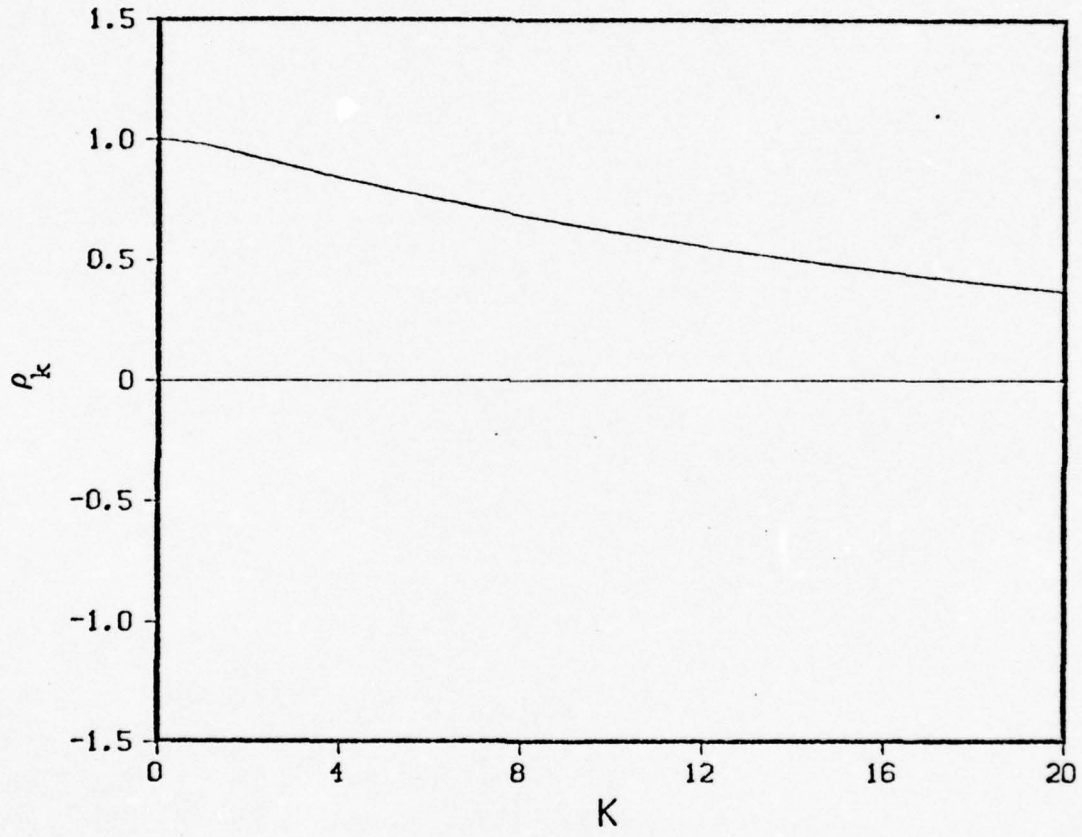

1.5
1.0
0.5
0
-0.5
-1.0
-1.5
ρ_k
0
4
8
12
16
20
K

FIGURE 4.3-8

Autocorrelation function of optimal ARMA(1, 1) input for the first order system, $\varphi = -.7$.

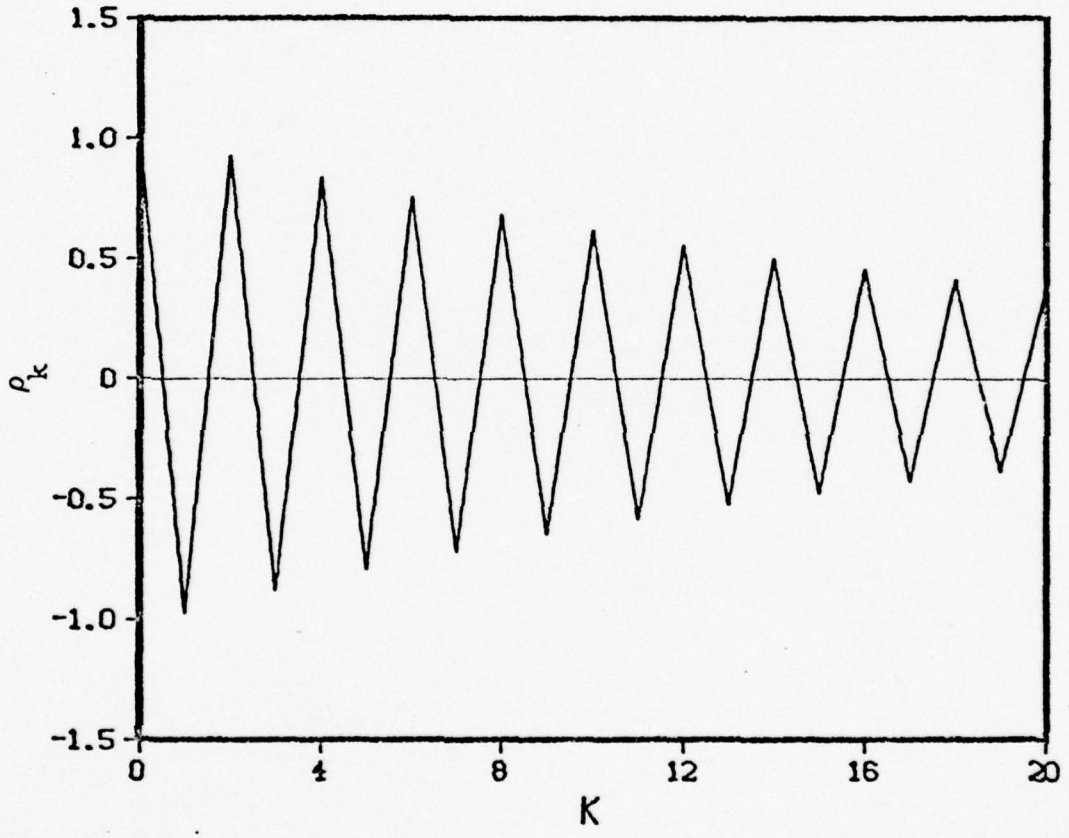

1.5
1.0
0.5
0
-0.5
-1.0
-1.5
ρ_k
0
4
8
12
16
20
K

FIGURE 4.3-9

Comparison of the frequency response of second order system and the spectral densities of AR(2), MA(2) and ARMA(1, 1) inputs

——— Frequency response of second order system

····· Spectral density of AR(2) input

------ Spectral density of MA(2) input

——·—— Spectral density of ARMA(1, 1) input.

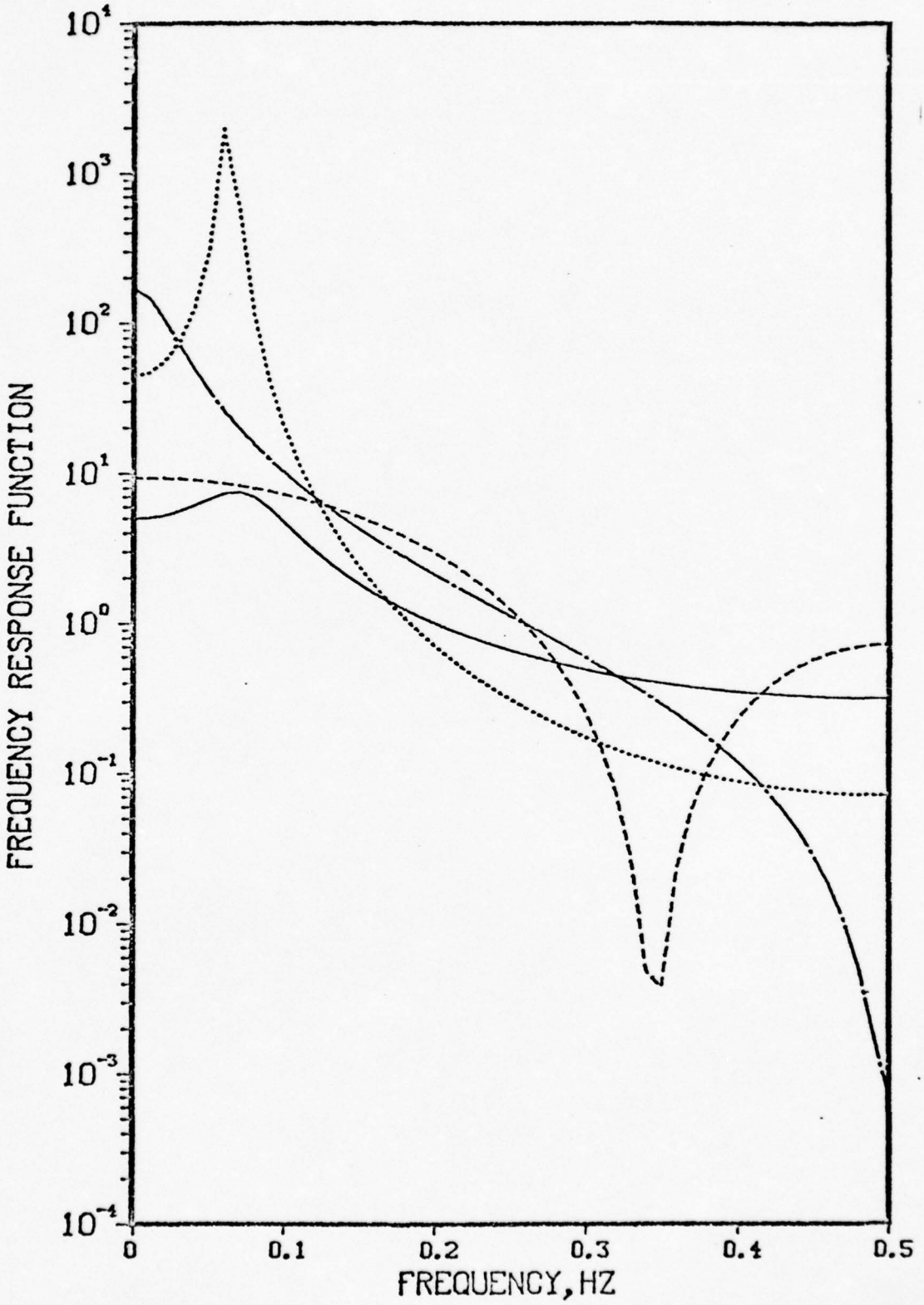

FREQUENCY RESPONSE FUNCTION
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

FIGURE 4.3-10

Comparison of the frequency response of first order system ($\varphi = .7$) and the spectral densities of AR(2), MA(2) and ARMA(1, 1) inputs.

——— Frequency response of first order system

····· Spectral density of AR(2) input

----- Spectral density of MA(2) input

—·— Spectral density of ARMA(1, 1) input.

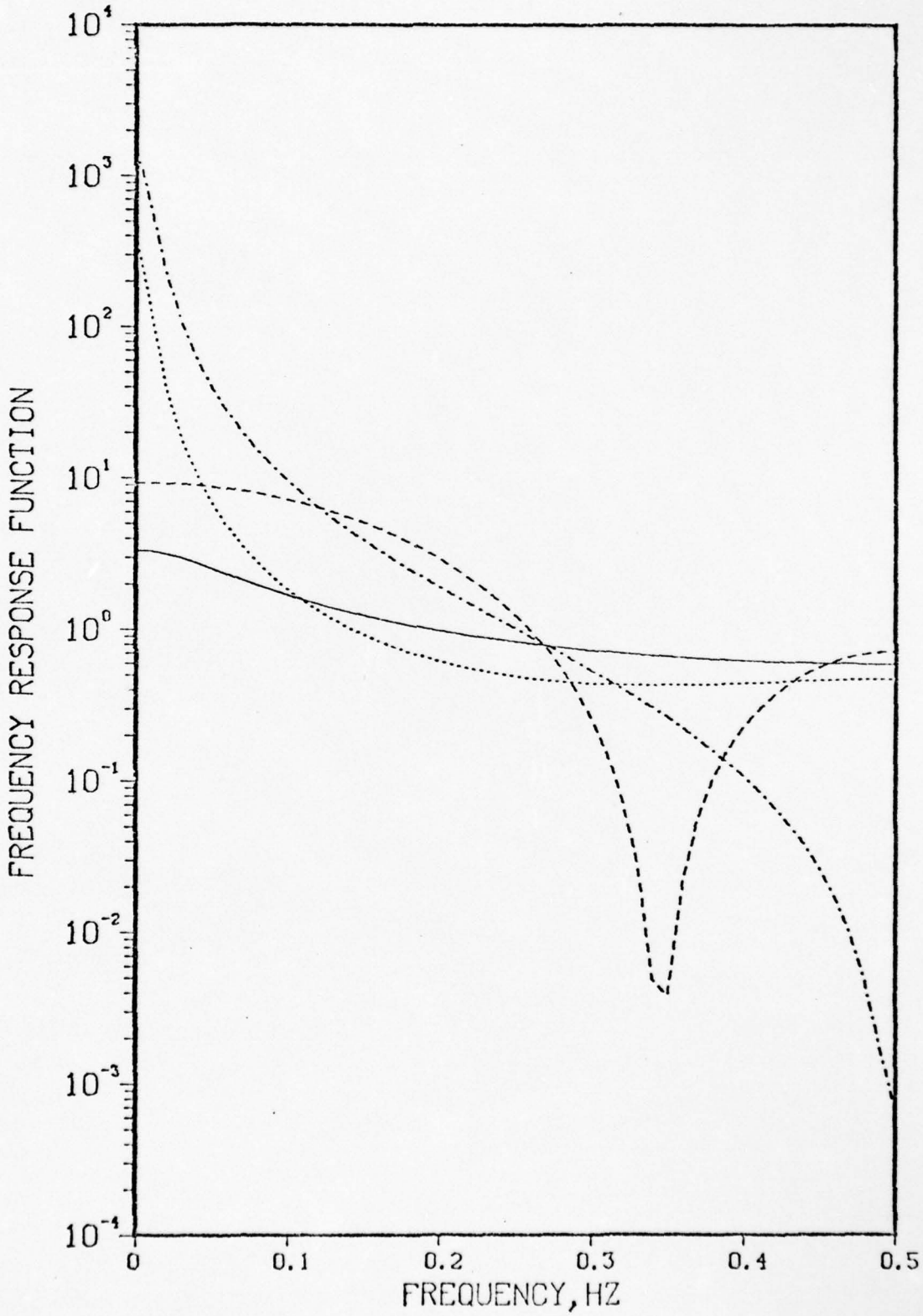

FREQUENCY RESPONSE FUNCTION
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

FIGURE 4.3-11

Comparison of frequency response of second order system and spectral densities of AR(2) inputs for the discrete parameter sets m = 3, 5, 10, 15.

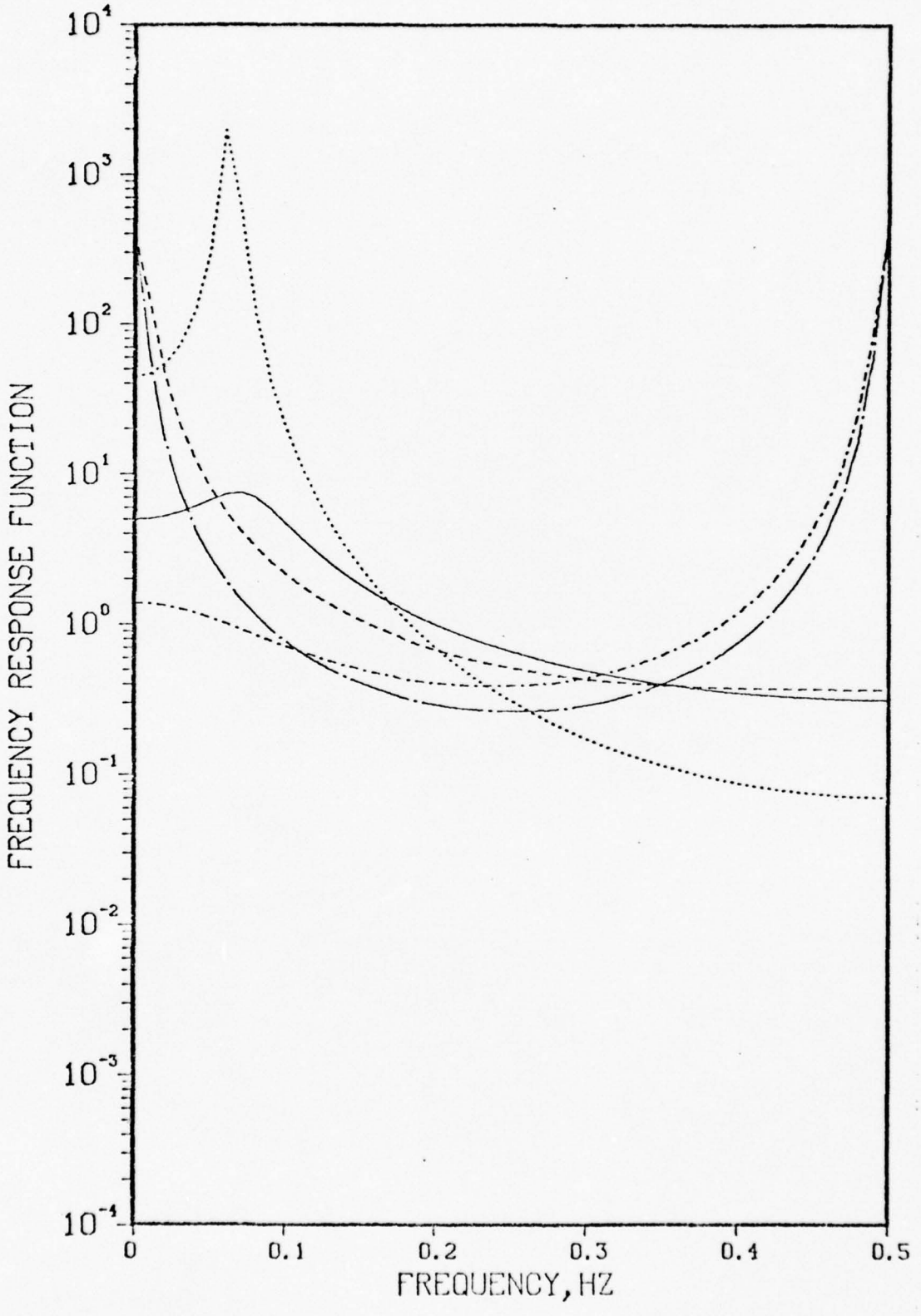
——— Frequency response of second order system

·····Spectral density of AR(2) input, m = 3

------ Spectral density of AR(2) input, m = 5

-·-·-·- Spectral density of AR(2) input, m = 10

——·—— Spectral density of AR(2) input, m = 15.


FREQUENCY RESPONSE FUNCTION
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
FREQUENCY, HZ

CHAPTER V

DESIGN OF FEEDBACK SIGNALS

The optimal inputs obtained in Chapter 3 do not make use of the measurement information $\{\underline{Z}^{N-1}\}$. The inputs obtained this way are called open-loop inputs and have a mapping of the form

$$u_k^{OL} = u_k^{OL}[\varphi, B_N], \quad k=0, 1, \ldots, N-1 \tag{5.1}$$

where φ is the prior information about the system parameters and the state, and B_N is the optimality criterion. In this chapter we will consider feedback signals, that is signals that are generated with the knowledge of measurements. In particular we consider open-loop feedback signals. At each stage k, the future inputs are obtained by taking another measurement and proceeding successively. As we will see the feedback is not always active when the unknown parameters are contained only in the gain parameters. For the linear Gaussian case the interpretation is obtained as the sum of open-loop input at stage k and a correction term. The idea of open-loop feedback controllers was introduced by Dreyfus[23] and since then many investigators have applied this control policy (which is better than the open-loop policy but not as efficient as the closed-loop policy) in the solution of control problems[8, 77].

Lopez-Toledo[56] used feedback signals generated from affine laws of the form

$$u_k = \sum_{i=1}^{s} g_i(k) z_{k-i} + h(k) \tag{5.2}$$

where $g_i(k)$ and $h(k)$ are deterministic quantities.

In Section 5.1 we review classification of optimal inputs. Section 5.2 presents the synthesis of open-loop feedback signals. Sections 5.3 and 5.4 discuss the characteristics of feedback inputs and comparison with the open-loop inputs. An example is presented in Section 5.5 illustrating the sysnthesis of feedback inputs.

5.1 Classification of Feedback Inputs

Let the measurements upto time k be denoted as

$$\underline{Z}^k \triangleq \underline{Z}^k(\underline{U}^{k-1}) \triangleq \left\{\underline{z}_i\right\}_{i=1}^{k}, \quad 1 \leq k \leq N \tag{5.1.1}$$

The a priori information on the state and parameters is given by

$$\varphi \triangleq \left\{P(\underline{\theta}),\ p(\underline{x}_0)\right\} \tag{5.1-2}$$

Depending on the amount of information used by each type of input sequence we have the following ramification.

(i) Open-loop Inputs: The input is given by the following mapping-

$$u_k^{OL} = u_k^{OL}[\varphi\ ;\ B_N],\ k=0, 1, \ldots, N-1 \tag{5.1-3}$$

The inputs are defined based on the a priori information only.

(ii) Feedback Inputs: Let the present time be denoted by k. Let us assume that the input $\underline{U}^{k-1}$ has been applied and the output $\underline{Z}^{k}$ has been observed. We want to determine the future inputs u_i based on observations upto time k. This open-loop feedback input has the following mapping-

$$u_k^{OLF} = u_k^{OLF}[\ \underline{Z}^k, \underline{U}^{k-1}; \theta; B_N\ ],\ k=0, 1, \ldots, N-1 \qquad (5.1\text{-}4)$$

This assumes that no future observations will be made and that the measurements upto time k are used in the computation of present and future inputs.

(iii) Closed-loop Inputs: This signal anticipates the fact that future learning is possible, that is, the knowledge that the loop will stay closed through the end of the process is fully utilized[78]. These inputs have the mapping-

$$u_k^{CL} = u_k^{OL}[\ \underline{Z}^k, \underline{U}^{k-1}; \xi_{k,N}; \theta; B_N\ ], k=0, 1, \ldots, N-1 \quad (5.1\text{-}5)$$

where $\xi_{k,N}$ represents the future observation program and the associated statistics. For a complete discussion of various control policies in stochastic control problems see [9]. In dual control problems and nonlinear problems the dependence of the controller on the future observation program comes into play because the covariance appearing in the expression for the optimal cost-to-go is a function of the control itself.

In the current discussion of the input synthesis problem, the dynamic programming formulation of the closed-loop problem shows that the closed form solution of the problem is not feasible. Hence we resort to the open-loop feedback formulation of the constrained optimization problem. Computationally this method is much easier.

5.2 Synthesis of Open-loop Feedback (OLF) Inputs

The closed-loop optimal policy has the following characteristics:

(i) It uses the knowledge of the observations $(\underline{Z}^k, \underline{U}^{k-1})$ upto the present time k.

(ii) It takes into account the future observation program $\xi_{k,N}$ and its associated statistics. Thus

$$u_k = u_k(\underline{Z}^k, \xi_{k,N}), \quad k=0,1,\ldots,N-1 \qquad (5.2\text{-}1)$$

The set of a posteriori probabilities $P(\underline{\theta}_i \mid \underline{Z}^k)$, $i=1,2,\ldots,m$; $k=1,2,\ldots,N-1$ form a sufficient statistic for the closed-loop problem and the optimal sequence can be derived. The resulting solution is complicated for the foloowing reasons:

(i) The functional formulation of the dynamic programming equation is difficult to solve; the constrained maximization involved in solving this equation makes the resulting optimization quite complicated.

(ii) The implementation of the closed-loop inputs requires the exact knowledge of the a posteriori probabilities $P(\underline{\theta}_i \mid \underline{Z}^k)$, $i=1,2,\ldots,m$; $k=1,2,\ldots,N-1$. The successive optimizations will result in input

sequences that are nonlinear functions of the a posteriori probabilities. Because of this a closed-form solution is not feasible.

We use the approximation given by the open-loop feedback approach, detailed below.

Let $\underline{U}^{k,N} \triangleq \left\{u_k, u_{k+1}, \ldots, u_{N-1}\right\}$ (5.2-2)

$$\mathcal{J}_{Nk} \triangleq \left\{k, k+1, \ldots, N-1\right\} \tag{5.2-3}$$

Denote the current time by k. Let us assume that the input sequence $\underline{U}^{k-1^*}$ has been applied and the corresponding observation $\left\{z_i\right\}_{i=1}^{k}$ is obtained. Required is the determination of future inputs $\underline{U}^{k,N}$ based on the information at time k. Denote the optimization after k stages as

$$B^*_{N-k} = \max_{\substack{u_k \in U_k \\ k \in \mathcal{J}_{Nk}}} \sum_{i-1}^{m-1} \sum_{j=i+1}^{m} E_{\underline{\theta}_i, \underline{\theta}_j} \left[B_{ij}(\underline{\theta}_i, \underline{\theta}_j, \underline{U}^{k-1^*}, \underline{U}^{k,N}) | \underline{Z}^k \right] \tag{5.2-4}$$

This can be written as

$$B^*_{N-k} = \max_{\substack{u_k \in U_k \\ k \in \mathcal{J}_{Nk}}} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P(\underline{\theta}_i | \underline{Z}^k)\, P(\underline{\theta}_j | \underline{Z}^k)\, B_{ij}(\underline{\theta}_i, \underline{\theta}_j, \underline{U}^{k-1^*}, \underline{U}^{k,N}) \tag{5.2-5}$$

The optimization carried out by (5.2-5) gives the following sequence of inputs

$$u_{j|k},\ k=0,1,\ldots,N-1,\quad j=k,k+1,\ldots,N-1 \tag{5.2-6}$$

The optimum OLF sequence results from (5.2-6) by choosing those inputs for which $j=k$

$$u_k^* = u_{k|k},\ k=0,1,\ldots,N-1 \tag{5.2-7}$$

The feedback is employed in the sense that at each stage of the open-loop synthesis an observation is made, thus introducing feedback. In the next section we will study the characteristics of OLF inputs.

5.3 Characterization of Feedback Signals for Linear Systems

For the linear system of (5.1-1) and (5.1-2) the feedback inputs can be interpreted as the sum of open-loop signal and a correction term. Let

$$w_k \sim G(0,Q) \tag{5.3-1}$$

$$v_k \sim G(0,R) \tag{5.3-2}$$

Using the results of Section (3.2), at stage k the OLF cost can be written as

$$B_{N-k} = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} P(\underline{\theta}_i|\underline{Z}^k)\,P(\underline{\theta}_j|\underline{Z}^k)\,B_{ij}(\underline{U}^{k-1^*},\underline{U}^{k,N}) \tag{5.3-3}$$

Using (3.2-20) this can be written as

$$B_{N-k} = \frac{1}{4} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P(\underline{\theta}_i | \underline{Z}^k)\, P(\underline{\theta}_j | \underline{Z}^k) \cdot$$

$$\cdot\, (\underline{U}^{k-1*},\, \underline{U}^{k,N})^T \left[M_{ij}^{-1}\, \Sigma_{ij}^{-1}\, M_{ij} \right] (\underline{U}^{k-1*},\, \underline{U}^{k,N}) \tag{5.3-4}$$

Let

$$C_k \triangleq \frac{1}{4} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P(\underline{\theta}_i | \underline{Z}^k)\, P(\underline{\theta}_j | \underline{Z}^k) \left[M_{ij}^T\, \Sigma_{ij}^{-1}\, M_{ij} \right] \tag{5.3-5}$$

C_k is $N \times N$ positive definite matrix. Then

$$B_{N-k} = \left[\underline{U}^{k-1*},\, \underline{U}^{k,N} \right]^T \left[\begin{array}{c|c} C_{11,k} & C_{12,k} \\ \hline C_{12,k}^T & C_{22,k} \end{array}\right] \left[\begin{array}{c} \underline{U}^{k-1*} \\ \hline \underline{U}^{k,N} \end{array}\right] \tag{5.3-6}$$

Now this can be written as

$$B_{N-k} = \underline{U}^{k-1*}\, C_{11,k}\, \underline{U}^{k-1*} + \underline{U}^{k,N^T}\, C_{22,k}\, \underline{U}^{k,N} + 2\, \underline{U}^{k,N^T}\, C_{12,k}^T \cdot \underline{U}^{k-1*} \tag{5.3-7}$$

with the constraints

$$v_{k,N} = v_k \otimes \cdots \otimes v_{N-1} = \left\{ \underline{U}^{kN} \mid \|\underline{U}^{kN}\|^2 \leq Y_k = Y - \|U^{*\,k-1}\|^2 \right\}$$

(5.3-8)

The necessary condition for optimality is given by

$$C_{22,k}\, \underline{U}^{k,N} = \lambda\, \underline{U}^{kN} + C_{12,k}^T\, \underline{U}^{k-1^*} \tag{5.3-9}$$

$C_{22,k}$ is $(N-k) \times (N-k)$ symmetric positive definite matrix and $\lambda > 0$. (5.3-9) is a nonhomogeneous eigevalue problem. Now a set of $(N-k)$ orthonormal vectors $e_k \epsilon R^{N-k}$ can be obtained from the eigenvectors of $C_{22,k}$, thus spanning R^{N-k}. In a finite dimensional vector space this forms a complete set of orthonormal vectors [4, p 467] and hence the solution to (5.3-9) can be written as

$$\underline{U}^{k,N^*} = \sum_{i=1}^{N-k} \alpha_i\, \underline{e}_{ik} \tag{5.3-10}$$

where

$$C_{22,k}\, \underline{e}_{ik} = \lambda_i\, \underline{e}_{ik}\,, \quad i = 1, 2, \ldots, N-k \tag{5.3-11}$$

$$\langle \underline{e}_{ik}\, \underline{e}_{jk} \rangle = \delta_{ij} \tag{5.3-12}$$

Using (5.3-10) - (5.3-12) we have

$$\alpha_i = \frac{\langle \underline{e}_{ik}\,, \underline{b}_k \rangle}{\lambda - \lambda_i}\,, \quad \lambda \neq \lambda_i\,, \quad i = 1, 2, \ldots, N-k \tag{5.3-13}$$

where

$$\underline{b}_k = C_{12,k}^T \underline{U}^{k-1^*} \in R^{N-k} \tag{5.3-14}$$

The constant λ is such that the resulting solution must satisfy the constraints (5.3-8). (5.3-10) can be written as

$$\underline{U}^{k,N^*} = \alpha_1 \underline{e}_{1k} + \sum_{i=2}^{N-k} \alpha_i \underline{e}_{ik} \tag{5.3-15}$$

where $\underline{e}_{1k}$ is the normalized eigenvector corresponding to the largest eigenvalue λ_1 of $C_{22,k}$. $\underline{e}_{1k}$ can be considered as the solution to the open-loop problem at stage k. Rewriting (5.3-15) as

$$\underline{U}^{kN^*} = \alpha_1 \left[\underline{e}_{1k} + \sum_{i=2}^{N-k} \frac{\alpha_i}{\alpha_1} \underline{e}_{ik}\right] \tag{5.3-16}$$

The term

$$U_c^{kN^*} = \sum_{i=2}^{N-k} \frac{\alpha_i}{\alpha_1} \underline{e}_{ik} \tag{5.3-17}$$

is a correction to the open-loop inputs $\underline{e}_{1k}$. Thus we can write $\underline{U}^{kN^*}$ as

$$U^{*kN} = \alpha_1 \left[\underline{U}_{ol}^{kN^*} + \underline{U}_c^{kN^*}\right] \tag{5.3-18}$$

The effectiveness of the feedback inputs depends on the correction term $\underline{U}_c^{kN^*}$. Initially, for a few stages, the feedback inputs follow

the open-loop signal and then the correction comes into effect. Towards the end of the sequence the energy of the inputs becomes small and the difference between feedback and open-loop inputs reduces. Thus the difference in the two signals is dominant during the stages that are away from the initial and final stages of the signal history.

The feedback enters the optimization via the a posteriori probabilities $P(\underline{\theta}_i | Z^k)$, $i=1,2,\ldots, m$ and $k=1,2,\ldots, N$; and the past inputs $\underline{U}^{k-1^*}$ in the form of the vector $C^T_{12,k}\,\underline{U}^{k-1^*}$. Note that as the learning approaches the correct parameters, that is, $P(\underline{\theta}_i | \underline{Z}^k) \rightarrow 1$ for $\underline{\theta}_i = \underline{\theta}_0$, the remaining probabilities become small. Thus the cross terms

$$a_{ij} = P(\underline{\theta}_i | Z^k)\, P(\underline{\theta}_j | Z^k) \quad \text{where} \quad \underline{\theta}_i \neq \underline{\theta}_0 \;\forall_i \tag{5.3-19}$$

become small compared to the terms

$$a_{i0} = P(\underline{\theta}_i | Z^k)\, P(\underline{\theta}_0 | Z^k) \quad \text{where} \quad \underline{\theta}_i \neq \underline{\theta}_0 \;\forall_i \quad . \tag{5.3-20}$$

Then $a_{io} >> a_{ij}$ and the discriminant function B_{ij} multiplying a_{i0} will dominate those multiplying a_{ij} . This shows that the distance functions B_{ij} which do not contain the right parameters will have minimum effect on the cost function. This is the major change introduced by the feedback synthesis, which is the desired one. To summarize, the feedback inputs are synthesized by using all the

available past and present information. The optimal cost function reduces the effect of distance measures that do not contain the true set of parameters, thus making it a function of distance $B(\underline{\theta}_0, {}_i\underline{\theta})$ where ${}_i\underline{\theta} \neq \underline{\theta}_0$, as one would desire.

5.4 Comparison of Open-Loop and Feedback Signals When the Unknown Parameters are Contained in the Gain Vector

Here let us consider the case where the unknown parameters are contained only in the gain vector $\underline{\beta}$. This is equivalent to the transfer function model with known poles and unknown zeros.

Consider a scalar system as follows:

$$x_{k+1} = \varphi x_k + \beta u_k \tag{5.4-1}$$

$$z_k = x_k + v_k \tag{5.4-2}$$

$$v_k \sim G(0, \sigma_v^2) \ , \quad k=1, 2, \ldots, N \tag{5.4-3}$$

The pairwise distance is given by

$$B_{ij} = \frac{1}{\sigma_v^2} \left[\underline{\hat{Z}}_i^N - \underline{\hat{Z}}_j^N \right]^T \left[\underline{\hat{Z}}_i^N - \underline{\hat{Z}}_j^N \right] \tag{5.4-4}$$

where

$$Z_i^N = \begin{bmatrix} \beta_i & & & \\ \varphi\beta_i & \beta_i & & \\ \vdots & \vdots & \ddots & \\ \varphi^{N-1}\beta_i & \varphi^{N-2}\beta_i & \cdots & \beta_i \end{bmatrix} \underline{U}^{N-1} \qquad (5.4\text{-}5)$$

Remiting β_{ij}

$$\beta_{ij} = \frac{(\beta_i - \beta_j)^2}{\sigma_v^2} \; \underline{U}^{N-1^T} A^T A \underline{U}^{N-1} \qquad (5.4\text{-}6)$$

where

$$A = \begin{bmatrix} 1 & & & \\ \varphi & 1 & & \\ \vdots & \vdots & \ddots & \\ \varphi^{N-1} & \varphi^{N-2} & \cdots & 1 \end{bmatrix} \qquad (5.4\text{-}7)$$

Consider the feedback optimization

$$B^{*}_{N-k} = \max_{\substack{u_k \in \upsilon_k \\ k \in \mathcal{I}_{Nk}}} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P(\beta_i | \underline{Z}^k)\, P(\beta_j | \underline{Z}^k)(\beta_i - \beta_j)^2$$

$$(\underline{U}^{k-1^*}, \underline{U}^{kN})^T A^T A\, (\underline{U}^{k-1^*}, \underline{U}^{kN}) \qquad (5.4\text{-}8)$$

This shows that the maximization does not depend on the past observation program; the solution coincides with the open-loop case. Thus for the case with scalar gain the solution to the OLF problem coincides with the solution for the open-loop synthesis since both maximizations are given by

$$\max_{\underline{U}^{N-1} \in \upsilon} \underline{U}^{N-1^T} A^T A\, \underline{U}^{N-1} \qquad (5.4\text{-}9)$$

When the unknown gain $\underline{\beta} \in R^n$, the separation of the cost function as in (5.4-9) is not possible. This can be shown by considering the following system as in Section (4.1-2). Let

$$x_k = \sum_{i=1}^{m} \varphi_i x_{k-i} + \sum_{i=1}^{r} \beta_i u_{k-i} \quad , \quad k=1,2,\ldots, N \qquad (5.4\text{-}10)$$

$$z_k = x_k + v_k \quad , \quad k=1,2,\ldots, N \qquad (5.4\text{-}11)$$

$$x_k \sim G(0, \sigma_k^2) \qquad (5.4\text{-}12)$$

The cost function (4.1-29) becomes

$$B_{ij} = \underline{U}^{N-1^T} B^T_{N,ij} \Phi_N^{-1^T} R_N^{-1} \Phi_N^{-1} B_{N,ij} \underline{U}^{N-1} \qquad (5.4\text{-}13)$$

The feedback optimization is given by

$$B^*_{N-k} = \max_{\substack{u_k \epsilon \upsilon_k \\ k \epsilon \mathcal{J}_{N-k}}} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P(\underline{\beta}_i | \underline{Z}^k) P(\underline{\beta}_j | Z^k)$$

$$\cdot (\underline{U}^{k-1^*}, \underline{U}^{kN})^T B^T_{N,ij} \Phi_N^{-1^T} R_N^{-1} \Phi_N^{-1} B_{N,ij} (\underline{U}^{k-1^*}, \underline{U}^{kN})$$

$$k=1,2,\ldots, N-1 \qquad (5.4\text{-}14)$$

Now it is clear that the inputs $\underline{U}^{kN}$ are a function of the observation process $\underline{Z}^k$ ($\underline{U}^{k-1}$) and the separation of the cost function as for the scalar case cannot be attained.

When the Fisher information matrix is used as a criterion the open-loop and feedback inputs coincide for the vector gain matrix as shown by Lopez-Toledo [56]. The distance function exhibits that the feedback can be affected when the dimension of $\underline{\beta}$ is > 1. However, the amount of improvement achieved may not be considerable.

5.5 Numerical Example

We consider the second order example given by

$$x_k = \varphi_1 x_{k-1} + \varphi_2 x_{k-2} + \beta u_{k-1} \qquad (5.5\text{-}1)$$

$$z_k = x_k + v_k \qquad (5.5\text{-}2)$$

$$v_k \sim G(0, 1.) \; \forall_k \quad . \qquad (5.5\text{-}3)$$

This example is considered in Chapter 3 to study open-loop synthesis. Figure (5.5-1) shows the open-loop and feedback inputs for $N = 20$. The feedback input has a smaller attenuation overall, compared to the open-loop signal. Figure (5.5-2) is a plot of the a posteriori probabilities $P(\underline{\theta}_0 | \underline{Z}^k)$ with open-loop and feedback inputs. The transient response with the feedback signals is much smoother than with open-loop inputs. The small improvement in learning with feedback synthesis is obtained at a much higher cost, the feedback syntheis requiring approximately four times as much computation time as the open-loop case.

The improvement in learning achieved with feedback identification will vary with the type of system and the problem must be analyzed individually. The procedure of computing the feedback law is completely illustrated.

FIGURE 5.3-1

Open-loop and feedback inputs for the second order system.

------ Feedback inputs

——— Open-loop input.

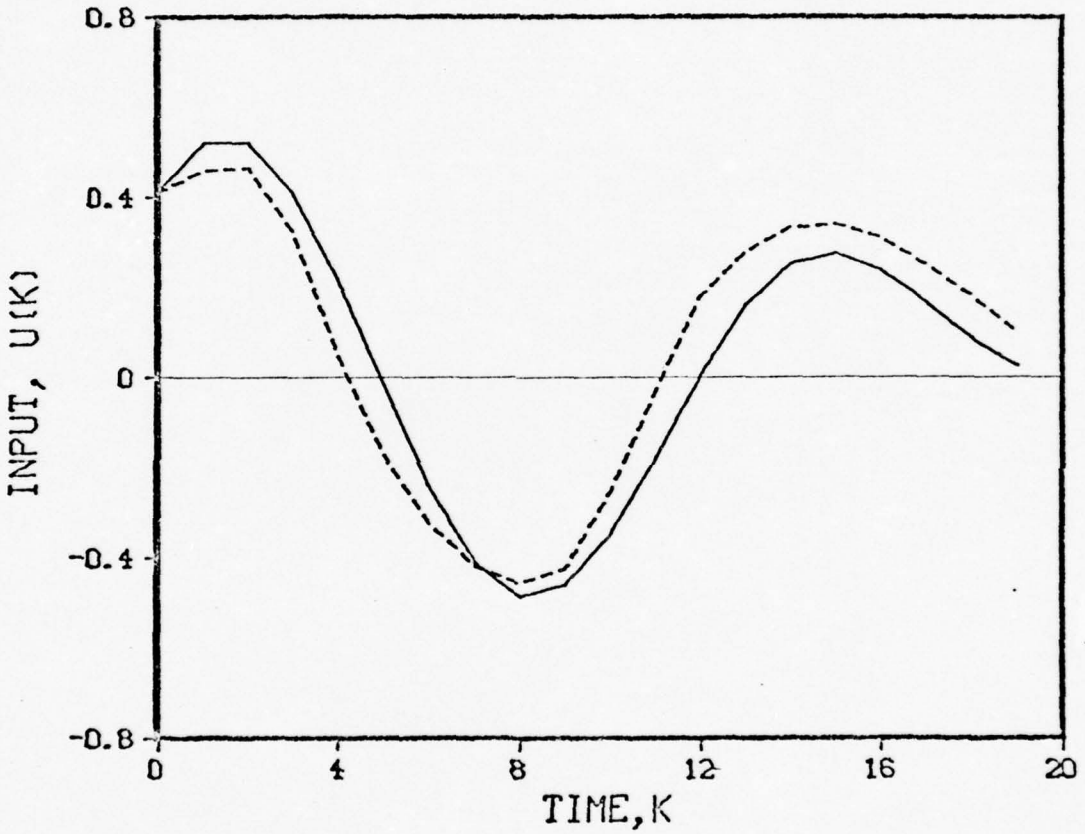
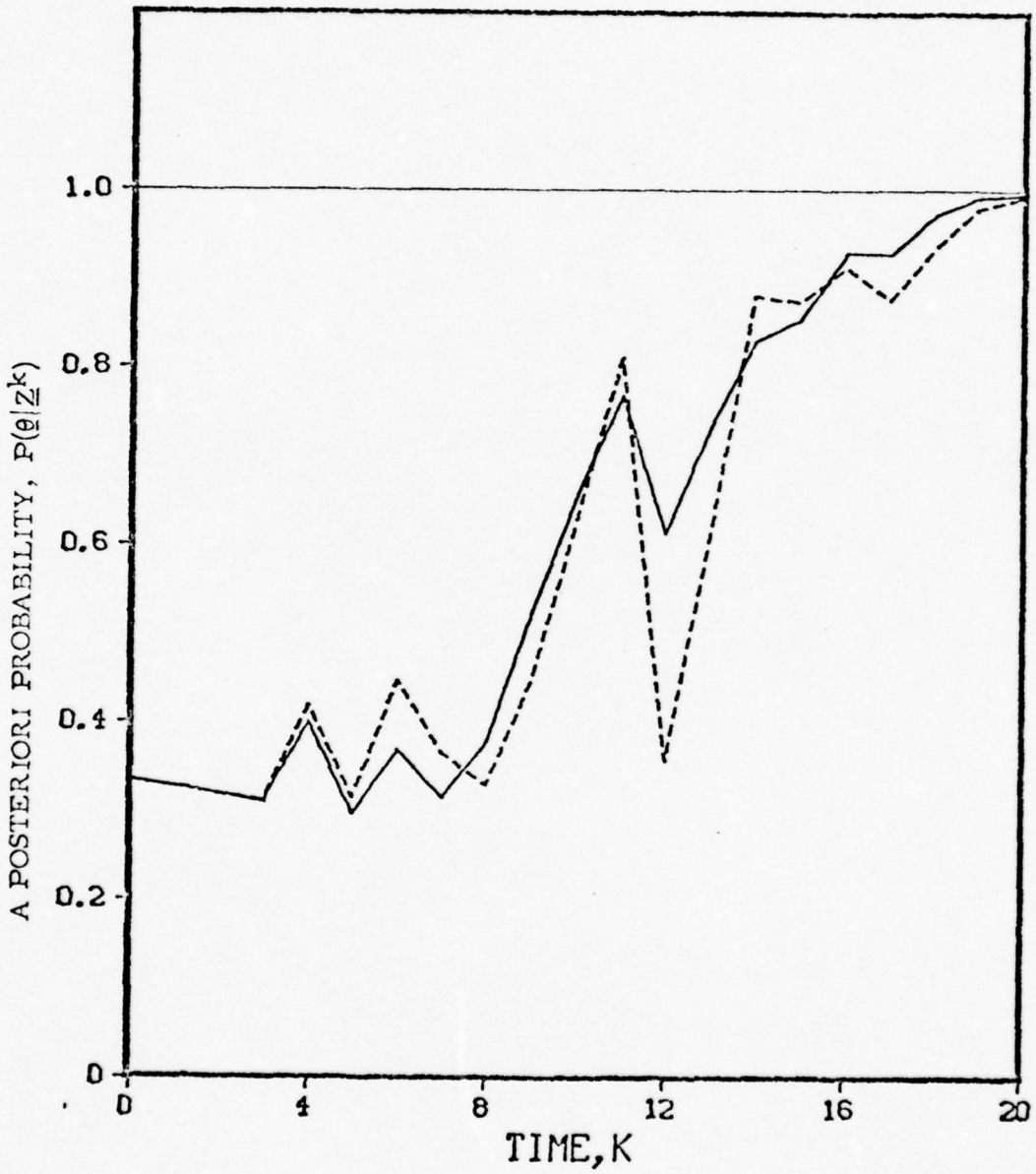

0.8
0.4
0
-0.4
-0.8
INPUT, U(K)
0
4
8
12
16
20
TIME,K

FIGURE 5.3-2

Comparison of a posteriori probabilities $P(\underline{\theta} \mid \underline{Z}^k)$ for the open-loop and feedback inputs.

——— $P(\underline{\theta} \mid \underline{Z}^k)$ with feedback inputs

------ $P(\underline{\theta} \mid \underline{Z}^k)$ with open-loop inputs.


A POSTERIORI PROBABILITY, P(θ|Z^K)
1.0
0.8
0.6
0.4
0.2
0
0
4
8
12
16
20
TIME, K

CHAPTER VI

SUMMARY AND CONCLUSIONS

A summary of the salient features of the dissertation presented in earlier chapters is given below. Some concluding remarks and possible extensions and suggestions for future research are contained in section (6-2).

6.1 Summary

It was pointed out that the system identification can be treated as the initial phase of a general control problem. Before the control policies are implemented a fast and efficient identification phase is preceded to determine the unknown parameters of the system. Such a procedure will avoid solution to a costly and time consuming dual control problem. This leads to the question of selecting optimal input signal sequence so that some measure of learning is minimized. The solution, set in an adaptive filtering framework, computes the a posteriori probability of the parameters, $P(\underline{\theta} \mid \underline{Z}^k)$, recursively. This is a sufficient statistic containing the information of the measurements about the unknown parameters. Distance functions representing the affinity of distributions characterized by distinct parameter sets are defined and used as optimality criteria.

In Chapter 2 the pairwise Bhattacharyya distance is introduced and several of its properties are studied. It was shown that the B-distance is monotonically nondecreasing as a function of time. The important ordering property of this measure with respect to the probability of error (in the sense of minimum Bayes' risk) is given by theorem (2.3-2). The distance measures are shown to be closely related to the mutual information $I(\underline{Z}^N; \underline{\theta})$. The divergence and B-distance form upper bounds on the mutual information. Using this fact a lower bound for the mean square error $E(\theta - \hat{\theta})^2$ is derived from Shannon's rate distortion theory.

The Bhattacharyya coefficient can be interpreted as the cosine of the angle between two points on a unit hypersphere with $p(\underline{Z}^N | \underline{\theta}_\alpha)^{\frac{1}{2}}$ and $p(\underline{Z}^N | \underline{\theta}_\gamma)^{\frac{1}{2}}$ being the direction cosines of the two points. Using this, the differential metric $B_{\underline{\theta}, \underline{\theta}+\Delta\underline{\theta}}$ is expressed in terms of the Fisher information, $\Delta\underline{\theta}$ being the perturbation of $\underline{\theta}$. Thus $B_{\underline{\theta}, \underline{\theta}+\Delta\underline{\theta}}$ is a function of the sensitivities of the observations with respect to the system parameters. The B-distance is applied for the first time in this thesis as a measure of signal selection. It is interesting to note that these discriminant functions which originally appeared in the context of statistical data analysis have a natural extension in application to more general dynamic systems. Furthermore the mutual

information can be replaced by the easily computable distance functions in practical applications.

In Chapter 3 solution to the input synthesis problem in linear dynamic systems is presented. The linear system is described in terms of the state space equations and the distance function is derived for the Gaussian noise statistics. It was shown that this quantity is quadratic in the inputs and the maximization problem under energy constraint has a global solution. The optimal input vector is given by the eigenvector of a symmetric positive definite matrix corresponding to its maximum eigenvalue. The calculation of eigenvectors and eigenvalues requires a computational time that increases as the square of the order of the matrix. A simple gradient technique is derived to compute the optimal input sequence. For the Bayesian learning scheme it was shown in theorem (3.3-1) that for a given positive integer N, $\epsilon > 0$ (ϵ sufficiently small), there exists a value of the input signal-to-noise ratio such that for all $n > N$, $p(\underline{\theta} \mid \underline{Z}^n) > 1 - \epsilon$. This gives a sufficient condition for the estimation error to be less than a given fidelity criterion.

The learning performances with inputs obtained from the B-distance, divergence and pulse inputs are compared. The results show that as a function of the plant noise covariance σ_w^2, the inputs obtained by the B-distance perform much better than that

obtained by the divergence measure, the difference increasing as a function of σ_w^2. This conclusion agrees with the studies of Grettenberg [34] and Kailath [41] in the selection of communication links. The open-loop inputs derived in Chapter 3 can be approximated by low order difference equations of the form

$$u_k = \sum_{i=1}^{s} \xi_i u_{k-i}$$

In the numerical examples presented in section (3-5) second, third and fourth order least square fits (i.e. $s = 2, 3, 4$) are obtained for first and second order systems. These are shown graphically and the corresponding distance functions are compared with the optimal value.

In Chapter 4 an alternate approach to the signal selection problem is presented. In particular we consider generating these inputs from white noise using linear stochastic processes of the autoregressive moving average type. These inputs are used in single-input single-output systems. To study the properties of these signals the sensitivity index is used and is shown to be related to the Bhattacharyya distance for the stochastic inputs. The properties of autoregressive, moving average and mixed process inputs are studied using the asymptotic properties of Toeplitz matrices. The input correlation matrix is approximated by

appropriate Toeplitz forms and using the results on asymptotic eigenvalue distribution of Toeplitz matrices it is shown that when the variance of the input white noise σ_ν^2 is fixed, among the class of all admissible input processes, the input process with the maximum cost function also has the largest value for the integral (4.2-11). Thus an optimal spectral shape is obtained. The resulting problem gives rise to a reduced optimization in the space of the process parameters. A condition to be satisfied for the above result to hold is that the spectral density $s_u(\lambda) > 0$. This is indeed true in all the linear input processes considered above. Numerical results show that the autoregressive AR(2) and ARMA(1, 1) inputs perform better than the moving average MA(2) inputs. Generation of these inputs requires only a few delay lines and unlike the binary sequences used by earlier investigators, the input sequence is much more smoother in its variation.

Synthesis of feedback inputs is the topic of Chapter 5. Open-loop feedback techniques are used to incorporate observation information in the input sequence. It is shown that at any stage k, the feedback input can be represented as the sum of open-loop signal at stage k and a correction term. The feedback becomes ineffective when the unknown parameter is contained only in the gain term. Numerical results indicate that the improvement in learning gained using feedback inputs is small compared to that with open-loop inputs.

6.2 Concluding Remarks and Suggestions for Future Research

The problem of identification and control is a subject of interest in many applications such as chemical industry processes, nuclear reactor control, aerospace applications, etc. These dual control problems require efficient solutions where time and cost of operation is important. It is suggested here that the dual control can be redefined as consisting of two phases - the identification phase and the control phase. To accomplish the first it is necessary to design input sequences that are constrained in their total energy and we use all the available information about the system, so that the identification phase will use only a short time span of the total operation time. The methods developed and presented here provide a promising approach to the solution of these problems.

The Bayesian approach to the identification problem requires a proper quantization of the parameter space so that the correct parameter set is also included in this dissertation. Thus it is necessary to develop methods to obtain efficient quantization of the parameter space. A dual problem of input design is the design of measurement subsystems. In practice this corresponds to the design of sensor locations in test set ups. The combined design of input and measurement system is another area where further investigation is necessary.

In section (2.6) we derived an expression for the differential metric $B_{\underline{\theta}, \theta+\Delta\underline{\theta}}$ in terms of the Fisher information. Use of this quantity as a criterion function is worth further study. Another important area of signal synthesis application is in distributed parameter systems. These belong to the category of thermal and chemical processes and nuclear reactors (see Kerlin [45]), where quantities such as temperature coefficients of reactivity and heat transfer coefficients are to be determined.

Further theoretical and experimental investigation is necessary to completely characterize the stochastic inputs discussed in Chapter 4. It is an interesting problem to study under what conditions the system frequency response and the input process frequency response match.

The feedback identification discussed in Chapter 5 showed that the synthesis of inputs via the open-loop feedback technique did not significantly improve the parameter learning over the open-loop synthesis. This will depend on the type of system and the experiment length, and further work should be carried out to study the effect of optimal closed-loop inputs and to classify systems whose performance may improve with feedback inputs.

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APPENDIX A

Optimal Value of t in $\rho_{\alpha\gamma}(t)$ for Gaussian Distributions With Equal Covariances

Consider

$$p_i(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(x-m_i)^2}{2\sigma^2} \right\}, \quad i=\alpha,\gamma$$

Define $\rho_{\alpha\gamma}(t)$ as

$$\rho_{\alpha\gamma}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{ - \frac{1}{2\sigma^2} \left[t(x-m_\alpha)^2 + (x-m_\gamma)^2 (1-t) \right] \right\} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{1}{2\sigma^2} (t\, m_\alpha^2 - t\, m_\gamma^2 + m_\gamma^2) \right\} *$$

$$\int_{-\infty}^{\infty} \exp\left\{ - \frac{1}{2\sigma^2} \left[x^2 - 2x (m_\alpha t - m_\gamma t + m_\gamma) \right] \right\} dx$$

$$= \exp\left\{ - \frac{1}{2\sigma^2} \left[t\, m_\alpha^2 - t\, m_\gamma^2 + m_\gamma^2 - (m_\alpha t - m_\gamma t + m_\gamma)^2 \right] \right\}$$

$\rho_{\alpha\gamma}(t)$ is minimized for $t = t^*$ when $\frac{d}{dt} \rho_{\alpha\gamma}(t) = 0$.

This gives

$$2\,t\,(m_1-m_2)^2 = (m_1-m_2)^2$$

$$\longrightarrow \quad t^* = \frac{1}{2} \quad , \quad m_1 \neq m_2$$

APPENDIX B

Bhattacharyya Distance for Gaussian, Gamma, Poisson and Multinominal Distributions

1. Multivariable Gaussian Distribution:

$$p(\underline{Z}^N|\underline{\theta}_\alpha) = G(\underline{\mu}_\alpha, \Sigma_\alpha) \quad , \quad p(Z^N|\underline{\theta}_\gamma) = G(\underline{\mu}_\gamma, \Sigma_\gamma)$$

$$B_{\alpha\gamma} = \frac{1}{8}(\underline{\mu}_\alpha - \underline{\mu}_\gamma)^T \sum^{-1} (\underline{\mu}_\alpha - \underline{\mu}_\gamma) + \frac{1}{2} \ln \left[\frac{\det \Sigma}{\left\{ \det \Sigma_1 \det \Sigma_2 \right\}^{\frac{1}{2}}} \right] \quad \text{(B-1)}$$

$$\Sigma = \frac{1}{2}(\Sigma_\alpha + \Sigma_\gamma) \quad .$$

2. Gamma Density

$$p(Z|\theta_\alpha) = \frac{c_\alpha^{b_\alpha+1}}{\Gamma(b_\alpha+1)} Z^{b_\alpha} e^{-c_\alpha Z} U(Z) \ , \ b_\alpha > 0 \ , \ c_\alpha > 0$$

$$\rho_{\alpha\gamma} = \int_0^\infty \left\{ \frac{c_\alpha^{b_\alpha+1} \, c_\gamma^{b_\gamma+1}}{\Gamma(b_\alpha+1)\,\Gamma(b_\gamma+1)} \right\}^{\frac{1}{2}} Z^{\frac{b_\alpha+b_\gamma}{2}} e^{-\frac{1}{2}(c_\alpha+c_\gamma)Z} \, dZ$$

$$= \left[\frac{c_\alpha^{b_\alpha+1} \, c_\gamma^{b_\gamma+1}}{\Gamma(b_\alpha+1)\,\Gamma(b_\gamma+1)} \right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{b_\alpha+b_\gamma}{2}+1\right)}{\left\{\frac{c_\alpha+c_\gamma}{2}\right\}^{\frac{b_\alpha+b_\gamma}{2}+1}} \quad \text{(B-2)}$$

$$B_{\alpha\gamma} = - \ln \rho_{\alpha\gamma} \tag{B-3}$$

Case (i) $b_\alpha = b_\gamma = b$

$$\rho_{\alpha\gamma} = \frac{(c_\alpha c_\gamma)^{\frac{1}{2}(b+1)}}{\left(\frac{c_\alpha + c_\gamma}{2}\right)^{b+1}}$$

$$B_{\alpha\gamma} = - (b+1) \ln \left[\frac{(c_\alpha c_\gamma)^{\frac{1}{2}}}{\frac{c_\alpha + c_\gamma}{2}} \right] \tag{B-4}$$

Case (ii) $c_\alpha = c_\gamma = c$

$$\rho_{\alpha\gamma} = \frac{\Gamma \frac{b_\alpha + b_\gamma}{2}}{\left[\Gamma (b_\alpha + 1) \Gamma (b_\gamma + 1)\right]^{\frac{1}{2}}} \tag{B-5}$$

3. Poisson Distribution

$$p(Z|\theta_\alpha) = e^{-\lambda_\alpha} \frac{(\lambda_\alpha)^n}{n!} \delta (z - n)$$

$$B_{\alpha\gamma} = \frac{1}{2} (\sqrt{\lambda_\alpha} - \sqrt{\lambda_\gamma})^2 \tag{B-6}$$

4. Multinominal Distribution

Let

$$p_\alpha(Z) = \sum_{i=1}^{N} p_{i\alpha}\,\delta(Z-i)$$

$$B_{\alpha\gamma} = -\ln\left[\sum_{i=1}^{N}\left\{p_{i\alpha}\,p_{i\gamma}\right\}^{\frac{1}{2}}\right] \qquad \text{(B-7)}$$

APPENDIX C

Numerical Method for the Solution of Energy Constrained Inputs

When the number of stages N is large, determination of optimum inputs requires finding eigenvectors of high dimensional matrices. This requires a computation time which increases as the square of the matrix dimension. In order to circumvent this a numerical scheme is presented for the energy constrained signals. This method is similar to the method of gradient projection [67].

C.1 Geometric Representation of the Problem

Consider the two dimensional quadratic cost function

$$f(u_1, u_2) = \frac{1}{2} \underline{u}^T Q \underline{u} + \underline{a}^T \underline{u} \qquad (C-1)$$

with constraint $u_1^2 + u_2^2 \leq \Gamma$ (C-2)

We want to maximize $f(u_1, u_2)$ subject to the constraint (C-2). The geometric aspects of the problem are shown in Figure (C-1).

Let $\underline{u}_0$ be the point where an initial cost contour intersects the constraint boundary. Then $\overrightarrow{A}$ is the gradient of constraint at D ($\nabla \underline{c}$) and DB is the gradient of cost function, $\nabla \underline{f}$. From the figure the angle θ ranges from 0 to $\frac{\pi}{2}$. These correspond to

maximum and minimum direction coincidence of the two gradients; when $\theta = 0$ we are at an extreme point of the function.

Consider a direction OB that makes an angle $\beta < \theta$ with OA. OB intersects the cost gradient $\nabla \underline{f}$ at B. The vector OB is given by

$$\overrightarrow{OB} = \underline{u}_0 + k \nabla \underline{f}(u_0) \tag{C-3}$$

When $\overrightarrow{OB}$ is scaled down to OE we get the new point E which satisfies the constraint.

$$\underline{\tilde{u}} = \overrightarrow{OE} = \overrightarrow{OB} \cdot \frac{\sqrt{\Gamma}}{\|OB\|} \tag{C-4}$$

The problem now is to determine k such that the new point E is as close to the extremum as possible.

C.2 The Method of Solution

The angle $\underline{\theta}$ between the gradient vectors $\nabla \underline{c}$ and $\nabla \underline{f}$ is calculated as follows

$$\cos \theta = \frac{\langle \nabla \underline{c} , \nabla \underline{f} \rangle}{\|\nabla \underline{c}\| \, \|\nabla \underline{f}\|} \tag{C-5}$$

Choose $\beta < \theta$. The new direction OB is obtained as

$$\overrightarrow{OB} = \underline{\tilde{u}} = u_0 + k \nabla \underline{f}(u_0) \tag{C-6}$$

The step size k is given by the solution of the following relationship

$$\cos \beta = \frac{\langle \underline{u}_0, \tilde{\underline{u}} \rangle}{\|\underline{u}_0\| \, \|\tilde{\underline{u}}\|} = \frac{\langle \underline{u}_0, \underline{u}_0 + k \, \underline{f}(u_0) \rangle}{\|\underline{u}_0\| \, \|\underline{u}_0 + k \, \underline{f}(u_0)\|} \tag{C-7}$$

In (C-7) all the quantities are known except k that can be obtained by a simple solution of a quadratic equation as follows.

Let $\cos \beta = \alpha$. Then from (C-7)

$$\alpha^2 = \frac{\left[\|\underline{u}_0\|^2 + k \langle \underline{u}_0 , \nabla \underline{f} \rangle \right]^2}{\|\underline{u}_0\|^2 \left[\|\underline{u}_0\|^2 + k^2 \|\nabla \underline{f}\|^2 + 2k \langle \underline{u}_0 , \nabla \underline{f} \rangle \right]} \tag{C-8}$$

or

$$\alpha^2 \|\underline{u}_0\|^2 + k^2 \|\nabla \underline{f}\|^2 + 2k \langle \underline{u}_0 , \nabla \underline{f} \rangle$$

$$= \|\underline{u}_0\|^2 + 2k \langle \underline{u}_0 , \nabla \underline{f} \rangle + \frac{k^2}{\|\underline{u}_0\|^2} \left| \langle \underline{u}_0 , \nabla \underline{f} \rangle \right|^2 \tag{C-9}$$

Simplifying (C-9) gives

$$k^2 \left[\frac{\left| \langle \underline{u}_0 , \nabla \underline{f} \rangle \right|^2}{\alpha^2 \|\underline{u}_0\|^2} - \|\nabla \underline{f}\|^2 \right] + k \left[\frac{2}{\alpha^2} \langle \underline{u}_0 , \nabla \underline{f} \rangle - 2 \langle \underline{u}_0 , \nabla \underline{f} \rangle \right]$$

$$+ \|\underline{u}_0\|^2 \left(\frac{1}{\alpha^2} \right) - 1 = 0 \tag{C-10}$$

Equation (C-10) is of the form

$$a k^2 + b k + g = 0 \tag{C-11}$$

where

$$a = \frac{|\langle \underline{u}_0 , \nabla \underline{f} \rangle|^2}{\alpha^2 \|\underline{u}_0\|^2} - \|\nabla \underline{f}\|^2 \qquad \text{(C-12)}$$

$$b = 2 \langle \underline{u}_0 , \nabla \underline{f} \rangle \left(\frac{1}{\alpha^2} - 1 \right) \qquad \text{(C-13)}$$

$$g = \|\underline{u}_0\|^2 \frac{1}{\alpha^2} - 1 \qquad \text{(C-14)}$$

Solving for k

$$k_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 a g}}{2 a} \qquad \text{(C-15)}$$

Choose $k = \max (k_1 , k_2)$ (C-16)

The angle β can be chosen as

$$\beta = t \theta \quad , \quad t \in (0, 1) \qquad \text{(C-17)}$$

The following search algorithm is adopted:

Step 1. Choose initial point $\underline{u}_1$ satisfying the constraint $\|u_1\|^2 = \Gamma$. Set i=1

Step 2. Compute θ_k using (C-5).

Set $\beta_i = t \theta_i$, $t \in (0, 1)$

Step 3. Compute step size k from (C-15) and (C-16).

Set $\underline{u}_{i+1} = \underline{u}_i + k \nabla \underline{f} (\underline{u}_i)$

Scale down $\underline{u}'_{i+1}$ such that $\underline{u}_{i+1} = \frac{\sqrt{r}}{\|\underline{u}'_{i+1}\|} \cdot \underline{u}_{i+1}$

Step 4. Set $i = i + 1$. Compute $\nabla \underline{f}_i$ and $\nabla \underline{c}_i$.

If $\|\nabla \underline{f}_i - \nabla \underline{c}_i\| < \epsilon$, Stop.

Step 5. Go to Step 2.

[X]

The convergence of the above procedure is guaranteed because of quadratic cost function and convex constraint region. The value of $t \in (0, 1)$ is selected after some experimentation. In the examples of Section (3.6) $t = .8$ gave satisfactory convergence rate. Note that the use of penalty functions is not necessary for the method outlined here. Table (C-1) gives the convergence of the algorithm for the example with $N = 20$.

Table C-1

Convergence of the Quadratic Maximization Problem

with Energy Constraint, N = 20 . Second Order System

ITERATION	θ	COST FUNCTION
1	6.35911E-01	17.8537
2	6.35872E-01	40.0499
3	3.17605E-01	68.9881
4	1.35247E-01	77.5470
5	6.56942E-02	79.2250
6	4.17047E-02	79.6626
7	3.19195E-02	79.8563
8	2.60171E-02	79.9742
9	2.15181E-02	80.0533
10	1.78435E-02	80.1076
11	1.47978E-02	80.1449
12	1.22680E-02	80.1706
13	1.01675E-02	80.1883
14	8.42464E-03	80.2004
15	6.97936E-03	80.2087
16	5.78134E-03	80.2144
17	4.78858E-03	80.2184
18	3.96607E-03	80.2210
19	3.28474E-03	80.2229
20	2.72036E-03	80.2242
21	2.25291E-03	80.2250
22	1.86577E-03	80.2256

ITERATION	θ	COST FUNCTION
23	1.54514E-03	80.2260
24	1.27958E-03	80.2263
25	1.05966E-03	80.2265
26	8.77546E-04	80.2266

Table C-2

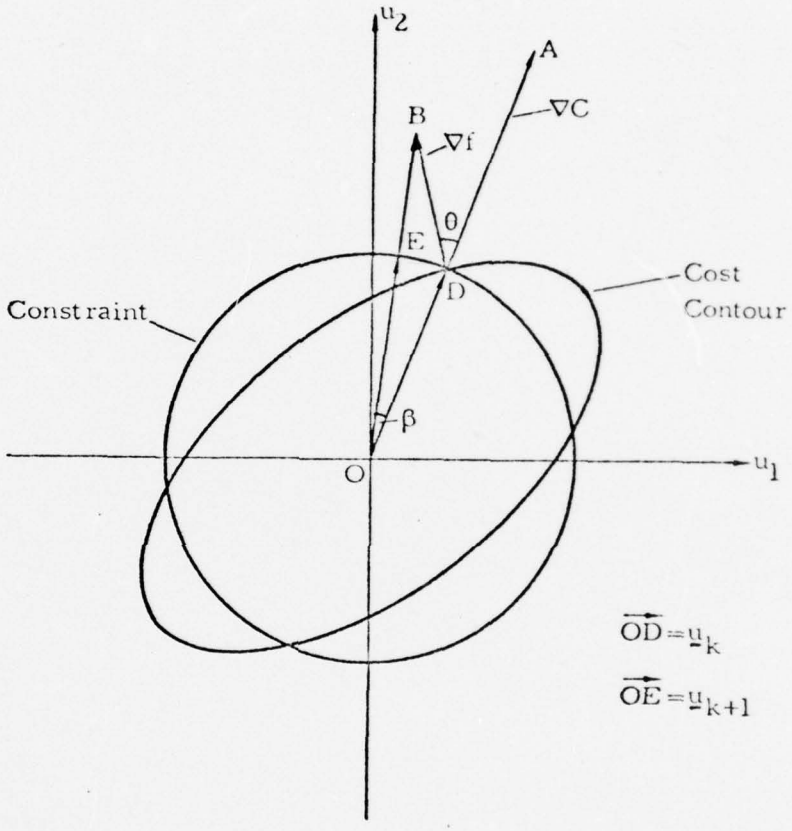
Optimal Point for $N = 20$ with $\|U\|^2 = 1$

Starting point $u_i = .22361$, $i = 1, 2, \ldots, 20$

u_1	.28974	u_{11}	-.24538
u_2	.36749	u_{12}	-.12598
u_3	.36705	u_{13}	.00362
u_4	.28611	u_{14}	.11321
u_5	.14979	u_{15}	.17916
u_6	-.11788	u_{16}	.19438
u_7	-.16898	u_{17}	.16812
u_8	-.28912	u_{18}	.11719
u_9	-.34445	u_{19}	.06069
u_{10}	-.32599	u_{20}	.01798

FIGURE C-1

Geometrical representation of two-dimensional constrained optimization problem.


u_2
A
∇C
B
∇f
θ
E
D
Cost
Contour
Constraint
β
O
u_1
$\overrightarrow{OD} = \underline{u}_k$
$\overrightarrow{OE} = \underline{u}_{k+1}$

APPENDIX D

Proof of Theorem 3.3-1

The a posteriori probability of $\underline{\theta}_0 = {}_{\ell}\underline{\theta}$ can be written as

$$P({}_{\ell}\underline{\theta}|\underline{Z}^N) = \frac{p(\underline{Z}^N|{}_{\ell}\underline{\theta})\, P_{\ell}}{\sum_{i=1}^{m} p(\underline{Z}^N|{}_{i}\underline{\theta})\, P_i} \tag{D-1}$$

Assuming equal prior probabilities this can be written as

$$P({}_{\ell}\underline{\theta}|\underline{Z}^N) = \frac{1}{1 + \dfrac{\sum_{i \neq \ell}^{m} p(\underline{Z}^N|{}_{i}\underline{\theta})}{p(\underline{Z}^N|{}_{\ell}\underline{\theta})}} \tag{D-2}$$

For ϵ sufficiently small we want

$$P({}_{\ell}\underline{\theta}|\underline{Z}^N) > 1 - \epsilon \tag{D-3}$$

Let

$$\delta = \frac{\sum_{i \neq \ell}^{m} p(\underline{Z}^N|{}_{\ell}\underline{\theta})}{p(\underline{Z}^N|{}_{\ell}\underline{\theta})} \tag{D-4}$$

Then

$$P({}_{\ell}\underline{\theta}|\underline{Z}^N) = (1 + \delta)^{-1} \tag{D-5}$$

Assuming $\delta \ll 1$ we get

$$P(_{\ell}\underline{\theta}|\underline{Z}^N) = 1 - \delta + \delta^2 - \ldots \simeq 1 - \delta$$

The condition (3.3-23) requires $1 - \delta > 1 - \epsilon$. Or

$$\delta < \epsilon \tag{D-6}$$

Using (D-4) δ can be written as

$$\delta = \frac{\sum_{i \neq \ell}^{m} \exp\left[-\sum_{k=1}^{N} \frac{(Z_k - \underline{h}^T{}_i \underline{x}_k)^2}{2\sigma_k^2} \right]}{\exp\left[-\sum_{k=1}^{N} \frac{(Z_k - \underline{h}^T{}_{\ell} \underline{x}_k)^2}{2\sigma_k^2} \right]} \tag{D-7}$$

$$\delta = \sum_{i \neq \ell}^{m} \exp\left\{ \sum_{k=1}^{N} \frac{1}{2\sigma_k^2} \left[(z_k - \underline{h}^T{}_{\ell} \underline{x}_k)^2 - (z_i - \underline{h}^T{}_i \underline{x}_k)^2 \right] \right\} \tag{D-8}$$

Dividing (D-8) by (m-1) and using the arithmetic geometric mean inequality [11, p 134] we get

$$\frac{\delta}{m-1} > \left\{ \prod_{i \neq \ell}^{m} \left[\exp\left\{ \sum_{k=1}^{N} \frac{1}{2\sigma_k^2} \left[(z_k - \underline{h}^T{}_{\ell} \underline{x}_k)^2 - (z_k - \underline{h}^T{}_i \underline{x}_k)^2 \right] \right\} \right] \right\}^{\frac{1}{m-1}} \tag{D-9}$$

Taking logarithms on both sides of (D-9)

$$\ln\left[\frac{\delta}{m-1}\right] > \frac{1}{m-1} \sum_{i\neq\ell}^{m} \left\{ \sum_{k=1}^{N} \frac{1}{2\sigma_k^2} \left[(z_k - \underline{h}^T{}_\ell \underline{x}_k)^2 - (z_k - \underline{h}^T{}_i \underline{x}_k)^2 \right] \right\}$$

(D-10)

The above inequality should also hold for the mean values of the respective random variables. Averaging with respect to z_k we get

$$\ln \hat{\delta} - \ln(m-1) > \frac{1}{m-1} \sum_{i\neq\ell}^{m} \left\{ -\sum_{k=1}^{N} \frac{1}{2\sigma_k^2} (h^T_\ell \underline{x}_k - h^T{}_i \underline{x}_k)^2 \right\}$$

(D-11)

For the system described by (3.3-16) and (3.3-17) the RHS can be written as

$$\frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\underline{h}^T{}_\ell \underline{x}_k - h^T{}_i \underline{x}_k)^2 = \frac{1}{2} \|U^{N-1}\|^2_{M_{\ell i}^T R_N^{-1} M_{\ell i}}$$

(D-12)

where $M_{\ell i} = M_\ell - M_i$, $i=1,\ldots,m$, $i\neq\ell$ (D-13)

and

$$M_i = \begin{bmatrix} \underline{h}^T \underline{\beta}_i & & & \\ \underline{h}^T \varphi_i \underline{\beta}_i & \underline{h}^T \underline{\beta}_i & & 0 \\ \vdots & \ddots & \ddots & \\ \underline{h}^T \varphi_i^{N-1} \underline{\beta}_i & \underline{h}^T \varphi_i^{N-2} \underline{\beta}_i & \cdots & \underline{h}^T \beta_i \end{bmatrix} \quad \text{is } (N \times N) \text{ matrix} \tag{D-14}$$

$$R_N = \text{diagonal } (\sigma_1^2, \sigma_2^2 \ldots, \sigma_N^2) \tag{D-15}$$

Using (D-12) in (D-11) it follows

$$\hat{\ln \delta} > \ln (m-1) = \frac{1}{2(m-1)} \sum_{i \neq \ell}^{m} \| U^{N-1} \|^2_{M_{\ell i}^T R_N^{-1} M_{\ell i}^T} \tag{D-16}$$

From (D-6) we require

$$\hat{\ln \delta} < \ln \epsilon \ , \quad \text{for given } \epsilon > 0 \tag{D-17}$$

Combining (D-16) and (D-17)

$$\ln (m-1) - \frac{1}{2(m-1)} \sum_{i \neq \ell}^{m} \| \underline{U}^{N-1} \|^2_{M_{\ell i}^T R_N^{-1} M_{\ell t}} < \ln \epsilon \tag{D-18}$$

Rearranging we get

$$\frac{1}{2\,(m-1)} \sum_{i \neq \ell}^{m} \frac{\| \underline{U}^{N-1} \|^2}{M_{\ell i}^T R_N^{-1} M_{\ell i}} > \ln \frac{m-1}{\epsilon} \quad , \quad m \geq 2 \tag{D-19}$$

The LHS in (D-19) is directly proportional to the ISNR . This quantity is a norm with respect to a symmetric positive definite matrix. Denoting this norm by $\| \underline{U}^{N-1} \|_A^2$ we can use an orthogonal transformation $T \in R^{N \times N}$ which does not alter the length of the norm.

Thus

$$\| \underline{U}^{N-1} \|_A^2 = \underline{U}^{N-1^T} T^T T A T^T t U^{N-1} \tag{D-20}$$

Now $T A T^T$ = diagonal $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ where λ_i are the eigenvalues of A matrix. So we have

$$\| U^{N-1} \|_A^2 = U^{N-1^T} T^T \Lambda T U^{N-1} = \sum_{i=1}^{N} \lambda_i \tilde{u}_{i-1}^2 \quad , \quad \lambda_i > 0 \tag{D-21}$$

where

$$\underset{\sim}{U}^{N-1} = T \underline{U}^{N-1} \tag{D-22}$$

Now it is clear that $\| \underline{U}^{N-1} \| = \| \underset{\sim}{U}^{N-1} \|$ and increasing $\| \underline{U}^{N-1} \|^2$ will also increase $\| \underline{U}^{N-1} \|_A^2$. From the definition of ISNR in (3.3-19), increasing the ISNR will also increase the norm in (D-19).

The quantity $\ln \frac{m-1}{\epsilon}$ is a nonnegative number independent of the parameter sets. Thus for a given N, $\epsilon > 0$ there exists an ISNR such that for all $n \geq N$

$$P(\underline{\theta}_0 | \underline{Z}^n) > 1 - \epsilon \qquad \text{(D-23)}$$

$\boxed{X}$

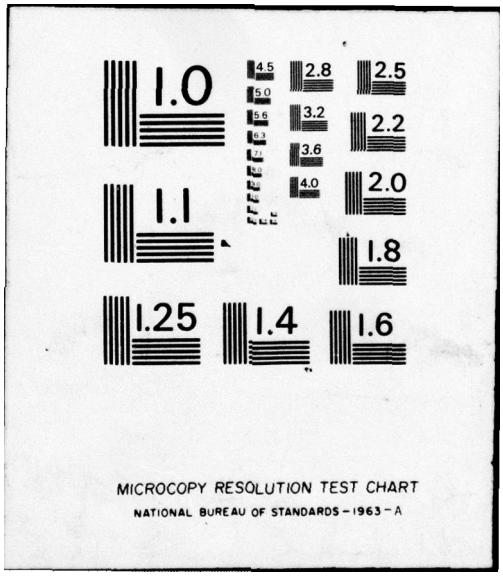
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APPENDIX E

Toeplitz Matrices and Asymptotic Eigenvalue Distribution

This appendix presents the main results on the theory of asymptotic eigenvalue distribution of Toeplitz matrices. The important theorems of Grenander and Szegö [33] and some generalizations and related theorems by Gray [31, 32] are given in this appendix without proof. The interested reader is referred to the above references for a complete exposé.

Let $f(x)$ be a real-valued, continuous, intergrable, bounded fuction on $[-\pi,\pi]$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\,x\,\ln f(x) > -\infty \qquad \text{(E-1)}$$

so that $f(x) > 0$ almost everywhere. Let us refer to this class of functions as L.

Definition E-1: The $N \times N$ Toeplitz matrix T is defined as follows

$$T_N = T[f(x)] = [t_{kj}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{i|k-j|x}\, dx \qquad \text{(E-2)}$$

The matrix T is a positive definite matrix and its eigenvalues $\lambda_{k,N}$ are strictly positive for all N.

Definition E-2: Consider the nonnegative sequences $\{a_{k,N}\}$ and $\{b_{k,N}\}$. Assume that for each k and N

$$a_{k,N} < R \ , \ b_{k,N} < R \tag{E-3}$$

where R is independent of k and N. The sets $\{a_{k,N}\}$ and $\{b_{k,N}\}$ are said to be asymptotically equally distributed as $N \to \infty$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left[F(a_{k,N}) - F(b_{k,N}) \right] = 0 \tag{E-4}$$

where $F(t)$ is an arbitrary continuous function on $[0, R]$.

[X]

The following definitions describe the concept of one matrix approximating another matrix.

Definition E-3: Let L_N be an $N \times N$ Hermitian matrix with entries $L_N(i,j)$ and eigenvalues $\lambda_{k,N}$, $k=1, 2, \ldots, N$. The strong norm of L_N is defined as

$$\|L_N\| = \max_k |\lambda_{k,N}| \ . \tag{E-5}$$

The weak norm of L_N is given by

$$|L_N| = \left\{ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |L_N(i,j)|^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{1}{N} \sum_{k=1}^{N} \lambda_{k,N}^2 \right\}^{\frac{1}{2}} \qquad \text{(E-6)}$$

[X]

Definition E-4: We say that two sequences of Hermitian matrices exhibit mutual approximation denoted by $L_N \sim G_N$ if they possess the following properties:

(i) Uniform boundedness in both norms, that is

$$\|L_N\|, \|G_N\|, |L_N|, |G_N| \leq M < \infty \qquad \text{(E-7)}$$

$$\lim_{N \to \infty} |L_N - G_N| = 0 \qquad \text{(E-8)}$$

[X]

Now we state the basic theorem on the asymptotic eigenvalue distribution of Toeplitz matrices (see [31, theorem 5.2, p 64]).

Theorem E-1: (Asymptotic eigenvalue distribution of Toeplitz matrices). Let $f(x)$ be a real-valued function of class L. Denote by m and M the essential lower and upper bounds of $f(x)$ respectively, and assume that m and M are finite. If $F(\lambda)$ is any

continuous function defined in the finite interval $m \leq \lambda \leq M$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F(\lambda_{k,N}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(x)]\, dx \qquad \text{(E-9)}$$

[X]

The next theorem is a slight modification of Theorem 7.4 of Grenander and Szegö ([33], p 105). This is the basic theorem for finding the asymptotic eigenvalue distribution of approximately Toeplitz matrices.

Theorem E-2: Given a Toeplitz matrix L_N and a Hermitian matrix G_N such that $L_N \sim G_N$. Let $\{\alpha_{k,N}\}_{k=1}^{N}$ and $\{\beta_{k,N}\}_{k=1}^{N}$ be the eigenvalues of L_N and G_N respectively. If either $\lim_{N \to \infty} |L_N|$ or $\lim_{N \to \infty} |G_N|$ is finite then $\{\alpha_{k,N}\}_{k=1}^{N}$ and $\{\beta_{k,N}\}_{k=1}^{N}$ are asymptotically equally distributed.

[X]

The following theorem of Gray gives a necessary and sufficient condition for the inverse of a Toeplitz matrix to itself approximate the Toeplitz matrix $T_N[1/f(x)]$.

Theorem E-3: Let $f(x) \in L$ and $f(x) \geq m > 0$. Then the eigenvalues of $T_N^{-1}[f(x)]$ are asymptotically distributed as $1/f(x), x \in [-\pi, \pi]$. Furthermore

$$T^{-1}[f(x)] \sim T[1/f(x)] \qquad \text{(E-10)}$$

that is, $T^{-1}[f(x)]$ approximates a Toeplitz matrix.

[X]

Corollary E-1: Let $f(x) > 0$ everywhere and $A_N \sim T_N[f]$ where A_N is bounded and Hermitian. Then

$$A_N^{-1} \sim T_N[1/f] \tag{E-11}$$

and from Theorem E-2 the eigenvalues of A_N^{-1} are asymptotically distributed as $1/f(x)$, $x \in [-\pi, \pi]$.

[X]

Corollary has the interpretation that if a matrix approximates a Toeplitz matrix generated by $f(x)$, then the inverse approximates another Toeplitz matrix generated by $1/f(x)$.

Theorem E-4: Let $f(x)$ and $g(x)$ belong to L. Then

$$\lim_{N\to\infty} |T_N[f]\, T_N[g] - T_N[f\,g]| = 0 \tag{E-12}$$

and the eigenvalues of $T_N[f]\, T_N[g]$ are asymptotically distributed as $f(x)\, g(x)$, $x \in [-\pi, \pi]$.

[X]

Corollary E-2: Let the bounded Hermitian matrices A_N and B_N approximate $T_N[f]$ and $T_N[g]$ respectively. Then

$$\lim_{N\to\infty} |A_N B_N - T_N[f\,g]| = 0 \tag{E-13}$$

and the eigenvalue of $A_N B_N$ are asymptotically distributed as

$f(x)\, g(x)$, $x \in [-\pi, \pi]$.

[X]

Theorem E-5: Let $f(x)$ and $g(x)$ belong to the class L with $g(x) > 0$ everywhere. Then

$$\lim_{N \to \infty} \left| T[f(x)]\, T^{-1}[g(x)] - T[f(x)/g(x)] \right| = 0 \qquad \text{(E-14)}$$

and the eigenvalues of $T[f(x)]\, T^{-1}[g(x)]$ are asymptotically distributed as $f(x)/g(x)$, $x \in [-\pi, \pi]$.

[X]

Corollary E-3: Define A_N and B_N as in Corollary E-2 then

$$\lim_{N \to \infty} \left| A_N B_N^{-1} - T_N[f/g] \right| = 0 \qquad \text{(E-15)}$$

and the eigenvalues of $A_N B_N^{-1}$ are asymptotically distributed as $f(x)/g(x)$, $x \in [-\pi, \pi]$.

[X]