AFOSR INTERIM REPORT

# MEAN-SQUARE RESPONSE OF A NONLINEAR SYSTEM TO NONSTATIONARY RANDOM EXCITATION



Hidekichi Kanematsu

William A. Nash

August, 1976

University of Massachusetts

Amherst, Massachusetts 01002



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cont.

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## ABSTRACT

The transient mean-square response of a nonlinear single-degree-of freedom mechanical system to nonstationary random excitation characterized by the product of an envelope function and a stationary Gaussian random process is determined by the equivalent linearization technique. A unit step envelope function is considered in conjunction with both correlated and white noise with zero mean.

It has been shown that for white noise modulated by a unit step function, the transient mean-square response never exceeds the stationary response. However, the mean-square response to correlated noise modulated by a unit step function may exceed its stationary value.

The analysis is extended to the multi-degree-of-freedom nonlinear system for the case of mutually uncorrelated noise.

# CHAPTER I

## INTRODUCTION

The transient mean-square response of a linear single-degree-of freedom mechanical system to certain types of nonstationary random excitation has been studied by several authors [1, 2, 3, 4]. The non-stationary input was taken in the form of a product of a well-defined envelope function, $A(t)$ and a stationary Gaussian noise with zero mean, $n(t)$.

T. K. Caughey and H. J. Stumpf [1] have examined the case in which the envelope function $A(t)$ was a unit step function and $n(t)$ was assumed to be either white noise or broad-band noise whose power spectral density has no sharp peaks. Results of their analysis were applied to the determination of the structural response to earthquake ground motion. V. V. Bolotin [2] has determined the mean-square response of a linear structure represented by a second order differential equation when that structure is subject to earthquake excitation. In his analysis, he considered the ground acceleration to be characterized by the product of an exponentially decaying harmonic correlation function and an envelope function, $A(t) = Ae^{-ct}$.

In a recent paper [3], R. L. Barnoski and J. R. Maurer have formulated the time varying mean-square response of a linear single-degree-of-freedom system in terms of the system frequency response function and the generalized spectral density function of the input excitation. They considered the envelope function to be either the unit step function or a rectangular step function. L. L. Bucciarelli and C. Kuo [4] have recently obtained an approximate expression for the mean-square response to excitation characterized by a general envelope function subject only to the restriction that the envelope function is slowly varying. Their work also gave an estimated maximum value of the mean-square response. In all the above studies, the systems treated were linear.

To date, the problem of response of a nonlinear system to nonstationary random excitation has been mentioned in only one place. There, R. H. Toland, C. Y. Yang, and C. Hsu [5] employed a random walk model to determine system response to stationary Gaussian white noise. The extension to the nonstationary case was discussed, but not carried through to completion. There are many systems whose motions are characterized by nonlinear differential equations, particularly when the motions are large. It is the purpose of this study to present an approximate solution to the transient mean-square response of a simple nonlinear system to a nonstationary random excitation. Only systems with geometric nonlinearities (rather than materials nonlinearities) involved are considered and the nonlinear differential equation is linearized by an equivalent linearization technique. All computation for obtaining the mean-square response of the system is straightforward. However, it is not generally possible to obtain a closed form solution of the mean-square response and the equation has to be solved numerically. First a single-degree-of-freedom system is treated in Chapter II and then a multidegree-of-freedom system is discussed for various special cases.

Results of the present investigation could, as an approximation, be applied to response determination of thin elastic plates and shells (regarded as single degree of freedom systems for any given mode) when these systems are subject to pressure fields that excite large amplitude oscillations. Jet engine sound pressures would be one example of such a pressure field.

# CHAPTER II

## A SINGLE-DEGREE-OF-FREEDOM NONLINEAR SYSTEM

### 2.1 Statement of the Problem

Consider a lightly damped single-degree-of-freedom mechanical system subjected to a random excitation and governed by the equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2(y(t) + g(y)) = f(t) \tag{2-1}$$

where

$\zeta$ = fraction of critical damping

$\omega_n$ = natural frequency of the corresponding linear system

$$g(y) = \sum_{k=1}^{N} \mu k y^{2k+1} \qquad \mu k \geq 0 \tag{2-2}$$

The nonstationary random excitation $f(t)$ is expressed by

$$f(t) = A(t)n(t) \tag{2-3}$$

where $A(t)$ is a well-defined envelope function and $n(t)$ is a Gaussian stationary random process with zero mean and autocorrelation function $R_n(\tau)$.

We are to determine the mean-square response $E[y^2(t)]$ to an input $f(t)$ when the envelope functions are a unit step function and exponential function, respectively, i.e.,

$$A(t) = u(t) \tag{2-4}$$

$$A(t) = \sum_{i=1}^{M} A_i \exp(-c_i t)u(t) \tag{2-5}$$

and n(t) has the following autocorrelation functions

$$R_n(\tau) = 2\pi k_o \delta(\tau) \tag{2-6}$$

$$R_n(\tau) = K_o \exp(-\alpha|\tau|)\cos\beta\tau \tag{2-7}$$

where u(t) is a unit step function and $\delta(\tau)$ is the Dirac delta function. Note that if we let all $A_i$ be zero except for $A_1 = 1$ and $c_i = 0$, Eq. (2-5) reduces to Eq. (2-4).

## 2.2 Response Formulation

Although various methods can be applied to determine the response of nonlinear systems, the equivalent linearization technique will be used here. This technique was developed by Krylov and Bogoliuvov for the treatment of nonlinear systems under deterministic excitations, and then R. C. Booton [6] and T. K. Caughey [7] applied this technique to problems of random vibrations.

We assume that an approximate solution to Eq. (2-1) can be obtained from the linearized equation

$$\ddot{y} + 2\beta_e\dot{y} + \omega_e^2 y = f(t) \tag{2-8}$$

where $\beta_e$ is the equivalent linear damping coefficient and $\omega_e^2$ is the equivalent linear stiffness. The error "e" due to linearization is given by the difference between Eqs. (2-1) and (2-8), i.e.,

$$e = 2(\zeta\omega_n - \beta_e)\dot{y} + (\omega_n^2 - \omega_e^2)y + g(y)\omega_n^2 \tag{2-9}$$

The variables $\beta_e$ and $\omega_e^2$ are selected so as to make the mean-square error $E[e^2]$ a minimum. The minimization of $E[e^2]$ require that:

$$\frac{\partial E[e^2]}{\partial \beta_e} = 0 \tag{2-10}$$

$$\frac{\partial E[e^2]}{\partial (\omega_e^2)} = 0$$

Substitution of Eq. (2-9) into Eqs. (2-10) and (2-11) gives

$$\frac{\partial E[e^2]}{\partial \beta_e} = -8(\zeta\omega_n - \beta_e)E[\dot{y}^2] - 4(\omega_n^2 - \omega_e^2)E[\dot{y}y]$$

$$- 4E[\dot{y}g(y)]\omega_n^2 = 0 \tag{2-12}$$

$$\frac{\partial E[e^2]}{\partial (\omega_e^2)} = -4(\zeta\omega_n - \beta_e)E[\dot{y}y] - 2(\omega_n^2 - \omega_e^2)E[y^2]$$

$$- 2E[yg(y)]\omega_n^2 = 0 \tag{2-13}$$

Solving for $\beta_e$ and $\omega_e^2$ from Eqs. (2-12) and (2-13), we have

$$2\beta_e = 2\zeta\omega_n + \omega_n^2 \frac{E[y^2]E[\dot{y}g(y)] - E[y\dot{y}]E[yg(y)]}{E[y^2]E[\dot{y}^2] - (E[y\dot{y}])^2} \tag{2-14}$$

$$\omega_e^2 = \omega_n^2 + \omega_n^2 \frac{E[\dot{y}^2]E[yg(y)] - E[y\dot{y}]E[\dot{y}g(y)]}{E[y^2]E[\dot{y}^2] - (E[y\dot{y}])^2} \tag{2-15}$$

From Eqs. (2-12) and (2-13),

$$\frac{\partial^2 E[e^2]}{\partial \beta_e^2} = 8E[\dot{y}^2] > 0 \tag{2-16}$$

$$\frac{\partial^2 E[e^2]}{\partial (\omega_e^2)^2} = 2E[y^2] > 0 \tag{2-17}$$

$$\frac{\partial^2 E[e^2]}{\partial\beta_e \partial(\omega_e^2)} = 4E[y\dot{y}] \tag{2-18}$$

and

$$\frac{\partial^2 E[e^2]}{\partial\beta_e^2}\ \frac{\partial^2 E[e^2]}{\partial(\omega_e^2)^2} - \left(\frac{\partial^2 E[e^2]}{\partial\beta_e \partial(\omega_e^2)}\right)^2 = 16\{E[y^2]E[\dot{y}^2] - (E[y\dot{y}])^2\}$$

$$= 16\det(K) \tag{2-19}$$

where det(K) is the determinant of the matrix of covariances. Since the upper bound for the nonstationary cross-covariance $E[y\dot{y}]$ is given by the inequality [8],

$$(E[y\dot{y}])^2 \leq E[y^2]E[\dot{y}^2] \tag{2-20}$$

then,

$$\frac{\partial^2 E[e^2]}{\partial\beta_e^2}\ \frac{\partial^2 E[e^2]}{\partial(\omega_e^2)^2} > \left(\frac{\partial^2 E[e^2]}{\partial\beta_e\ (\omega_e^2)}\right)^2$$

From Eqs. (2-16), (2-17), and (2-21), it can be seen that the conditions (2-14) and (2-15) truly give a minimum $E[e^2]$.

In order to express the right hand side of Eqs. (2-14) and (2-15) in terms of $E[y^2]$, $E[\dot{y}^2]$ and $E[y\dot{y}]$, it is necessary to know the probability density function p(y,y). In general, however, $p(y,\dot{y})$ is not known. If the input is Gaussian and the nonlinearities of the system are small, then the response of the linearized equation (2-8) is also Gaussian. Therefore, the assumption is made that the probability density function $p(y,\dot{y})$ is Gaussian with covariances to be determined.

Before constructing the probability density function, the ensemble average of $y$ and $\dot{y}$ is calculated by use of Duhamel's integral. Assuming that the system is at rest initially, we have the solution of Eq. (2-8) to be

$$y(t) = \int_0^t h(t-\tau)f(\tau)d\tau \tag{2-22}$$

where $h(t)$ is the impulse response of the system defined by

$$h(t) = \frac{e^{-\beta_e t}}{\omega_d} (\sin\omega_d t)u(t) \tag{2-23}$$

and

$$\omega_d^2 = \omega_e^2 - \beta_e^2 \tag{2-24}$$

The ensemble average of $y$ is obtained by taking the ensemble average of Eq. (2-22).

$$E[y] = \int_0^t h(t-\tau)E[f(\tau)]d\tau \tag{2-25}$$

Since we assumed that $E[f(t)] = 0$, then

$$E[y] = 0 \tag{2-26}$$

Similarly, the ensemble average of $\dot{y}$ is obtained by

$$E[\dot{y}] = \int_0^t \frac{\partial}{\partial t} h(t-\tau)E[f(\tau)]d\tau \quad = 0 \tag{2-27}$$

Because of Eqs. (2-26) and (2-27), the assumed Gaussian probability density function $p(y,\dot{y})$ takes the form:

$$p(y,\dot{y}) = \frac{1}{2\pi(\det(K))^{\frac{1}{2}}} \exp(-ay^2 + 2by\dot{y} - c\dot{y}^2) \tag{2-28}$$

where

$$\begin{aligned} a &= E[\dot{y}^2]/(2\det(K)) \\ b &= E[y\dot{y}]/(2\det(K)) \\ c &= E[y^2]/(2\det(K)) \\ \det(K) &= E[y^2]E[\dot{y}^2] - (E[y\dot{y}])^2 \end{aligned} \tag{2-29}$$

Expression for $E[yg(y)]$ and $E[\dot{y}g(y)]$ are obtained in the Appendix and are shown to be [9],

$$E[yg(y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yg(y)p(y,\dot{y})dyd\dot{y}$$

$$= \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!}(E[y^2])^{k+1} \tag{2-30}$$

$$E[yg(y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{y}g(y)p(y,\dot{y})dyd\dot{y}$$

$$= \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!}(E[y^2])^{k} E[y\dot{y}] \tag{2-31}$$

Substituting Eqs. (2-30) and (2-31) into Eqs. (2-14) and (2-15), we have

$$2\beta_e = 2\zeta\omega_n \tag{2-32}$$

$$\omega_e^2 = \omega_n^2 \left\{1 + \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!} (E[y^2])^k \right\} \tag{2-33}$$

It is interesting to observe that the above equivalent linear damping $2\beta_e$ and stiffness $\omega_e^2$ are identical to those found for a stationary process in which case $E[y\dot{y}]$ is equal to zero and Eqs. (2-14) and (2-15) are simplified as

$$2\beta_e = 2\zeta\omega_n + \omega_n^2 \frac{E[\dot{y}g(y)]}{E[\dot{y}^2]}$$

$$\omega_e^2 = \omega_n^2 + \omega_n^2 \frac{E[yg(y)]}{E[y^2]} \tag{2-34}$$

If instead the nonlinearity is involved only in the velocity term such as $g(y)$, we can easily show that the equivalent linear damping and stiffness for a nonstationary process are the same as those for a stationary process.

The mean-square response $E[y^2]$ at any instant of time, t, is obtained from the computation of the expected value of $(y(t))^2$ over the ensemble of the response. From Eq. (2-22),

$$E[y^2] = \int_0^t \int_0^t h(t-\tau')h(t-\tau)E[f(\tau')f(\tau)]d\tau d\tau' \tag{2-35}$$

Since $A(t)$ is well-defined function and $n(t)$ is stationary,

$$\begin{aligned} E[f(\tau)f(\tau')] &= A(\tau)A(\tau')E[n(\tau)n(\tau')] \\ &= A(\tau)A(\tau')R_n(\tau-\tau') \end{aligned} \tag{2-36}$$

where $R_n(\tau-\tau')$ is the autocorrelation function of $n(t)$. Substitution of Eqs. (2-23), (2-32) and (2-36) into Eq. (2-35) leads to

$$E[y^2] = \frac{1}{\omega_d^2} \int_0^t \int_0^t h(t-\tau')h(t-\tau)E[f(\tau)f(\tau')]d\tau d\tau'$$

$$\times \sin\omega_d(t-\tau)A(\tau)A(\tau')R_n(\tau-\tau')d\tau d\tau' \qquad (2\text{-}37)$$

From Eqs. (2-24), (2-32), and (2-33), $\omega_d^2$ becomes

$$\omega_d^2 = \omega_d^2 \left\{ 1 + \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!} (E[y^2])^k - \zeta^2 \right\} \qquad (2\text{-}38)$$

With $R_n(\tau)$ defined by Eqs. (2-6) or (2-7), and $A(t)$ defined by Eqs. (2-4) or (2-5), Eqs. (2-37) and (2-38) become simultaneous nonlinear algebraic equations for $E[y^2]$.

Alternatively, Eq. (2-37) can be expressed in terms of the power spectral density $\phi(\omega)$ of $n(t)$. Since the autocorrelation function $R_n(\tau-\tau')$ is given by

$$R_n(\tau-\tau') = \int_0^{\infty} \phi(\omega)\cos\omega(\tau-\tau')d\omega \qquad (2\text{-}39)$$

Then, substituting Eq. (2-39) into Eq. (2-37) and changing the order of integration, we have

$$E[y^2] = \int_0^\infty \frac{\phi(\omega)}{\omega_d^2} \int_0^t \int_0^t \exp\{-\zeta\omega_n(2t-\tau-\tau')\}\sin\omega_d(t-\tau)$$

$$\times \sin\omega_d(t-\tau')A(\tau)A(\tau')\cos\omega(\tau-\tau')d\tau d\tau' d\omega \qquad (2\text{-}40)$$

## 2.3 Response to Shaped White Noise

If the input is assumed to be white noise, then substitution of Eq. (2-6) into Eq. (2-37) leads to

$$E[y^2] = \frac{\pi K_o}{\omega_d^2} \int_0^t \{\exp -2\zeta\omega_n(t-\tau)\}A^2(\tau)\sin^2\omega_d(t-\tau)d\tau \qquad (2\text{-}41)$$

### 2.3.1 Unit Step Envelope Function

Let us first consider the case in which the envelope function is defined by Eq. (2-4). By performing the integration of Eq. (2-41), we obtain

$$E[y^2] = \frac{\pi K_o}{\omega_d^2} \int_0^t \exp\{-2\zeta\omega_n(t-\tau)\}\sin^2\omega_d(t-\tau)d\tau$$

$$= \frac{\pi K_o}{4\zeta\omega_n(\zeta^2\omega_n^2 + \omega_d^2)} [1-e^{-2\zeta\omega_n t}(1 + \frac{2\zeta^2\omega_n^2}{\omega_d^2} \sin^2\omega_d t$$

$$+ \frac{\zeta\omega n}{\omega_d} \sin 2\omega_d t)] \qquad (2\text{-}42)$$

Employing Eq. (2-38), Eq. (2-42) becomes a nonlinear algebraic equation for $E[y^2]$ since $\omega_d^2$ is a function of $E[y^2]$ in Eq. (2-38). This type of equation generally has more than one solution. However, from physical considerations the desired solution will be that one close to the solution of the corresponding linear system because only a weakly nonlinear system is being considered. Since the general procedure for solving the nonlinear algebraic equation is not available, we shall use Newton's method of tangents to obtain an approximate solution at instantaneous values of time, t, and then iterate. The solution by Newton's method sometimes does not converge if a poor initial value is chosen. However, since the mean-square response of the linear system $E[y_o^2]$ is assumed to be close to that of the nonlinear system, $E[y_o^2]$ is suitable for use as the initial trial solution for an iteration scheme. Throughout the present study, this iteration scheme together with Newton's method is used for obtaining an approximate solution.

As a numerical example, let us consider the simplest case, i.e.,

$$g(y) = \mu y^3 \tag{2-43}$$

For various values of $\mu$ and damping coefficient $\zeta$, $E[y^2]$ is computed and the normalized plots are shown in Figure 1 through 3. The normalization factor is determined by the stationary mean-square response of linear system

$$E[y_o^2]_s = \pi K_o / 4\zeta\omega_n^3 \tag{2-43}$$

The parameter $\mu$ is chosen in such a manner that given $\mu$ the stationary mean-square response reaches 40 percent, 60 percent and 80 percent of

$E[y_o^2]_s$. If the damping is small, Eq. (2-42) can be approximated by

$$E[y^2] \approx E[y_o^2]_s \frac{1 - e^{-2\zeta\omega_n t}}{1 + 3\mu E[y^2]} \tag{2-45}$$

from which the following approximate solution is obtained.

$$E[y^2] \approx \frac{1}{6\mu} \{[1 + 12\mu E[y_o^2]_s (1 - e^{-2\zeta\omega_n t})]^{\frac{1}{2}} - 1\} \tag{2-46}$$

In what follows, let us show that the transient mean-square response for both linear and nonlinear systems does not exceed the stationary mean-square response to white noise. That is,

$$\begin{aligned} E[y_o^2]_s &\geq E[y_o^2] \\ E[y^2]_s &\geq E[y^2] \end{aligned} \tag{2-47}$$

From Eq. (2-42), we have

$$E[y^2] = \frac{\pi K_o}{4\zeta\omega_n\omega_e^2} \left\{ 1 - e^{-2\zeta\omega_n t}[1 + (\frac{\zeta\omega_n}{\omega_d})^2 + \quad (\frac{\zeta\omega_n}{\omega_d})^2 + (\frac{\zeta\omega_n}{\omega_d})^4 \right.$$
$$\left. \times \sin(2\omega_d t - \theta)] \right\} \tag{2-48}$$

where

$$\theta = \tan^{-1}(\frac{\zeta\omega_n}{\omega_d})$$

Since

$$1 + (\frac{\zeta\omega_n}{\omega_d})^2 > [(\frac{\zeta\omega_n}{\omega_d})^2 + (\frac{\zeta\omega_n}{\omega_d})^4]^{\frac{1}{2}}$$

and

$$-1 \le \sin(2\omega_d t-\theta) \le 1$$

then

$$1-e^{-2\zeta\omega_n t}[1+(\frac{\zeta\omega_n}{\omega_d})^2 + \sqrt{(\frac{\zeta\omega_n}{\omega_d})^2 + (\frac{\zeta\omega_n}{\omega_d})^4}\ \sin(2\omega_d t-\theta)] \le 1 \qquad (2\text{-}49)$$

The equality holds for $t \to \infty$.

Therefore,

$$E[y^2] \le \pi K_o/4\zeta\omega_n^3(1+3\mu E[y^2]) \qquad (2\text{-}50)$$

Solving for $E[y^2]$, it is concluded that

$$E[y^2] \le \frac{1}{6\mu}\left\{[1 + 12(\frac{\pi K_o}{4\zeta\omega_n^3})]^{\frac{1}{2}} -1\right\} = E[y^2]_s \quad \mu\neq 0$$

For linear systems, substitution of $\mu = 0$ into Eq. (2-50) leads to the first equation of (2-47).

### 2.3.2 Exponential Envelope Function

If a white noise is modulated by the exponential envelope function described by Eq. (2-5), then Eq. (2-41) becomes

$$E[y^2] = \frac{\pi K_o}{\omega_d^2}\int_0^t \exp[-2\zeta\omega_n(t-\tau)]\sin^2\omega_d(t-\tau)$$

$$\times \sum_{i=1}^{M}\sum_{j=1}^{M} A_i A_j \exp[-(c_i+c_j)\tau]d\tau \qquad (2\text{-}51)$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{M}\frac{K_o A_i A_j e^{-2\zeta\omega_n t}}{4\lambda_{ij}(\lambda^2_{ij}+\omega_d^2)}[e^{2\lambda_{ij}t}-(1+\frac{2\lambda^2_{ij}}{\omega_d^2})\sin^2\omega_d t$$

$$+ \frac{\lambda_{ij}}{\omega_d} \sin 2\,{}_d t)] \tag{2-51 cont'd}$$

where

$$\lambda_{ij} = \zeta\omega_n - \tfrac{1}{2}(c_i + c_j) \tag{2-52}$$

Consider the special case in which the envelope function and nonlinear term $g(y)$ are the following:

$$\begin{aligned} A(t) &= Ae^{-ct} \\ g(y) &= \mu y^3 \end{aligned} \tag{2-53}$$

Then, Eq. (2-51) is simplified to the form

$$E[y^2] = \frac{\pi K_o A^2}{4\zeta\omega_n^3} \frac{e^{-2\zeta\tau}}{\delta^2 - 2\zeta\delta + x^2} \left\{ e^{2(\zeta-\delta)\tau} - [1 + \frac{2(\zeta-\delta)^2}{x^2-\delta^2} \sin^2 (x^2-\zeta^2)^{\frac{1}{2}} \right.$$

$$\left. + \frac{\zeta-\delta}{(x^2-\delta^2)^{\frac{1}{2}}} \sin 2(x^2-\zeta^2)^{\frac{1}{2}}\tau] \right\} \tag{2-54}$$

where

$$\begin{aligned} x^2 &= 1 + 3\mu E[y^2] \\ \tau &= \omega_n t \\ \delta &= c/\omega_n \end{aligned} \tag{2-55}$$

Equation (2-54) was solved by a numerical iteration method described in Chapter 2.3.1, and normalized plots are shown in Figure 4 and 5. The normalization factor is chosen in this case to be

$$\sigma_o^2 = \pi K_o A^2 / 4\zeta\omega_n^3 \tag{2-56}$$

which is the stationary mean-square response for the case of $c = 0$ and $\mu = 0$. Two cases $\delta = 2\zeta$ and $\delta = \frac{1}{2}\zeta$ are illustrated in those figures and the results show that for large $\delta$, i.e., rapidly decreasing amplitude of the input, the effect of the parameter $\mu$ is insignificant.

If $\zeta$, $\delta \ll 1$, Eq. (2-54) can be expressed approximately by

$$E[y^2] \approx \sigma_0^2 \frac{(e^{-2\delta\tau}-e^{-2\zeta\tau})}{(\zeta-\delta)(1+3\mu E[y^2])}$$

Solving for $E[y^2]$,

$$E[y^2] \approx \frac{1}{6\mu} \{[1+\frac{12\mu\sigma_0^2}{\zeta-\delta} (e^{-2\delta\tau}-e^{-2\zeta\tau})]^{\frac{1}{2}} -1\} \tag{2-57}$$

is obtained.

## 2.4 Response to Shaped Correlated Noise

If the input noise is assumed to be correlated as the damped harmonic form of Eq. (2-7), then Eq. (2-37) becomes

$$E[y^2] = \frac{K_0 e^{-2\zeta\omega_n t}}{\omega_d^2} \int_0^t \int_0^t \exp[\zeta\omega_n(\tau+\tau') - \alpha|\tau-\tau'|]A(\tau)A(\tau')$$

$$\sin\omega_d(t-\tau)\sin\omega_d(t-\tau')\cos\beta(\tau-\tau')d\tau d\tau' \tag{2-58}$$

### 2.4.1 Unit Step Envelope Function

Substituting Eq. (2-4) into Eq. (2-58), we find that Eq. (2-58) takes the form

$$E[y^2] = \frac{K_0 e^{-2\zeta\omega_n t}}{\omega_d^2} \left[ \int_0^t \exp[(\zeta\omega_n-\alpha)\tau]\sin\omega_d(t-\tau) \left\{ \int_0^{\tau'} \exp[(\zeta\omega_n+\alpha)\tau] \right.\right.$$

$$\left. \times \sin\omega_d(t-\tau)\cos\beta(\tau-\tau')d\tau \right\} d\tau' + \int_0^t \exp[(\zeta\omega_n+\alpha)\tau']\sin\omega_d(t-\tau')$$

$$\times \left\{ \int_{\tau'}^{t} \exp[(\zeta\omega_n-\alpha)\tau]\sin\omega_d(t-\tau)\cos\beta(\tau-\tau')d\tau \right\} d\tau' \Bigg] \tag{2-59}$$

The double integral in Eq. (2-59) may be evaluated after some tedious algebra to give,

$$E[y^2] = \frac{K_o}{\omega_d^2} e^{-2\zeta\omega_n t} \sum_{i=1}^{8} Q_i(t)T_i(t) \tag{2-60}$$

where

$$Q_1 = e^{p_1 t}(R_1-R_2)p_1/2R_1R_2$$

$$Q_2 = e^{p_1 t}(\Omega_1R_2+\Omega_2R_1)/2R_1R_2 \tag{2-61}$$

$$Q_3 = \frac{p_1(R_1+R_2)}{2R_1R_2} - \frac{p_2(R_3+R_4)}{2R_3R_4}$$

$$Q_4 = \frac{\Omega_1R_4+\Omega_2R_3}{2R_3R_4} - \frac{\Omega_1R_2+\Omega_2R_1}{2R_1R_2}$$

$$Q_5 = \frac{p_2}{2R_3}$$

$$Q_6 = -\frac{\Omega 1}{2R_3}$$

$$Q_7 = -\frac{p_2}{2R_4}$$

$$Q_8 = -\frac{\Omega_2}{2R_4}$$

$$T_1 = [p_2 e^{p_2 t}(R_4 - R_3) + R_4(p_2 \cos\Omega_1 t - \Omega_1 \sin\Omega_1 t) +$$

$$R_3(\Omega_2 \sin\Omega_2 t - p_2 \cos\Omega_2 t)]/2R_2 R_4$$

$$T_2 = [e^{p_2 t}(\Omega_1 R_4 + \Omega_2 R_3) - R_4(p_2 \sin\Omega_1 t + \Omega_1 \cos\Omega_1 t) -$$

$$R_3(p_2 \sin\Omega_2 t + \Omega_2 \cos\Omega_2 t)]/2R_3 R_4$$

$$T_3 = (1 - e^{p_3 t})/2p_3 + p_3 e^{p_3 t}/2R_5 -$$

$$p_3 \cos 2\omega_d t - 2\omega_d \sin 2\omega_d t)/2R_5$$

$$T_4 = (2\omega_d e^{p_3 t} - p_3 \sin 2\omega_d t - 2\omega_d \cos 2\omega_d t)/2R_5$$

$$T_5 = -e^{p_1 t}(p_1 \sin\Omega_1 + p_1 \cos\Omega_1 t)/2R_1 + p_1/2R_1 -$$

$$(\Omega_2 \sin\Omega_1 t - p_1 \cos\Omega_1 t)/2R_2 - (\Omega_2 \sin 2\omega_d t - p_1 \cos 2\omega_d t)/2R_2$$

$$T_6 = e^{p_1 t}(\Omega_1 \cos\Omega_1 t - p_1 \sin\Omega_1 t)/2R_1 - \Omega_1/2R_1 +$$

$$e^{p_1 t}(p_1 \sin\Omega_1 t + \Omega_2 \cos\ _1 t)/2R_2 -$$

$$(p_1 \sin 2\omega_d t + \Omega_2 \cos 2\omega_d t)/2R_2$$

$$T_7 = e^{p_1 t}[(\Omega_1 R_2 + \Omega_2 R_1)\sin\Omega_2 t + p_1(R_1 - R_2)\cos\Omega_2 t]/2R_1 R_2 -$$

$$[\Omega_1 R_2 \sin 2\omega_d t - p_1(R_2 \cos 2\omega_d t - R_1)]/2R_1 R_2 \qquad \text{(2-61) cont'd}$$

$$T_8 = e^{p_1 t}(p_x \sin\Omega_2 t + \Omega_1 \cos\Omega_2 t)/2R_1 - \Omega_2/2R_2 -$$

$$(p_1 \cos 2\omega_d t + \Omega_1 \cos 2\omega_d t)/2R_1 -$$

$$e^{p_1 t}(p_1 \sin\Omega_2 t - 2\cos\omega_d t)2R_2$$

$$p_1 = \zeta\omega_n - \alpha$$

$$p_2 = \zeta\omega_n + \alpha$$

$$p_3 = p_1 + p_2$$

$$\Omega_1 = \omega_d - \beta$$

$$\Omega_2 = \omega_d + \beta$$

$$R_1 = p_1^2 + \Omega_1^2$$

$$R_2 = p_1^2 + \Omega_2^2$$

$$R_3 = p_2^2 + \Omega_1^2$$

$$R_4 = p_2^2 + \Omega_2^2 \qquad \text{(2-61) cont'd}$$

$$R_5 = (p_1 + p_2)^2 + 4\omega_d^2$$

From Eqs. (2-60) and (2-61) it is seen that the mean-square response depends upon an interrelationship which involves the system damping $\zeta$, the corresponding linear system natural circular frequency $\omega_n$, the decaying constant $\alpha$ and the frequency $\beta$ of the correlation function. Determining the solution $E[y^2]$ requires much algebra even for the simplest case of $g(y) = \mu y^3$.

When the input is white noise, only the value of damping of the system affects how quickly stationarity is attained as seen in Section 2.4. However, for a correlated noise input, the time required for the response to reach a stationary value is influenced not only by the system damping coefficient $\zeta$ but also by the decay constant $\alpha$ of the input noise. Normalized plots for $g(y) = \mu y^3$ are shown in Figures 6 through 8 for various value of $\alpha$ with fixed $\beta$. The normalization factor was chosen as the stationary mean-square response $E[y_o^2]_s$ of the linear system which can be obtained by letting $t \to \infty$ and $\mu = 0$ in Eq. (2-60). Instead of finding the stationary mean-square value from Eq. (2-60), we obtain it from

$$E[y_o^2]_s = \int_0^\infty \phi(\omega) \frac{d\omega}{(\omega_n^2-\omega^2)+(2\zeta\omega_n\omega)^2} \tag{2-62}$$

where $\phi(\omega)$ is the one-sided power spectral density of $n(t)$:

$$\phi(\omega) = \frac{K_o\alpha}{\pi}\left[\frac{1}{\alpha^2+(\omega+\beta)^2} + \frac{1}{\alpha^2+(\omega+\beta)^2}\right] \tag{2-63}$$

By contour integration, $E[y^2]_s$ becomes

$$E[y_o^2]_s = \frac{K_o\alpha'}{4\omega_n^4}\left[\frac{A_1(1-{}^2)^{\frac{1}{2}}-B_1\zeta}{\zeta(1-\zeta^2)^{\frac{1}{2}}(A_1^2+B_1^2)} + \frac{A_2(1-\zeta^2)^{\frac{1}{2}}-B_2\zeta}{\zeta(1-\zeta^2)^{\frac{1}{2}}(A_2^2+B_2^2)} + \right.$$

$$\left.\frac{4C}{\alpha'(C^2+D^2)}\right] = C_o \tag{2-64}$$

where

$$A_1 = \alpha'^2+\beta'^2+1-2\zeta^2+2\beta'(1-\zeta^2)^{\frac{1}{2}}$$

$$B_1 = 2\zeta[(1-\zeta^2)^{\frac{1}{2}}+\beta']$$

$$A_2 = \alpha'^2+\beta'^2+1-2\zeta^2-2\beta'(1-\zeta^2)^{\frac{1}{2}}$$

$$B_2 = 2\zeta[(1-\zeta^2)^{\frac{1}{2}}-\beta']$$

$$C = 1+(\beta'^2-\alpha'^2)^2-4\alpha'^2\beta'^2+2(\beta'^2-\alpha'^2)(2\zeta^2-1)$$

$$D = 4\alpha'\beta'(1-2\zeta^2+\alpha'^2-\beta'^2) \tag{2-65}$$

$$\alpha' \quad \alpha/\omega_n$$

$$\beta' \quad \beta/\omega_n$$

This was done for checking purposes. Since the normalized mean-square value must asymptotically approach unity for large t, we can check the results of Eq. (2-60) if we let $\mu=0$. Results show that as $\alpha$ decreases, i.e., the power spectral density has a sharp peak at some frequency, then the transient response tends to exceed the stationary value. Another interesting result is that the nonlinear response becomes greater than the corresponding linear response under certain conditions even if the system has hardening spring-type nonlinearity. One such example is shown in Figure 12.

### 2.4.2 Exponential Envelope Function

For the exponential envelope function expressed by Eq. (2-5), Eq. (2-58) is of the form

$$E[y^2] = \frac{K_0 e^{-2\zeta\omega_n t}}{\omega_d^2} \int_0^t \int_0^t \sin\omega_d(t-\tau)\sin\omega_d(t-\tau')\cos\beta(\tau-\tau') \times$$

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \exp[(\zeta\omega_n-c_i)\tau+(\zeta\omega_n-c_j)\tau'-\alpha|\tau-\tau'|]d\tau d\tau'$$

$$\frac{K_o e^{-2\zeta\omega_n t}}{\omega_d^2}\Bigg[\sum_i^M\sum_j^M A_iA_j\int_0^t \exp[(\zeta\omega_n-\alpha-c_j)\tau']\sin\omega_d(t-\tau')\ \times$$

$$\left\{\int_0^{\tau'} \exp[(\zeta\omega_n+\alpha-c_i)\tau]\sin\omega_d(t-\tau)\cos\beta(\tau-\tau')d\tau\right\}d\tau' +$$

$$\sum_i^M\sum_j^M A_iA_j\int_0^t \exp[(\zeta\omega_n+\alpha-c_j)\tau']\sin\omega_d(t-\tau')\ \times \tag{2-66}$$

$$\left\{\int_{\tau'}^{t} \exp(\zeta\omega_n-\alpha-c_i)\tau\ \sin\omega_d(t-\tau)\cos\beta(\tau-\tau')d\tau\right\}d\tau'\Bigg]$$

After some tedious algebra, we have

$$E[y^2] = \frac{K_o e^{-2\zeta\omega_n t}}{\omega_d^2}\sum_{i=1}^{M}\sum_{j=1}^{M} A_iA_j\sum_{k=1}^{8} Q_{ijk}T_{ijk} \tag{2-67}$$

where $Q_{ijk}$ and $T_{ijk}$ are defined in Eq. (2-61). However, $p_1$, $p_2$, and $p_3$ are replaced by the following.

$$\begin{aligned} p_{1i} &= \zeta\omega_n-\alpha-c_i \\ p_{1j} &= \zeta\omega_n-\alpha-c_j \\ p_{2i} &= \zeta\omega_n+\alpha-c_i \\ p_{3i} &= p_{1i}+p_{2i} \\ p_{3j} &= p_{1j}+p_{2j} \end{aligned} \tag{2-68}$$

Also all other $Q_1,\dots Q_8$, $T_1,\dots T_8$, $R_1,\dots R_5$ are denoted by $Q_{1ij},\dots$ $R_{5ij}$. Equations (2-67) and (2-38) may be solved simultaneously for $E[y^2]$.

As a numerical example, the simplest case M=1 in Eq. (2-5), that is, $A(t)=e^{-ct}$ is shown. In Figures 9 through 11, the normalized mean-square value is plotted for various value of $\alpha$. Here, $\alpha$, $\beta$ and normalization factor $C_0$ are the same as those used in the previous Chapter 2.4.1. The exponential decay constant c of the envelope function is taken as $c = \frac{1}{2}\delta$.

# CHAPTER III

## MULTIDEGREE-OF-FREEDOM NONLINEAR SYSTEM

### 3.1 Statement of the Problem

Now, consider the N-degree-of-freedom system governed by

$$\ddot{y}_i+2\zeta_i\omega_i\dot{y}_i+\omega_i^2[1+\mu\sum_{j=1}^{N}\omega_j^2y_j^2]y_i = f_i(t) \quad i=1,2,\ldots,N \tag{3-1}$$

Or, this may be written as

$$\ddot{y}_i+2\zeta_i\omega_i\dot{y}_i + \frac{\partial V}{\partial y_i} = f_i(t) \tag{3-2}$$

where V is the total potential energy per unit mass expressed by

$$V = \frac{1}{2}\sum_{k=1}^{N}\omega_k^2y_k^2 + \frac{\mu}{4}\sum_{k=1}^{N}\sum_{j=1}^{N}\omega_k^2\omega_k^2y_k^2y_j^2 \tag{3-3}$$

The forcing function $f_i(t)$ is assumed to be represented by a product of a well-defined envelope function $A_i(t)$ and the uncorrelated stationary random process $n_i(t)$ with zero mean, that is,

$$f_i(t) = A_i(t)n_i(t) \tag{3-4}$$

$$E[n_i(t)] = 0 \tag{3-5}$$

$$E[n_i(t)n_j(t)] = 0 \qquad i\neq j \tag{3-6}$$

Furthermore it is assumed that $n_i(t)$ has a power spectral density $\phi_i(\omega)$ which is a smooth function of $\omega$, having no sharp peaks. The stationary response of the system of Eq. (3-1) to uncorrleated white random processes has been studied by Caughey [7]. We are to determine the approximate mean square response $Ey_i^2(t)]$ subject to the assumptions given in Eqs. (3-4) through (3-6) by equivalent linearization technique. The envelope functions $A_i(t)$

considered in this Chapter are the unit step function and the exponentia'
function, i.e.,

$$A_i(t) = u(t) \tag{3-7}$$

$$A\ (t) = A_i e^{-c_i t} \tag{3-8}$$

3.2 Response Formulation

Let us assume that an approximate solution to Eq. (3-1) can be obtained from the linearized equation

$$\ddot{y}_i + 2\beta_{ie}\dot{y}_i + \omega_{ie}^2 y_i = f_i(t) \tag{3-9}$$

where $\beta_{ie}$ is the equivalent linear damping coefficient and $\omega_{ie}^2$ is the equivalent linear stiffness. Then the error caused by this linearization is obtained from

$$e_i(t) = 2(\zeta_i\omega_i - \beta_{ie})y_i + (\omega_i^2 - \omega_{ie}^2)y_i - \mu\frac{\partial U}{\partial y_i} \tag{3-10}$$

where

$$U = \tfrac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N} \omega_j^2\omega_k^2 y_j^2 y_k^2 \tag{3-11}$$

Minimizing $E[e_i^2]$ with respect to $\beta_{ie}$ and $\omega_{ie}$, we obtain

$$2\beta_{ie} = 2\zeta_i\omega_i + \mu\,\frac{E[y_i^2]E[y_i\,{}^{\partial U}/_{\partial yi}] - E[y_i y_i]E[y_i\,{}^{\partial U}/_{\partial yi}]}{E[y_i^2]E[y_i^2] - (E[y_i y_i])^2} \tag{3-12}$$

$$\omega_{ie}^2 = {}^{2}_{i} + \frac{E[\dot{y}_i^2]E[y_i\ {}^{\partial U}/_{\partial y_i}] - E[y_i\dot{y}_i]E[\dot{y}_i\,{}^{\partial U}/_{\partial y_i}]}{E[y_i^2]E[\dot{y}_i^2] - (E[y_i\dot{y}_i])^2} \tag{3-13}$$

It is easy to show that the conditions given by Eqs. (3-12) and (3-13) yield the true minimum of $E[e_i^2]$ if we apply the same argument used in Chapter 2.2.

In order to express the right hand side of Eqs. (3-12) and (3-13) in terms of the mean-square of the displacement and the velocity, we must know the joint probability density function $p(y_1, y_2, \ldots, y_N, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_N)$. Since the inputs $f_i(t)$ are Gaussian and the nonlinearities of the system are assumed to be small, the outputs of the linearized system are also assumed to be Gaussian. The displacement and the velocity of the i-th mode are:

$$y_i(t) = \int_0^t h_i(t-\tau)f_i(\tau)d\tau \tag{3-14}$$

$$\dot{y}_i(t) = \int \frac{h_i(t-\tau)}{\partial t} f_i(\tau)d\tau \tag{3-15}$$

where

$$h_i(t) = \frac{e^{-\beta_{ie}t}}{\omega_{id}} (\sin\omega_{id}t)u(t) \tag{3-16}$$

$$\omega_{id}^2 = \omega_{ie}^2 - \beta_{ie}^2 \tag{3-17}$$

$$E[y_i] = \int_0^t h_i(t-\tau)E[y_i(\tau)]d\tau = 0 \tag{3-18}$$

Similarly,

$$E[\dot{y}_i] = 0 \tag{3-19}$$

Next, let us find the covariances $E[y_i(t)y_j(t)]$ and $E[y_i(t)\dot{y}_j(t)]$.

$$E[y_i(t)y_j(t)] = \int_0^t \int_0^t h_i(t-\tau)h_j(t-\tau')E[f_i(\tau)f_j(\tau')]d\tau d\tau' \qquad (3\text{-}20)$$

since $f_i(t)$ and $f_j(t)$ are uncorrelated, for $i \neq j$, i.e.,

$$E[f_i(t)f_j(t)] = A_i(t)A_j(t)E[n_i(t)n_j(t)] = 0 \quad i \neq j \qquad (3\text{-}21)$$

it is concluded that

$$E[y_i(t)y_j(t)] = 0 \quad i \neq j \qquad (3\text{-}22)$$

Similarly, we can show that

$$E[y_i(t)\dot{y}_j(t)] = \int_0^t \int_0^t h_i(t-\tau) \frac{\partial h_j(t-\tau')}{\partial t} E[f_i(\tau)f_j(\tau')]d\tau d\tau' = 0 \qquad (3\text{-}23)$$

The displacements and velocities between different modes are mutually uncorrelated. Therefore, the covariance matrix becomes:

$$K = \begin{bmatrix} E[y_1^2] & E[y_1\dot{y}_1] & 0 & & \cdots & 0 & 0 \\ E[y_1\dot{y}_1] & E[\dot{y}_1^2] & 0 & & \cdots & 0 & 0 \\ 0 & 0 & E[y_2^2] & E[y_2\dot{y}_2] & \cdots & 0 & 0 \\ 0 & 0 & E[y_2\dot{y}_2] & E[\dot{y}_2^2] & & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & E[y_N^2] & E[y_N\dot{y}_N] \\ 0 & 0 & 0 & 0 & \cdots & E[y_N\dot{y}_N] & E[\dot{y}_N^2] \end{bmatrix} \qquad (3\text{-}24)$$

For the covariance matrix expressed by Eq. (3-24), the 2N-fold probability density function is given by

$$p(y_1, y_2, \ldots y_N, \dot{y}_1, \dot{y}_2, \ldots \dot{y}_N) = \prod_{k=1}^{N} p_k(y_k, \dot{y}_k) \tag{3-25}$$

where $p_k(y_k, \dot{y}_k)$ is the probability density function associated with the k-th mode defined by

$$p_k(y_k, \dot{y}_k) = \frac{1}{2\pi\sqrt{\det(K_k)}} \exp(-a_k y_k^2 + 2b_k y_k \dot{y}_k - c_k \dot{y}_k^2) \tag{3-26}$$

$$\det(K_k) = E[y_k^2]E[\dot{y}_k^2] - (E[y_k \dot{y}_k])^2$$

$$a_k = \frac{E[y_k^2]}{2\det(K_k)}$$

$$b_k = \frac{E[y_k \dot{y}_k]}{2\det(K_k)} \tag{3-27}$$

$$c_k = \frac{E[\dot{y}_k^2]}{2\det(K_k)}$$

Using Eqs. (3-25) and (3-26), $E[y_i \frac{\partial U}{\partial y_i}]$ and $E[y_i \frac{\partial U}{\partial y_i}]$ can be evaluated as

$$E[y_i \frac{\partial U}{\partial y_i}] = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2N\text{-fold}} y_i \frac{\partial U}{\partial y_i} \prod_{k=1}^{N} p_k(y_k \dot{y}_k) dy_k d\dot{y}_k$$

$$= \omega_i^2 E[y_i^2] \{ \sum_{\substack{k=1 \\ k \neq i}}^{N} \omega_k^2 E[y_k^2] + 3\omega_i^2 E[y_i^2] \}$$

$$= \omega_i^2 E[y_i^2] \{ \sum_{k=1}^{N} \omega_k^2 E[y_k^2] + 2\omega_i^2 E[y_i^2] \} \tag{3-28}$$

Similarly,

$$E[\dot{y}_i \frac{\partial U}{\partial y_i}] = \underbrace{\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty}}_{2N\text{-fold}} \dot{y}_i \frac{\partial U}{\partial y_i} \sum_{k=1}^{N} p_k(y_k, \dot{y}_k) dy_k d\dot{y}_k$$

$$= \omega_i^2 E[y_i \dot{y}_i] \{\sum_{k=1}^{N} \omega_k^2 E[y_k^2] + 2\omega_i^2 E[y_i^2]\} \tag{3-29}$$

Substituting Eqs. (3-28) and (3-29) into Eqs. (3-12) and (3-13), we obtain,

$$2\beta_{ie} = 2\zeta_i \omega_i \tag{3-30}$$

$$\omega_{ie}^2 = \omega_i^2[1+\mu(\sum_{k=1}^{N} \omega_k^2 E[y_k^2] + 2\omega_i^2 E[y_i^2])] \tag{3-31}$$

These results are the same as the equivalent linear damping and stiffness for the stationary processes.

In this Chapter, the mean-square response is formulated by use of Eq. (2-40). The mean-square response of the i-th mode is given by,

$$E[y_i^2] = \int_0^{\infty} \frac{\phi_i(\omega)}{\omega_{id}^2} \int_0^t \int_0^t \exp\{-\zeta_i\omega_i(2t-\tau-\tau')\} \sin\omega_{id}(t-\tau) \times$$

$$\sin\omega_{id}(t-\tau')A_i(\tau)A_i(\tau')\cos\omega(\tau-\tau')d\tau d\tau' d\omega \tag{3-32}$$

where $\omega_{id}^2$ is defined by Eqs. (3-17). Now by substitution of Eqs. (3-30) and (3-31) into Eq. (3-17), $\omega_{id}^2$ is expressed as follows:

$$\omega_{id}^2 = \omega_i^2[1-\zeta_i^2 + \mu(\sum_{k=1}^{N} \omega_k^2 E[y_k^2] + 2\omega_i^2 E[y_i^2])] \quad (3\text{-}33)$$

3.2.1 Unit Step Envelope Function

Substituting Eq. (3-7) into Eq. (3-32) and integrating, Eq. (3-32) reduces to

$$E[y_i^2] = \int_0^\infty \phi_i(\omega)|H_i(\omega)|^2 K_i(\omega,t)d\omega \quad (3\text{-}34)$$

where

$$K_i(\omega,t) = 1+e^{-2\zeta_i\omega_i t}(1+\frac{\zeta_i\omega_i}{\omega_{id}}\sin 2\omega_{id}t)+e^{-2\zeta_i\omega_i t}\sin^2\omega_{id}t \times$$

$$(\frac{\zeta_i^2\omega_i^2-\omega_{id}^2+\omega^2}{\omega_{id}^2})-2e^{-\zeta_i\omega_i t}(\cos\omega_{id}t+\frac{\zeta_i\omega_i}{\omega_{id}}\sin\omega_{id}t) \times$$

$$\cos\omega t-\frac{2e^{-\zeta_i\omega_i t}}{\omega_{id}}\sin\omega_{id}t\sin\omega t \quad (3\text{-}35)$$

$$|H_i(\omega)|^2 = \frac{1}{(\omega_{ie}^2-\omega^2)^2+(2\omega\omega_i\zeta_i)^2} \quad (3\text{-}36)$$

Since $\phi_i(\omega)$ has been assumed to be a smooth function of $\omega$, having no sharp peaks, and if $\zeta_i$ is small, then the integral of Eq. (3-34) can be evaluated approximately by [1]:

$$E[y_i^2] \approx \frac{\phi_i(\omega_{ie})}{4\zeta_i\omega_i(\omega_i^2\zeta_i^2+\omega_{id}^2)}[1-e^{-2\zeta_i\omega_i t}(1+\frac{2\zeta_i^2\omega_i^2}{\omega_{id}^2}\sin^2\omega_{id}t+\frac{\zeta_i\omega_i}{\omega_{id}}\sin 2\omega_{id}t)]$$

(3-37)

If $\phi_i(\omega)$ is given and Eqs. (3-31) and (3-33) are substituted into Eq. (3-37), we have N simultaneous nonlinear algebraic equations for $E[y_i^2]$ ($i = 1, 2, \ldots, N$). These can be solved numerically by the Newton-Raphson method.

Now consider the particular case in which $n_i(t)$ is white noise. If we denote $\phi_i(\omega) = \text{constant} = K$, Eq. (3-38) becomes

$$E[y_i^2] = \frac{\pi K}{4\zeta_i\omega_i\omega_{ie}^2}\left[1-e^{-2\zeta_i\omega_i t}\left(1+\frac{2\zeta_i^2\omega_i^2}{\omega_{id}^2}\sin^2\omega_{id}t+\frac{\zeta_i\omega_i}{\omega_{id}}\sin 2\omega_{id}t\right)\right] \tag{3-38}$$

Let us now show that the transient mean-square response $E[y_i^2]$ does not exceed the stationary mean-square response $E[y_i^2]_s$ if $n_i(t)$ is white noise. The stationary mean-square response is obtained by letting $t \to \infty$.

$$E[y_i^2]_s = \frac{\pi K}{4\zeta_i\omega_i\omega_{ie}^2} \tag{3-39}$$

Substituting Eq. (3-31) into Eq. (3-39), we find:

$$E[y_i^2]_s\,\{1+\mu\left(\sum_{k=1}^{N}\omega_k^2E[y_k^2]_s+2\omega_i^2E[y_i^2]_s\right)\} - \frac{\pi K}{4\zeta_i\omega_i^3} = 0 \tag{3-40}$$

Using Eq. (2-47), we have the following inequality:

$$E[y_i^2] \leq \frac{\pi K}{4\zeta_i\omega_i\omega_{ie}^2} \tag{3-41}$$

Substituting Eq. (3-31) into Eq. (3-41) and rearranging the terms we obtain

$$E[y_i^2]\,\{1+\mu(\sum_{k=1}^{N}\omega_k^2E[y_k^2]+2\omega_i^2E[y_i^2])\}-\frac{\pi K}{4\zeta_i\omega_i^3}\leq 0 \tag{3-42}$$

After eliminating $\pi K/4\zeta_i\omega_i^3$ from Eqs. (3-40) and (3-42) and after some manipulation,

$$(E[y_i^2]-E[y_i^2]_s)\{1+2\mu\omega_i^2[\sum_{k=1}^{N}\omega_k^2(E[y_k^2]+E[y_k^2]_s)]+\mu\sum_{k=1}^{N}\omega_k^2E[y_k^2]\}\leq$$

$$E[y_i^2]_s\,\{\sum_{k=1}^{N}\omega_k^2(E[y_k^2]_s-E[y_k^2])\} \tag{3-43}$$

Suppose

$$E[y_i^2]\geq E[y_i^2]_s \tag{3-44}$$

Equation (3-44) implies

$$\sum_{k=1}^{N}\omega_k^2(E[y_k^2]-E[y_k^2]_s)\geq 0 \tag{3-45}$$

Then the right hand side of the inequality (3-43) is negative so the left hand side must be negative, too.

$$(E[y_i^2]-E[y_i^2]_s)\leq 0$$

This is a contradiction of Eq. (3-44).

Hence,

$$E[y_i^2]\leq E[y_i^2]_s \tag{3-46}$$

Thus, it has been proven that if $n_i(t)$ is white noise and the envelope function is the unit step function, then the transient mean-square response does not exceed the stationary mean-square response.

### 3.2.2 Exponential Envelope Function

For the exponential envelope function $A_i(t) = e^{-c_i t}$, Eq. (3-32) now becomes, after double integration,

$$E[y_i^2(t)] = \int_0^\infty \phi_i(\omega)|H_{iA}(\omega)|^2 W_i(\omega,t)d\omega \tag{3-47}$$

where

$$W_i(\omega,t) = e^{-2c_i t}\{1+\lambda_1(t)+\lambda_2(t)[\frac{r_i^2-\omega_{id}^2+\omega^2}{\omega_{id}^2}]$$

$$-2\lambda_3(t)\cos\ t-2\lambda_4(t)\frac{\omega}{\omega_{id}}\sin\omega t\}$$

$$\lambda_1(t) = e^{-2r_i t}(1+\frac{r_i}{\omega_{id}}\sin 2\omega_{id}t)$$

$$\lambda_2(t) = e^{-2r_i t}\sin^2\omega_{id}t \tag{3-48}$$

$$\lambda_3(t) = e^{-2r_i t}(\cos\omega_{id}t+\frac{r_i}{\omega_{id}}\sin\omega_{id}t)$$

$$\lambda_4(t) = e^{-r_i t}\sin\omega_{id}t$$

$$r_i = \zeta_i\omega_i-c_i$$

$$|H_{iA}(\omega)|^2 = \frac{1}{(\omega_{id}^2+r_i^2-\omega^2)^2+(2r_i\omega)^2}$$

If $c_i$ is either the same order of magnitude as $\zeta_i\omega_i$ or smaller, then the integration of Eq. (3-47) may be approximated by the following expression:

$$E[y_i^2(t)] = \frac{\pi\phi_i(\omega_{id}^2+r_i^2)}{4r_i(\omega_{id}^2+r_i^2)} \; e^{-2c_i t}$$

$$\times \left[1-e^{-2r_i t}\left(1+\frac{r_i}{\omega_{id}}\sin\omega_{id}t+2\frac{r_i^2}{\omega_{id}^2}\sin^2\omega_{id}t\right)\right] \qquad (3\text{-}49)$$

Letting $c_i \to 0$, Eq. (3-49) reduces to Eq. (3-38).

# CHAPTER IV

## CONCLUSIONS

In Chapter II, the time varying mean-square response of a nonlinear single-degree-of-freedom mechanical system to nonstationary random excitation characterized by the product of an envelope function and a stationary Gaussian random process has been considered. A unit step envelope function and an exponential envelope function are considered in conjunction with both correlated and white noise with zero mean. The nonlinear governing equation was linearized by the method of equivalent linearization.

For the nonstationary process, it has been shown that the equivalent linear damping coefficient and the equivalent linear stiffness for the system with nonlinearities involved only in displacements or only in velocities are the same as those for the stationary process.

The mean-square response depends upon the coefficients of the system equation, the shape of the envelope function, and the parameters of the autocorrelation of the process n(t). It was proved that for white noise modulated by a unit step function, the transient mean-square response never exceeds the stationary response. However, the mean-square response to correlated noise modulated by a unit step function may exceed its stationary value, especially when the power spectral density of the process n(t) has a sharp peak, and its maximum value becomes several times the stationary value.

It has also been shown that the mean-square response of the system with cubic hardening spring-type nonlinearity may be greater than the corresponding linear system response under certain conditions.

In Chapter III, the analysis has been extended to the N-degree-of-freedom nonlinear system for the case of mutually uncorrelated noise.

## NOMENCLATURE

| | |
|---|---|
| $A(t)$ | = envelope function |
| $A_i$ | = constants of the exponential envelope function |
| $a,b,c$ | = functions of the correlation function $E[y^2]$, $E[y\dot{y}]$, and $E[\dot{y}^2]$ defined by Eq. (2-29). |
| $c_i$ | = decay coefficients of the exponential envelope function. |
| $C_o$ | = normalization factor |
| $E[\ ]$ | = expected value of [ ]. |
| $E[y^2]$ | = time varying mean-square response |
| $E[y_o^2]$ | = time varying mean-square response of the linear system. |
| $E[y_o^2]_s$ | = stationary mean-square response of the linear system |
| $E[y^2]_s$ | = stationary mean-square response of the nonlinear system |
| $e$ | = difference between a nonlinear system and its equivalent linear system |
| $h(\tau)$ | = impulse response function or weighting function of the equivalent linear system |
| $H(\omega)$ | = transfer function of the equivalent linear system |
| $\det(K)$ | = determinant of the correlation matrix |
| $K_o$ | = constant |
| $K_i(\omega,t)$ | = modulation function due to unit step function |
| $n(t)$ | = input random process |
| $p(\dot{y},y)$ | = probability density function |
| $r_i$ | = $\zeta_i\omega_i - c_i$ |
| $R_n(\tau)$ | = autocorrelation function of input noise $n(t)$ |
| $t$ | = time |

| | |
|---|---|
| $u(t)$ | = unit step function |
| $W_1(\omega,t)$ | = modulation function due to exponential envelope function |
| $y(t)$ | = displacement response |
| $\alpha$ | = decay coefficient of noise correlation function |
| $\beta$ | = frequency of noise correlation function |
| $\beta_e$ | = equivalent linear damping |
| $\delta(t)$ | = Dirac delta function |
| $\lambda_k(t)$ | = functions defined by Eq. (3-48). $k = 1,2,\ldots,4$ |
| $\lambda_{ij}$ | = constants defined by Eq. (2-52) |
| $\mu_k$ | = coefficients of the nonlinear terms of $y(t)$ |
| $\zeta$ | = system damping coefficient |
| $\omega_n$ | = circular natural frequency of the corresponding linear system |
| $\omega_{ne}^2$ | = equivalent linear stiffness |
| $\omega_d^2$ | = $\omega_{ne}^2 - \beta_e^2$ |
| $\phi(\omega)$ | = power spectral density of input noise $n(t)$ |
| $(\dot{\ })$ | = $d(\ )/dt$ |
| $\Pi$ | = product |
| $\tau$ | = $\omega_n t$ |
| $\Sigma$ | = summation |
| $\approx$ | = approximately equal to |

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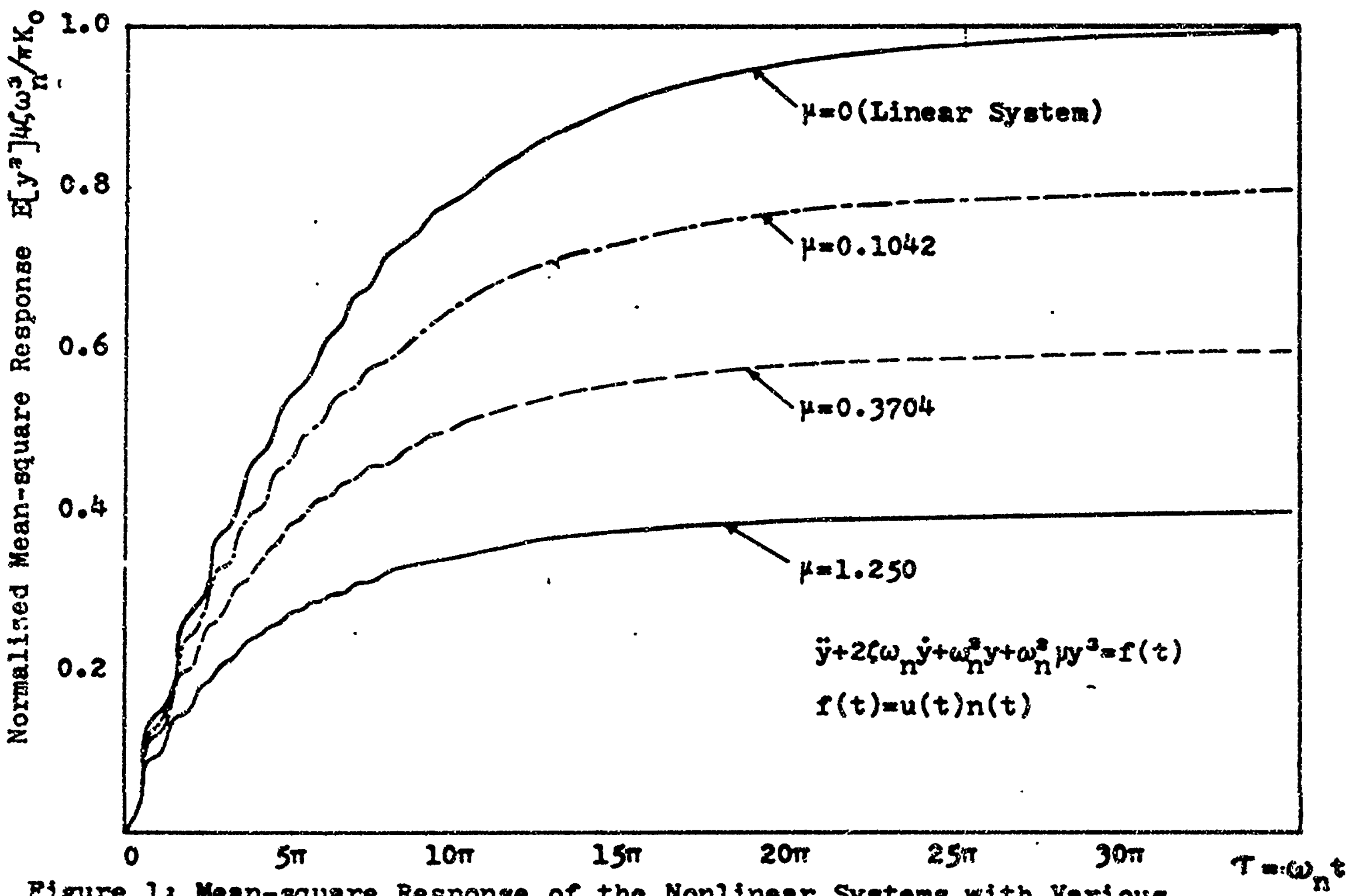


Figure 1: Mean-square Response of the Nonlinear Systems with Various Nonlinearities to White Noise Modulated by a Unit Step Function. System Damping $\zeta$=0.025

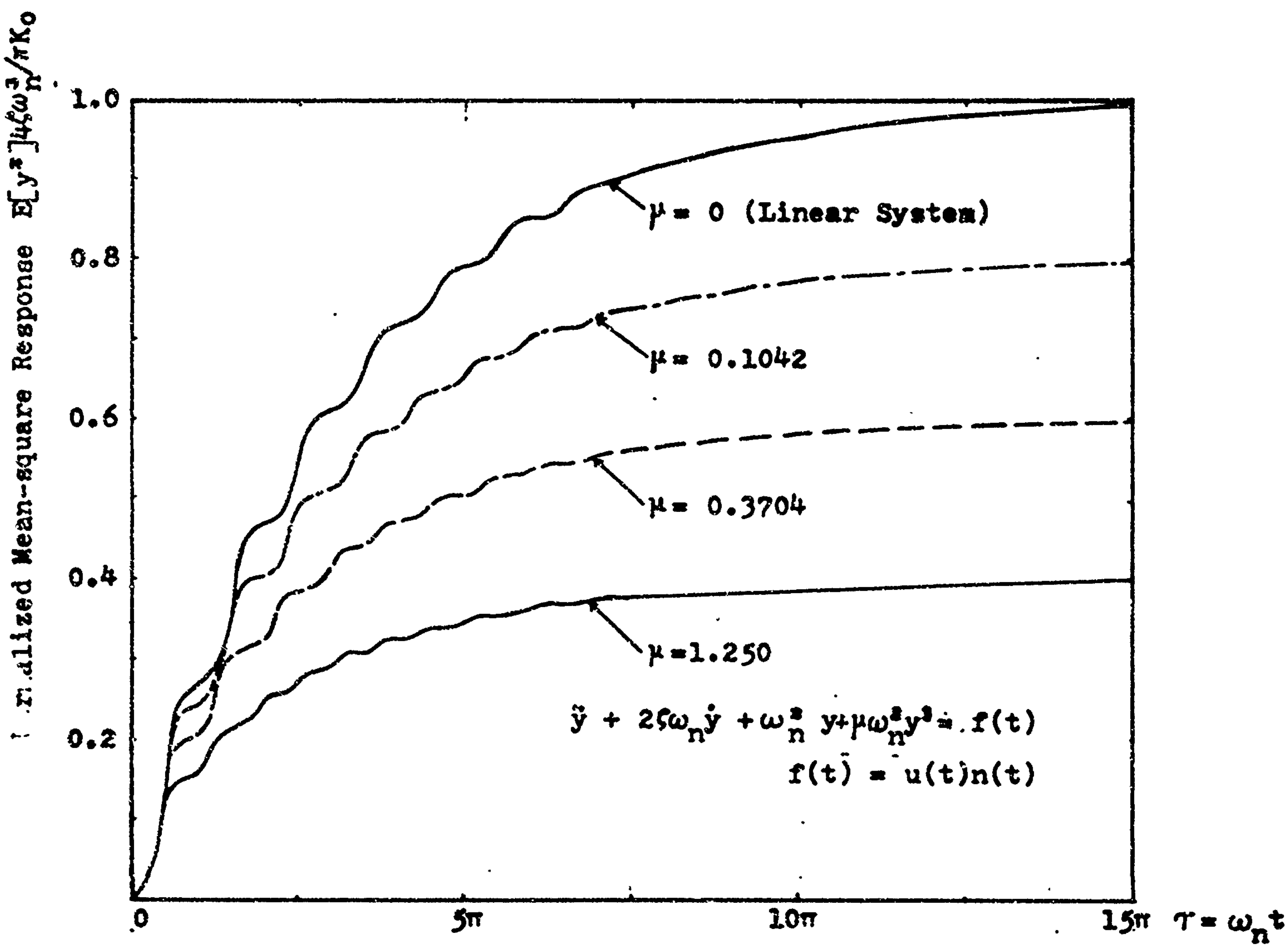


Figure 2; Mean-square Response of the Nonlinear Systems with various Nonlinearities to White Noise Modulated by a Unit Step Function. System Damping $\zeta=0.05$

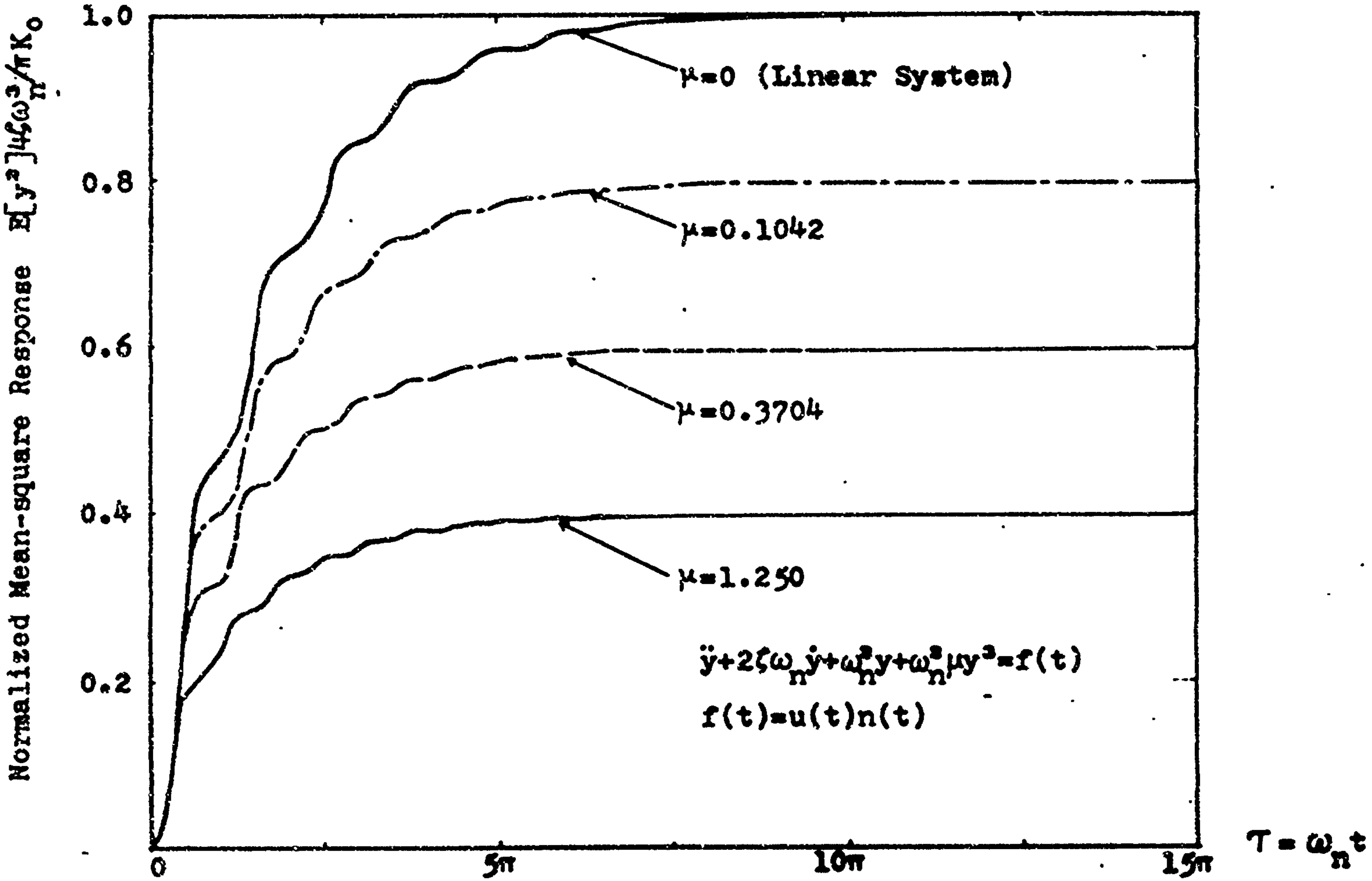


Figure 3: Mean-square Response of the Nonlinear Systems with Various Nonlinearities to White Noise Modulated by a Unit Step Function. System Damping $\zeta$=0.10

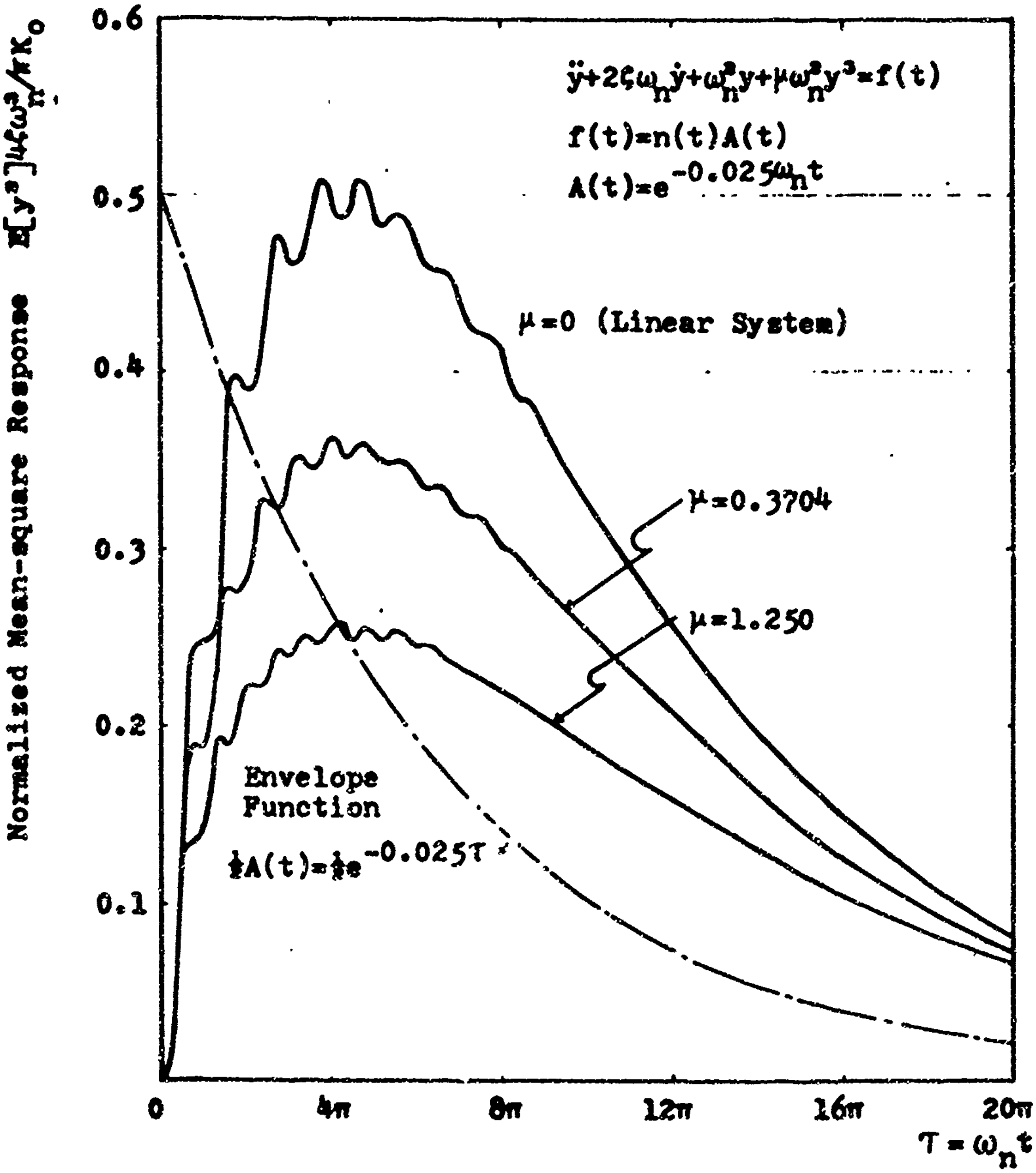


Figure 4: Mean-square Response of the Nonlinear System to White Noise Modulated byExponential Envelope Function. System Damping $\zeta=0.05$

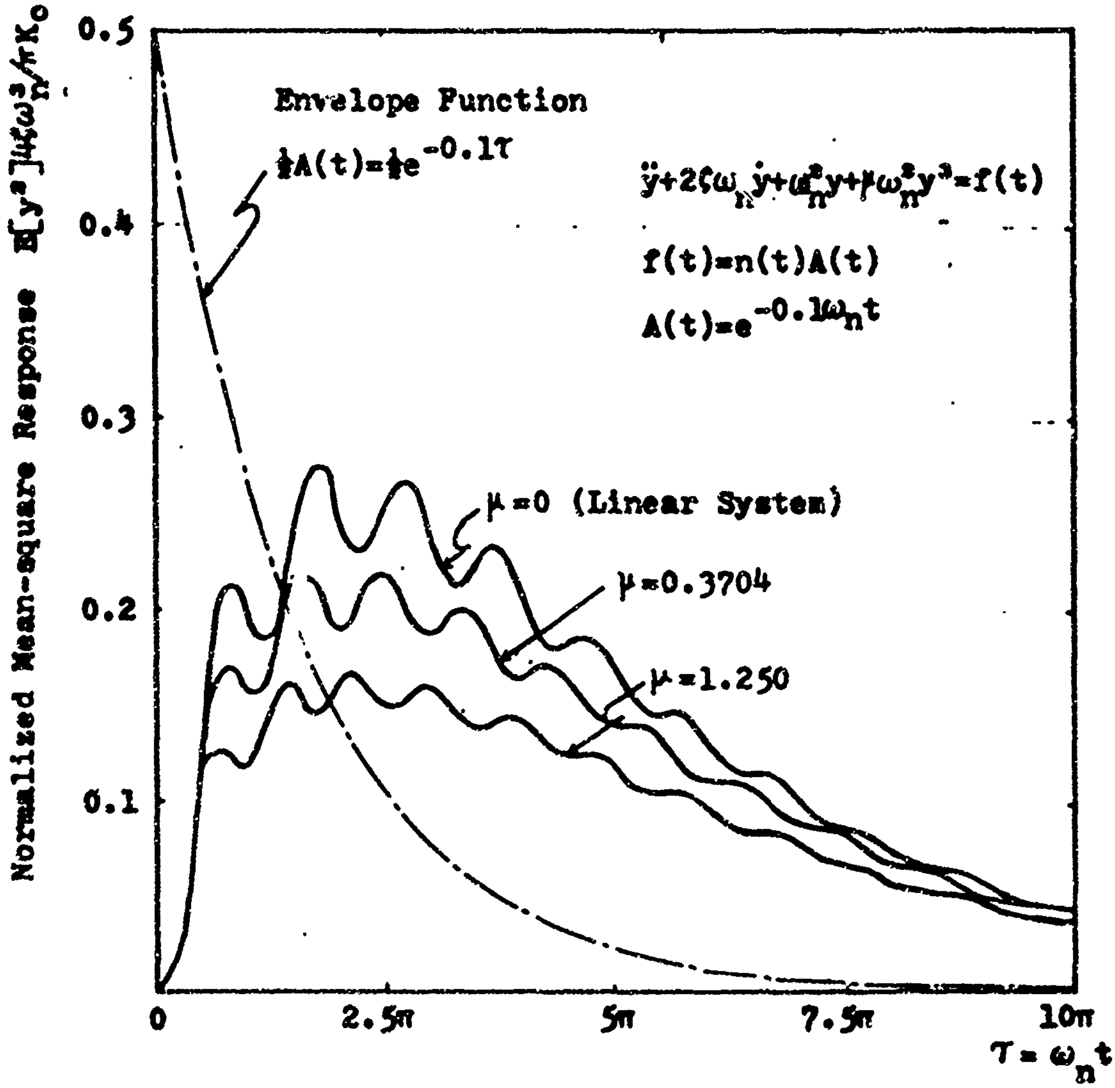


Figure 5: Mean-square Response of the Nonlinear System to White Noise Modulated by Exponential Envelope Function. System Damping $\zeta=0.05$

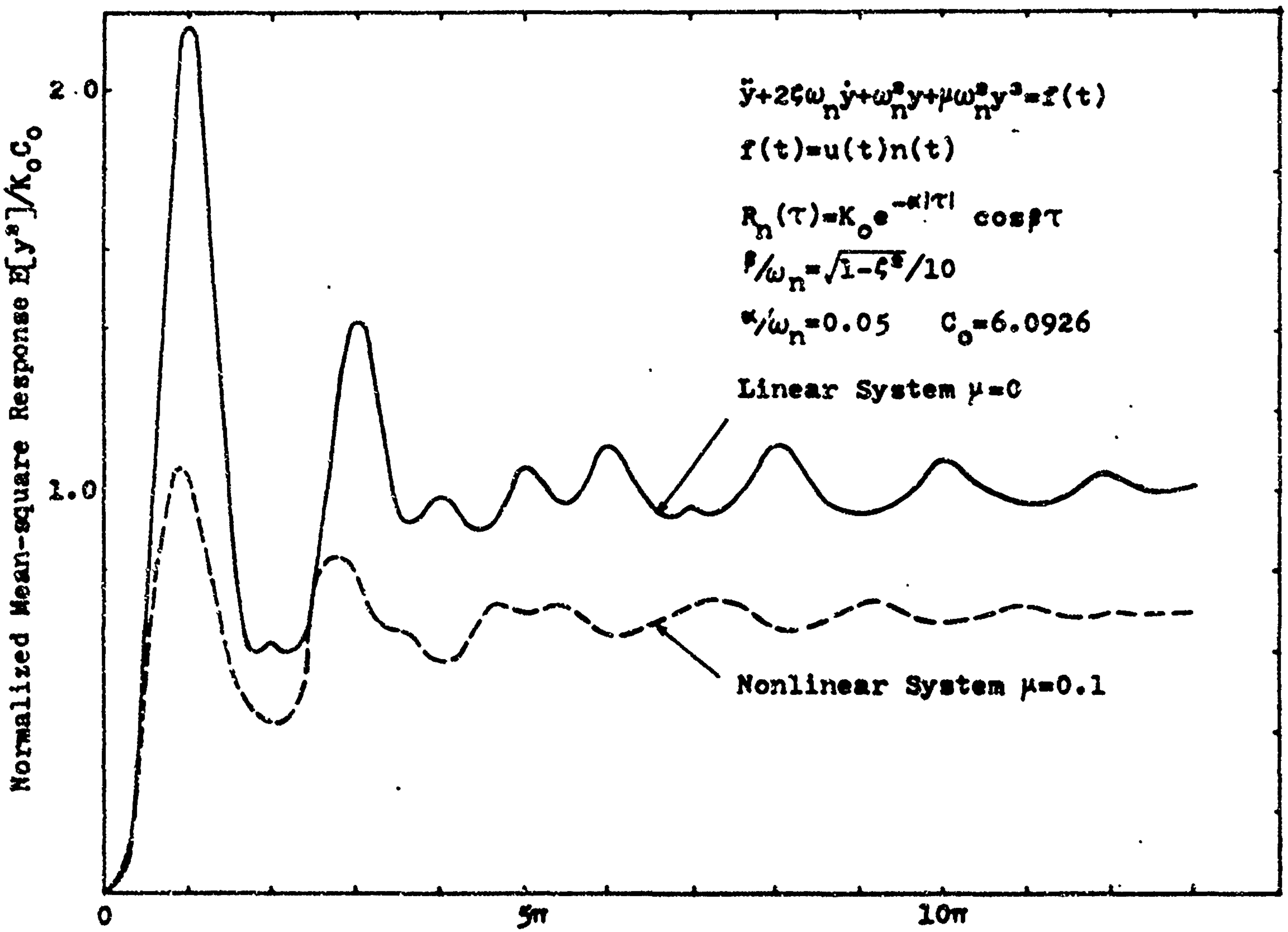


Figure 6: Mean-square Response of the Nonlinear System to

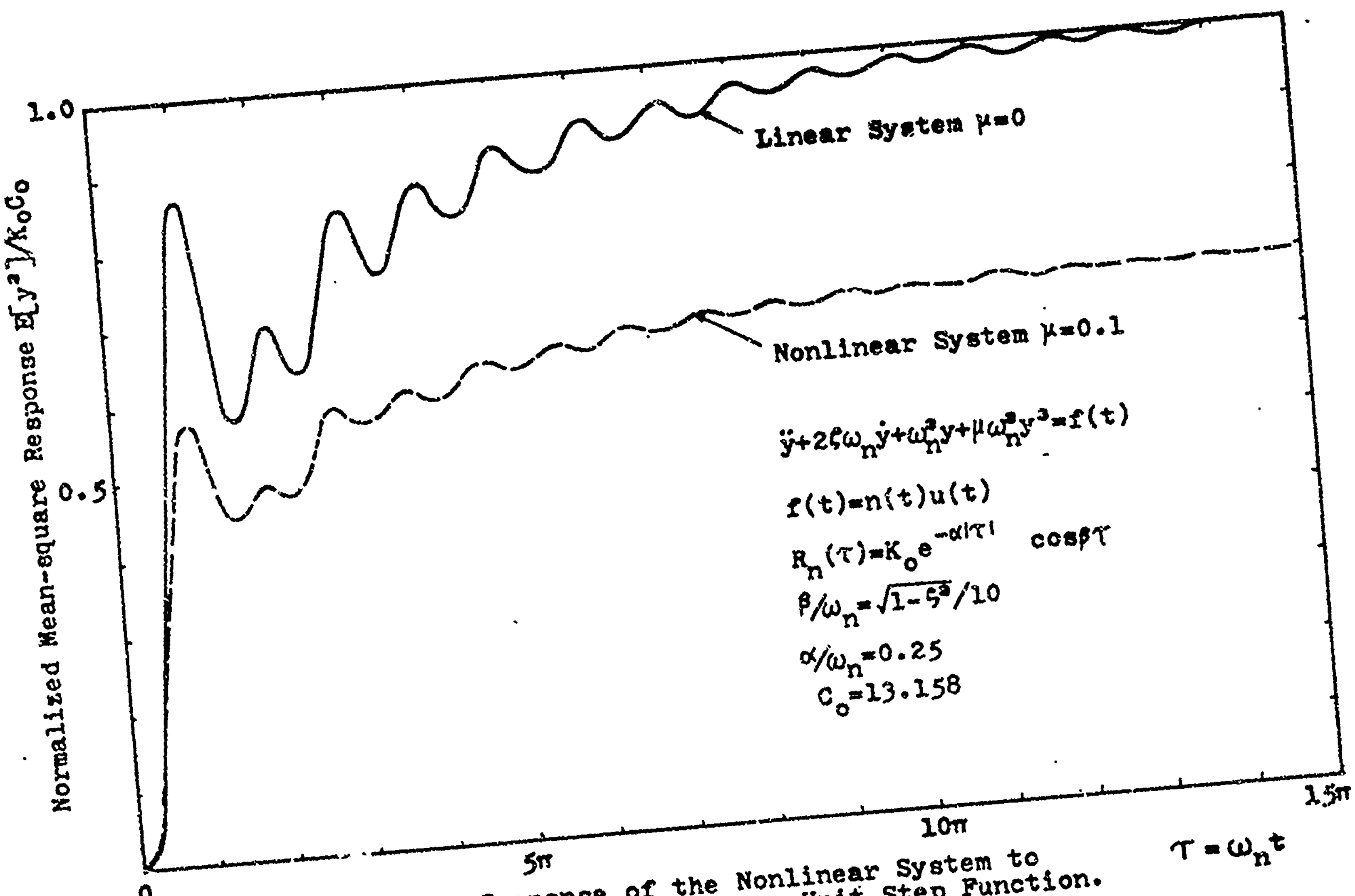


Figure 7: Mean-square Response of the Nonlinear System to Correlated Noise Modulated by a Unit Step Function. System Damping $\zeta=0.05$

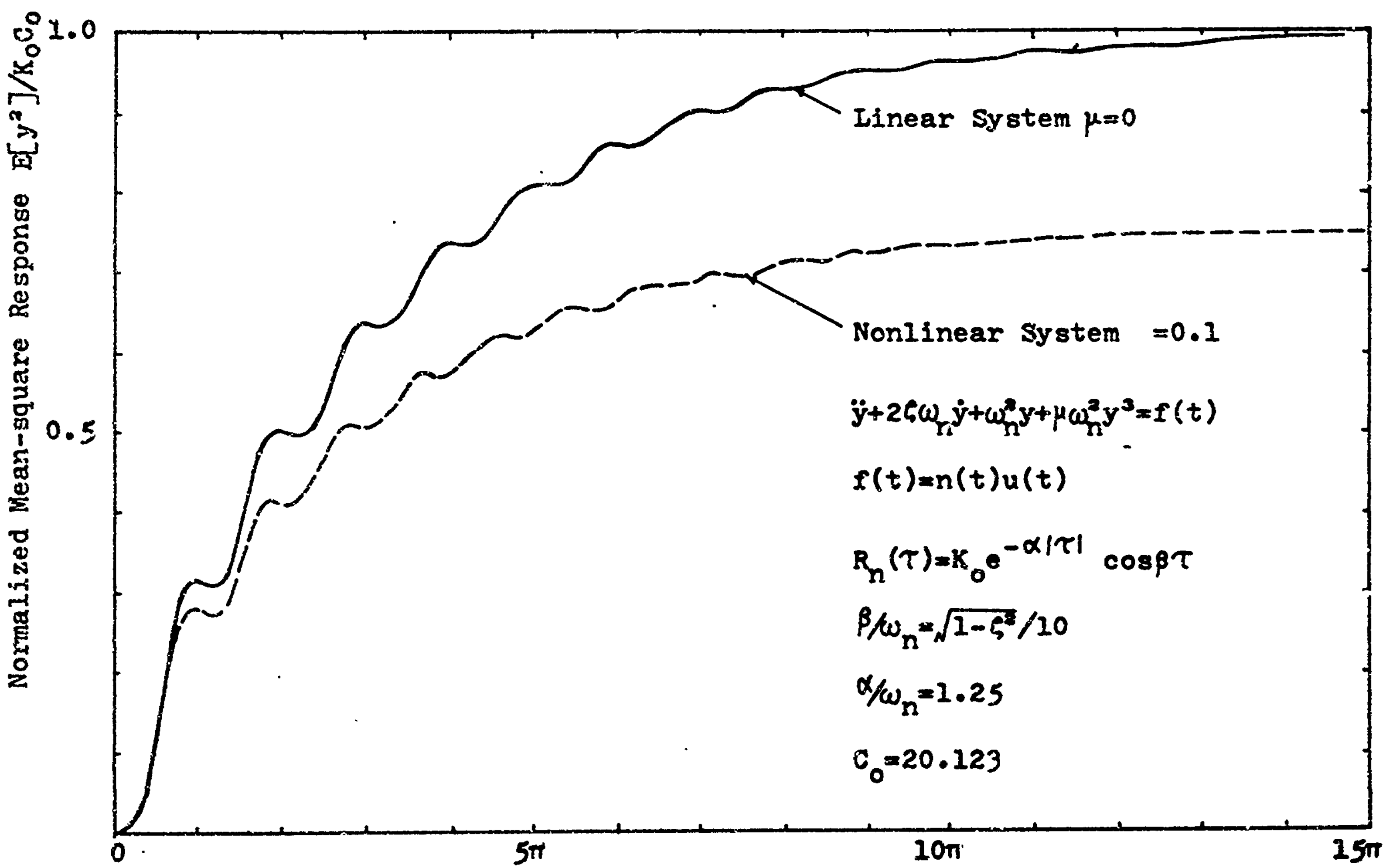


Figure 8: Mean-square Response of the Nonlinear System to Correlated Noise Modulated by a Unit Step Function. $\tau = \omega_n t$
System Damping $\zeta$=0.05

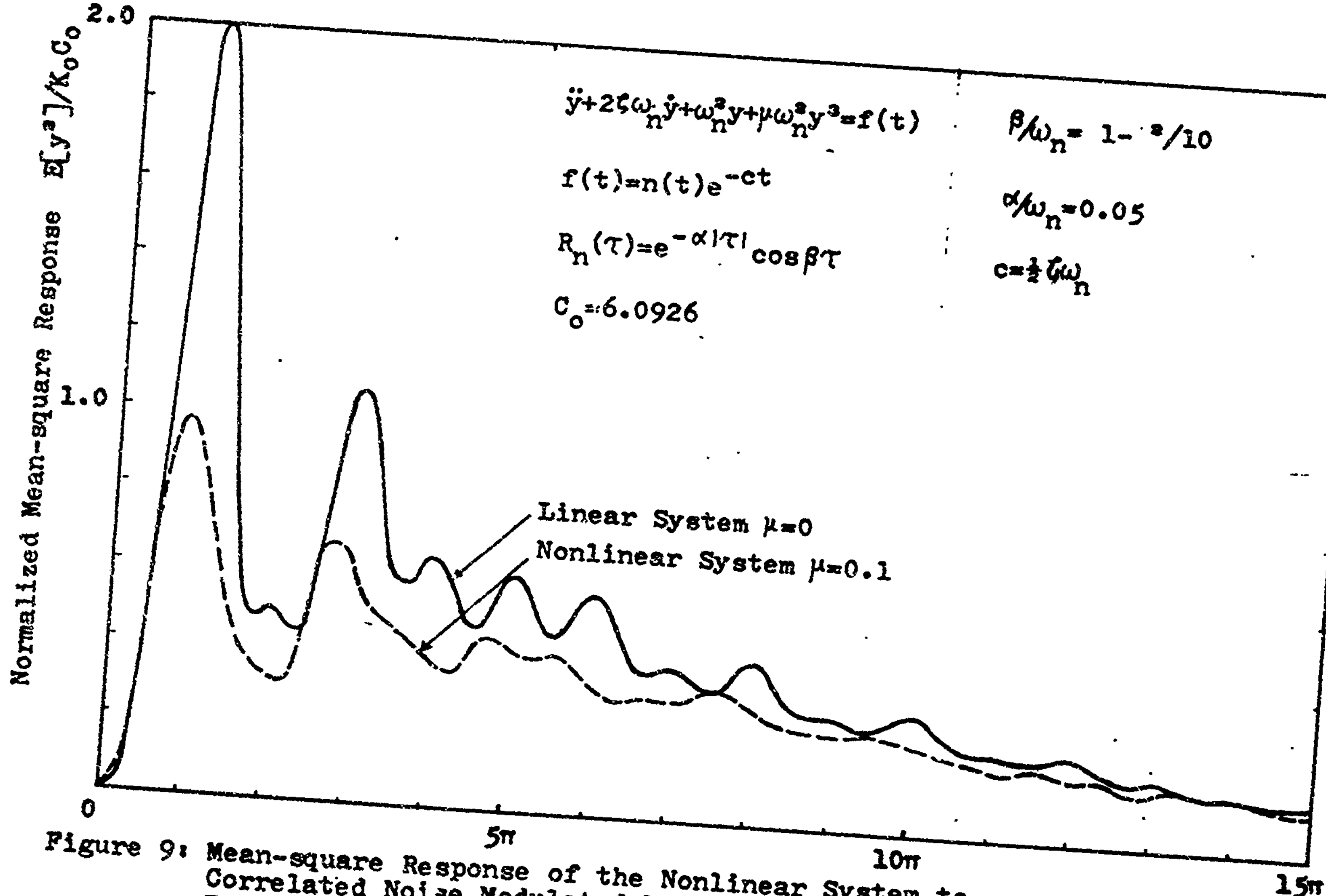


Figure 9: Mean-square Response of the Nonlinear System to Correlated Noise Modulated by an Exponential Envelope Function. System Damping $\zeta=0.05$

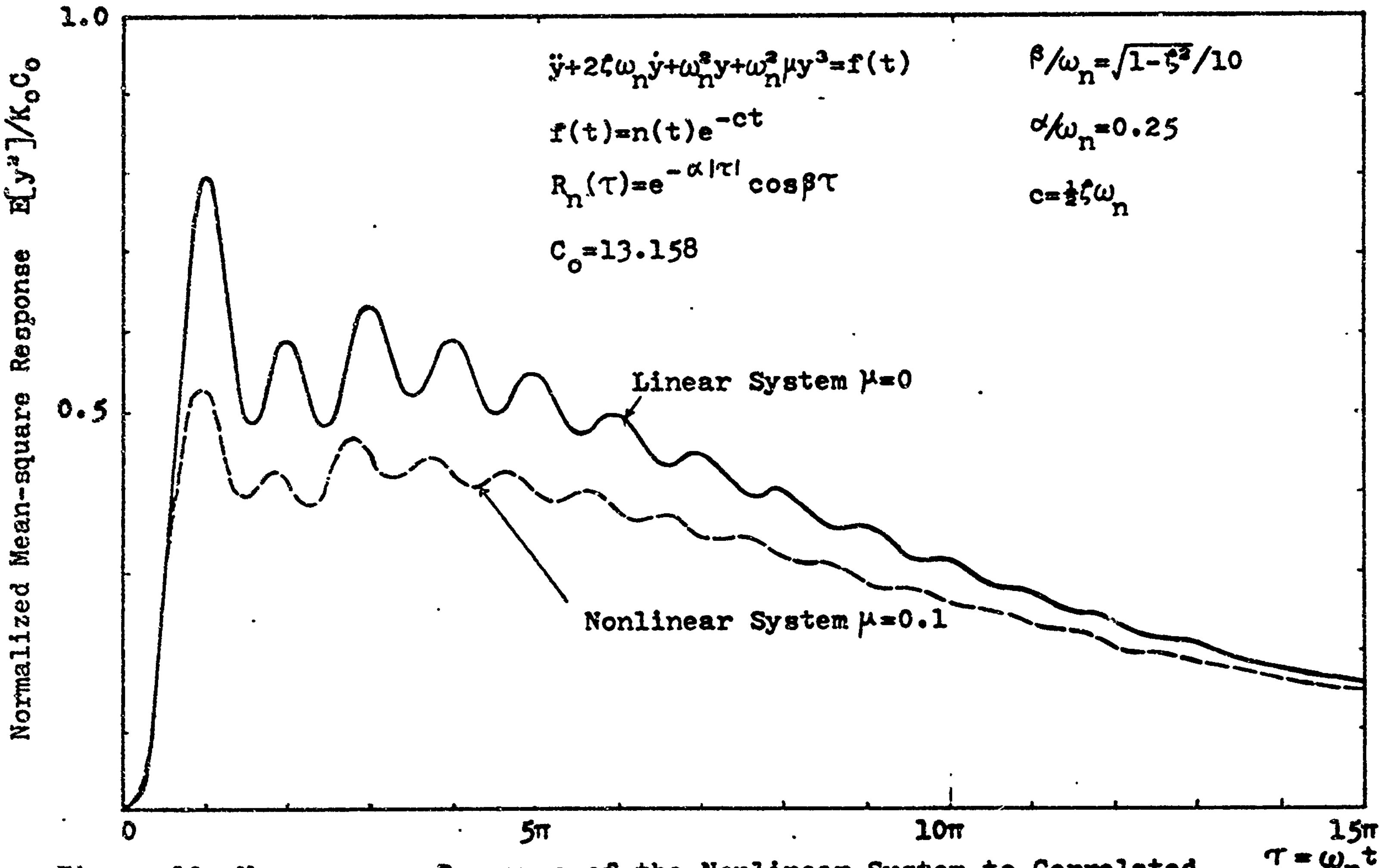


Figure 10: Mean-square Response of the Nonlinear System to Correlated Noise Modulated by an Exponential Envelope Function. System Damping $\zeta=0.05$

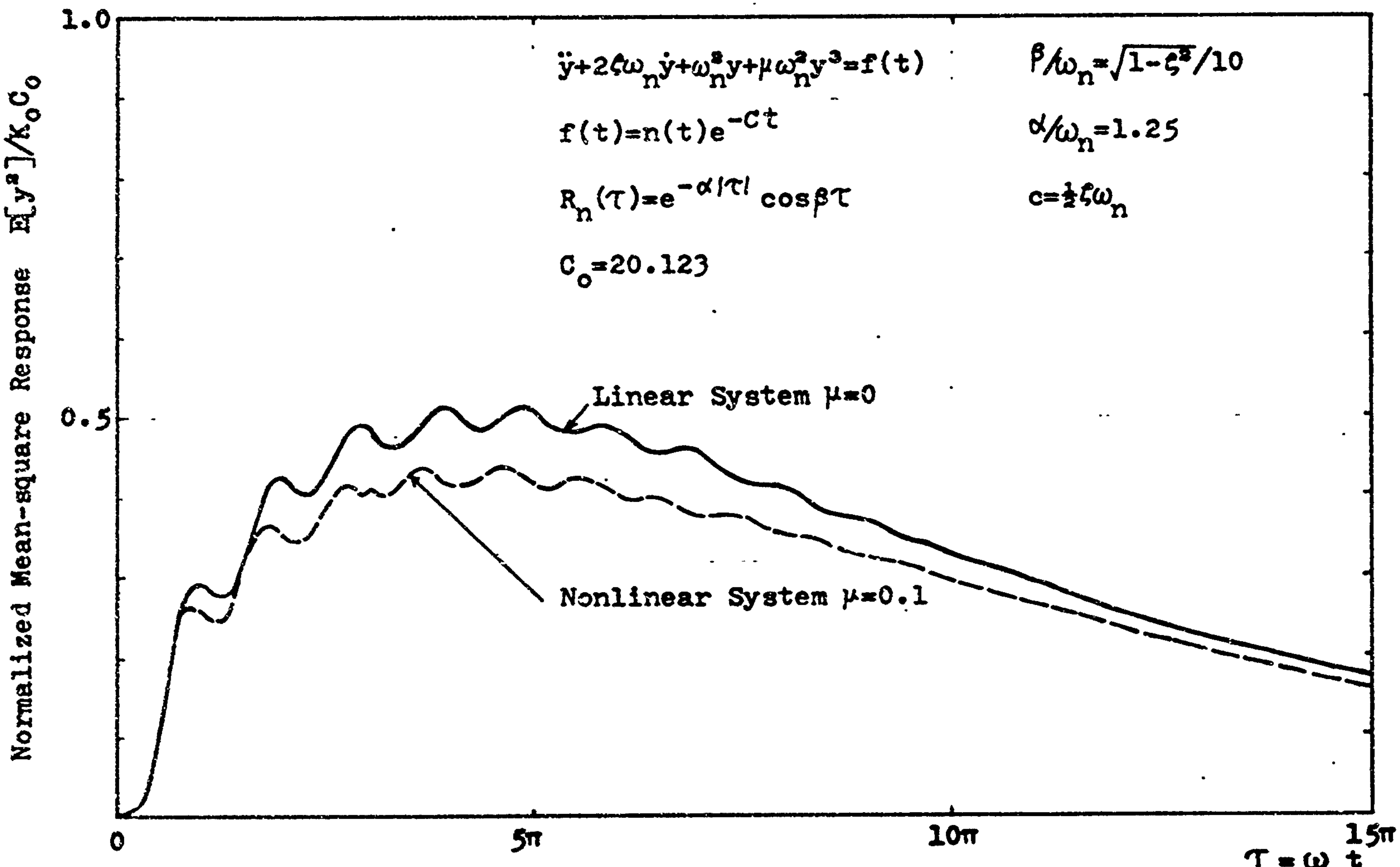


Figure 11: Mean-square Response of the Nonlinear System to Correlated Noise Modulated by an Exponential Envelope Function. System Damping $\zeta=0.05$

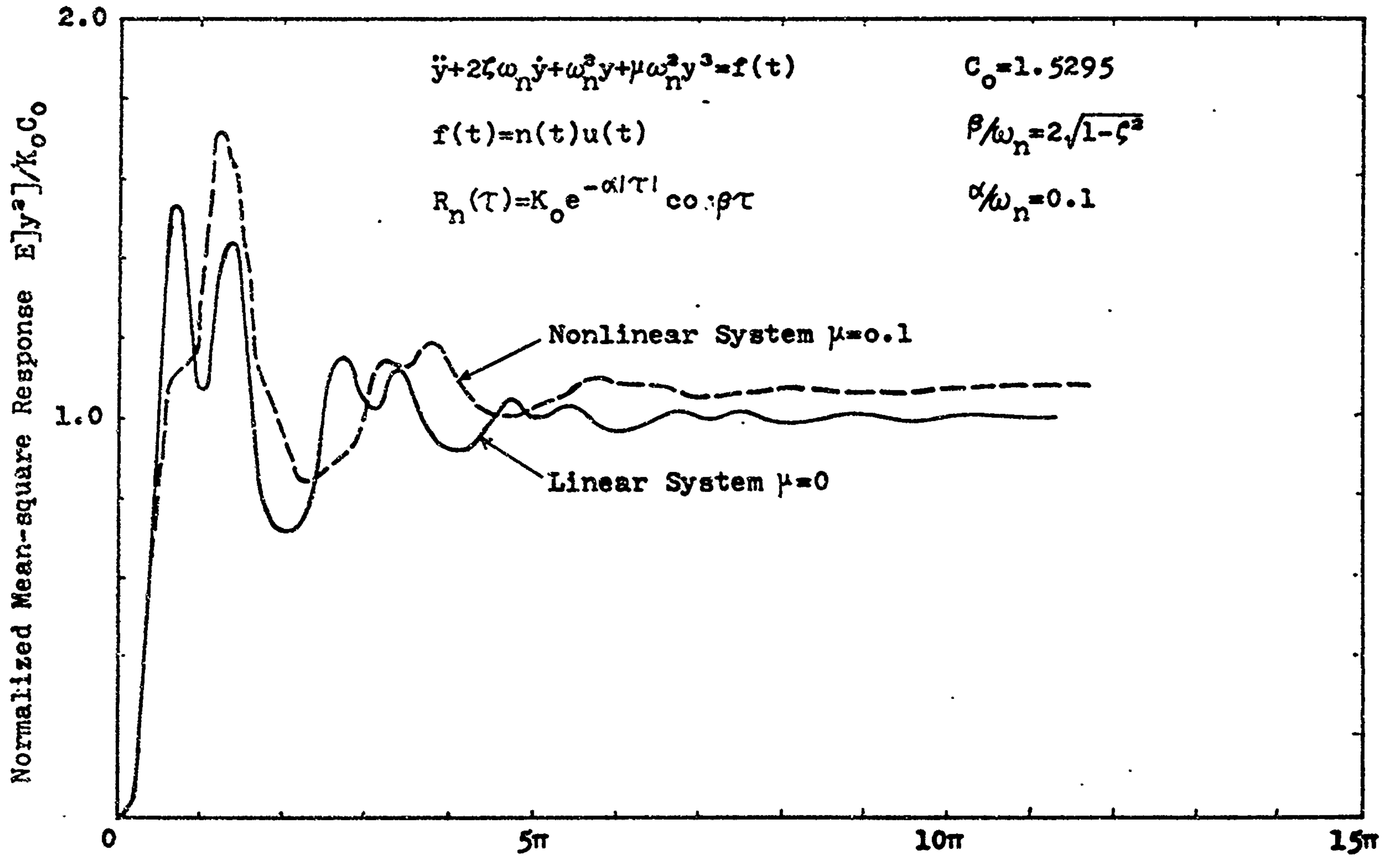


Figure 12: Mean-square Response of the Systems to Correlated Noise modulated by a Unit Step Function. System Damping $\zeta=0.1$

# APPENDIX

## Computation of $E[\dot{y}y^{2m+1}]$ and $E[y^{2m+2}]$

$$E[\dot{y}y^{2m+1}] = \frac{1}{2\pi[\det(K)]^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{y}y^{2m+1}\exp(-ay^2+2by\dot{y}-c\dot{y}^2)\,d\dot{y}\,dy$$

$$= \frac{1}{2\pi[\det(K)]^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot{y}\exp(-c\dot{y}^2)\left(\frac{b}{a}\,\dot{y}\right)^{2m+1}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$

$$\times \exp\left(\frac{b^2\dot{y}^2}{a}\right) \sum_{\nu=0}^{m} \frac{(2m+1;-1;2\nu)}{\nu!} \left(\frac{a}{4b^2}\right)^{\nu}\dot{y}^{-2\nu}\,d\dot{y} \qquad (1)$$

where

$$(2m+1;-1;2\nu) = (2m+1)(2m)(2m-1)\cdots(2m-2\nu+1) \quad \nu=1,2,\cdots$$

and

$$(2m+1;-1;0) = 1$$

$$E[\dot{y}y^{2m+1}] = \frac{1}{2\pi[\det(K)]^{\frac{1}{2}}} \sqrt{\frac{\pi}{a}}\left(\frac{b}{a}\right)^{2m+1} \frac{1}{a}\sqrt{\frac{\pi}{\alpha}}\,2^{-m-1}$$

$$\left[\sum_{=0}^{m} \frac{m!}{(m-\nu)!\nu!}\left(\frac{1}{\alpha}\right)^{m-\nu}\left(\frac{a}{b^2}\right)^{\nu}\right] \frac{(2m+1)!}{2^m m!} \qquad (2)$$

where

$$\alpha = (ac-b^2)/a$$

Noting that

$$\sum_{\nu=0}^{m} \frac{m!}{(m-\nu)!\nu!} \left(\frac{1}{\alpha}\right)^{m-\nu} \left(\frac{a}{b^2}\right)^{\nu} = \left(\frac{1}{\alpha} + \frac{a}{b^2}\right)^{m}$$

Eq. (2) reduces to

$$E[\dot{y}y^{2m+1}] = (E[y^2])^m \, E[y\dot{y}] \, \frac{(2m+1)!}{2^m m!} \tag{3}$$

$$E[y^{2m+2}] = \frac{1}{2\pi[\det(K)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^{2m+2} \exp(-ay^2+2by\dot{y}-c\dot{y}^2)\, dy\, d\dot{y}$$

$$= \frac{1}{2\pi[\det(K)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left[\left(-c+\frac{b^2}{a}\right)\dot{y}^2\right]\left(\frac{by}{a}\right)^{2m+2}$$

$$\times \sqrt{\frac{\pi}{a}} \sum_{\nu=0}^{m} \frac{(2m+2;-1;2\nu)}{\nu!}\left(\frac{a}{4b^2}\right)\dot{y}^{-2\nu} d\dot{y}$$

$$= (E[y^2])^{m+1} \, \frac{(2m+1)!}{2^m \, m!} \tag{4}$$

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may exceed its stationary value The analysis is extended to the multi-degree-of-freedom nonlinear system for the case of mutually uncorrelated noise.