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SAMPLING FROM THE GAMMA DISTRIBUTION ON A COMPUTER. (U)

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OPERATIONS RESEARCH AND SYSTEMS ANALYSIS

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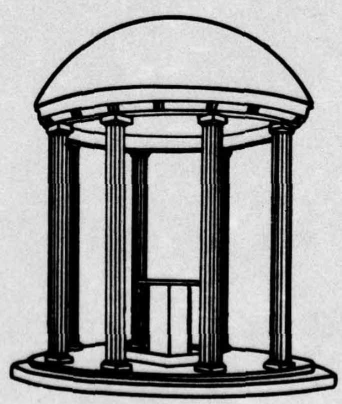

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6 SAMPLING FROM THE GAMMA DISTRIBUTION ON A COMPUTER .

10 George S./Fishman

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Abstract

This paper describes a method of generating gamma variates that appears to be less costly than other methods. a recently suggested method in [3]. For large shape parameter alpha α the cost of computation is proportional to square root of alpha $\sqrt{\alpha}$, whereas the other method in [3] is proportional to alpha α. Experimentation in [2] also indicates that for small alpha α the method suggested here also dominates other ~~methods~~ recently suggested methods, in [1], albeit those methods dominate for large alpha α. The method suggested here uses the rejection technique.

1. Introduction[†]

This paper describes a new technique (method 1) for sampling from the gamma distribution on a digital computer and compares it with an alternative technique (method 2) that Wallace has suggested in [3]. A gamma variate X has the probability density function[††] (p.d.f.)

$$(1) \qquad f_X(x) = \begin{cases} x^{\alpha-1} e^{-x}/\Gamma(\alpha) & 0 \leq x \leq \alpha, \quad \alpha > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Both methods use the rejection method and apply for $\alpha \geq 1$.

2. Rejection Method

Let X be a nonnegative valued continuous random variable with bounded p.d.f. representable in the form

$$(2) \qquad f_X(x, \alpha) = \begin{cases} c(\alpha, \beta)a(\alpha, \beta)g(x, \alpha, \beta)h(x, \alpha, \beta) & 0 \leq x \leq \infty \\ 0 & \text{elsewhere} \end{cases}$$

$$0 \leq h(x, \alpha, \beta), \qquad \int_0^\infty h(x, \alpha, \beta)dx = 1,$$

$$0 < g(x, \alpha, \beta) < \infty, \qquad a(\alpha, \beta) \geq 1/g(x, \alpha, \beta),$$

$$1/c(\alpha, \beta) = a(\alpha, \beta) \int_0^\infty g(x, \alpha, \beta)h(x, \alpha, \beta)dx.$$

Let X' denote a random variable with p.d.f. h and let U be a uniform deviate on (0, 1). If $U \leq a(\alpha, \beta)g(X', \alpha, \beta)$ then X' has the p.d.f.

† I am grateful to Mr. Hunter McDaniel for programming methods 1 and 2 in PL/1.

†† Here we assume a unit scale parameter without loss of generality.

f_X in (2). This result follows from

$$(3)\quad f_{X'}(x \mid U \le a(\alpha, \beta)g(x, \alpha, \beta)) = \frac{pr[U \le a(\alpha, \beta)g(x, \alpha, \beta) \mid X' = x]h(x, \alpha, \beta)}{pr[U \le a(\alpha, \beta)g(x, \alpha, \beta)]}$$

$$= f_X(x, \alpha).$$

Since

$$(4)\qquad pr[U \le a(\alpha, \beta)g(x, \alpha, \beta)] = 1/c(\alpha, \beta)$$

$c(\alpha, \beta)$ denotes the mean number of trials to obtain an X from (2). For a given X' from a specified h we want the probability of success to be as close to unity as possible. This feature requires

$$(5)\qquad 1/a^*(\alpha, \beta) = \max_x g(x, \alpha, \beta).$$

For any X' we want (4) to be as large as possible, which implies.

$$c(\alpha, \beta^*) = \min_\beta c(\alpha, \beta) = \min_\beta[1/a^*(\alpha, \beta)Eg(X', \alpha, \beta)]$$

$$= \min_\beta[\max_x g(x, \alpha, \beta)/Eg(X', \alpha, \beta)]$$

$$Eg(X', \alpha, \beta) = \int_0^\infty g(x, \alpha, \beta)h(x, \alpha, \beta)dx.$$

The distinction between methods 1 and 2 lies in the choice of h. Table 1 shows relevant quantities for each proposal. To make an appropriate comparison between methods we need to consider the mean number of trials $c_j(\alpha, \beta^*)$ for each <u>and</u> the mean number of required random numbers.

Table 1

Gamma Generation* for $\alpha \geq 1$

Method i	$h_i(x, \alpha, \beta)$	$g_i(x, \alpha, \beta)$	$a_i^*(\alpha, \beta)$	β_i^*	$c_j(\alpha, \beta^*)$
1	$\beta^{-1}e^{-x/\beta}$	$x^{\alpha-1}e^{(1/\beta-1)x}$	$(e/\alpha)^{\alpha-1}$	α	$\alpha^{\alpha}e^{1-\alpha}/\Gamma(\alpha)$
2	$\dfrac{x^{\gamma-1}e^{-x}[(1-\beta)\gamma+\beta x]}{\Gamma(\gamma+1)}$	$\dfrac{x^{\gamma'}}{(1-\beta)\gamma+\beta x}$	$\dfrac{\gamma(1-\beta)}{1-\gamma'}\left[\dfrac{\beta(1-\gamma')}{\gamma(1-\beta)\gamma'}\right]^{\gamma'}$	γ'	$\Gamma(\gamma)\gamma^{1-\gamma'}/\Gamma(\gamma)$
	$\gamma=\langle\alpha\rangle$	$\gamma'=\alpha-\gamma$			

* $\langle\theta\rangle$ denotes the largest integer in θ.

3. Method 1

Conceptually, method 1 implies 4 steps:

1. Generate an exponential deviate† X'.
2. Generate a uniform deviate U.
3. If $U \leq (X'/e^{X'+1})^{\alpha-1}$ then $X = \alpha X'$ has the p.d.f. in (2).
4. Otherwise, return to step 1.

If we use the inverse transform method to generate X' then each trial requires 2 random numbers. Therefore, the mean number of random numbers needed to generate X from (2) is $2\alpha^{\alpha}/\Gamma(\alpha)e^{\alpha-1}$. For large α this quantity is approximately $e(2\alpha/\pi)^{1/2}$, an appealing result. Notice that for large integral α, using method 1

† An exponential deviate has unit mean.

requires fewer random numbers than the conventional method which uses

$$X = -\ln \prod_{i=1}^{b} (U_i), \tag{7}$$

$U_1, \ldots, U_\alpha$ being a sequence of independent uniform deviates. For small integral α one can show that (7) is superior. Our experiments indicate that method one prevails for nonintegral $\alpha < 7$ and all $\alpha > 7$.

4. Method 2

For integral α method 2 uses (7). For nonintegral α the 6 steps are:

1. Generate a uniform deviate U.
2. If $U \leq 1-\alpha+\langle\alpha\rangle$ generate X' from (7) using $b = \langle\alpha\rangle$.
3. Otherwise, generate X' from (7) using $b = \langle\alpha+1\rangle$.
4. Generate a uniform deviate U.
5. If $U \leq (X'/\gamma)^{\gamma'}/(1-\gamma'+X'/\gamma)$ then X' has the p.d.f. (2).
6. Otherwise, go to step 1.

These steps require $\alpha+2$ random numbers on average per trial. Therefore, for nonintegral α, method 2 uses $(\alpha+2)\Gamma(\gamma)\gamma^{1-\gamma'}/\Gamma(\alpha)$ random numbers on average. This quantity is approximately $\alpha+2$ for $\alpha>5$.

5. Comparison of Methods

PL/1 programs were prepared using algorithm G1 for method 1 and using the steps given in [3] for method 2.

Algorithm G1

Given: α

1. $\alpha' \leftarrow \alpha-1$.
2. Generate a uniform deviate U.

(continued)

3. $V \leftarrow -\ln U$.

4. Generate a uniform deviate U.

5. $W \leftarrow -\ln U$.

6. If $W \geq \alpha'(V - \ln V - 1)$, $X \leftarrow \alpha V$ and return with X.

7. Otherwise, go to 2.

Table 2 displays the results for generation of 10,000 gamma variates for each selected value of† α.

Table 2

Comparison of Methods

α	mean CPU time (in μsec.) Method 1	Method 2	Ratio
1.25	723	1093	.661
2.25	988	1300	.760
3.25	1193	1542	.774
4.25	1352	1787	.757
5.25	1541	2039	.756

Based on these results we computed expressions for T_i, the mean CPU time for method i, as a function of α. These expressions are in microseconds.

$$T_1 = 140 + 624\alpha^{\alpha}/\Gamma(\alpha)e^{\alpha-1}$$

(8)

$$T_2 = -88+(752+254\alpha)\Gamma(\gamma)\gamma^{1-\alpha+\gamma}/\Gamma(\alpha).$$

† The programs were run on the IBM 360/75 computer at the University of North Carolina Computer Center at Chapel Hill as single stream inputs. This procedure minimized the error due to monitoring in a multiprogram mode.

For large α $T_1/T_2 \sim 624/254\sqrt{\alpha} = 2.46/\sqrt{\alpha}$. For example, $\alpha = 30$ gives $T_1/T_2 \sim 0.45$ and $\alpha = 50$, $T_1/T_2 \sim 0.35$.

One modification to method 1 makes it at least as good as method 2 for all integral α, while preserving its superiority for nonintegral α. Experimentation with method 1 revealed that it is superior to method 2 for all $\alpha \geq 7$. Addition of the statement:

0. If $\alpha \leq 7$ and $\langle\alpha\rangle = \alpha$, return with $X = -\ln(\prod_{i=1}^{\alpha} U_i)$

prior to statement 1 in algorithm G1 modifies the flow appropriately.

5. New Prospects

Upon conclusion of the work presented here the writer learned of research by Dieter and Ahrens in [1] on gamma generation 1) using a truncated noncentral Cauchy distribution for h and 2) exploiting the relationship between the gamma and normal distributions for large α. The most notable feature of their work is that computation time goes to a fixed limit as α increases. Although this property makes the Dieter and Ahrens procedures more attractive for large α, Robinson and Lewis [2] have recently prepared a gamma generation program in which a variant of algorithm G1 dominates all competitors for $1.2 \leq \alpha \geq 2.9$. Since this is a commonly encountered range in practice the significance of method 1 remains.

Since the work in [2] generates exponential variates by a more efficient method than inverse transformation does, it is not presently clear to the writer what the range of superiority would be using algorithm G1. This issue is a legitimate concern since simulation languages such as SIMSCRIPT and SIMPL/1 use the inverse approach.

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1. Dieter, V. and J.H. Ahrens, "Acceptance-Rejection Techniques for Sampling from the Gamma and Beta Distributions," Department of Statistics, Stanford University, Technical Report No. 83, May 29, 1974.

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3. Wallace, N.D., "Computer Generation of Gamma Random Variates with Non-integral Shape Parameters," Comm. ACM, Vol. 17, No. 12, December 1974, pp. 691-695.

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This paper describes a method of generating gamma variates that appears to be less costly than a recently suggested method in [3]. For large shape parameter α the cost of computation is proportional to $\sqrt{\alpha}$, whereas the method in [3] is proportional to α. Experimentation in [2] indicates that for small α the method suggested here also dominates methods recently suggested in [1], albeit those methods dominate for large α. The method suggested here uses the rejection technique.

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