THE RECURRENCE CLASSIFICATION OF RISK AND STORAGE PROCESSES

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1. Introduction.

We consider in this paper two Markov processes $X = \{X(t),\ t \geq 0\}$ and $X^{\#} = \{X^{\#}(t),\ t \geq 0\}$ having stationary transition probabilities and state space $S = [0,\infty)$. Both X (called the storage process) and $X^{\#}$ (called the risk process) are defined in terms of a function $r : S \to [0,\infty)$ and a compound Poisson process $A = \{A(t),\ t \geq 0\}$ having positive jumps. We denote by λ and $F(\cdot)$ the jump rate and jump size distribution respectively of A , so that $0 < \lambda < \infty$ and $F(0) = 0$. We assume that $r(0) = 0$ and that $r(\cdot)$ is strictly positive, is left continuous, and has a strictly positive right limit $r^{\#}(\cdot)$ on $(0,\infty)$. Thus r is bounded away from zero on any compact subset of $(0,\infty)$. We further assume the behavior of r near zero to be such that

$$(1) \qquad R(x) = \int_0^x \frac{1}{r(y)}\, dy < \infty , \qquad x \geq 0 .$$

(Throughout the paper, integrals are understood to be over open intervals unless we explicitly indicate otherwise.)

The process $X = \{X(t), t \geq 0\}$, which we formally define by construction as in [4], satisfies the storage equation

$$X(t) = X(0) + A(t) - \int_0^t r(X(s))ds \ , \qquad t \geq 0 \ .$$

We interpret $X(t)$ as the content at time t of a storage system such as a dam, $A(t)$ as the input to the system during the interval $[0,t]$, and $r(x)$ as the rate at which material flows out of the system when its content is x. Thus $\int_0^t r(X(s))ds$ represents the total output from the system during $[0,t]$, and the storage equation says simply that current content equals initial content plus cumulative input minus cumulative output. The paths of X are absolutely continuous and non-increasing between positive jumps generated by A.

The quantity $R(x)$ represents the time required for X to move from state x down to state zero in the absence of any input (jumps). Thus, assumption (1) requires that state zero be accessible from any starting state.

In our previous paper [4], we calculated the generator of X, provided a necessary and sufficient condition for positive recurrence, and calculated the unique stationary distribution in the positive recurrent case. In Section 2 of this paper we complete the recurrence classification of X, providing necessary and sufficient conditions for null recurrence and transience.

The risk process $X^{\#} = \{X^{\#}(t), t \geq 0\}$ is defined by construction in Section 3. One may interpret $X^{\#}(t)$ as the size of an insurance company's risk reserve at time t. The paths of $X^{\#}$ are strictly increasing and absolutely continuous between jumps, the instantaneous rate of increase being $r(X^{\#}(t))$ at time t. The jumps of $X^{\#}$ are all downward. They occur at the same Poisson times and have the same absolute magnitude as those of A, except that jumps are truncated as necessary to keep $X^{\#}$ non-negative. The downward jumps correspond to the occurrence of claims against the company. The continuous upward movement of $X^{\#}$ between jumps corresponds to the receipt of premium payments from policyholders. By allowing the instantaneous income rate r to depend on the size of the risk reserve, we are able to represent a situation where management changes the fraction of premium income that is channelled into the risk reserve (rather than invested) as the size of the reserve fluctuates. In traditional collective risk theory, attention is focused on the probability that $X^{\#}$ will ever hit zero (the probability of ruin), and our truncation of those jumps that would drive the process negative is irrelevant for this calculation.

As we shall demonstrate in Section 4, the analytical problems which arise in the recurrence classification of $X^{\#}$ are identical to those encountered in the classification of X. It will be shown that $X^{\#}$ is positive recurrent iff X is transient, null recurrent iff X is null recurrent, and transient iff X is positive recurrent. Furthermore, in the case where $X^{\#}$ is transient, the probability of ruin (viewed as a function of the initial risk reserve) has the same density as the stationary distribution of X . A similar duality between the two processes is found in the case where $X^{\#}$ is positive recurrent and X is transient.

In Section 5 we consider the special case where the jumps of A are exponentially distributed. There all of our results can be made entirely explicit, and we are able to give a counterexample for a conjecture of Çinlar and Pinsky [3] concerning the recurrence classification of storage processes. Section 6 contains some concluding remarks about the recurrence classification of storage processes for which state zero is inaccessible.

2. Recurrence Classification of the Storage Process.

The following notation is necessary for what follows and generally conforms to conventions used in [4] with exceptions to be noted. Denote by $P_x(\cdot)$ the distribution on the path space of X corresponding to initial state $x \in S$ and by $E_x(\cdot)$ the associated expectation operator. The generator of X is $\mathfrak{A}$ and the domain of $\mathfrak{A}$ is $\mathfrak{D}$ (cf. Breiman [2], p. 341). With $Q(x) = \lambda(1 - F(x))$ define $K(x,y) = Q(x-y)/r(x)$, $0 \leq y \leq x$. The iterates of K are defined by $K_1(x,y) = K(x,y)$ and $K_{n+1}(x,y) = \int_y^x K_n(x,z)K(z,y)dz = \int_y^x K(x,z)K_n(z,y)dz$ for $0 \leq y \leq x < \infty$. From Section 3 of [4] we have $K_{n+1}(x,y) \leq \lambda^{n+1}[R(x)-R(y)]^n/r(x)n!$ so the kernel $K^*(x,y) = \Sigma_{n=1}^{\infty} K_n(x,y)$ is well defined.

Let $T = \inf\{t > 0 | X(t) = 0$ and $X(s) > 0$ for some $s \in (0,t)\}$. We shall say that X is <u>recurrent</u> if $P_0(T < \infty) = 1$ and <u>transient</u> otherwise. In the recurrent case, we say that X is <u>positive recurrent</u> if $E_0(T) < \infty$ and <u>null recurrent</u> otherwise. This definition of positive recurrence differs from that used in [4], but Proposition 7 of [4] shows the two definitions to be equivalent. The following is a restatement of Theorem 2 of [4].

<u>Theorem 1</u>. X is positive recurrent iff $\int_0^{\infty} K^*(x,0)dx < \infty$.

In differentiating between the null recurrent and transient cases, a key role is played by the integral equation

$$r(x)u(x) = \int_x^{\infty} Q(y-x)u(y)dy \ , \qquad x > 0 \ . \tag{2}$$

We shall say that a function $u : (0,\infty) \to [0,\infty)$ satisfying (2) is a positive integrable solution if $0 < \int_0^\infty u(y)dy < \infty$.

Proposition 1. If there exists a positive integrable solution u of (2), then

$$P_x(T=\infty) = \int_0^x u(y)dy \Big/ \int_0^\infty u(y)dy > 0 , \qquad \forall x > 0 ,$$

and any other positive integrable solution of (2) differs from u only by a multiplicative constant.

Proof: Let u be a positive integrable solution of (2) and define $U(x) = \int_0^x u(y)dy/\int_0^\infty u(y)dy$ for $x \geq 0$. We begin by showing that U is in the domain $\mathcal{D}$ of the generator $\mathfrak{A}$ of X. From (2) it is immediate that

$$r(x)u(x) \leq \int_x^\infty \lambda\, u(y)dy \leq \lambda \int_0^\infty u(y)dy , \qquad x > 0 ,$$

so $r(\cdot)u(\cdot)$ is bounded. Now let $z > 0$ be fixed. For $x < z$ we have

$$\begin{aligned} |r(x)u(x)-r(z)u(z)| &= \left|\int_x^\infty Q(y-x)u(y)dy - \int_z^\infty Q(y-z)u(y)dy\right| \\ &\leq \left|\int_z^\infty Q(y-x)u(y)dy - \int_z^\infty Q(y-z)u(y)dy\right| \\ &\quad + \left|\int_z^\infty Q(y-x)u(y)dy - \int_x^\infty Q(y-x)u(y)dy\right| = a(x) + b(x) . \end{aligned}$$

Clearly $b(x) = |\int_x^z Q(y-x)u(y)dy| \to 0$ as $x \uparrow z$, and $a(x) = |\int_z^\infty [Q(y-x) - Q(y-z)]u(y)dy|$. The integrand is bounded by $\lambda\, u(y)$, which is integrable, and $Q(y-x) - Q(y-z) \to 0$ for all y as $x \uparrow z$, since Q is right continuous. Thus $b(x) \to 0$ as $x \uparrow z$ by dominated convergence, and we conclude that $r(\cdot)u(\cdot)$ is left continuous. Since r is left continuous and strictly positive on $(0,\infty)$, it follows that u is left continuous. From Proposition 4 of [4] and its corollary, we then have $U \varepsilon \mathfrak{D}$ and

$$\mathfrak{A}U(x) = \int_x^\infty Q(y-x)u(y)dy - r(x)u(x) = 0 , \qquad x > 0 .$$

Now suppose $b > 0$, let $\tau(b) = \inf\{t \geq 0 | X(t) \geq b\}$, and let $T^* = \tau(b) \wedge T$. Proceeding as in the proof of Theorem 3 of [4], one can show $E_x(T^*) < \infty$ for $0 < x < b$, so from Dynkin's identity (Ito [5], p. 2.11.1) we have

$$(3) \qquad E_x U(X(T^*)) = U(x) + E_x \int_0^{T^*} \mathfrak{A}U(X(t))dt = U(x) .$$

Since $U(X(T)) = U(0) = 0$ on $\{T < \infty\}$, we have

$$(4) \qquad E_x U(X(T^*)) = E_x[U(X(\tau(b))) \ ; \ T > \tau(b)] .$$

It is easy to show that $\tau(b) \to \infty$ P_x- a.s. as $b \to \infty$ so $P_x(T > \tau(b)) \downarrow P_x(T = \infty)$ as $b \to \infty$. Since from (4) we have (recall $U(\infty) = 1$)

$$1\, P_x(T > \tau(b)) \geq E_x[U(X(\tau(b))); T > \tau(b)]$$

$$\geq U(b)\, P_x(T > \tau(b))$$

we conclude from (3) and (4) that $U(x) = P_x(T = \infty)$ for $x > 0$. Thus, $P_x(T = \infty) > 0$ for sufficiently large x. Combining this with the strong Markov property of X and Theorem 3 of [4] and its corollary, it follows that $P_x(T = \infty) > 0 \ \forall\, x > 0$.

Let u^* be any other positive integrable solution of (2) and let $U^*(x) = \int_0^x u^*(y)dy / \int_0^\infty u^*(y)dy$ for $x \geq 0$. By repeating the argument above, we get $U^*(x) = P_x(T = \infty) = U(x)$ for all $x > 0$, and the last statement of the proposition follows directly, since u and u^* are left continuous.

Proposition 2. If there exists no positive integrable solution of (2), then $P_x(T = \infty) = 0$ for all $x > 0$.

Proof: Let $U(x) = P_x(T = \infty)$ for $x > 0$. Proceeding exactly as in the proof of Theorem 3 of [4] by conditioning on the time of the first jump, one can show that $U(x) = \int_0^\infty u(y)dy$, where u is the left derivative of U on $(0,\infty)$ and satisfies

$$(5) \qquad r(x)u(x) = \lambda \int_0^\infty [U(x+z) - U(x)]F(dz)\,, \qquad x > 0\,.$$

From the path structure of X it is immediate that U is non-decreasing, so u is non-negative. We can rewrite $U(x+z) - U(x)$ as an integral of u in (5) and reverse the order of integration by Fubini's Theorem. The result shows that u satisfies (2). If there exists no positive integrable solution of (2), it follows that $u \equiv 0$ and hence $U \equiv 0$, which completes the proof.

Theorem 2. X is transient iff there exists a positive integrable solution u of (2).

Proof: Defining $U(x) = P_x(T = \infty)$ for $x > 0$, the strong Markov property of X gives us

$$P_0(T = \infty) = E_0[U(X(t_1))] = \int_0^\infty U(x)F(dx) \ ,$$

and the theorem is then immediate from Propositions 1 and 2.

Remark: Combining Theorems 1 and 2, we see that there cannot exist a positive integrable solution of (2) when $\int_0^\infty K^*(x,0)dx < \infty$. This can be seen analytically as follows: Setting $\phi(x) = r(x)u(x)$ we rewrite (2) as

$$\phi(x) = \int_x^\infty \phi(y)\ K(y,x)dy = \int_x^\infty \phi(y)\ K_n(y,x)dy \ . \tag{2'}$$

If $u(x)$ is an integrable solution of (2) we will show that $\int_0^\infty K^*(z,0)dz < \infty$ requires $u \equiv \phi \equiv 0$. Rewriting (2) as $\phi(x) = \int_0^\infty (U(x+s) - U(x))\lambda\ F(ds)$ shows that $\phi(x) \leq \lambda\ U(\infty)$, $\forall x$ so that from (2')

$$\phi(x) \leq \lambda\ U(\infty) \int_x^\infty K_n(y,x)dy \ .$$

It is not hard to see that $\int_0^\infty K^*(z,0)dz < \infty$ implies $\forall x > 0 \int_x^\infty K^*(y,x)dy < \infty$ and the desired result follows by letting $n \to \infty$.

Theorem 3. X is null recurrent iff $\int_0^\infty K^*(x,0)dx = \infty$, and there exists no positive integrable solution of (2).

Proof: Immediate from Theorems 1 and 2.

3. The Risk Process and its Generator.

Throughout this section and the next, we assume that $R(\infty) = \infty$. Since $R(\cdot)$ is strictly increasing on S, we can speak unambiguously of $R^{-1}(t)$ for $t \geq 0$. We define $q^{\#}(x,t) = R^{-1}(R(x)+t)$ for $x \geq 0$ and $t \geq 0$, so

$$(6) \qquad q^{\#}(x,t) = x + \int_0^t r(q^{\#}(x,s))ds = x + \int_0^t r^{\#}(q^{\#}(x,s))ds$$

for $x \geq 0$ and $t \geq 0$. The second equality in (6) follows from the fact that $q(x,\cdot)$ is strictly increasing and r can differ from its right limit $r^{\#}$ at only countably many points. Suppose that $t_1, t_2, \ldots$ are the Poisson jump times of A and $Y_1, Y_2, \ldots$ are the successive jump sizes. Then the risk process $X^{\#}$ with starting state $x \in S$ is constructively defined by

$$\begin{aligned}
X^{\#}(t) &= q^{\#}(x,t) &&, \quad 0 \leq t < t_1 , \\
X^{\#}(t_1) &= [q^{\#}(x,t_1) - Y_1]^+ &&, \\
X^{\#}(t_1 + t) &= q^{\#}(X^{\#}(t_1),t) &&, \quad 0 \leq t < t_2 - t_1 , \\
X^{\#}(t_2) &= [q^{\#}(X^{\#}(t_1), t_2 - t_1) - Y_2]^+ &&,
\end{aligned}$$

and so forth. Note that $t_n \to \infty$ as $n \to \infty$, so $X^{\#}(t)$ is well defined for all $t \geq 0$. Also, observe that assumption (1) and the requirement $R(\infty) = \infty$ are necessary for the construction to make sense. If (1) does not hold, then $X^{\#}$ cannot get away from state zero. If $R(\infty) < \infty$, then explosions in $X^{\#}$ are possible. (More precisely, an explosion is inevitable.)

One can show that $X^{\#}$ is a strong Markov process with standard, stationary transition probabilities. This can be done by brute force or by using Corollary 2 of Ito [5], p. 2. 12. 13. together with (11) below.

We denote by $P_x^{\#}(\cdot)$ the distribution on the path space of $X^{\#}$ corresponding to initial state $x \in S$ and by $E_x^{\#}(\cdot)$ the corresponding expectation operator. This notation, and all of that to come, parallels precisely the notation for X used in [4].

Let $\mathcal{L}^{\#}$ be the set of bounded, measurable $f : S \to R$ such that $E_x^{\#} f(X^{\#}(t)) \to f(x)$ as $t \downarrow 0$ $\forall x \in S$. Also, let $\mathcal{D}^{\#}$ be the set of $f \in \mathcal{L}^{\#}$ such that $[E_x^{\#} f(X^{\#}(t)) - f(x)]/t$ converges boundedly pointwise on S as $t \downarrow 0$ to a function (denoted $\mathfrak{A}^{\#}f$) in $\mathcal{L}^{\#}$. We call the operator $\mathfrak{A}^{\#}$ thus defined on $\mathcal{D}^{\#}$ the generator of $X^{\#}$. Finally, for each $\alpha > 0$ and bounded, measurable $f : S \to R$, we define the resolvent

$$R_\alpha^{\#} f(x) = E_x^{\#} \int_0^\infty e^{-\alpha t} f(X^{\#}(t))dt , \qquad x \in S .$$

The following is a standard result in the theory of Markov processes, cf. Breiman [2], p. 342.

<u>Proposition 3</u>. For all $\alpha > 0$, $R_\alpha^{\#} \mathcal{L}^{\#} = \mathcal{D}^{\#}$ and

$$h(x) + \mathcal{A}^{\#} R_{\alpha}^{\#} h(x) - \alpha R_{\alpha}^{\#} h(x) = 0 \quad \forall x \in S,\ h \in \mathcal{L}^{\#} . \tag{7}$$

Proposition 4. The set $\mathcal{L}^{\#}$ consists of all bounded functions $f : S \to R$ that are right continuous. The domain $\mathcal{D}^{\#}$ consists of all bounded, absolutely continuous functions $f : S \to R$ that have a right derivative f' on $(0,\infty)$ such that $r^{\#}(x)f'(x)$ is bounded and right continuous on $(0,\infty)$ and approaches a finite limit as $x \downarrow 0$. Furthermore, for $f \in \mathcal{D}^{\#}$,

$$\begin{aligned} \mathcal{A}^{\#} f(x) &= r^{\#}(x)f'(x) - \int_0^x Q(x-y)f'(y)dy \quad \forall x > 0 , \\ \mathcal{A}^{\#} f(0) &= \lim_{x\downarrow 0} r^{\#}(x)f'(x) \end{aligned} \tag{8}$$

Remark: Since $r^{\#}$ is strictly positive and right continuous on $(0,\infty)$, it follows that f' is right continuous on $(0,\infty)$ if $f \in \mathcal{D}^{\#}$. Also, if the right limit $r^{\#}(0)$ exists and is positive, then each $f \in \mathcal{D}^{\#}$ must have a (finite) right derivative at zero.

Proof: The characterization of $\mathcal{L}^{\#}$ follows from the fact that $q^{\#}(x,t) \downarrow x$ as $t \downarrow 0$ and $E_x^{\#} f(X^{\#}(t)) = f(q^{\#}(x,t)) \exp(-\lambda t) + o(1)$ as $t \downarrow 0$.

Suppose $f \in \mathcal{D}^{\#}$. Then $f = R_{\alpha}^{\#} h$ for some $\alpha > 0$ and $h \in \mathcal{L}^{\#}$ by Proposition 3. Using the strong Markov property of $X^{\#}$, we have

$$f(x) = E_x^{\#} \int_0^{t_1} e^{-\alpha t} h(q^{\#}(x,t))dt + E_x^{\#} e^{-\alpha t_1} f([q^{\#}(x,t_1) - Y_1]^{+}) \tag{9}$$

$\forall x \in S$. Let $W(y) = \int_0^y f(y-z)F(dz) + f(0)[1-F(y)]$ for $y \geq 0$. (Thus, $W(0) = f(0)$.) Using the definitive relationship (6) for $q^{\#}(\cdot,\cdot)$ and the independence of t_1 and Y_1 , we can manipulate (9) to obtain

$$(10) \qquad f(x) = \int_x^\infty \frac{1}{r(y)}[h(y) + \lambda\, W(y)]e^{-(\lambda+\alpha)(R(y)-R(x))}dy \quad \forall x \in S .$$

Now rewrite (10) as

$$(11) \qquad f(x) = e^{(\lambda+\alpha)R(x)} \int_x^\infty \frac{1}{r(y)}[h(y) + \lambda\, W(y)]e^{-(\lambda+\alpha)R(y)}dy$$

$\forall x \in S$, and observe that both factors are absolutely continuous. Since f is the product of absolutely continuous functions, it is absolutely continuous. From the definition of $R(\cdot)$ it follows that the first factor on the right side of (11) has right derivative $\exp[(\lambda+\alpha)R(x)]/r^{\#}(x)$ at $x > 0$. Using the continuity and boundedness of f , one can easily show that W is right continuous and bounded, and h is right continuous and bounded by the first part of the proposition. Thus, for $x > 0$, the integrand in (11) approaches

$$(12) \qquad [h(x) + \lambda\, W(x)]e^{-(\lambda+\alpha)R(x)}\big/r^{\#}(x)$$

as $y \downarrow x$, implying that the integral factor is right differentiable at x and its right derivative is the negative of (12). Thus, right differentiating (11), f has right derivative

$$(13)\qquad f'(x) = (\lambda+\alpha)f(x)/r^{\#}(x) - [h(x) + \lambda\, W(x)]/r^{\#}(x) \quad \forall x > 0 .$$

(We have re-used (11) to simplify the first term on the right in (13).) Multiplying through by $r^{\#}(x)$, we have

$$(14)\qquad h(x) = \alpha\, f(x) - r^{\#}(x)f'(x) + \lambda[f(x)-W(x)] \quad \forall x > 0 .$$

It follows that $r^{\#}(\cdot)f'(\cdot)$ is bounded and right continuous on $(0,\infty)$, since all other terms in (14) are bounded and right continuous. Furthermore, observing that $W(x) \to W(0) = f(0)$ as $x \downarrow 0$, we let $x \downarrow 0$ in (14) to obtain

$$(15)\qquad \lim_{x\downarrow 0} r^{\#}(x)f'(x) = \alpha\, f(0) - h(0) .$$

Noting that f' is a version of the density of f, we use the definition of W and Fubini's Theorem to obtain

$$(16)\qquad \lambda[f(x)-W(x)] = \lambda \int_0^x [f(x)-f(x-y)]F(dy) + \lambda[1-F(x)][f(x)-f(0)]$$

$$= \lambda \int_0^x \int_{x-y}^x f'(z)dz\, F(dy) + \lambda[1-F(x)][f(x)-f(0)]$$

$$= \lambda \int_0^x \int_{x-z}^x F(dy)f'(z)dz + \lambda[1-F(x)][f(x)-f(0)]$$

$$= \lambda \int_0^x [1-F(x-z)]f'(z)dz = \int_0^x Q(x-y)f'(y)dy \quad \forall x > 0 .$$

Substituting (16) into (14) then yields

$$(17)\qquad h(x) = \alpha\, f(x) - [r^{\#}(x)f'(x) - \int_0^x Q(x-y)f'(y)dy] \quad \forall x > 0 .$$

We know from Proposition 3 that $h(x) + \mathfrak{A}^{\#}f(x) - \alpha\, f(x) = 0 \;\; \forall x \in S$, and combining this with (15) and (17) proves that $\mathfrak{A}^{\#}f$ is given by (8).

We have now shown that every function $f \in \mathfrak{D}^{\#}$ has the properties enumerated in the proposition and that $\mathfrak{A}^{\#}f$ is given by (9). It remains to show that every function $f : S \to R$ with the enumerated properties satisfies $f = R_{\alpha}^{\#}h$ for some $\alpha > 0$ and $h \in \mathcal{L}^{\#}$. This we do by reversing the logic of the previous paragraph. Let f be any such function, pick $\alpha > 0$ arbitrarily, define W in terms of f as above, and define h in terms of f by (14) and (15). Then $h \in \mathcal{L}^{\#}$. Dividing (14) through by $r^{\#}(x)$ for $x > 0$ gives (13), and integrating this over $(0,x)$ gives (11). This is equivalent to (9), from which it follows easily that $R_{\alpha}^{\#}h = f$, completing the proof.

4. Recurrence Classification of the Risk Process.

Let $T^{\#} = \inf\{t > 0 | X^{\#}(t) = 0\}$ and observe that $P_0^{\#}(T^{\#} > 0) = 1$. We shall say that $X^{\#}$ is recurrent if $P_0^{\#}(T^{\#} < \infty) = 1$ and transient otherwise. In the recurrent case, we shall say that $X^{\#}$ is positive recurrent if $E_0^{\#}(T^{\#}) < \infty$ and null recurrent otherwise. For $x \geq 0$, let $T^{\#}(x) = \inf\{t > 0 | X^{\#}(t) = x\}$, so $T^{\#} = T^{\#}(0)$. If $0 \leq x < b$, it is quite easy to show that $P_x^{\#}(T^{\#}(b) < \infty) = 1$, using the fact that $X^{\#}$ cannot get above level b without hitting it.

Proposition 5. If $0 \leq x < b < \infty$ then

$$P_x^{\#}(T^{\#} < T^{\#}(b)) = \int_x^b K^*(y,0)dy \Big/ \left[1 + \int_0^b K^*(y,0)dy\right] .$$

Proof: Consider a storage process $X_0 = \{X_0(t), t \geq 0\}$ on S with release rule $r_0(\cdot)$ satisfying $r_0(0) = 0$ and $r_0(x) = r^{\#}(b-x)$ for $0 < x < b$. The form of $r_0(\cdot)$ on $[b,\infty)$ will be immaterial for our purpose. Let the input process for X_0 be A and let the initial state be $X_0(0) = b-x$ so that $X_0(t) = b-X^{\#}(t)$ for $0 \leq t < T^{\#} \wedge T^{\#}(b)$ and $P_x^{\#}(T^{\#} < T^{\#}(b))$ is the probability that X_0 hits or exceeds level b before hitting level zero. The current proposition is then a direct application of Theorem 3 of Harrison and Resnick [4].

Theorem 4. If $\int_0^\infty K^*(y,0)dy = \infty$, then $P_x^{\#}(T^{\#} < \infty) = 1 \ \forall x \geq 0$. Otherwise

$$P_x^{\#}(T^{\#} < \infty) = \int_x^\infty K^*(y,0)dy \Big/ \left[1 + \int_0^\infty K^*(y,0)dy\right] < 1 \quad \forall x \geq 0 .$$

Thus $X^{\#}$ is transient iff $\int_0^\infty K^*(y,0)dy < \infty$.

Proof: With $X^{\#}(0) = x < b$, it is immediate from our construction that $T^{\#}(b) \geq R(b) - R(x) \to \infty$ as $b \to \infty$. Thus, $P_x^{\#}(T^{\#} < T^{\#}(b)) \uparrow P_x^{\#}(T^{\#} < \infty)$ as $b \uparrow \infty$. The theorem is then immediate from Proposition 5.

We say that a probability measure $\gamma^{\#}$ on $(S,\mathcal{S})$ is a stationary distribution for $X^{\#}$ if

$$\int_S P_x^{\#}(X^{\#}(t) \in B)\gamma^{\#}(dx) = \gamma^{\#}(B) \quad \forall t > 0,\ B \in \mathcal{S} .$$

Proposition 6. There exists a stationary distribution for $X^{\#}$ iff there exists a positive integrable solution u of (2), in which case the unique stationary distribution is absolutely continuous with density proportional to u .

Proof: It is known that $\gamma^{\#}$ is a stationary distribution for $X^{\#}$ iff

$$\int_S \mathfrak{U}^{\#} f(x)\gamma^{\#}(dx) = 0 \qquad \forall f \in \mathcal{D}^{\#} , \tag{18}$$

cf. Breiman [2], p. 346, or Azema, Duflo and Revuz [1], Lemma 1.

Assume $f \in \mathcal{D}^{\#}$ with $r^{\#}(x)f'(x) \to a$ as $x \downarrow 0$. If $\gamma^{\#}$ is a probability measure, then (9) and Fubini's Theorem give us

$$(19)\quad \int_S \mathcal{G}^{\#} f(x)\gamma^{\#}(dx) = a\,\gamma^{\#}\{0\} + \int_0^\infty r^{\#}(x)f'(x)\gamma^{\#}(dx)$$

$$- \int_0^\infty \int_0^x Q(x-y)f'(y)dy\gamma^{\#}(dx)$$

$$= a\,\gamma^{\#}\{0\} + \int_0^\infty r^{\#}(x)f'(x)\gamma^{\#}(dx) - \int_0^\infty \int_x^\infty Q(y-x)\gamma^{\#}(dy)f'(x)dx\ .$$

Suppose that u is a positive integrable solution of (2) and that $\gamma^{\#}$ is an absolutely continuous probability measure with density proportional to u. Then $\gamma^{\#}\{0\} = 0$, and it is immediate from (2) and (19) that (18) holds. Thus $\gamma^{\#}$ is a stationary distribution.

Now suppose conversely that $\gamma^{\#}$ is a stationary distribution, so (18) holds. Assume $b > 0$ and let $I = (0,b)$. Let $f : S \to R$ be absolutely continuous with density $f'(x) = 1/r^{\#}(x)$ on I and $f'(x) = 0$ otherwise. Then $r^{\#}(x)f'(x) \to 1$ as $x \downarrow 0$, and it follows from Proposition 4 that $f \in \mathcal{D}^{\#}$. From (18) and (19) we have

$$0 = \gamma^{\#}\{0\} + \int_0^b \gamma^{\#}(dx) - \int_0^b \left[\frac{1}{r^{\#}(x)} \int_x^\infty Q(y-x)\gamma^{\#}(dy)\right]dx$$

Letting $b \downarrow 0$ shows that $\gamma\{0\} = 0$, and since b is arbitrary we have

$$\gamma^{\#}(dx) = \left[\frac{1}{r^{\#}(x)} \int_x^\infty Q(y-x)\gamma^{\#}(dy)\right]dx\ .$$

Thus, $\gamma^{\#}$ has a density $u^{\#}$ on $(0,\infty)$ satisfying

$$r^{\#}(x)u^{\#}(x) = \int_x^{\infty} Q(y-x)u^{\#}(y)dy , \qquad x > 0 .$$

Let $u(x) = \int_x^{\infty} Q(y-x)u^{\#}(y)dy/r(x)$ for $x > 0$. Then $u(x) = u^{\#}(x)$ almost everywhere, since r differs from $r^{\#}$ at only countably many points, and it follows both that u is a positive integrable solution of (2) and that u is a version of the density of $\gamma^{\#}$. The uniqueness of the stationary distribution follows from the last statement of Proposition 1.

Theorem 5. $X^{\#}$ is positive recurrent iff there exists a positive integrable solution of (2), in which case

$$P_x^{\#}(X^{\#}(t) \in B) \to \gamma^{\#}(B) \quad \text{as} \quad t \to \infty \quad \forall x \in S, B \in \mathcal{S} ,$$

where $\gamma^{\#}$ is the unique stationary distribution of $X^{\#}$.

Proof: The argument is virtually identical to the proofs of Proposition 8 and Theorem 2 of [4], so we only sketch it. The times at which $X^{\#}$ hits zero constitute a sequence of regeneration points. The duration of a regenerative cycle is distributed as T when $X^{\#}(0) = 0$. It is easy to show that this distribution is non-arithmetic. It has finite expectation iff $X^{\#}$ is positive recurrent. If $X^{\#}$ is positive recurrent, then Smith's Theorem for regenerative processes shows that $P_x^{\#}(X^{\#}(t) \in B) \to \pi(B)$ as $t \to \infty$, and it is easily established that the limit distribution π is a stationary distribution for $X^{\#}$. If $X^{\#}$ is not positive recurrent, then a standard renewal theoretic argument shows that $P_x^{\#}(X^{\#}(t) \varepsilon B) \rightarrow 0$ as $t \rightarrow \infty$ $\forall x \varepsilon S$ and bounded $B \varepsilon \mathcal{S}$, and it follows directly that $X^{\#}$ does not have a stationary distribution. Combining these facts with Proposition 6 proves the theorem.

<u>Theorem 6</u>. $X^{\#}$ is null recurrent iff $\int_0^\infty K^*(y,0)dy = \infty$ and there exists no positive integrable solution of (2).

<u>Proof</u>: Immediate from Theorems 4 and 5.

Thus, we see that X is positive recurrent iff $X^{\#}$ is transient, X is null recurrent iff $X^{\#}$ is null recurrent and X is transient iff $X^{\#}$ is positive recurrent.

5. The Case of Exponential Jumps.

Suppose that F is an exponential distribution with mean $1/\mu$, so $Q(x) = \lambda \exp(-\mu x)$ for $x \geq 0$. In Harrison and Resnick [4] it is shown that

$$K^*(x,0) = \mu\, \rho(x) \exp\left[-\mu \int_0^x (1-\rho(y))dy\right], \qquad x \geq 0, \tag{20}$$

where $\rho(x) = \lambda/\mu r(x)$ for $x > 0$. Also, our integral equation (2) becomes in this case

$$r(x)u(x) = \int_x^\infty \lambda\, e^{-\mu(y-x)}u(y)dy, \qquad x > 0. \tag{21}$$

Setting $\phi(x) = \int_x^\infty e^{-\mu y}u(y)dy$, we can rewrite (21) as $\lambda\, \phi(x) = -r(x)\phi'(x)$ for $x > 0$. Solving this elementary differential equation, we find that the only non-trivial, non-negative solutions of (2) are of the form

$$u(x) = c\, \rho(x) \exp\left[\mu \int_0^x (1-\rho(y))dy\right], \qquad x > 0, \tag{22}$$

where $c < 0$. Combining this with Theorem 4 gives an explicit criterion for recurrence of the risk process and a closed-form solution for the probability of ruin in the transient case.

In the context of storage processes, an interesting case is that where $\rho(x) \downarrow 1$ as $x \uparrow \infty$. From Theorem 1 and (20) we see that X

cannot then be positive recurrent. Cinlar and Pinsky [3] have conjectured that X is null recurrent, but from (22) and Theorems 2 and 3 it is clear that this need not be the case. The storage process may be either transient or null recurrent (u may be either integrable or not), depending on the speed at which $\rho(\cdot)$ approaches unity.

6. Concluding Remarks.

Consider the storage process X, and suppose that the release rule r fails to satisfy (1). State zero is then inaccessible, and the recurrence classification used in Section 3 (focusing on the distribution of the return time to state zero) is inappropriate. Denoting by $T(y)$ the first hitting time of state $y > 0$, it is quite easy to show that either

(a) $P_x(T(y) < \infty) < 1 \ \forall\ 0 < y < x < \infty$, or

(b) $P_x(T(y) < \infty) = 1$ and $E_x(T(y)) = \infty \ \forall\ 0 < y < x < \infty$, or

(c) $P_x(T(y) < \infty) = 1$ and $E_x(T(y)) < \infty \ \forall\ 0 < y < x < \infty$.

Furthermore, in the case where zero is accessible, conditions (a), (b), and (c) correspond to transience, null recurrence and positive recurrence as we have defined them in Section 2. When assumption (1) fails, we may simply define the recurrence classifications by (a)-(c).

Given that (a)-(c) are mutually exclusive and exhaustive, it is clear that the recurrence classification of X depends only on $\{r(x) : x \geq a\}$ where $a > 0$ is arbitrary. More particularly, choosing $a > 0$ arbitrarily, X must have the same recurrence classification as another storage process X_a having input A and release rule $r_a(x) = r(x+a)$ for $x > 0$. Since $r_a^{\#}(0) = r^{\#}(a) > 0$, the release rule r_a satisfies assumption (1), and thus Theorems 1-3 can be used for the recurrence classification of X_a . Theorem 1 shows that X_a is positive recurrent iff $K_a^*(x,0)$ is integrable, where $K_a^*(\cdot,\cdot)$ is defined in terms of r_a and Q in the obvious way, and it is easy to show that $K_a^*(x,0) = K^*(a+x,a)$ for all $x > 0$. Thus, our necessary and sufficient condition for positive recurrence is

$$\int_a^\infty K^*(x,a)dx < \infty \quad .$$

Theorem 2 shows that X_a is transient iff there exists a positive integrable solution u_a of equation (2) with r_a in place of r . But it is easy to show that any such u_a has the form $u_a(x) = u(a+x)$, $x > 0$, where u is a positive integrable solution of (2) over the restricted range $x > a$. Thus, the necessary and sufficient condition for transience is the existence of a non-negative solution u of (2) such that

$$0 < \int_a^\infty u(x)dx < \infty \ .$$

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20. ABSTRACT. J. Michael Harrison and Sidney Resnick

Consider a storage process $X = \{X(t), t \geq 0\}$ with compound Poisson input process $A = \{A(t), t \geq 0\}$ and release rule $r(\cdot)$ which is arbitrary except for the requirement that state zero be accessible. In a previous paper, we derived necessary and sufficient conditions for the Markov process X to be positive recurrent. Here we complete the recurrence classification of X, providing necessary and sufficient conditions for null recurrence and transience.

Closely related to X is a Markov process $X^{\#} = \{X^{\#}(t), t \geq 0\}$ which increases at rate $r(X^{\#}(t))$ between negative jumps. The jumps are generated by A except that they are truncated as necessary to keep the process non-negative. It is shown that the risk process $X^{\#}$ is transient iff X is positive recurrent, null recurrent iff X is null recurrent, and positive recurrent iff X is transient. In the transient case, we calculate the probability that $X^{\#}$ ever hits level zero (the probability of ruin) as a function of the initial state $X^{\#}(0)$.

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