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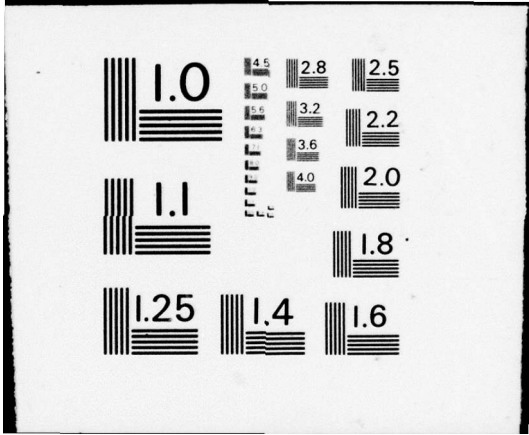
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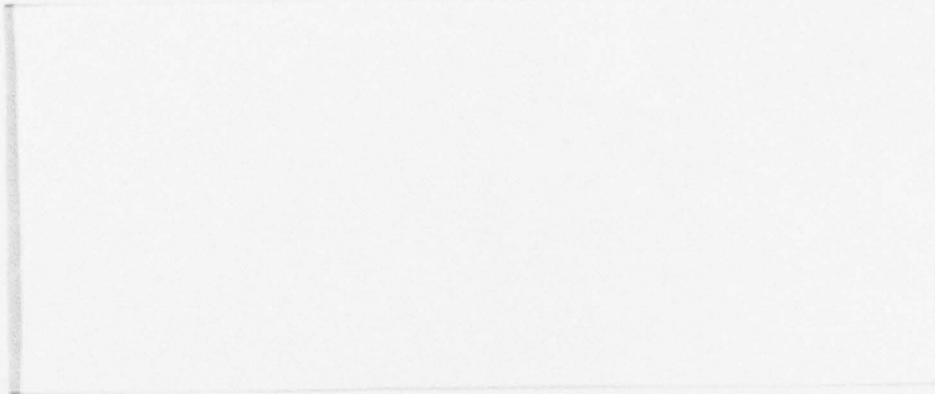
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SOME RESULTS ON SUBSET SELECTION PROBLEMS\*

by

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### INTRODUCTION

The shortcomings of the classical tests of homogeneity, i.e., testing the hypothesis of equality of parameters, have long been known. The only question answered by such a test is whether there is any difference at all among the available populations. Bahadur [2], Mosteller [58] and Paulson [61] were among the earliest research workers to recognize this and to formulate the problem as a multiple decision problem concerned with the selection and ordering of  $k$  populations.

In the two decades since these early papers, ranking and selection problems have become an active area of statistical research. There have been two approaches to these problems, the 'indifference zone' approach and the 'subset selection' approach. In the first approach, a single population (or a fixed number of populations) is chosen and is guaranteed to be the one of interest with a fixed probability  $P^*$  whenever the unknown parameters lie outside some subset, or zone of indifference, of the entire parameter space. This formulation is due to Bechhofer [11]. Other contributions to this problem are Bechhofer and Sobel [16], Bechhofer, Dunnett and Sobel [14], Sobel and Huyett [75], Chambers and Jarratt [20], Barr and Rizvi [7], Eaton [26], and Mahamunulu [56]. A quite adequate bibliography may be found in Santner [69] and Bechhofer, Kiefer and Sobel [15].

The second approach assumes no a priori information about the parameter space. A single population is not necessarily chosen; rather a subset of the given  $k$  populations is selected depending on the outcome of the experiment. It is guaranteed to contain the population of interest with probability  $P^*$ , the basic probability requirement in these procedures. This 'subset selection' formulation is due to Gupta [38], [41]. Some recent contributions to this aspect of the problem are Gnanadesikan [30], Gnanadesikan and Gupta [31], Gupta and McDonald [44], McDonald [57], Panchapakesan [60], Gupta and Panchapakasan [45], Gupta and Studden [46], Santner [69], Huang [50], Huang [49], Gupta and Huang [42].

The sequential and multistage aspects of the ranking and selection problems have been explored by Bechhofer, Dunnett and Sobel [14], Bechhofer [12], Bechhofer and Blumenthal [13], and Paulson [62], [63], [64], [65]. Nearly all of this work in sequential and multistage procedures is based on the indifference zone approach. Barron and Gupta [9] and Huang [50] consider sequential procedures using the subset selection approach.

An optimum theory was developed for the first approach by Bahadur [2], Bahadur and Goodman [3], Lehmann [55] and Eaton [26]. Contributions toward optimum properties of subset selection approach have also been made by Goel and Rubin [33], Govindarajulu and Harvey [36], Gupta [39], Deely and Gupta [25], Lehmann [54], Robbins [68], Seal [70], [71], [72] and Studden [77].

The main purpose of this thesis is to study some problems using the subset selection approach and make some contributions.

Chapter I deals with some selection and ranking procedures for the smallest unknown parameter of  $k$  Poisson populations. In Section 1.2, a procedure is derived to select a subset containing the best of several Poisson populations. In Section 1.3, a procedure conditioned on the total sum of the observations is proposed. A different selection procedure of the type suggested by Seal is considered in Section 1.4. In Section 1.5, selection of populations better than a standard is discussed. An application to a test of homogeneity is described in Section 1.6. Tables related to the selection procedures are given at the end of this chapter. These tables give the necessary constants to carry out the procedure and also evaluate the efficiency of the procedure in terms of the probability of a correct selection and the expected proportion in the selected subset under specified configurations of parameters.

Chapter II discusses some results on subset selection procedures for double exponential (Laplace) distributions. Section 2.1 deals with some characteristics and use of this distribution as a model. In Section 2.2, a selection procedure for the location parameters is proposed and studied using the subset selection approach. Also selection with respect to largest location parameter using the indifference zone approach is considered in Section 2.3. Section 2.4 gives a discussion of selecting the  $t$ -best populations. In Section 2.5 a procedure is proposed for subset selection with respect to the scale parameter. In Section 2.6, a test of homogeneity is given which is based on the sample median range. The distribution of a statistic associated with the procedure in Section 2.2 is considered in Section 2.7. Tables

of the upper percentage points of  $Y = \max_{1 \leq i \leq p} (X_i - X_0)$  where  $X_0, X_1, \dots, X_p$  are independent and identically distributed Laplace random variables with scale parameter unity are given at the end of this chapter.

In Chapter III, the subset selection approach is used to the problem of classification of  $k$  univariate normal populations. In Section 3.2, two classification rules with respect to the mean are proposed according as the  $k$  populations have (i) common known variance  $\sigma^2$  and (ii) common unknown variance  $\sigma^2$ . In Section 3.3, a classification rule with respect to the variance is given. These rules might not classify  $\pi_0$  as any one of the  $k$  populations. Hence different classification procedures, with respect to the mean and the reciprocal of the coefficient of variation, which classify  $\pi_0$  as at least one of the  $k$  populations are proposed and studied in Section 3.4 and Section 3.5, respectively.

Chapter IV deals with some selection procedures for the negative binomial populations. A statistic of type  $c \max_{1 \leq j \leq k} X_j - X_i$  is used in Section 4.2 where  $X_i$  denotes the number of failures before the  $r_i$ th success from the  $i$ th negative binomial population. In Section 4.3, a rule based on the same statistic as in Section 4.2 but conditioned on  $\sum X_i$ , the total number of observations, is investigated. The problem of selecting all populations better than a standard is considered in Section 4.4. An application is given in Section 4.5. For  $k = 2$  and various values of  $r$ ,  $t$  and  $P^*$ , the tables of the constants  $c_{15}(t)$  required for the procedure in Section 4.3 are given at the end of the chapter.

CHAPTER I  
ON SUBSET SELECTION PROCEDURES FOR POISSON POPULATIONS

1.1 Introduction

Poisson distribution has been used as a model in several statistical problems. As early as 1898, Bortkiewicz [18] used it to fit the data pertaining to the deaths by kicks from horses in a regiment. Poisson process is used as a model in many applied probability problems, for example, for the waiting time, for arrivals of calls at a telephone exchange, for arrivals of radioactive particles at a Geiger counter, etc.

In this chapter our object is to study the problem of comparing  $k$  Poisson distributions. Not much work has been done on this problem. More specifically, we consider the problem of selecting a subset of  $k$  Poisson populations including the best which is associated with the smallest value of the parameter. Gupta and Huang [43] have considered the selection problem according to the largest value of the parameter. However, a procedure of the type proposed by them does not work for the problem of selection with respect to the smallest parameter. Goel [32] has shown that the usual type of selection procedures do not exist for some values of the probability  $P^*$  of a correct selection. In this chapter, we propose a procedure different from that of Gupta and Huang [43] for subset selection which exists for all  $P^*$ . The rule is based on a result of Chapman [21] who showed that there is no unbiased estimator of the

ratio  $\frac{\lambda_1}{\lambda_2}$  with finite variance, where  $\lambda_1, \lambda_2$  are expected values of two independent random variables  $X_1, X_2$  with Poisson distributions, but that the estimator  $\frac{X_1}{X_2+1}$  is "almost unbiased".

Let  $\pi_1, \pi_2, \dots, \pi_k$  be  $k$  independent Poisson populations, i.e.  $\pi_i$  has a Poisson distribution with unknown parameter  $\lambda_i, i = 1, 2, \dots, k$ . Suppose that we take  $n$  independent observations  $X_{i1}, \dots, X_{in}$  from each population  $\pi_i, i=1, \dots, k$ . A sufficient statistic for  $\lambda_i$  is  $\sum X_{ij}$ , hence without loss of generality we will assume the sample size to be one. Let

$\lambda_{[1]} \leq \lambda_{[2]} \leq \dots \leq \lambda_{[k]}$  be the ordered values of the parameters; it is assumed that there is no a priori information available about the correct pairing of the ordered  $\lambda_{[i]}$  and the  $k$  given populations from which observations are taken.

Given any  $P^* (\frac{1}{k} < P^* < 1)$ , we wish to select a non-empty (small) subset of these  $k$  populations such that the subset contains the population corresponding to the parameter  $\lambda_{[1]}$  with probability at least  $P^*$ , no matter what the configuration of  $\lambda_1, \lambda_2, \dots, \lambda_k$  is. We use the notation CS for correct selection where CS means that the selected subset includes the best population. Therefore we are interested in defining a selection procedures  $R$  such that

$$\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(\text{CS}|R) \geq P^* \quad (1.1.1)$$

where  $\Omega$  is the set of all  $k$ -tuples  $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k), \lambda_i > 0, i = 1, 2, \dots, k$ .

Let  $X_1, X_2, \dots, X_k$  denote the independent observations from populations  $\pi_1, \pi_2, \dots, \pi_k$ , respectively. Let  $X_{(i)}$  be that value of  $X_1, \dots, X_k$  which is associated with  $\lambda_{[i]}$ ; of course  $X_{(i)}$  is unknown.

In Section 1.2, we discuss a subset selection rule so as to satisfy the basic probability requirement (1.1.1), and to find an upper bound for the expected subset size. In Section 1.3, we consider a conditional selection procedure conditioned on the total sum of the observations; a method for constructing the constants (conservative) and an upper bound for the expected subset size are derived for this conditional rule. Section 1.4 deals with a different selection procedure of the type suggested by Seal for the normal means problem. We also discuss the Seal type procedure conditioning on the total sum of the observation, in which case the selection constant can be determined precisely so as to satisfy the basic probability requirement. An exact expression for the expected subset size of the conditional Seal type procedure is stated in Theorem 1.4.5. Selection procedures for selecting a subset which contains all populations better than a standard are considered in Section 1.5. An application to a test of homogeneity is mentioned in Section 1.6. Tables related to the selection procedures are given at the end of this chapter.

## 1.2 The Unconditional Selection Procedure $R_1$

### (A) The Rule $R_1$ and the Probability of Correct Selection

$R_1$ : Select the population  $\pi_i$  in the subset if and only if

$$X_i \leq c_1 \min_{1 \leq j \leq k} x_j + c_1$$

where  $c_1 \geq 1$  is to be chosen so as to satisfy the basic probability requirement (1.1.1).

For  $i = 1, 2, \dots, k$ , let  $p_{\underline{\lambda}}(i) = P_{\underline{\lambda}}(\text{select population } \pi_{(i)} | R_1)$ .

Theorem 1.2.1.  $P_{\underline{\lambda}}(i)$  is a decreasing function in  $\lambda_{[i]}$  when all other  $\lambda$ 's are fixed and  $p_{\underline{\lambda}}(i)$  is an increasing function in  $\lambda_{[j]}$ ,  $j \neq i$ , when all other  $\lambda$ 's are fixed.

Proof. 
$$p_{\underline{\lambda}}(i) = P_{\underline{\lambda}}(\text{select population } \pi_{(i)} | R_1)$$

$$= P_{\underline{\lambda}}(X_{(i)} \leq c_1 \min_{1 \leq j \leq k} X_{(j)} + c_1)$$

$$= P_{\underline{\lambda}}(X_{(i)} \leq c_1 X_{(j)} + c_1, \quad j = 1, 2, \dots, k, j \neq i)$$

$$= \sum_{x=0}^{\infty} e^{-\lambda_{[i]}} \frac{\lambda_{[i]}^x}{x!} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^k \sum_{\ell = \langle \frac{x}{c_1} - 1 \rangle}^{\infty} e^{-\lambda_{[j]}} \frac{\lambda_{[j]}^{\ell}}{\ell!} \right\}$$

where  $\langle \alpha \rangle$  is the smallest integer  $\geq \alpha$ .

$$= \sum_{x=0}^{\infty} e^{-\lambda_{[i]}} \frac{\lambda_{[i]}^x}{x!} \prod_{\substack{j=1 \\ j \neq i}}^k \int_0^{\lambda_{[j]}} \frac{1}{\Gamma(\langle \frac{x}{c_1} - 1 \rangle + 1)} y^{\langle \frac{x}{c_1} - 1 \rangle} e^{-y} dy$$

$$= \sum_{x=0}^{\infty} f(x; \lambda_{[1]}, \dots, \hat{\lambda}_{[i]}, \dots, \lambda_{[k]}) e^{-\lambda_{[i]}} \frac{\lambda_{[i]}^x}{x!}$$

$$= E_{\lambda_{[i]}} f(x; \lambda_{[1]}, \dots, \hat{\lambda}_{[i]}, \dots, \lambda_{[k]}) \quad (1.2.1)$$

where  $f(x; \lambda_{[1]}, \dots, \hat{\lambda}_{[i]}, \dots, \lambda_{[k]}) = \prod_{\substack{j=1 \\ j \neq i}}^k \int_0^{\lambda_{[j]}} \frac{1}{\Gamma(\langle \frac{x}{c_1} - 1 \rangle + 1)} y^{\langle \frac{x}{c_1} - 1 \rangle} e^{-y} dy$

and  $\hat{a}$  denote that  $a$  is deleted. From (1.2.1), it is obvious that  $p_{\underline{\lambda}}(i)$  is increasing in  $\lambda_{[j]}$ ,  $j \neq i$ , when all other  $\lambda$ 's are fixed. On the other hand, for fixed  $\lambda_{[j]}$ ,  $j \neq i$ ,  $f$  is a decreasing function in  $x$  and Poisson distribution belongs to the SI (Stochastically increasing) family, so by a lemma on P. 112 Lehmann [53],

$P_{\underline{\lambda}}(i) = E_{\lambda_{[i]}} f(x; \lambda_{[1]}, \dots, \hat{\lambda}_{[i]}, \dots, \lambda_{[k]})$  is a decreasing function in  $\lambda_{[i]}$  when other  $\lambda$ 's are fixed. Hence, the Theorem is proved.

Let  $\Omega_0 = \{\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k), \lambda_1 = \lambda_2 = \dots = \lambda_k = \lambda, \lambda > 0\}$ .

Corollary 1.2.1.  $\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_1) = \inf_{\underline{\lambda} \in \Omega_0} P_{\underline{\lambda}}(CS|R_1)$

$$= \inf_{\lambda > 0} \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \left\{ \sum_{\ell=\langle \frac{x}{c_1} - 1 \rangle}^{\infty} e^{-\lambda} \frac{\lambda^{\ell}}{\ell!} \right\}^{k-1}.$$

Proof. The proof follows directly from Theorem 1.2.1.

It should be pointed out that the infimum depends on the common unknown  $\lambda$ ,  $0 < \lambda$ . In Section 1.7, we discuss numerical methods to determine this infimum and the constant for the selection rule.

Under the parameter space, the joint distribution of  $X_1, X_2, \dots, X_k$ , given  $\sum_{i=1}^k X_i = t$ , is a multinomial distribution with parameters  $t; \theta_1, \dots, \theta_k$ , where  $\theta_j = \frac{\lambda_j}{\sum \lambda_j}$ ,  $j = 1, \dots, k$ , i.e.

$$P(X_1 = x_1, \dots, X_k = x_k | \sum_{i=1}^k X_i = t) = \frac{t!}{x_1! \dots x_k!} \theta_1^{x_1} \dots \theta_k^{x_k} \quad (1.2.2)$$

Lemma 1.2.1. For any  $t$ ,  $t \geq 0$ ,  $c_1(t) \geq 1$  and for  $\underline{\lambda} \in \Omega_0$

$$P_{\underline{\lambda}}(X_1 \leq c_1(t) \min_{2 \leq j \leq k} X_j + c_1(t) | \sum_{i=1}^k X_i = t) \\ = \sum \frac{t!}{x_1! \dots x_k!} \left(\frac{1}{k}\right)^t$$

where the summation is over all  $x_i$ 's such that  $x_1 \leq c_1(t) \min_{2 \leq j \leq k} x_j + c_1(t)$ ,  $x_i \geq 0$ , for  $i = 1, 2, \dots, k$ ,  $\sum_{i=1}^k x_i = t$ .

$$\text{Let } A(k, t, c_1(t)) = \sum_{\substack{x_1 < c_1(t) \min_{2 \leq j \leq k} x_j + c_1(t) \\ x_i \geq 0, \sum x_i = t}} \frac{t!}{x_1! \dots x_k!} \left(\frac{1}{k}\right)^t \quad (1.2.3)$$

Theorem 1.2.2. For given  $P^*$ , any  $t, t \geq 0$ , let  $c_1(t)$  be the smallest non-negative number such that  $A(k, t, c_1(t)) \geq P^*$ . If  $c_1 = \sup \{c_1(t) : t \geq 0\}$ , then

$$\inf_{\lambda \in \Omega} P_{\lambda}(CS|R_1) \geq P^*.$$

Proof. For  $\lambda \in \Omega_0$ ,  $P_{\lambda}(CS|R_1) = P_{\lambda}(X_{(1)} \leq c_1 \min_{1 \leq j \leq k} X_{(j)} + c_1)$

$$\begin{aligned} &\geq P_{\lambda}(X_{(1)} \leq c_1(t) \min_{1 \leq j \leq k} X_{(j)} + c_1(t)) \\ &= \sum_{t=0}^{\infty} P_{\lambda}(X_{(1)} \leq c_1(t) \min_{2 \leq j \leq k} X_{(j)} + c_1(t) | \sum_{i=1}^k X_i = t) \\ &\quad \cdot P_{\lambda}(\sum_{i=1}^k X_i = t) \\ &= \sum_{t=0}^{\infty} A(k, t, c_1(t)) P_{\lambda}(\sum_{i=1}^k X_i = t) \\ &\geq P^* \end{aligned}$$

Thus, the Theorem follows from Corollary 1.2.1.

(B) An Upper Bound on the Expected Subset Size Associated with  $R_1$

Let  $S$  denote the size of the selected subset, then  $S$  is a random variable taking values  $1, 2, \dots, k$ . Let us consider the special case  $\lambda_{[1]} = \delta\lambda$ ,  $\delta < 1$ ,  $\lambda_{[2]} = \dots = \lambda_{[k]} = \lambda$ ,  $\lambda > \lambda_0 > 0$ , and denote the space of all slippage configurations of this type by  $\Omega_1$ . We discuss the expected subset size as follows.

Theorem 1.2.3.  $E_{\Omega_1}(S|R_1) \leq k - \left[ \inf_{t \geq [c_1]+1} g(t) + (k-1) \inf_{t \geq [c_1]+1} h(t) \right]$

$$\int_0^{(1+\delta)\lambda_0} \frac{1}{([c_1])!} y^{[c_1]} e^{-y} dy$$

where

$$g(t) = \sum_{s=0}^{\lfloor \frac{t-c_1}{1+c_1} \rfloor} \binom{t}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{t-s} \quad (1.2.4)$$

and

$$h(t) = \sum_{s=0}^{\lfloor \frac{t-c_1}{1+c_1} \rfloor} \binom{t}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{t-s} \quad (1.2.5)$$

Proof. For  $\lambda \in \Omega_1$ ,

$$\begin{aligned} E_{\lambda}(S|R_1) &= P_{\lambda}(X_{(1)} \leq c_1, \min_{2 \leq j \leq k} X_{(j)} + c_1) + (k-1) P_{\lambda}(X_{(k)} \leq c_1, \min_{1 \leq j \leq k-1} X_{(j)} + c_1) \\ &= k - P_{\lambda}(X_{(1)} > c_1, \min_{2 \leq j \leq k} X_{(j)} + c_1) - (k-1) P_{\lambda}(X_{(k)} > c_1, \min_{1 \leq j \leq k-1} X_{(j)} + c_1) \\ &\leq k - P_{\lambda}(X_{(1)} > c_1, X_{(2)} + c_1) - (k-1) P_{\lambda}(X_{(k)} > c_1, X_{(1)} + c_1) \\ &= k - \sum_{t=0}^{\infty} P_{\lambda}(X_{(1)} > c_1, X_{(2)} + c_1 \mid X_{(1)} + X_{(2)} = t) P_{\lambda}(X_{(1)} + X_{(2)} = t) \\ &\quad - (k-1) \sum_{t=0}^{\infty} P_{\lambda}(X_{(k)} > c_1, X_{(1)} + c_1 \mid X_{(1)} + X_{(k)} = t) P_{\lambda}(X_{(1)} + X_{(k)} = t) \\ &= k - \sum_{t=0}^{\infty} P_{\lambda}(X_{(2)} < \frac{t-c_1}{1+c_1} \mid X_{(1)} + X_{(2)} = t) P_{\lambda}(X_{(1)} + X_{(2)} = t) \\ &\quad - (k-1) \sum_{t=0}^{\infty} P_{\lambda}(X_{(1)} < \frac{t-c_1}{1+c_1} \mid X_{(1)} + X_{(k)} = t) P_{\lambda}(X_{(1)} + X_{(k)} = t) \end{aligned}$$

$$\begin{aligned}
&= k - \sum_{t=[c_1]+1}^{\infty} \sum_{s=0}^{\lfloor \frac{t-c_1}{1+c_1} \rfloor} \binom{t}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{t-s} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!} \\
&\quad - (k-1) \sum_{t=[c_1]+1}^{\infty} \sum_{s=0}^{\lfloor \frac{t-c_1}{1+c_1} \rfloor} \binom{t}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{t-s} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!}
\end{aligned}$$

where  $[a]$  is the greatest integer  $\leq a$ .

$$\begin{aligned}
&\leq k - \inf_{t \geq [c_1]+1} g(t) \sum_{t=[c_1]+1}^{\infty} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!} \\
&\quad - (k-1) \inf_{t \geq [c_1]+1} h(t) \sum_{t=[c_1]+1}^{\infty} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!} \\
&= k - \left[ \inf_{t \geq [c_1]+1} g(t) + (k-1) \inf_{t \geq [c_1]+1} h(t) \right] \int_0^{(1+\delta)\lambda} \frac{1}{([c_1]!)!} y^{[c_1]} e^{-y} dy \\
&\leq k - \left[ \inf_{t \geq [c_1]+1} g(t) + (k-1) \inf_{t \geq [c_1]+1} h(t) \right] \int_0^{(1+\delta)\lambda_0} \frac{1}{([c_1]!)!} y^{[c_1]} e^{-y} dy
\end{aligned}$$

This completes the proof.

### 1.3 The Conditional Procedure $R_2$

$R_2$ : Select the population  $\pi_i$  in the subset if and only if

$$X_i \leq c_2(t) \min_{1 \leq j \leq k} X_j + c_2(t), \text{ given } \sum_1^k X_i = t \quad (1.3.1)$$

where  $t \geq 0$  and  $c_2(t) \geq 1$  is the smallest non-negative number chosen to satisfy the basic probability requirement (1.1.1).

For this rule  $R_2$  we obtain an exact result for  $k = 2$  in Theorem 1.3.1. For  $k \geq 3$ , we have a lower bound for the probability of a correct selection in Theorem 1.3.4.

(A) Property of the Rule  $R_2$ 

A monotonicity property of the rule  $R_2$  is discussed in the following theorem. As before, let  $p_{\lambda}(i)$  denote the probability of selecting population  $\pi_{(i)}$  using rule  $R_2$ .

Theorem 1.3.1. For  $i < j$ , we have  $p_{\lambda}(i) \geq p_{\lambda}(j)$ .

Proof. Since

$$\begin{aligned}
 p_{\lambda}(i) &= P_{\lambda}(\text{select population } \pi_{(i)}) \\
 &= P_{\lambda}(X_{(i)} \leq c_2(t) \min_{\ell \neq i} X_{(\ell)} + c_2(t) \mid \sum_{\ell=1}^k X_{(\ell)} = t) \\
 &= \sum_{x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k} P_{\lambda}(x_i \leq \frac{c_2(t)(x_i + x_j) + c_2(t)}{1 + c_2(t)} \mid \sum_{\ell=1}^k x_{\ell} = t) \\
 &= \binom{x_i + x_j}{x_i} \left( \frac{p_{[i]}}{p_{[i]} + p_{[j]}} \right)^{x_i} \left( \frac{p_{[j]}}{p_{[i]} + p_{[j]}} \right)^{x_j} \\
 &= \binom{x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k, x_i + x_j}{x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k} \frac{p_{[i]}^{x_i} p_{[j]}^{x_j}}{(p_{[i]} + p_{[j]})^{x_i + x_j}}
 \end{aligned}$$

where  $p_{[i]} = \frac{\lambda_{[i]}}{\sum_{\ell=1}^k \lambda_{[\ell]}}$  and  $\hat{x}_i$  denote that  $x_i$  is deleted. Note that

when  $x_i$  and  $x_j$  are interchanged, the second part in the above summand remains unchanged, and Binomial distribution belongs to the SI family, hence  $p_{\lambda}(i)$  is decreasing in  $\frac{p_{[i]}}{p_{[i]} + p_{[j]}}$ . So, if  $i < j$ , we have

$$p_{\lambda}(i) \geq p_{\lambda}(j).$$

(B) The Probability of a Correct Selection for  $R_2$ 

Theorem 1.3.2. For a given  $P^*$ ,  $\frac{1}{k} < P^* < 1$ ,  $k = 2$  and any  $t \geq 0$ , let  $c_2(t)$  be the smallest value such that

$$P_{\Omega_0}(X_1 \leq \frac{c_2(t)(1+t)}{1+c_2(t)} \mid X_1 + X_2 = t) \geq P^*. \quad (1.3.2)$$

Then,  $\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{\underline{\lambda} \in \Omega_0} P_{\underline{\lambda}}(CS|R_2) \geq P^*$ .

Proof. For  $\underline{\lambda} \in \Omega$ ,

$$\begin{aligned} P_{\underline{\lambda}}(CS|R_2) &= P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(2)} + c_2(t) \mid X_{(1)} + X_{(2)} = t) \\ &= P_{\underline{\lambda}}(X_{(1)} \leq \frac{c_2(t)(1+t)}{1+c_2(t)} \mid X_{(1)} + X_{(2)} = t) \\ &= \sum_{x=0}^t \binom{t}{x} \left( \frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}} \right)^x \left( 1 - \frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}} \right)^{t-x} \end{aligned}$$

For fixed  $\lambda_{[2]}$ ,  $\frac{1}{1 + \frac{\lambda_{[2]}}{\lambda_{[1]}}}$  increases with  $\lambda_{[1]}$  to  $\frac{1}{2}$ , and since Binomial

distribution belongs to the SI family, so the right hand side of the above expression decreases as  $\frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}}$  increases to the value  $\frac{1}{2}$ .

Hence,  $\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{\underline{\lambda} \in \Omega_0} P_{\underline{\lambda}}(CS|R_2)$ . Thus complete the proof.

For  $k \geq 3$ , we need the following definitions in order to discuss the least favorable configuration of the probability of a correct selection of the conditional rule.

Definition 1.3.1. If  $a_1 \geq a_2 \geq \dots \geq a_m$ ,  $b_1 \geq b_2 \geq \dots \geq b_m$ ,

$\sum_{i=1}^r a_i \geq \sum_{i=1}^r b_i$  for  $r = 1, 2, \dots, m-1$ , and  $\sum_{i=1}^m a_i = \sum_{i=1}^m b_i$ , then

$\underline{a} = (a_1, a_2, \dots, a_m)$  is said to majorize  $\underline{b} = (b_1, b_2, \dots, b_m)$ , written  $\underline{a} \succ \underline{b}$  or equivalently  $\underline{b} \prec \underline{a}$ .

**Definition 1.3.2.** If a function  $\varphi$  satisfies the property that  $\varphi(\underline{x}) \leq \varphi(\underline{y})$  ( $\varphi(\underline{x}) \geq \varphi(\underline{y})$ ) whenever  $\underline{x} \succ \underline{y}$ , then  $\varphi$  is called a Schur-concave (Schur-convex) function.

We need the following lemma, due to Rinott [67], which is stated below without proof.

**Lemma 1.3.1.** Let  $\underline{X} = (X_1, \dots, X_k)$  have the multinomial distribution

$$P(\underline{X} = \underline{x}) = \binom{N}{x_1, \dots, x_k} \prod_{i=1}^k \theta_i^{x_i}$$

where  $\underline{x} = (x_1, \dots, x_k)$ ,  $\sum_{i=1}^k x_i = N$ , and  $\sum_{i=1}^k \theta_i = 1$ .

Let  $\phi(x_1, \dots, x_k)$  be a Schur function. Then  $E_{\underline{\theta}} \phi(\underline{X})$  is a Schur function.

Let  $\Omega_2 = \{\underline{\lambda} = (\lambda_1, \dots, \lambda_k) : \lambda_{[1]} = \dots = \lambda_{[k-1]} = \lambda, \lambda_{[k]} = \lambda', 0 < \lambda < \lambda'\}$ .

**Theorem 1.3.3.**  $\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{\underline{\lambda} \in \Omega_2} P_{\underline{\lambda}}(CS|R_2)$

**Proof.**  $P_{\underline{\lambda}}(CS|R_2) = P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) \mid \min_{2 \leq j \leq k} X_{(j)} + c_2(t) \mid \sum_{i=1}^k X_i = t)$

$$= \sum_{y_1=0}^t \binom{t}{y_1} p_1^{y_1} (1-p_1)^{t-y_1} \cdot \sum_{y_2, \dots, y_k} \binom{t-y_1}{y_2, \dots, y_k} \prod_{j=2}^k \left(\frac{p_j}{1-p_1}\right)^{y_j}$$

where  $p_i = \frac{\lambda_{[i]}}{\sum_{j=1}^k \lambda_{[j]}}$ ,  $i = 1, \dots, k$  and the summation is over all those  $y_2, \dots, y_k$  such that  $y_j \geq \frac{y_1 - c_2(t)}{c_2(t)}$ ,  $j = 2, \dots, k$  and  $\sum_{j=2}^k y_j = t - y_1$ .

Let

$$\phi_{y_1}(y_2, \dots, y_k) = \begin{cases} 1 & \text{if } \min_{2 \leq i \leq k} y_i \geq \frac{y_1 - c_2(t)}{c_2(t)} \\ 0 & \text{otherwise} \end{cases}$$

then  $P_{\underline{\lambda}}(CS|R_2)$  can be written as  $E_{Y_1}[E[\phi_{Y_1}(Y_2, \dots, Y_k) | Y_1 = y_1]]$  where the joint distribution of  $Y_1, \dots, Y_k$  is a multinomial distribution with parameters  $t; p_1, \dots, p_k$ . It is easy to see that for a fixed  $y_1$ ,

$\phi_{y_1}(y_2, \dots, y_k)$  is a Schur-concave function. By Lemma 1.3.1.,  $E[\phi_{Y_1}(Y_2, \dots, Y_k) | Y_1 = y_1]$  is a Schur-concave function in  $\frac{p_2}{1-p_1}, \dots, \frac{p_k}{1-p_1}$ .

Now, since  $p_j \geq p_1, j = 2, \dots, k$  and  $p_2 + \dots + p_k = 1 - p_1$ , we have

$(\frac{p_2}{1-p_1}, \dots, \frac{p_k}{1-p_1}) \succ (\frac{p_1}{1-p_1}, \dots, \frac{p_1}{1-p_1}, \frac{1-(k-1)p_1}{1-p_1})$ . Hence  $P_{\underline{\lambda}}(CS|R_2)$  is minimized

when  $p_1 = p_2 = \dots = p_{k-1}$  and  $p_k = 1 - (k-1)p_1$ , or when

$\lambda_{[1]} = \dots = \lambda_{[k-1]} = \lambda, \lambda_{[k]} = \lambda' > \lambda$ . Thus the proof is completed.

Under the parameter space  $\Omega_2$ , the joint distribution of  $X_1, \dots, X_k$  given  $\sum_{i=1}^k X_i = t$ , is a multinomial distribution with parameters  $t; p_1, \dots, p_k$ ; where  $p_1 = \dots = p_{k-1} = \frac{\lambda}{(k-1)\lambda + \lambda'} = p$ ,

$$p_k = \frac{\lambda'}{(k-1)\lambda + \lambda'} = q, p < q, \text{ i.e.,}$$

$$P_{\underline{\lambda}}(X_1 = x_1, \dots, X_k = x_k | \sum_{i=1}^k X_i = t) = \frac{t!}{x_1! \dots x_k!} p^{\sum_{i=1}^{k-1} x_i} q^{x_k}.$$

Theorem 1.3.4.

$$\inf_{\lambda \in \Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{0 < \lambda < \lambda'} \sum_{\substack{x_1 < c_2(t) \\ j \neq 1 \\ x_i \geq 0, \sum x_i = t}} \frac{t!}{x_1! \dots x_k!} \left[ \frac{1}{k-1 + \frac{\lambda'}{\lambda}} \right]^t \left( \frac{\lambda'}{\lambda} \right)^{x_k}$$

Proof. For  $\underline{\lambda} \in \Omega_2$ ,

$$\begin{aligned} P_{\underline{\lambda}}(CS|R_2) &= P_{\underline{\lambda}}\{X(1) \leq c_2(t) \min_{j \neq 1} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t\} \\ &= \sum_{\substack{x_1 \leq c_2(t) \min_{j \neq 1} x_j + c_2(t) \\ x_i \geq 0, \sum x_i = t}} \frac{t!}{x_1! \dots x_k!} \left[ \frac{\lambda}{(k-1)\lambda + \lambda'} \right]^{\sum_{i=1}^{k-1} x_i} \left[ \frac{\lambda'}{(k-1)\lambda + \lambda'} \right]^{x_k} \end{aligned}$$

The Theorem follows from Theorem 1.3.3. after simplification.

Theorem 1.3.5. For  $k \geq 3$ , and for any  $P^*$ , let  $P_2^* = 1 - \frac{1-P^*}{k-1}$ ,  $0 \leq r \leq t$ , let  $c_2(r)$  be the smallest value such that

$$P_{\Omega_0}(X_1 \leq c_2(r) X_2 + c_2(r) \mid X_1 + X_2 = r) \geq P_2^* .$$

If  $c_2(t) = \max \{c_2(r) : 0 \leq r \leq t\}$ , then

$$\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_2) \geq P^*$$

Proof. For  $\underline{\lambda} \in \Omega$ ,

$$\begin{aligned} P_{\underline{\lambda}}(CS|R_2) &= P_{\underline{\lambda}}\{X(1) \leq c_2(t) \min_{2 \leq j \leq k} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t\} \\ &\geq 1 - \sum_{j=2}^k P_{\underline{\lambda}}\{X(1) > c_2(t) X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t\} \\ &= 1 - \sum_{j=2}^k [1 - P_{\underline{\lambda}}\{X(1) \leq c_2(t) X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t\}] \\ &= 2 - k + \sum_{j=2}^k P_{\underline{\lambda}}\{X(1) \leq c_2(t) X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t\} \end{aligned}$$

now, we note that for fixed  $j$  ( $j = 2, \dots, k$ ),

$$\begin{aligned}
P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)} + c_2(t) \mid \sum_{i=1}^k X_i = t) &= \sum_{r=0}^t P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)} \\
&\quad + c_2(t) \mid X_{(1)} + X_{(j)} = r) \cdot P_{\underline{\lambda}}(X_{(1)} + X_{(j)} = r) \\
&= \sum_{r=0}^t P_{\underline{\lambda}}(X_{(1)} \leq \frac{c_2(t)(1+r)}{1+c_2(t)} \mid X_{(1)} + X_{(j)} = r) P_{\underline{\lambda}}(X_{(1)} + X_{(j)} = r) \\
&= \sum_{r=0}^t \left[ \frac{c_2(t)(1+r)}{1+c_2(t)} \right] \sum_{s=0}^r \binom{r}{s} \left( \frac{p_1}{p_1+p_j} \right)^s \left( \frac{p_j}{p_1+p_j} \right)^{r-s} P_{\underline{\lambda}}(X_{(1)} + X_{(j)} = r)
\end{aligned}$$

where  $p_{\ell} = \frac{\lambda_{[\ell]}}{\sum_{i=1}^k \lambda_{[i]}}$ , but Binomial distribution belongs to the SI family,

so infimum of the expression

$$\left[ \frac{c_2(t)(1+r)}{1+c_2(t)} \right] \sum_{s=0}^r \binom{r}{s} \left( \frac{p_1}{p_1+p_j} \right)^s \left( \frac{p_j}{p_1+p_j} \right)^{r-s}$$

takes place when  $p_1 = p_j$ , i.e., when  $\lambda_{[1]} = \lambda_{[j]}$ . Hence,

$$\begin{aligned}
\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)} + c_2(t) \mid \sum_{i=1}^k X_i = t) &= \inf_{\underline{\lambda} \in \Omega_0} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)} \\
&\quad + c_2(t) \mid \sum_{i=1}^k X_i = t)
\end{aligned}$$

and for  $\underline{\lambda} \in \Omega_0$ ,

$$\begin{aligned}
P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) X_{(j)} + c_2(t) \mid \sum_1^k X_i = t) \\
&= \sum_{r=0}^t P_{\underline{\lambda}}(X_1 \leq c_2(t) X_2 + c_2(t) \mid X_1 + X_2 = r) \cdot P_{\underline{\lambda}}(X_1 + X_2 = r) \\
&\geq \sum_{r=0}^t P_{\underline{\lambda}}(X_1 \leq c_2(r) X_2 + c_2(r) \mid X_1 + X_2 = r) \cdot P(X_1 + X_2 = r) \\
&\geq P_2^*
\end{aligned}$$

Thus, we have the result.

Hence, for each  $k$  and  $P^*$ , Theorem 1.3.5. guarantees the existence of  $c_2(t)$  and gives a method to find  $c_2(t)$  for given  $\sum_1^k X_i = t$  such that  $P_{\underline{\lambda}}(CS|R_2) \geq P^*$  for any  $\underline{\lambda} \in \Omega$ .

(C) An Upper Bound on the Expected Subset Size for  $R_2$

For the procedure  $R_2$ , the subset size  $S$  of the selected subset is a random variable which can take on only integer values from 1 to  $k$ , inclusively. For any fixed values of  $k$  and  $P^*$ , the expected size of the selected subset is a function of the true configuration  $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ . Now, consider the special case,  $\lambda_{[1]} = \delta\lambda$ ,  $\delta < 1$ ,  $\lambda_{[2]} = \dots = \lambda_{[k]} = \lambda$ ,  $\lambda > 0$ . Let us denote the space of all slippage configuration of the type discussed here by  $\Omega_3$ . We investigate the expected subset size as follows:

$$\begin{aligned}
\text{Theorem 1.3.6. } E_{\Omega_3}(S|R_2) &\leq k - \sum_{r=0}^t \frac{[ \frac{r-c_2(t)}{1+c_2(t)} ] - 1}{\sum_{s=0}^r} \binom{r}{s} \{ \delta^{r-s} + (k-1)\delta^s \} \cdot \\
&\quad \binom{t}{r} \cdot \frac{(k-2)t-r}{(k-1+\delta)^t}
\end{aligned}$$

Proof.

$$\begin{aligned}
 E_{\Omega_3}(S|R_2) &= P_{\underline{\lambda}}(\pi(1) \text{ is selected}) + \sum_{j=2}^k P_{\underline{\lambda}}(\pi(j) \text{ is selected}) \\
 &= P_{\underline{\lambda}}(X(1) \leq c_2(t) \min_{2 \leq j \leq k} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &\quad + (k-1) P_{\underline{\lambda}}(X(k) \leq c_2(t) \min_{1 \leq j \leq k-1} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &= k - P_{\underline{\lambda}}(X(1) > c_2(t) \min_{2 \leq j \leq k} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &\quad - (k-1) P_{\underline{\lambda}}(X(k) > c_2(t) \min_{1 \leq j \leq k-1} X(j) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &\leq k - P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &\quad - (k-1) P_{\underline{\lambda}}(X(k) > c_2(t) X(1) + c_2(t) \mid \sum_{i=1}^k X_i = t) \\
 &= k - \frac{1}{P_{\underline{\lambda}}(\sum_{i=1}^k X_i = t)} P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t), \sum_{i=1}^k X_i = t) \\
 &\quad - \frac{k-1}{P_{\underline{\lambda}}(\sum_{i=1}^k X_i = t)} P_{\underline{\lambda}}(X(k) > c_2(t) X(1) + c_2(t), \sum_{i=1}^k X_i = t).
 \end{aligned}$$

Now, we note that

$$\begin{aligned}
 &P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t), \sum_{i=1}^k X_i = t) \\
 &= \sum_{r=0}^t P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t), X(1) + X(2) = r, \sum_{i \neq 1,2} X_i = t-r) \\
 &= \sum_{r=0}^t P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t), \\
 &\quad X(1) + X(2) = r \mid \sum_{i \neq 1,2} X_i = t-r) \cdot P_{\underline{\lambda}}(\sum_{i \neq 1,2} X_i = t-r) \\
 &= \sum_{r=0}^t P_{\underline{\lambda}}(X(1) > c_2(t) X(2) + c_2(t), X(1) + X(2) = r) \cdot P_{\underline{\lambda}}(\sum_{i \neq 1,2} X_i = t-r)
 \end{aligned}$$

since the event  $\{ \sum_{i \neq 1,2} X(i) = t-r \}$  is independent of the event

$$\{X(1) > c_2(t) X(2) + c_2(t), X(1) + X(2) = r\}.$$

$$= \sum_{r=0}^t P_{\lambda} (X(1) > c_2(t) X(2) + c_2(t) | X(1) + X(2) = r) \cdot P_{\lambda} (X(1) + X(2) = r) \\ \cdot P_{\lambda} ( \sum_{i \neq 1,2} X(i) = t-r )$$

and a similar expression for  $P_{\lambda} (X(k) > c_2(t) X(1) + c_2(t), \sum_{i=1}^k X_i = t)$ . So,

$$E_{\Omega_3} (S | R_2) \leq k - \frac{1}{P_{\lambda} ( \sum_{i=1}^k X_i = t )} \sum_{r=0}^t P_{\lambda} (X(1) > c_2(t) X(2) + c_2(t) | X(1) + X(2) = r) \\ \cdot P_{\lambda} (X(1) + X(2) = r) \cdot P_{\lambda} ( \sum_{i \neq 1,2} X(i) = t-r ) \\ - \frac{k-1}{P_{\lambda} ( \sum_{i=1}^k X_i = t )} \sum_{r=0}^t P_{\lambda} (X(k) > c_2(t) X(1) + c_2(t) | X(1) + X(k) = r) \\ \cdot P_{\lambda} (X(1) + X(k) = r) \cdot P_{\lambda} ( \sum_{i \neq 1,k} X(i) = t-r ) \\ = k - \frac{1}{P_{\lambda} ( \sum_{i=1}^k X_i = t )} \sum_{r=0}^t P_{\lambda} (X(2) < \frac{r - c_2(t)}{1 + c_2(t)} | X(1) + X(2) = r) \\ \cdot P_{\lambda} (X(1) + X(2) = r) \cdot P_{\lambda} ( \sum_{i \neq 1,2} X(i) = t-r ) \\ - \frac{k-1}{P_{\lambda} ( \sum_{i=1}^k X_i = t )} \sum_{r=0}^t P_{\lambda} (X(1) < \frac{r - c_2(t)}{1 + c_2(t)} | X(1) + X(k) = r) \\ \cdot P_{\lambda} (X(1) + X(k) = r) \cdot P_{\lambda} ( \sum_{i \neq 1,k} X(i) = t-r )$$

$$\begin{aligned}
&= k - \frac{t!}{[(k-1+\delta)\lambda]^t e^{-(k-1+\delta)\lambda}} \sum_{r=0}^t \sum_{s=0}^{\left[\frac{r-c_2(t)}{1+c_2(t)}\right]-1} \binom{r}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{r-s} \\
&\quad \frac{[(1+\delta)\lambda]^r e^{-(1+\delta)\lambda}}{r!} \frac{[(k-2)\lambda]^{t-r} e^{-(k-2)\lambda}}{(t-r)!} \\
&- \frac{(k-1)t!}{[(k-1+\delta)\lambda]^t e^{-(k-1+\delta)\lambda}} \sum_{r=0}^t \sum_{s=0}^{\left[\frac{r-c_2(t)}{1+c_2(t)}\right]-1} \binom{r}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{r-s} \\
&\quad \frac{[(1+\delta)\lambda]^r e^{-(1+\delta)\lambda}}{r!} \cdot \frac{[(k-2)\lambda]^{t-r} e^{-(k-2)\lambda}}{(t-r)!}
\end{aligned}$$

After simplifying, we have the result.

#### 1.4 A Different Selection Procedure

##### (A) The Selection Procedure $R_3$ and Its Expected Subset Size

In this section we consider a selection procedure of the type suggested by Seal [70].

$R_3$ : Select population  $\pi_i$  if and only if

$$X_i \leq c_3 + \frac{c_3}{k-1} \sum_{j \neq i} X_j$$

where  $c_3 \geq 1$  is the smallest constant determined from  $P(CS|R_3) \geq P^*$ .

Theorem 1.4.1.  $\inf_{\lambda \in \Omega} P_{\lambda}(CS|R_3) = \inf_{\lambda \in \Omega_0} P_{\lambda}(CS|R_3)$

Proof. For  $\lambda \in \Omega$ ,

$$\begin{aligned}
P_{\lambda}(CS|R_3) &= P_{\lambda}\left\{X_{(1)} \leq c_3 + \frac{c_3}{k-1} \sum_{j=2}^k X_{(j)}\right\} \\
&= \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \sum_{\ell = \langle (k-1)\left(\frac{i}{c_3} - 1\right) \rangle}^{\infty} e^{-\sum_{j=2}^k \lambda[j]} \frac{\left(\sum_{j=2}^k \lambda[j]\right)^{\ell}}{\ell!}
\end{aligned}$$

$$= \sum_{i=0}^{\infty} e^{-\lambda_{[1]}} \frac{\lambda_{[1]}^i}{i!} \left[ \sum_{j=2}^k \lambda_{[j]} \frac{1}{\Gamma(\langle (k-1)(\frac{i}{c_3} - 1) \rangle)} \int_0^{\langle (k-1)(\frac{i}{c_3} - 1) \rangle - 1} e^{-y} dy \right].$$

The proof follows by observing that the expression in the square brackets is a monotone increasing function of  $\sum_{j=2}^k \lambda_{[j]}$ .

By using an analogous argument as in the proof of Theorem 1.2.2., we have the following theorem.

**Theorem 1.4.2.** For any  $P^*$ , any  $t$ ,  $t \geq 0$ , let  $c_3(t)$  be the smallest value such that

$$\left[ \frac{(k-1)c_3(t) + tc_3(t)}{k-1 + c_3(t)} \right] \sum_{i=0}^{\infty} \binom{t}{i} \left(\frac{1}{k}\right)^i \left(\frac{k-1}{k}\right)^{t-i} \geq P^*.$$

If  $c_3 = \sup \{c_3(t) : t \geq 0\}$ , then  $\inf_{\lambda \in \Omega} P_{\lambda}(CS|R_3) \geq P^*$ .

Consider the special configuration  $\lambda_{[1]} = \delta\lambda$ ,  $\delta < 1$ ;  
 $\lambda_{[2]} = \dots = \lambda_{[k]} = \lambda$ ,  $\lambda > \lambda_0 > 0$ . Using the same notation as in Section 1.2, the space of all such slippage configuration is denoted by  $\Omega_1$ . In the following theorem, we give an upper bound for the expected subset size  $S$ .

**Theorem 1.4.3.**  $E_{\Omega_1}(S|R_3) \leq \sup_{r \geq 2} g(r) + (k - \sup_{r \geq 2} g(r))(1 + (k-1+\delta)\lambda_0) e^{-(k-1+\delta)\lambda_0}$

where

$$g(r) = \sum_{s=0}^{\lfloor \frac{c_3(r+k-1)}{c_3+k-1} \rfloor} \left\{ \binom{r}{s} \left( \frac{\delta}{k-1+\delta} \right)^s \left( 1 - \frac{\delta}{k-1+\delta} \right)^{r-s} \right. \\ \left. + (k-1) \binom{r}{s} \left( \frac{1}{k-1+\delta} \right)^s \left( 1 - \frac{1}{k-1+\delta} \right)^{r-s} \right\}$$

Proof.

$$\begin{aligned} E_{\Omega_1}(S|R_3) &= P\{X_{(1)} \leq c_3 + \frac{c_3}{k-1} \sum_{j=2}^k X_{(j)}\} + (k-1) P\{X_{(k)} \leq c_3 + \frac{c_3}{k-1} \sum_{j=1}^{k-1} X_{(j)}\} \\ &= \sum_{r=0}^{\infty} P\{X_{(1)} \leq c_3 + \frac{c_3}{k-1} \sum_{j=2}^k X_{(j)} \mid \sum_{i=1}^k X_i = r\} P\{\sum_{i=1}^k X_i = r\} \\ &\quad + (k-1) \sum_{r=0}^{\infty} P\{X_{(k)} \leq c_3 + \frac{c_3}{k-1} \sum_{j=1}^{k-1} X_{(j)} \mid \sum_{i=1}^k X_i = r\} P\{\sum_{i=1}^k X_i = r\} \\ &= \sum_{r=0}^{\infty} P\{X_{(1)} \leq \frac{c_3(r+k-1)}{c_3+k-1} \mid \sum_{i=1}^k X_i = r\} P\{\sum_{i=1}^k X_i = r\} \\ &\quad + (k-1) \sum_{r=0}^{\infty} P\{X_{(k)} \leq \frac{c_3(r+k-1)}{c_3+k-1} \mid \sum_{i=1}^k X_i = r\} P\{\sum_{i=1}^k X_i = r\} \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\lfloor \frac{c_3(r+k-1)}{c_3+k-1} \rfloor} \left\{ \binom{r}{s} \left( \frac{\delta}{k-1+\delta} \right)^s \left( 1 - \frac{\delta}{k-1+\delta} \right)^{r-s} \right. \\ &\quad \left. + (k-1) \binom{r}{s} \left( \frac{1}{k-1+\delta} \right)^s \left( 1 - \frac{1}{k-1+\delta} \right)^{r-s} \right\} \\ &\quad \cdot e^{-(k-1+\delta)\lambda} \frac{[(k-1+\delta)\lambda]^r}{r!} \\ &\leq k \{ e^{-(k-1+\delta)\lambda} + e^{-(k-1+\delta)\lambda} (k-1+\delta)\lambda \} + \sup_{r \geq 2} g(r) \\ &\quad \cdot \sum_{r=2}^{\infty} e^{-(k-1+\delta)\lambda} \frac{[(k-1+\delta)\lambda]^r}{r!} \end{aligned}$$

$$\begin{aligned}
&= \sup_{r \geq 2} g(r) + (k - \sup_{r \geq 2} g(r)) \{e^{-(k-1+\delta)\lambda_0} + e^{-(k-1+\delta)\lambda} (k-1+\delta)\lambda\} \\
&\leq \sup_{r \geq 2} g(r) + (k - \sup_{r \geq 2} g(r)) [1 + (k-1+\delta)\lambda_0] e^{-(k-1+\delta)\lambda_0}
\end{aligned}$$

The proof is completed.

(B) A Conditional Procedure  $R_4$

We also consider a conditional rules as follows.

$R_4$ : Select the population  $\pi_i$  if and only if

$$X_i \leq c_4(t) + \frac{c_4(t)}{k-1} \sum_{j \neq i} X_j \quad \text{given} \quad \sum_{i=1}^k X_i = t.$$

We know that  $X_1, \dots, X_k$  given  $\sum_{i=1}^k X_i = t$  is distributed as multinomial

with parameters  $t$ ;  $\frac{\lambda_1}{\sum_{j=1}^k \lambda_j}, \dots, \frac{\lambda_k}{\sum_{j=1}^k \lambda_j}$ , and the marginal distribution of  $X_i$  given  $\sum_{i=1}^k X_i = t$  is binomial with parameters  $t$  and  $\frac{\lambda_i}{\sum_{j=1}^k \lambda_j}$ .

Theorem 1.4.4.  $\inf_{\underline{\lambda} \in \Omega} P_{\underline{\lambda}}(CS|R_4) = \inf_{\underline{\lambda} \in \Omega_0} P_{\underline{\lambda}}(CS|R_4)$

Proof. For  $\underline{\lambda} \in \Omega$ ,

$$\begin{aligned}
P_{\underline{\lambda}}(CS|R_4) &= P_{\underline{\lambda}}\{X_{(1)} \leq c_4(t) + \frac{c_4(t)}{k-1} \sum_{j=2}^k X_{(j)} \mid \sum_{i=1}^k X_i = t\} \\
&= P_{\underline{\lambda}}\{X_{(1)} \leq D(t) \mid \sum_{i=1}^k X_i = t\}
\end{aligned}$$

where  $D(t) = \frac{c_4(t)(t+k-1)}{c_4(t)+k-1}$ . Since  $X_{(1)}$  given  $\sum_{i=1}^k X_i = t$  is distributed

as  $B(t, \frac{\lambda_{[1]}}{\sum_{j=1}^k \lambda_{[j]}})$  which belongs to the SI family, hence

$$\inf_{\lambda \in \Omega} P_{\lambda}(CS|R_4) = \inf_{\lambda \in \Omega_0} P_{\lambda}(CS|R_4) = \sum_{i=0}^{[D(t)]} \binom{t}{i} \left(\frac{1}{k}\right)^i \left(\frac{k-1}{k}\right)^{t-i}.$$

Note that the infimum of the probability of a correct selection is independent of the common value  $\lambda$  and  $c_4(t)$  is the smallest constant determined from the following inequality

$$\sum_{i=0}^{[D(t)]} \binom{t}{i} \left(\frac{1}{k}\right)^i \left(\frac{k-1}{k}\right)^{t-i} \geq p^*.$$

Theorem 1.4.5. 
$$E_{\Omega_1}(S|R_4) = \sum_{s=0}^{[D(t)]} \left\{ \binom{t}{s} \left(\frac{\delta}{k-1+\delta}\right)^s \left(\frac{k-1}{k-1+\delta}\right)^{t-s} + (k-1) \binom{t}{s} \left(\frac{1}{k-1+\delta}\right)^s \left(\frac{k-2+\delta}{k-1+\delta}\right)^{t-s} \right\}$$

Proof. 
$$E_{\Omega_1}(S|R_4) = P_{\lambda}\{X_{(1)} \leq c_4 + \frac{c_4}{k-1} \sum_{j=2}^k X_{(j)} \mid \sum_{i=1}^k X_i = t\} + (k-1) P\{X_{(k)} \leq c_4(t) + \frac{c_4(t)}{k-1} \sum_{j=1}^{k-1} X_{(j)} \mid \sum_{i=1}^k X_i = t\} = P_{\lambda}\{X_{(1)} \leq D(t) \mid \sum_{i=1}^k X_i = t\} + (k-1) P\{X_{(k)} \leq D(t) \mid \sum_{i=1}^k X_i = t\}$$

The theorem follows easily.

### 1.5 Selecting a Subset which Contains All Populations

#### Better Than a Standard

In this section, we discuss a related problem.

Let  $\pi_0, \pi_1, \dots, \pi_k$  be  $k+1$  independent Poisson populations with parameters  $\lambda_0, \lambda_1, \dots, \lambda_k$  respectively. We use the same notation and definitions as in Section 1.2. Population  $\pi_i$  is said to be better than the standard if  $\lambda_i \leq \lambda_0$ . The procedure described in this section control

the probability that the selected subset contains all those populations better than the standard and guarantees this probability of such a correct decision to be at least  $P^*$ . Let  $X_i$  denote the observation from  $\pi_i$  ( $i = 1, 2, \dots, k$ ). Let  $r_1$  and  $r_2$  denote the number of populations with  $\lambda \leq \lambda_0$  and  $\lambda > \lambda_0$  respectively. We discuss the following cases:

Case (i): Known Standard

We assume  $\lambda_0$  is known, and propose a procedure as follows:

$R_{d_1}$ : Retain in the selected subset those and only those populations  $\pi_i$  for which

$$X_i \leq d_1(\lambda_0 + 1) \quad (1.5.1)$$

where  $d_1 \geq 1$  is the smallest number to be determined below.

The probability  $P_i$  of a correct decision is given by

$$\begin{aligned} P_1 &= \prod_{i=1}^{r_1} P\{X(i) \leq d_1(\lambda_0 + 1)\} \\ &\geq \prod_{i=1}^{r_1} P_{\lambda_0}\{X(i) \leq d_1(\lambda_0 + 1)\} \\ &\geq \left[ \sum_{j=0}^{[d_1(\lambda_0+1)]} e^{-\lambda_0} \frac{\lambda_0^j}{j!} \right]^k \end{aligned} \quad (1.5.2)$$

Remark 1.5.1. By solving for the smallest value  $d_1$  satisfying

$$\left[ \sum_{j=0}^{[d_1(\lambda_0+1)]} e^{-\lambda_0} \frac{\lambda_0^j}{j!} \right]^k \geq P^*$$

we obtain the procedure.

Case (ii):  $\lambda_0$  unknown.

Let  $X_0$  be an observation from  $\pi_0$ . Then we propose the following procedure.

$R_{d_2}$ : Retain in the selected subset those and only those populations  $\pi_i$  for which

$$X_i \leq d_2(X_0 + 1) \quad (1.5.3)$$

where  $d_2 \geq 1$  is the smallest value to be determined below.

Then the probability  $P_2$  of a correct decision is given by

$$\begin{aligned} P_2 &= P\{X_{(i)} \leq d_2(X_0 + 1), i=1, \dots, r_1\} \\ &= \sum_{x=0}^{\infty} \left\{ \prod_{j=1}^{r_1} \sum_{x_j=0}^{[d_2(x+1)]} e^{-\lambda_{[j]}} \frac{\lambda_{[j]}^{x_j}}{x_j!} \right\} e^{-\lambda_0} \frac{\lambda_0^x}{x!} \\ &\geq \sum_{x=0}^{\infty} \left\{ \sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_0} \frac{\lambda_0^y}{y!} \right\}^{r_1} e^{-\lambda_0} \frac{\lambda_0^x}{x!} \\ &\geq \left\{ \sum_{x=0}^{\infty} \left( \sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_0} \frac{\lambda_0^y}{y!} \right) e^{-\lambda_0} \frac{\lambda_0^x}{x!} \right\}^{r_1} \\ &\geq \left\{ \sum_{x=0}^{\infty} \left( \sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_0} \frac{\lambda_0^y}{y!} \right) e^{-\lambda_0} \frac{\lambda_0^x}{x!} \right\}^k \quad (1.5.4) \\ &= \{P(X_1 \leq d_2(X_0 + 1))\}^k \end{aligned}$$

where  $X_0, X_1$  are iid Poisson with parameter  $\lambda_0$ .

$$= \left\{ \sum_{r=0}^{\infty} P(X_1 \leq d_2(X_0 + 1) \mid X_0 + X_1 = r) \cdot P(X_0 + X_1 = r) \right\}^k$$

For any fixed  $r \geq 0$ , let  $d_2(r)$  be the smallest number such that

$$A(2, r, d_2(r)) \geq k\sqrt{p^*}$$

where  $A(2, r, d_2(r))$  is defined in (1.2.3). Let  $d_2 = \sup\{d_2(r) : r \geq 0\}$ , we have  $P_2 \geq P^*$ .

### 1.6 Applications to a Test of Homogeneity

$$\underline{\text{for } \lambda_1 = \lambda_2 = \dots = \lambda_k}$$

In some practical situations one wishes to know whether  $\lambda_i$  are significantly different or not. This is the problem of the test of homogeneity of the Poisson populations. In order to test the homogeneity of  $k$  experiments, i.e. to test  $H: \lambda_1 = \lambda_2 = \dots = \lambda_k = \lambda_0$  against the  $H_A: \text{not } H$ , we proposed the following rules  $\phi_1$  and  $\phi_2(T)$ ,

(1) The procedure  $\phi_1$ :  $H$  is accepted if, and only if,

$$X_{\max} - c X_{\min} \leq c, \text{ where } c \text{ is some constant depending on } k, \lambda_0, \text{ and the level of significance } \alpha.$$

(2) The procedure  $\phi_2(T)$ :  $H$  is accepted if and only if

$$X_{\max} - c(t) X_{\min} \leq c(t), \text{ given that } T = \sum_{i=1}^k X_i = t.$$

For the procedure  $\phi_1$ , if we choose  $c = \sup\{c(t) : t \geq 0\}$ , where for any  $t, t \geq 0$ ,  $c(t)$  is the smallest constant such that

$$A(k, t, c(t)) \geq 1 - \frac{\alpha}{k},$$

then, for  $\underline{\lambda} \in \Omega_0$ , i.e., when  $H$  is true

$$\begin{aligned} & P_{\underline{\lambda}} \{ \max_{1 \leq j \leq k} X_j - c \min_{1 \leq j \leq k} X_j \leq c \} \\ &= 1 - P_{\underline{\lambda}} \{ \max_{1 \leq j \leq k} X_j > c \min_{1 \leq j \leq k} X_j + c \} \\ &\geq 1 - \sum_{i=1}^k P_{\underline{\lambda}} \{ X_i > c \min_{1 \leq j \leq k} X_j + c \} \end{aligned}$$

$$\begin{aligned}
&= 1 - k + \sum_{i=1}^k P_{\lambda} \{X_i \leq c \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c\} \\
&= 1 - k + \sum_{i=1}^k \sum_{t=0}^{\infty} P_{\lambda} \{X_i \leq c \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c \mid \sum_{i=1}^k X_i = t\} P(\sum_{i=1}^k X_i = t) \\
&\geq 1 - k + \sum_{i=1}^k \sum_{t=0}^{\infty} P_{\lambda} \{X_i \leq c(t) \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c(t) \mid \sum_{i=1}^k X_i = t\} P(\sum_{i=1}^k X_i = t) \\
&\geq 1 - \alpha
\end{aligned}$$

by Lemma 1.2.1. Hence  $P_{\lambda \in \Omega_0} [\text{Reject } H] \leq \alpha$ .

The probability of the error of the first kind for  $\phi_2(T)$  is then given by

$$\begin{aligned}
E(\phi_2(T) | H, t) &= P(\max_{1 \leq j \leq k} X_j - c(t) \min_{1 \leq j \leq k} X_j > c(t) \mid \sum_{i=1}^k X_i = t) \\
&= P(X_i - c(t) \min_{1 \leq j \leq k} X_j > c(t) \text{ for some } i \mid \sum_{i=1}^k X_i = t) \\
&\leq \sum_{i=1}^k P(X_i > c(t) \min_{1 \leq j \leq k} X_j + c(t) \mid \sum_{i=1}^k X_i = t) \\
&= k P(X_1 > c(t) \min_{1 \leq j \leq k} X_j + c(t) \mid \sum_{i=1}^k X_i = t) \\
&= k[1 - A(k, t, c(t))]
\end{aligned}$$

by Lemma 1.2.1. Hence, for given significance level  $\alpha$ , we can find  $c(t)$  such that  $E(\phi_2(T) | H, \alpha) \leq \alpha$ .

### 1.7 Explanations of the Tables

- (1) Tables I, II and III list the infimum of the probability of a correct selection (approximate value) for the rules  $R_1$ ,  $R_3$  and

$R_1'$ .  $R_1$  and  $R_3$  are proposed and studied in this paper and  $R_1'$  is the selection procedure for selecting a subset to include the population associated with  $\lambda_{[k]}$  discussed in Gupta and Huang [43]. It should be pointed out that the probability of a correct selection for all these rules is decreasing with  $\lambda$  when  $\lambda$  is small and then it is increasing again with  $\lambda$ . Hence, the approximate infimum can be determined numerically by computing the probability as a function of  $\lambda$ , for fixed values of  $c$ . For given  $k$  and  $P^*$ , the selection constants (approximately) can be found from these tables. For example, for  $P^* = .8504$ , and  $k = 4$ , the approximate value of  $c$  associated with  $R_1$  is 2.4.

- (2) In Tables IVA, IVB, IVC and IVD, the first entry denotes the probability of selecting the best population, the second entry denotes the probability of selecting a non-best population and the third entry is the expected proportion, all under the slippage configuration  $\lambda_{[1]} = \delta\lambda$ ,  $\delta < 1$ ;  $\lambda_{[2]} = \dots = \lambda_{[k]} = \lambda$ , when the rule  $R_1$  is used. The three entries in Table VA, VB, VC, VD define the same quantities for the rule  $R_3$ . For example, from Table IVC, we find that  $\lambda$  for the rule  $R_1$  if  $\lambda = 2.50$  and  $c = 1.50$ , ( $k=5$  and  $\delta=.3$ , the probability of a correct selection is .9461, the probability of selecting a non-best population is .4879 and the expected proportion of populations in the selected subset is .5796.

### 1.8 Some Remarks on the Comparison of $R_1$ and $R_3$

We define a rule  $R$  to be better than another rule  $R'$  if the expected proportion for  $R$  is smaller than the expected proportion for  $R'$ . We compare the performance of the rules  $R_1$  and  $R_3$  in the aspect. For example, when  $k = 5$ ,  $P^* = 0.92$ , we obtain the approximate values of selection constants for  $R_1$  and  $R_3$  as  $c_1 = 3.0$ ,  $c_3 = 1.55$  from Table I and Table II respectively. For these constants Tables IV, V show that if  $\delta$  is kept fixed  $R_3$  seems to be better than  $R_1$  when  $\lambda$  is small, while  $R_1$  performs better than  $R_3$  for large values of  $\lambda$ .

Table I  
 Table of  $\inf P(CS|R_1)$  (Approximate) Using the Rule  $R_1$

$k \setminus c$	1.50	1.60	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	3.50	4.00	4.50	5.00
2	0.8543	0.8577	0.8756	0.8762	0.8762	0.9353	0.9353	0.9391	0.9515	0.9517	0.9771	0.9792	0.9902	0.9906	0.9956
3	0.7588	0.7627	0.7892	0.7895	0.7895	0.8845	0.8845	0.8904	0.9116	0.9118	0.9566	0.9604	0.9811	0.9817	0.9913
4	0.6897	0.6936	0.7245	0.7246	0.7246	0.8431	0.8431	0.8504	0.8782	0.8784	0.9380	0.9433	0.9724	0.9733	0.9872
5	0.6369	0.6394	0.6740	0.6740	0.6740	0.8076	0.8076	0.8151	0.8478	0.8484	0.9209	0.9277	0.9643	0.9654	0.9832
6	0.5937	0.5963	0.6311	0.6313	0.6313	0.7769	0.7769	0.7845	0.8207	0.8212	0.9053	0.9135	0.9566	0.9578	0.9793
8	0.5266	0.5322	0.5643	0.5644	0.5644	0.7263	0.7263	0.7341	0.7747	0.7750	0.8774	0.8881	0.9425	0.9439	0.9720
10	0.4769	0.4807	0.5144	0.5144	0.5144	0.6858	0.6858	0.6943	0.7372	0.7374	0.8532	0.8641	0.9289	0.9314	0.9651

For given  $k$  and  $c$ , this table represents the minimum value (approximately) of

$$P_\lambda \left[ X_k \leq c \min_{1 \leq j \leq k-1} X_j + c \right] = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \left\{ \sum_{j=c-i}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \right\}^{k-1}$$

where  $X_1, \dots, X_k$  are i.i.d. Poisson variables with parameter  $\lambda$ .

Table 11  
 Table of  $\inf P(CS|R_3)$  (Approximate) Using the Rule  $R_3$

$\frac{c_3}{k}$	1.50	1.60	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	3.50	4.00	4.50	5.00
2	0.8542	0.8577	0.8756	0.8762	0.8762	0.9353	0.9353	0.9391	0.9515	0.9517	0.9771	0.9792	0.9902	0.9906	0.9956
3	0.8924	0.8996	0.9244	0.9407	0.9407	0.9575	0.9726	0.9772	0.9887	0.9887	0.9950	0.9965	0.9989	0.9990	0.9996
4	0.8949	0.9201	0.9331	0.9452	0.9624	0.9730	0.9802	0.9826	0.9924	0.9937	0.9953	0.9985	0.9995	0.9997	0.9999
5	0.9160	0.9260	0.9445	0.9573	0.9657	0.9733	0.9824	0.9889	0.9925	0.9955	0.9979	0.9993	0.9995	0.9998	0.9999
6	0.9212	0.9389	0.9550	0.9611	0.9703	0.9796	0.9852	0.9911	0.9951	0.9964	0.9982	0.9993	0.9998	0.9999	0.9999
8	0.9272	0.9453	0.9579	0.9676	0.9752	0.9828	0.9896	0.9938	0.9961	0.9973	0.9987	0.9995	0.9999	0.9999	0.9999
10	0.9277	0.9465	0.9598	0.9678	0.9789	0.9845	0.9902	0.9940	0.9969	0.9981	0.9987	0.9997	0.9999	0.9999	0.9999

For given  $k$  and  $c_3$ , this table represents the minimum value (approximately) of

$$\begin{aligned}
 P_\lambda \left[ X_k \leq \frac{c_3}{k-1} \sum_{j=1}^{k-1} X_j + c_3 \right] &= \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \left\{ \sum_{j=(k-1)(\frac{i}{c_3}-1)}^{\infty} e^{-(k-1)\lambda} \frac{((k-1)\lambda)^j}{j!} \right\} \\
 &= \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \int_0^{(k-1)\lambda} \frac{1}{\Gamma((k-1)(\frac{i}{c_3}-1)+1)} y^{<(k-1)(\frac{i}{c_3}-1)>-1} e^{-y} dy
 \end{aligned}$$

where  $X_1, \dots, X_k$  are i.i.d. Poisson variables with parameter  $\lambda$ .

Table III

Table of  $\inf P(CS | R_1^c)$  (Approximate) Using the Rule  $R_1^c: X_i + 1 \geq c' \max_{1 \leq j \leq k} X_j$

$k \backslash c'$	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
2	0.9902	0.9771	0.9517	0.9391	0.9353	0.9352	0.8762	0.8577	0.8542	0.8103	0.8031	0.7396	0.7184	0.6761
3	0.9821	0.9591	0.9176	0.8976	0.8922	0.8922	0.8015	0.7740	0.7692	0.7063	0.6961	0.6090	0.5795	0.5254
4	0.9753	0.9448	0.8904	0.8651	0.8596	0.8596	0.7505	0.7150	0.7103	0.6372	0.6255	0.5261	0.4951	0.4376
5	0.9687	0.9328	0.8664	0.8391	0.8341	0.8340	0.7123	0.6705	0.6651	0.5866	0.5737	0.4689	0.4357	0.3800
6	0.9629	0.9211	0.8467	0.8182	0.8130	0.8130	0.6772	0.6356	0.6289	0.5448	0.5315	0.4257	0.3929	0.3358
8	0.9529	0.9021	0.8170	0.7859	0.7793	0.7792	0.6281	0.5785	0.5756	0.4869	0.4698	0.3625	0.3307	0.2757
10	0.9449	0.8875	0.7967	0.7575	0.7526	0.7525	0.5960	0.5366	0.5330	0.4489	0.4269	0.3212	0.2890	0.2368

For given  $k$  and  $c'$ , this table represents the minimum value (approximately) of

$$P_\lambda \left[ X_k + 1 \geq c' \max_{1 \leq j \leq k-1} X_j \right] = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \left\{ \sum_{j=0}^{\lfloor \frac{i+1}{c'} \rfloor} e^{-\lambda} \frac{\lambda^j}{j!} \right\}$$

where  $X_1, \dots, X_k$  are i.i.d. Poisson variables with parameter  $\lambda$ .

Table IVA

Using the rule  $R_1$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $[(a)+(k-1)(b)]/k$ .

$$k = 3, \delta = 0.3$$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9913	0.9913	0.9995	0.9995	0.9999	0.9999	0.9999	0.9999	0.9999
	0.9146	0.9146	0.9863	0.9864	0.9983	0.9983	0.9998	0.9998	0.9999
	0.9402	0.9402	0.9907	0.9907	0.9988	0.9988	0.9988	0.9998	0.9999
0.75	0.9842	0.9842	0.9988	0.9988	0.9999	0.9999	0.9999	0.9999	0.9999
	0.8443	0.8443	0.9636	0.9637	0.9934	0.9934	0.9990	0.9990	0.9998
	0.8909	0.8909	0.9754	0.9754	0.9956	0.9956	0.9993	0.9993	0.9999
1.00	0.9777	0.9777	0.9978	0.9978	0.9998	0.9998	0.9999	0.9999	0.9999
	0.7761	0.7761	0.9322	0.9327	0.9841	0.9841	0.9969	0.9969	0.9995
	0.8433	0.8433	0.9541	0.9544	0.9893	0.9893	0.9979	0.9979	0.9996
1.50	0.9695	0.9696	0.9956	0.9956	0.9995	0.9995	0.9999	0.9999	0.9999
	0.6654	0.6658	0.8580	0.8615	0.9526	0.9527	0.9866	0.9866	0.9967
	0.7668	0.7671	0.9038	0.9062	0.9682	0.9683	0.9910	0.9910	0.9978
2.00	0.9678	0.9679	0.9940	0.9941	0.9991	0.9991	0.9999	0.9999	0.9999
	0.5889	0.5912	0.7857	0.7974	0.9114	0.9125	0.9678	0.9678	0.9898
	0.7152	0.7168	0.8551	0.8630	0.9406	0.9413	0.9785	0.9785	0.9932
2.50	0.9699	0.9702	0.9932	0.9935	0.9988	0.9988	0.9998	0.9998	0.9999
	0.5333	0.5405	0.7239	0.7491	0.8698	0.8734	0.9434	0.9437	0.9783
	0.6788	0.6837	0.8137	0.8306	0.9128	0.9152	0.9622	0.9624	0.9855

Table IVA (continued)

$$k = 3, \delta = 0.3$$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.9736	0.9742	0.9932	0.9938	0.9986	0.9986	0.9997	0.9997	0.9999
	0.4880	0.5034	0.6729	0.7146	0.8332	0.8415	0.9179	0.9190	0.9632
	0.6499	0.6604	0.7797	0.8077	0.8883	0.8939	0.9452	0.9459	0.9755
3.50	0.9775	0.9786	0.9937	0.9945	0.9986	0.9986	0.9997	0.9997	0.9999
	0.4478	0.4741	0.6304	0.6890	0.8029	0.8182	0.8947	0.8972	0.9469
	0.6244	0.6423	0.7515	0.7908	0.8681	0.8783	0.9297	0.9314	0.9646
4.50	0.9811	0.9827	0.9944	0.9954	0.9987	0.9987	0.9997	0.9997	0.9999
	0.4111	0.4495	0.5945	0.6680	0.7783	0.8020	0.8752	0.8803	0.9314
	0.6011	0.6272	0.7278	0.7771	0.8518	0.8676	0.9167	0.9201	0.9542
5.00	0.9866	0.9890	0.9960	0.9971	0.9990	0.9991	0.9997	0.9997	0.9999
	0.3481	0.4083	0.5360	0.6307	0.7411	0.7822	0.8480	0.8609	0.9075
	0.5609	0.6018	0.6893	0.7528	0.8271	0.8545	0.8986	0.9072	0.9383
6.00	0.9904	0.9931	0.9973	0.9983	0.9993	0.9994	0.9998	0.9998	0.9999
	0.2980	0.3722	0.4892	0.5961	0.7134	0.7679	0.8313	0.8537	0.8940
	0.5288	0.5791	0.6586	0.7302	0.8087	0.8451	0.8874	0.9024	0.9293
8.00	0.9952	0.9973	0.9988	0.9994	0.9997	0.9998	0.9999	0.9999	0.9999
	0.2252	0.3052	0.4168	0.5400	0.6723	0.7400	0.8108	0.8466	0.8837
	0.4818	0.5359	0.6108	0.6931	0.7815	0.8266	0.8739	0.8977	0.9225
10.00	0.9977	0.9988	0.9995	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	0.1738	0.2479	0.3616	0.4996	0.6421	0.7186	0.7984	0.8382	0.8797
	0.4485	0.4982	0.5742	0.6663	0.7614	0.8124	0.8655	0.8921	0.9198
15.00	0.9996	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	0.0955	0.1614	0.2647	0.4257	0.5901	0.8920	0.7838	0.8337	0.8810
	0.3969	0.4409	0.5098	0.6171	0.7267	0.7947	0.8559	0.8891	0.9206

Table IVB

Using the rule  $R_1$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$$k = 3, \delta = 0.5$$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9775	0.9775	0.9981	0.9981	0.9998	0.9998	0.9999	0.9999	0.9999
	0.9174	0.9174	0.9868	0.9868	0.9984	0.9984	0.9998	0.9998	0.9999
	0.9375	0.9375	0.9906	0.9906	0.9988	0.9988	0.9998	0.9998	0.9999
0.75	0.9601	0.9601	0.9951	0.9952	0.9995	0.9995	0.9999	0.9999	0.9999
	0.8541	0.8541	0.9660	0.9661	0.9939	0.9939	0.9991	0.9991	0.9998
	0.8894	0.8894	0.9757	0.9758	0.9957	0.9957	0.9993	0.9993	0.9999
1.00	0.9452	0.9452	0.9913	0.9913	0.9989	0.9989	0.9998	0.9998	0.9999
	0.7871	0.7971	0.9388	0.9395	0.9857	0.9857	0.9972	0.9972	0.9995
	0.8465	0.8465	0.9563	0.9568	0.9901	0.9901	0.9981	0.9981	0.9996
1.50	0.9281	0.9282	0.9835	0.9838	0.9971	0.9971	0.9995	0.9995	0.9999
	0.7159	0.7168	0.8809	0.8857	0.9609	0.9612	0.9890	0.9890	0.9973
	0.7866	0.7873	0.9151	0.9184	0.9730	0.9731	0.9925	0.9925	0.9982
2.00	0.9249	0.9256	0.9782	0.9794	0.9951	0.9952	0.9990	0.9990	0.9998
	0.6679	0.6728	0.8322	0.8467	0.9334	0.9347	0.9760	0.9761	0.9924
	0.7536	0.7570	0.8809	0.8910	0.9540	0.9549	0.9837	0.9837	0.9949
2.50	0.9284	0.9306	0.9761	0.9786	0.9938	0.9939	0.9985	0.9985	0.9997
	0.6349	0.6483	0.7960	0.8248	0.9105	0.9146	0.9619	0.9623	0.9854
	0.7327	0.7424	0.8560	0.8761	0.9383	0.9410	0.9741	0.9744	0.9902

Table IVB (continued)

$$k = 3, \delta = 0.5$$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.9339	0.9384	0.9762	0.9802	0.9933	0.9935	0.9981	0.9981	0.9995
	0.6066	0.6332	0.7695	0.8138	0.8945	0.9029	0.9501	0.9512	0.9779
	0.7157	0.7350	0.8384	0.8692	0.9274	0.9331	0.9661	0.9668	0.9851
3.50	0.9394	0.9466	0.9777	0.9828	0.9935	0.9939	0.9980	0.9980	0.9994
	0.5805	0.6227	0.7495	0.8076	0.8841	0.8982	0.9419	0.9442	0.9713
	0.7001	0.7307	0.8255	0.8660	0.9206	0.9301	0.9606	0.9621	0.9807
4.00	0.9443	0.9542	0.9798	0.9857	0.9941	0.9946	0.9980	0.9980	0.9993
	0.5570	0.6146	0.7338	0.8028	0.8777	0.8977	0.9372	0.9413	0.9666
	0.6861	0.7278	0.8158	0.8638	0.9165	0.9300	0.9574	0.9602	0.9775
5.00	0.9534	0.9667	0.9842	0.9904	0.9957	0.9964	0.9984	0.9985	0.9994
	0.5187	0.6004	0.7108	0.7940	0.8714	0.9007	0.9345	0.9431	0.9629
	0.6636	0.7225	0.8019	0.8594	0.9128	0.9326	0.9558	0.9616	0.9751
6.00	0.9617	0.9756	0.9881	0.9936	0.9971	0.9978	0.9989	0.9990	0.9995
	0.4897	0.5845	0.6943	0.7871	0.8695	0.9037	0.9359	0.9483	0.9641
	0.6470	0.7149	0.7923	0.8559	0.9120	0.9351	0.9569	0.9652	0.9759
8.00	0.9750	0.9861	0.9936	0.9970	0.9988	0.9992	0.9996	0.9997	0.9998
	0.4454	0.5482	0.6719	0.7827	0.8713	0.9084	0.9417	0.9563	0.9696
	0.6219	0.6942	0.7791	0.8542	0.9138	0.9386	0.9610	0.9708	0.9797
10.00	0.9839	0.9917	0.9966	0.9986	0.9995	0.9997	0.9998	0.9999	0.9999
	0.4110	0.5221	0.6568	0.7837	0.8761	0.9158	0.9484	0.9621	0.9750
	0.6020	0.6786	0.7700	0.8554	0.9172	0.9438	0.9655	0.9747	0.9833
15.00	0.9947	0.9979	0.9993	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	0.3474	0.4849	0.6325	0.7909	0.8911	0.9347	0.9634	0.9759	0.9853
	0.5631	0.6559	0.8548	0.8605	0.9274	0.9564	0.9756	0.9839	0.9902

Table IVC

Using the rule  $R_1$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$$k = 5, \delta = 0.3$$

$\lambda \backslash c_1$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	0.9900	0.9900	0.9995	0.9995	0.9999	0.9999	0.9999	0.9999	0.9999
	0.9105	0.9105	0.9857	0.9857	0.9982	0.9982	0.9998	0.9998	0.9999
	0.9264	0.9264	0.9884	0.9884	0.9986	0.9986	0.9998	0.9998	0.9999
0.75	0.9798	0.9798	0.9985	0.9985	0.9999	0.9999	0.9999	0.9999	0.9999
	0.8315	0.8315	0.9606	0.9606	0.9929	0.9929	0.9989	0.9989	0.9998
	0.8612	0.8612	0.9682	0.9682	0.9943	0.9943	0.9991	0.9991	0.9998
1.00	0.9689	0.9689	0.9969	0.9969	0.9997	0.9997	0.9999	0.9999	0.9999
	0.7518	0.7518	0.9247	0.9249	0.9822	0.9822	0.9965	0.9965	0.9994
	0.7952	0.7952	0.9391	0.9393	0.9857	0.9857	0.9972	0.9972	0.9995
1.50	0.9516	0.9516	0.9930	0.9930	0.9992	0.9992	0.9999	0.9999	0.9999
	0.6221	0.6221	0.8382	0.8406	0.9453	0.9454	0.9845	0.9845	0.9963
	0.6880	0.6880	0.8692	0.8711	0.9561	0.9562	0.9876	0.9876	0.9970
2.00	0.9447	0.9447	0.9896	0.9897	0.9985	0.9985	0.9998	0.9998	0.9999
	0.5399	0.5408	0.7568	0.7665	0.8975	0.8985	0.9626	0.9627	0.9882
	0.6208	0.6216	0.8034	0.8112	0.9177	0.9185	0.9700	0.9701	0.9906
2.50	0.9461	0.9463	0.9877	0.9882	0.9978	0.9978	0.9996	0.9996	0.9999
	0.4879	0.4919	0.6915	0.7152	0.8515	0.8550	0.9352	0.9354	0.9751
	0.5796	0.5828	0.7508	0.7698	0.8808	0.8836	0.9481	0.9483	0.9800

Table IVC (continued)

$$k = 5, \delta = 0.3$$

$\lambda \backslash c_1$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	0.9518	0.9524	0.9874	0.9882	0.9975	0.9975	0.9995	0.9995	0.9999
	0.4487	0.4595	0.6408	0.6827	0.8135	0.8221	0.9078	0.9088	0.9587
	0.5493	0.5581	0.7101	0.7438	0.8503	0.8572	0.9261	0.9270	0.9669
3.50	0.9585	0.9598	0.9881	0.9893	0.9974	0.9974	0.9994	0.9994	0.9999
	0.4139	0.4353	0.6004	0.6609	0.7840	0.8002	0.8841	0.8868	0.9416
	0.5228	0.5402	0.6780	0.7266	0.8267	0.8396	0.9072	0.9094	0.9532
4.00	0.9648	0.9670	0.9894	0.9910	0.9975	0.9975	0.9994	0.9994	0.9998
	0.3812	0.4154	0.5671	0.6437	0.7612	0.7865	0.8654	0.8708	0.9260
	0.4980	0.5257	0.6516	0.7132	0.8084	0.8287	0.8922	0.8966	0.9408
5.00	0.9748	0.9787	0.9923	0.9943	0.9981	0.9982	0.9995	0.9998	0.9998
	0.3242	0.3825	0.5139	0.6122	0.7278	0.7715	0.8406	0.8643	0.9032
	0.4542	0.5017	0.6096	0.6886	0.7819	0.8168	0.8724	0.9321	0.9225
6.00	0.9818	0.9866	0.9948	0.9966	0.9987	0.9988	0.9996	0.9996	0.9998
	0.2794	0.3530	0.4718	0.5817	0.7034	0.7603	0.8261	0.8496	0.8912
	0.4199	0.4797	0.5764	0.6647	0.7624	0.8080	0.8608	0.8796	0.9129
8.00	0.9908	0.9946	0.9978	0.9988	0.9995	0.9996	0.9998	0.9998	0.9999
	0.2141	0.2940	0.4062	0.5315	0.6668	0.7360	0.8084	0.8448	0.8825
	0.3694	0.4341	0.5245	0.6250	0.7334	0.7888	0.8467	0.8758	0.9060
10.00	0.9955	0.9977	0.9991	0.9996	0.9998	0.9999	0.9999	0.9999	0.9999
	0.1673	0.2408	0.3552	0.4948	0.6392	0.7166	0.7972	0.8374	0.8792
	0.3329	0.3922	0.4840	0.5957	0.7113	0.7733	0.8377	0.8699	0.9034
15.00	0.9992	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	0.0937	0.1594	0.2628	0.4244	0.5859	0.6917	0.7837	0.8336	0.8809
	0.2748	0.3274	0.4102	0.5395	0.6716	0.7533	0.8269	0.8669	0.9047

Table IV D

Using the rule  $R_1$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$k = 5, \delta = 0.5$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9741	0.9741	0.9978	0.9978	0.9998	0.9998	0.9999	0.9999	0.9999
	0.9109	0.9109	0.9858	0.9858	0.9982	0.9982	0.9998	0.9998	0.9999
	0.9236	0.9236	0.9882	0.9982	0.9985	0.9985	0.9998	0.9998	0.9999
0.75	0.9492	0.9492	0.9938	0.9938	0.9994	0.9994	0.9999	0.9999	0.9999
	0.8342	0.8342	0.9613	0.9613	0.9930	0.9930	0.9989	0.9989	0.9998
	0.8572	0.8572	0.9678	0.9678	0.9943	0.9943	0.9991	0.9991	0.9998
1.00	0.9239	0.9239	0.9878	0.9878	0.9985	0.9985	0.9998	0.9998	0.9999
	0.7601	0.7601	0.9273	0.9276	0.9829	0.9828	0.9967	0.9967	0.9994
	0.7929	0.7929	0.9394	0.9396	0.9860	0.9860	0.9973	0.9973	0.9995
1.50	0.8873	0.8874	0.9739	0.9742	0.9953	0.9953	0.9993	0.9993	0.9999
	0.6516	0.6519	0.8518	0.8551	0.9503	0.9505	0.9860	0.9860	0.9966
	0.6988	0.6989	0.8762	0.8789	0.9593	0.9595	0.9886	0.9886	0.9973
2.00	0.8748	0.8751	0.9629	0.9643	0.9915	0.9916	0.9983	0.9983	0.9997
	0.5954	0.5973	0.7899	0.8029	0.9139	0.9151	0.9687	0.9688	0.9902
	0.6513	0.6529	0.8245	0.8351	0.9294	0.9304	0.9747	0.9747	0.9921
2.50	0.8776	0.8792	0.9576	0.9610	0.9887	0.9888	0.9973	0.9973	0.9994
	0.5651	0.5730	0.7478	0.7772	0.8853	0.8895	0.9507	0.9511	0.9811
	0.6276	0.6342	0.7898	0.8140	0.9060	0.9094	0.9600	0.9603	0.9848

Table IVD (continued)

$k = 5, \delta \approx 0.5$

$\lambda \backslash c_1$	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.8861	0.8904	0.9569	0.9628	0.9875	0.9878	0.9965	0.9965	0.9991
	0.5415	0.5615	0.7199	0.7684	0.8674	0.8770	0.9367	0.9379	0.9718
	0.6104	0.6273	0.7673	0.8073	0.8914	0.8992	0.9486	0.9496	0.9773
3.50	0.8950	0.9035	0.9589	0.9673	0.9876	0.9882	0.9961	0.9962	0.9989
	0.5186	0.5554	0.7008	0.7664	0.8577	0.8744	0.9282	0.9309	0.9645
	0.5939	0.6251	0.7524	0.8065	0.8837	0.8971	0.9418	0.9440	0.9713
4.00	0.9030	0.9163	0.9623	0.9724	0.9886	0.9895	0.9961	0.9962	0.9988
	0.4971	0.5520	0.6869	0.7654	0.8531	0.8769	0.9244	0.9294	0.9598
	0.5783	0.6249	0.7420	0.8068	0.8802	0.8995	0.9387	0.9427	0.9676
5.00	0.9171	0.9381	0.9701	0.9814	0.9916	0.9929	0.9969	0.9970	0.9988
	0.4634	0.5474	0.6687	0.7622	0.8512	0.8856	0.9247	0.9349	0.9576
	0.5541	0.6256	0.7289	0.8060	0.8793	0.9071	0.9391	0.9473	0.9659
6.00	0.9306	0.9543	0.9772	0.9875	0.9944	0.9957	0.9979	0.9981	0.9991
	0.4402	0.5395	0.6573	0.7599	0.8534	0.8923	0.9288	0.9429	0.9605
	0.5383	0.6225	0.8213	0.8055	0.8816	0.9130	0.9426	0.9539	0.9682
8.00	0.9536	0.9737	0.9875	0.9942	0.9976	0.9985	0.9992	0.9994	0.9996
	0.4061	0.5123	0.6439	0.7644	0.8614	0.9019	0.9381	0.9538	0.9681
	0.5156	0.6045	0.7126	0.8103	0.8886	0.9212	0.9503	0.9629	0.9744
10.00	0.9696	0.9839	0.9933	0.9973	0.9990	0.9994	0.9997	0.9998	0.9999
	0.3799	0.4934	0.6358	0.7714	0.8701	0.9123	0.9466	0.9609	0.9743
	0.4979	0.5915	0.7073	0.8166	0.8959	0.9298	0.9573	0.9687	0.9794
15.00	0.9897	0.9959	0.9986	0.9996	0.9999	0.9999	0.9999	0.9999	0.9999
	0.3298	0.4700	0.6224	0.7864	0.8895	0.9340	0.9631	0.9758	0.9852
	0.4618	0.5752	0.6977	0.8291	0.9116	0.9472	0.9705	0.9806	0.9882

Table VA

Using the rule  $R_3$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$$k = 3, \delta = 0.3$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
.50	.9960	.9962	.9998	.9998	.9999	.9999	.9999	.9999	.9999
	.9477	.9521	.9918	.9924	.9990	.9990	.9999	.9999	.9999
	.9638	.9668	.9945	.9948	.9993	.9993	.9999	.9999	.9999
.75	.9945	.9950	.9996	.9996	.9999	.9999	.9999	.9999	.9999
	.9179	.9305	.9817	.9843	.9971	.9972	.9995	.9995	.9999
	.9434	.9520	.9877	.9894	.9981	.9981	.9997	.9997	.9999
1.00	.9939	.9948	.9994	.9995	.9999	.9999	.9999	.9999	.9999
	.8931	.9165	.9701	.9766	.9945	.9947	.9989	.9990	.9998
	.9267	.9426	.9799	.9842	.9963	.9965	.9993	.9993	.9998
1.50	.9940	.9957	.9992	.9994	.9999	.9999	.9999	.9999	.9999
	.8530	.8974	.9461	.9662	.9891	.9903	.9973	.9973	.9993
	.9000	.9302	.9638	.9773	.9927	.9935	.9981	.9982	.9995
2.00	.9948	.9969	.9991	.9995	.9999	.9999	.9999	.9999	.9999
	.8176	.8782	.9227	.9605	.9848	.9881	.9958	.9960	.9987
	.8766	.9178	.9482	.9735	.9898	.9921	.9972	.9973	.9991
2.50	.9956	.9977	.9992	.9996	.9999	.9999	.9999	.9999	.9999
	.7855	.8583	.9010	.9559	.9814	.9876	.9949	.9955	.9983
	.8555	.9048	.9337	.9705	.9875	.9917	.9966	.9970	.9988

Table VA (continued)

$$k = 3, \delta = 0.3$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9963	.9983	.9993	.9997	.9999	.9999	.9999	.9999	.9999
	.7590	.8412	.8828	.9511	.9784	.9875	.9945	.9957	.9981
	.8381	.8936	.9216	.9673	.9856	.9916	.9963	.9971	.9987
3.50	.9969	.9987	.9994	.9998	.9999	.9999	.9999	.9999	.9999
	.7387	.8279	.8692	.9463	.9758	.9871	.9941	.9962	.9982
	.8248	.8849	.9126	.9641	.9839	.9914	.9961	.9974	.9988
4.00	.9975	.9990	.9995	.9999	.9999	.9999	.9999	.9999	.9999
	.7235	.8175	.8599	.9419	.9739	.9865	.9937	.9966	.9983
	.8149	.8780	.9064	.9612	.9826	.9910	.9958	.9977	.9988
5.00	.9985	.9995	.9997	.9999	.9999	.9999	.9999	.9999	.9999
	.7006	.7996	.8503	.9365	.9724	.9857	.9931	.9970	.9985
	.7999	.8663	.9001	.9577	.9816	.9905	.9954	.9980	.9990
6.00	.9991	.9997	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.6818	.7826	.8471	.9363	.9734	.9864	.9933	.9972	.9986
	.7876	.8550	.8980	.9575	.9822	.9909	.9955	.9981	.9990
8.00	.9997	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
	.6537	.7637	.8475	.9426	.9780	.9900	.9951	.9979	.9990
	.7690	.8425	.8983	.9617	.9853	.9933	.9967	.9986	.9993
10.00	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000
	.6336	.7641	.8514	.9494	.9827	.9930	.9969	.9987	.9994
	.7557	.8427	.9009	.9662	.9884	.9953	.9979	.9991	.9996
15.00	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	.6003	.7607	.8658	.9647	.9909	.9973	.9991	.9997	.9999
	.7335	.8404	.9105	.9764	.9939	.9982	.9994	.9998	.9999

Table VB

Using the rule  $R_3$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$$k = 3, \delta = 0.5$$

$c_3$ $\lambda$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9894	.9901	.9991	.9992	.9999	.9999	.9999	.9999	.9999
	.9520	.9565	.9925	.9931	.9991	.9991	.9999	.9999	.9999
	.9644	.9677	.9947	.9951	.9994	.9994	.9999	.9999	.9999
0.75	.9852	.9873	.9982	.9984	.9998	.9998	.9999	.9999	.9999
	.9268	.9394	.9838	.9864	.9975	.9976	.9996	.9996	.9999
	.9463	.9554	.9886	.9904	.9983	.9983	.9997	.9997	.9999
1.00	.9830	.9868	.9975	.9980	.9997	.9997	.9999	.9999	.9999
	.9068	.9294	.9743	.9806	.9955	.9957	.9991	.9991	.9998
	.9322	.9485	.9820	.9864	.9969	.9970	.9994	.9994	.9999
1.50	.9818	.9883	.9964	.9977	.9996	.9996	.9999	.9999	.9999
	.8750	.9160	.9554	.9740	.9917	.9928	.9979	.9980	.9995
	.9106	.9401	.9691	.9819	.9943	.9950	.9986	.9986	.9996
2.00	.9819	.9901	.9958	.9981	.9995	.9996	.9999	.9999	.9999
	.8472	.9019	.9377	.9711	.9891	.9919	.9971	.9973	.9991
	.8921	.9313	.9571	.9801	.9926	.9944	.9980	.9981	.9994
2.50	.9825	.9914	.9956	.9985	.9996	.9997	.9999	.9999	.9999
	.8235	.8884	.9224	.9687	.9873	.9920	.9968	.9972	.9989
	.8765	.9227	.9468	.9787	.9914	.9946	.9978	.9981	.9993

Table VB (continued)

$$k = 3, \delta = 0.5$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9837	.9925	.9957	.9989	.9997	.9998	.9999	.9999	.9999
	.8059	.8782	.9111	.9663	.9858	.9923	.9967	.9976	.9989
	.8652	.9163	.9393	.9771	.9904	.9948	.9978	.9983	.9993
3.50	.9854	.9936	.9961	.9991	.9997	.9998	.9999	.9999	.9999
	.7941	.8713	.9040	.9639	.9847	.9924	.9966	.9980	.9990
	.8479	.9121	.9347	.9756	.9897	.9948	.9977	.9986	.9993
4.00	.9873	.9946	.9966	.9993	.9998	.9999	.9999	.9999	.9999
	.7860	.8662	.9005	.9622	.9942	.9923	.9965	.9983	.9991
	.8531	.9090	.9325	.9746	.9894	.9948	.9977	.9988	.9994
5.00	.9908	.9963	.9978	.9995	.9998	.9999	.9999	.9999	.9999
	.7743	.8568	.8996	.9617	.9849	.9926	.9966	.9986	.9993
	.8464	.9033	.9323	.9743	.9899	.9950	.9977	.9990	.9995
6.00	.9934	.9974	.9986	.9997	.9999	.9999	.9999	.9999	.9999
	.7655	.8489	.9023	.9645	.9867	.9937	.9970	.9988	.9994
	.8414	.8984	.9344	.9762	.9911	.9958	.9980	.9992	.9996
8.00	.9965	.9987	.9995	.9999	.9999	.9999	.9999	.9999	.9999
	.7554	.8491	.9107	.9719	.9907	.9962	.9983	.9993	.9997
	.8358	.8989	.9403	.9812	.9938	.9975	.9988	.9995	.9998
10.00	.9982	.9994	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.7507	.8588	.9198	.9781	.9937	.9978	.9991	.9996	.9998
	.8332	.9057	.9465	.9853	.9958	.9985	.9994	.9997	.9999
15.00	.9996	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000
	.8488	.8742	.9402	.9886	.9977	.9994	.9998	.9999	.9999
	.8324	.9161	.9601	.9924	.9985	.9996	.9999	.9999	.9999

Table VC

Using the rule  $R_3$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations  $([(a)+(k-1)(b)]/k)$ .

$$k = 5, \delta = 0.3$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9982	.9983	.9996	.9999	.9999	.9999	.9999	.9999	1.0000
	.9721	.9739	.9886	.9961	.9995	.9996	.9999	.9999	.9999
	.9773	.9788	.9908	.9969	.9996	.9997	.9999	.9999	.9999
0.75	.9979	.9982	.9992	.9998	.9999	.9999	.9999	.9999	.9999
	.9562	.9635	.9759	.9929	.9988	.9993	.9998	.9999	.9999
	.9646	.9704	.9805	.9943	.9990	.9994	.9998	.9999	.9999
1.00	.9977	.9983	.9990	.9998	.9999	.9999	.9999	.9999	.9999
	.9394	.9540	.9655	.9899	.9979	.9990	.9996	.9998	.9999
	.9510	.9629	.9722	.9919	.9983	.9992	.9997	.9998	.9999
1.50	.9976	.9986	.9991	.9998	.9999	.9999	.9999	.9999	.9999
	.9097	.9359	.9561	.9861	.9962	.9987	.9991	.9997	.9999
	.9272	.9485	.9647	.9889	.9970	.9990	.9993	.9997	.9999
2.00	.9979	.9989	.9994	.9999	.9999	.9999	.9999	.9999	.9999
	.8871	.9218	.9526	.9849	.9953	.9985	.9989	.9996	.9999
	.9092	.9372	.9620	.9879	.9962	.9988	.9991	.9997	.9999
2.50	.9983	.9992	.9996	.9999	.9999	.9999	.9999	.9999	.9999
	.8704	.9150	.9505	.9845	.9950	.9983	.9989	.9996	.9999
	.8960	.9318	.9603	.9875	.9960	.9987	.9991	.9997	.9999

Table VC (continued)

$$k = 5, \delta = 0.3$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9987	.9994	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.8597	.9140	.9499	.9845	.9953	.9983	.9991	.9997	.9999
	.8875	.9311	.9599	.9876	.9962	.9986	.9993	.9998	.9999
3.50	.9990	.9996	.9998	.9999	.9999	.9999	.9999	.9999	.0000
	.8539	.9154	.9506	.9853	.9957	.9984	.9993	.9998	.9999
	.8829	.9322	.9604	.9882	.9966	.9987	.9994	.9998	.9999
4.00	.9993	.9998	.9999	.9999	.9999	.9999	.9999	.9999	1.0000
	.8508	.9167	.9519	.9865	.9963	.9987	.9994	.9998	.9999
	.8805	.9333	.9615	.9892	.9970	.9989	.9995	.9999	.9999
5.00	.9997	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000
	.8462	.9176	.9555	.9892	.9973	.9992	.9997	.9999	.9999
	.8769	.9341	.9644	.9914	.9979	.9993	.9997	.9999	.9999
6.00	.9998	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000
	.8437	.9193	.9596	.9914	.9981	.9995	.9998	.9999	.9999
	.8750	.9355	.9677	.9931	.9985	.9996	.9998	.9999	.9999
8.00	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	.8446	.9265	.9676	.9946	.9991	.9998	.9999	.9999	.9999
	.8757	.9412	.9741	.9957	.9993	.9998	.9999	.9999	.9999
10.00	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.8484	.9357	.9744	.9967	.9996	.9999	.9999	.9999	.9999
	.8787	.9486	.9974	.9996	.9999	.9999	.9999	.9999	.9999
15.00	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.8624	.9534	.9861	.9990	.9999	.9999	.9999	.9999	1.0000
	.8899	.9627	.9889	.9992	.9999	.9999	.9999	.9999	1.0000

Table VD

Using the rule  $R_3$  and under the configuration  $(\delta\lambda, \lambda, \dots, \lambda)$ , this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected population  $([(a)+(k-1)(b)]/k)$ .

$k = 5, \delta = 0.5$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9948	.9952	.9985	.9996	.9999	.9999	.9999	.9999	.9999
	.9737	.9756	.9890	.9964	.9995	.9996	.9999	.9999	.9999
	.9779	.9795	.9909	.9971	.9996	.9997	.9999	.9999	.9999
0.75	.9930	.9944	.9970	.9994	.9999	.9999	.9999	.9999	.9999
	.9589	.9663	.9773	.9935	.9989	.9994	.9998	.9999	.9999
	.9657	.9720	.9813	.9947	.9991	.9995	.9998	.9999	.9999
1.00	.9915	.9941	.9961	.9993	.9999	.9999	.9999	.9999	.9999
	.9435	.9580	.9686	.9910	.9982	.9992	.9996	.9998	.9999
	.9531	.9652	.9741	.9927	.9985	.9993	.9997	.9998	.9999
1.50	.9898	.9940	.9963	.9993	.9999	.9999	.9999	.9999	.9999
	.9175	.9424	.9617	.9883	.9968	.9990	.9993	.9997	.9999
	.9320	.9527	.9687	.9905	.9974	.9992	.9994	.9998	.9999
2.00	.9898	.9943	.9972	.9995	.9999	.9999	.9999	.9999	.9999
	.8985	.9313	.9596	.9876	.9962	.9988	.9991	.9997	.9999
	.9168	.9439	.9672	.9900	.9970	.9990	.9993	.9998	.9999
2.50	.9905	.9952	.9979	.9996	.9999	.9999	.9999	.9999	.9999
	.8854	.9274	.9587	.9876	.9962	.9987	.9992	.9997	.9999
	.9064	.9410	.9665	.9900	.9969	.9990	.9994	.9998	.9999

Table VD (continued)

$$k = 5, \delta = 0.5$$

$\lambda \backslash c_3$	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9917	.9963	.9985	.9997	.9999	.9999	.9999	.9999	.9999
	.8781	.9284	.9592	.9881	.9965	.9988	.9994	.9998	.9999
	.9009	.9420	.9671	.9904	.9972	.9990	.9995	.9995	.9999
3.50	.9931	.9973	.9989	.9998	.9999	.9999	.9999	.9999	.9999
	.8752	.9307	.9606	.9891	.9970	.9989	.9995	.9998	.9999
	.8988	.9440	.9683	.9912	.9976	.9991	.9996	.9999	.9999
4.00	.9945	.9980	.9992	.9999	.9999	.9999	.9999	.9999	.9999
	.8740	.9326	.9625	.9903	.9975	.9991	.9996	.9999	.9999
	.8981	.9457	.9698	.9922	.9980	.9993	.9997	.9999	.9999
5.00	.9964	.9989	.9996	.9999	.9999	.9999	.9999	.9999	.9999
	.8726	.9350	.9666	.9927	.9983	.9995	.9998	.9999	.9999
	.8974	.9478	.9732	.9941	.9986	.9996	.9998	.9999	.9999
6.00	.9976	.9993	.9998	.9999	.9999	.9999	.9999	1.0000	1.0000
	.8733	.9382	.9708	.9944	.9989	.9997	.9999	.9999	.9999
	.8982	.9504	.9766	.9955	.9991	.9998	.9999	.9999	.9999
8.00	.9990	.9998	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000
	.8787	.9470	.9783	.9969	.9995	.9999	.9999	.9999	.9999
	.9028	.9576	.9826	.9975	.9996	.9999	.9999	.9999	.9999
10.00	.9996	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000
	.8858	.9560	.9840	.9983	.9998	.9999	.9999	.9999	.9999
	.9086	.9648	.9872	.9986	.9998	.9999	.9999	.9999	.9999
15.00	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.9046	.9720	.9927	.9996	.9999	.9999	.9999	1.0000	1.0000
	.9237	.9776	.9941	.9997	.9999	.9999	.9999	1.0000	1.0000

CHAPTER II  
SOME RESULTS ON SUBSET SELECTION  
PROCEDURES FOR DOUBLE EXPONENTIAL POPULATIONS

2.1 Introduction

In this chapter we study the selection problems and some other related statistical inference problems for the  $k$  double exponential (Laplace) populations. Before we do this, we give some discussion of the Laplace distribution, its characteristics (vs. normal, logistic and Cauchy) and its use as a model in statistics and probability.

The double exponential distribution arises as a model in some statistical problems as explained later. This distribution is also considered in robustness studies, which suggests that it provides a model with different characteristics than some of the other commonly used models such as the normal distribution. In particular, the tails of the double exponential distribution are thicker than the tails of the normal or logistic, but not as thick as the Cauchy (see p. 43, Hajék [47]). Yet the double exponential has not been used very extensively as a model. This could be due in part to the lack of available statistical techniques for this distribution, although it is likely that the experimenter has shied away from using the double exponential because it has a sharp peak in the center. However, many applications would be primarily concerned with tail probabilities, and it would seem

that the double exponential would be a useful model if exponential tails are required.

The double exponential has some application as a model in the area of Actuarial Science, and it has been suggested as a model for the distribution of the strength of flaws in materials by Epstein [27]. Using the weakest link principle, the strength of the material should decrease as the number of flaws or volume increases. In particular, from extreme-value theory the double exponential assumption leads to the result that the mode or most probable strength decreases in proportion to  $\log n$ , where  $n$  represents the size or number of flaws of the material. In comparison, the assumption of a normal model leads to a decrease in proportion to  $(\log n)^{1/2}$ . For most applications to material strength, only the minimum flaw strength would ordinarily be observable; however, Epstein [27] suggests that there may be many other types of problems, such as a system of components in series, which might be similar from a statistical point of view. Other possible applications of the double exponential are suggested by the fact that the difference of two independent (not necessary identical) two parameter exponential variables follows the double exponential distribution, and that the logarithm of the ratios of uniform or Pareto variables follows the double exponential distribution.

In classical theory, once having assumed the form of the parent distribution, we can derive a criterion which is appropriate to this assumption. For example, under the assumption of normality, for the comparison of two means we would derive the t-statistic. It is then customary to justify the use of such a normal theory criterion in the

practical circumstance in which normality cannot be guaranteed by arguing that the distribution of the characteristic is but little affected by non-normality of the parent distribution - that is, it is robust under non-normality. However, this argument ignores the fact that if the parent distribution really differed from the normal, the appropriate criterion would no longer be the normal-theory statistic. Box and Tiao [19] reconsidered the analysis of Darwin's paired data on the heights of self and cross-fertilized plants quoted by Fisher in "The Design of Experiments (1935)". In this development the parent distribution is not assumed to be normal, but only a member of the following class of symmetric distributions

$$p(y|\theta, \sigma, \beta) = \frac{1}{\Gamma[1+\frac{1}{2}(1+\beta)] 2^{1+\frac{1}{2}(1+\beta)} \sigma} \exp \left\{ -\frac{1}{2} \left| \frac{y-\theta}{\sigma} \right|^{2/(1+\beta)} \right\} \quad (2.1.1)$$

where  $-\infty < y < \infty$ ,  $0 < \sigma < \infty$ ,  $-\infty < \theta < \infty$ ,  $-1 < \beta \leq 1$ . This class of distributions includes the normal ( $\beta=0$ ) and the double exponential ( $\beta=1$ ), and its kurtosis parameter is  $\beta$ .

If the probability density function of the double exponential is given by

$$f(x, \theta, \sigma) = \frac{1}{2\sigma} e^{-\left| \frac{x-\theta}{\sigma} \right|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty, \quad \sigma > 0 \quad (2.1.2)$$

then the mode of the distribution is  $x = \theta$  where it has a sharp peak. The expected value and standard deviation of (2.1.2) are  $\theta$  and  $\sqrt{2} \sigma$  respectively. Moments of the standardized double exponential order statistics can be obtained by using the closed-form expressions for the

moments of the standardized negative exponential order statistics derived by Epstein and Sobel [28]. Govindarajulu [34] has given the expressions for these moments.

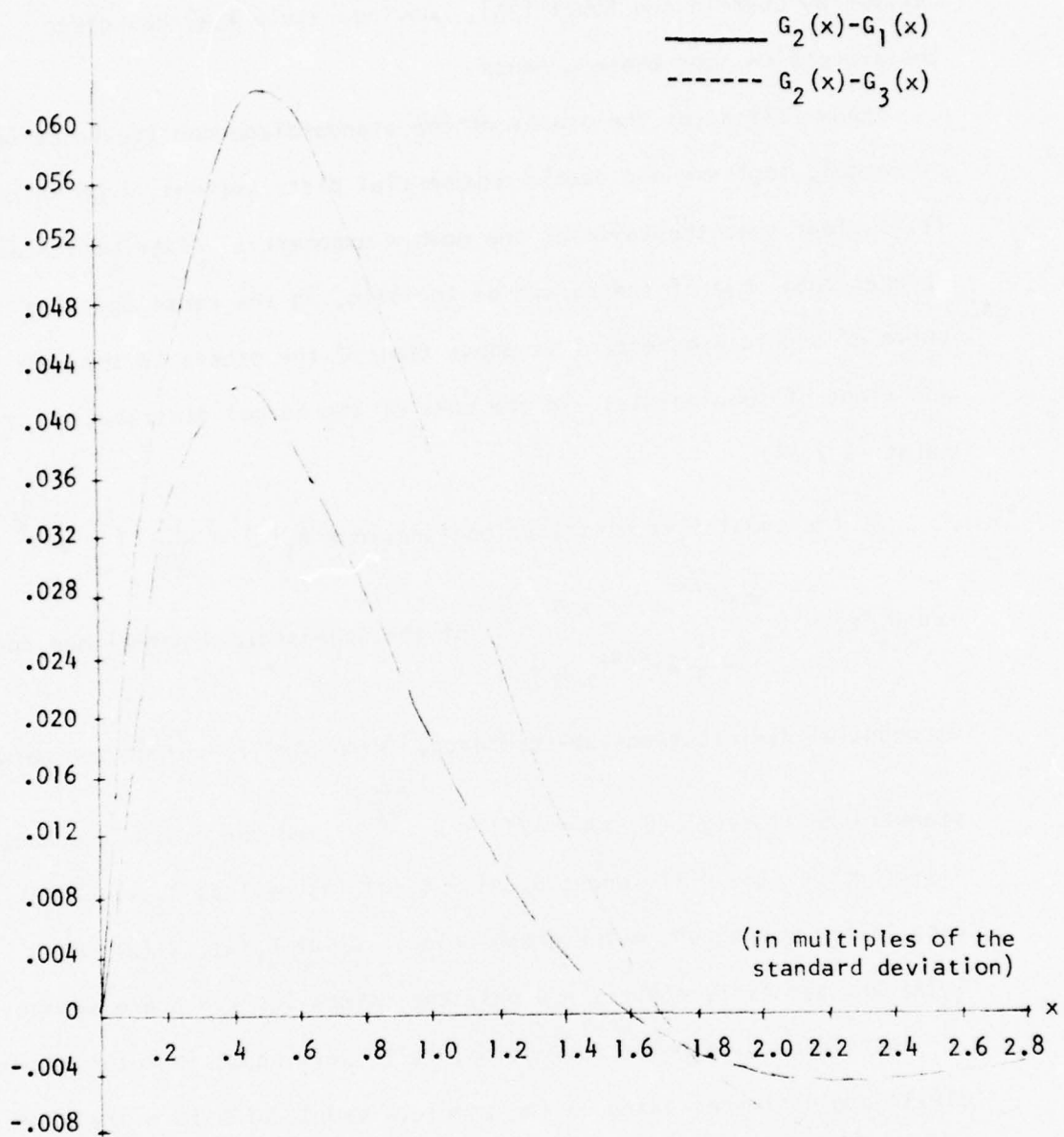
Chew [23] gives the graphs of the standardized density functions of normal, logistic and double exponential distributions, from which it is clear that the tails of the double exponential distribution are thicker than that of the normal or logistic, in the sense that the curve of double exponential is above that of the others to the left and right of some points. In the case of the normal distribution this point is 2.64.

If the cumulative distribution functions  $G_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}u^2} du$  and  $G_2(x) = \begin{cases} \frac{1}{2} e^{\sqrt{2}x} & , x < 0 \\ 1 - \frac{1}{2} e^{-\sqrt{2}x} & , x \geq 0 \end{cases}$  of the standardized normal and double

exponential distributions are compared, (also similar comparison between standardized logistic  $G_3(x) = 1/(1 + e^{-\frac{\pi}{\sqrt{3}}x})$  and the double exponential distribution) the differences  $G_2(x) - G_1(x)$  (as well as  $G_2(x) - G_3(x)$ ) vary in the way shown in the graph below. Since  $G_1(x)$ ,  $G_2(x)$  and  $G_3(x)$  are symmetric about  $x = 0$  only the values for  $x \geq 0$  are shown.

With regard to point estimation, it is well known that the maximum likelihood estimates based on the complete sample of size  $n$  are given by  $\hat{\theta} = \tilde{X}$  and  $\hat{\sigma} = \frac{1}{n} \sum_{i=1}^n |X_i - \tilde{X}|$ , where  $\tilde{X}$  denotes the sample median.

Also best linear estimators (based on order statistics) under symmetric censoring are given by Govindarajulu [35] for sample sizes up to 20, and some alternate estimates are suggested by Raghunandan and Srinivasan [66]. Interval estimation for the parameters of the



two-parameter double exponential distribution is considered by Bain and Engelhardt [4].

Now we discuss the problem of comparison of  $k$  ( $\geq 2$ ) double exponential distributions. First we study the selection problem for the largest mean (location).

## 2.2 Selecting a Subset Containing the Best of Several Double Exponential Populations with Respect to the Location Parameter

### (A) Formulation of the Problem

Let  $X_i$ ,  $i = 1, 2, \dots, k$  be  $k$  independent random variables from double exponential population  $\pi_i$ ,  $i = 1, 2, \dots, k$  respectively, with probability density function

$$f(X; \theta_i, \sigma) = \frac{1}{2\sigma} \exp[-|X - \theta_i|/\sigma], \quad -\infty < x < \infty, \quad -\infty < \theta_i < \infty, \quad \sigma > 0$$

where  $\sigma$  is a common, known constant for each of the  $k$  populations. We may, without loss of generality, assume  $\sigma$  to be one. The ranked parameters are denoted by  $\theta_{[1]} \leq \theta_{[2]} \leq \dots \leq \theta_{[k]}$ . As before, it is assumed that there is no a priori information available about the correct pairing of the ordered  $\theta_{[i]}$  and the  $k$  given populations from which observations are taken. Any population whose parameter value equals  $\theta_{[k]}$  will be defined as a best population. A correct selection (CS) is defined as the selection of any subset of the  $k$  given populations which contains at least one best population.

Suppose we take  $(2n+1)$  independent observations from  $\pi_i$ ,  $i = 1, 2, \dots, k$ ; the sample size  $(2n+1)$  is assumed to be given in the primary problem below. Let  $P^*(\frac{1}{k} < P^* < 1)$  be a preassigned constant.

Let  $P(\text{CS}; k, n, \underline{\theta}, R)$  denote the probability of a correct selection when the procedure  $R$  is used with the given  $k, n$  and when the true configuration of parameter values is  $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$ ; let the space of all possible values of  $\underline{\theta}$  be denoted by  $\Omega$ .

The problem of primary interest is to define a procedure  $R$  which selects a subset of the  $k$  given populations that is small, never empty, and large enough so that it contains the best population with probability at best  $P^*$ , regardless of the true configurations  $\underline{\theta}$  in  $\Omega$ , i.e., so that

$$\inf_{\Omega} P(\text{CS}; k, n, \underline{\theta}, R) \geq P^* . \quad (2.2.1)$$

After having defined a particular procedure  $R = R(k, n, P^*)$  for each possible set of values of  $k, n$  and  $P^*$ , we discuss the expected size  $E\{S; k, n, \underline{\theta}, P^*, R\}$  of the selected subset when the procedure  $R$  is used with the given  $k, n, P^*$  and where  $\underline{\theta}$  is the true parameter configuration in  $\Omega$ .

Let  $Y_i$  denote the sample median of the  $(2n+1)$  observations  $X_{i1}, \dots, X_{i,2n+1}$ ; from the  $i$ th population, and let  $Y_{(i)}$  denote that unknown variable which is associated with  $\theta_{[i]}$ . The probability density  $g_n(\cdot)$  and the cumulative distribution  $G_n(\cdot)$  of  $Y_i$  are given by

$$g_n(y; \theta_i) = \frac{(2n+1)!}{n!n!} \left(\frac{1}{2} e^{-|y-\theta_i|}\right)^{n+1} \left(1 - \frac{1}{2} e^{-|y-\theta_i|}\right)^n \quad (2.2.2)$$

$$G_n(y; \theta_i) = \begin{cases} 1 - \sum_{j=0}^n \binom{2n+1}{j} \left(\frac{1}{2} e^{y-\theta_i}\right)^j \left(1 - \frac{1}{2} e^{y-\theta_i}\right)^{2n+1-j} & , y < \theta_i \\ \sum_{j=0}^n \binom{2n+1}{j} \left(\frac{1}{2} e^{-(y-\theta_i)}\right)^j \left(1 - \frac{1}{2} e^{-(y-\theta_i)}\right)^{2n+1-j} & , y \geq \theta_i \end{cases} \quad (2.2.3)$$

Now, we propose the selection procedure  $R_5$  as follows:

$R_5$ : Retain in the selected subset only those populations  $\pi_i$  for which

$$Y_i \geq \max_{1 \leq j \leq k} Y_j - d \quad (2.2.4)$$

where  $d = d(k, n, P^*)$  is the smallest non-negative constant to be determined that will satisfy the basic probability requirement (2.2.1) for all configurations  $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$ .

(B) Probability of a Correct Selection and Its Infimum

The following result concerning the rule  $R_5$  can be proved.

Theorem 2.2.1.  $\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R_5) = \inf_{\underline{\theta} \in \Omega_0} P_{\underline{\theta}}(CS|R_5) = \int_{-\infty}^{\infty} G_n^{k-1}(y+d) g_n(y) dy$

where  $\Omega_0 = \{\underline{\theta} = (\theta_1, \dots, \theta_k) : \theta_1 = \theta_2 = \dots = \theta_k = \theta\}$ ,  $G_n(y)$ ,  $g_n(y)$  are the cdf and pdf of the sample median of  $(2n+1)$  observations from the standard double exponential distribution.

Proof. For  $\underline{\theta} \in \Omega$ ,

$$\begin{aligned} P_{\underline{\theta}}(CS|R_5) &= P_{\underline{\theta}}\{Y_{(k)} \geq \max_{1 \leq j \leq k} Y_{(j)} - d\} \\ &= P_{\underline{\theta}}\{Y_{(k)}^{-\theta_{[k]}} \geq Y_{(j)}^{-\theta_{[j]}} + \theta_{[j]}^{-\theta_{[k]}} - d, j=1,2,\dots,k-1\} \\ &= \int_{-\infty}^{\infty} \left[ \prod_{j=1}^{k-1} \int_{-\infty}^{y+\theta_{[k]} - \theta_{[j]} + d} g_n(z) dz \right] g_n(y) dy \quad (2.2.5) \end{aligned}$$

Note that  $\theta_{[k]} - \theta_{[j]} \geq 0$  for  $j = 1, \dots, k-1$ ; thus the result follows.

Hence, if we choose  $d$  to be the smallest constant to satisfy

$$\int_{-\infty}^{\infty} G_n^{k-1}(y+d) g_n(y) dy = P^*, \quad (2.2.6)$$

then we have determined the constant  $d$  for which

$$\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R_5) = P^* . \quad (2.2.7)$$

(c) Some Properties of  $R_5$

For  $\underline{\theta} \in \Omega$  and  $\underline{\theta} = (\theta_{[1]}, \dots, \theta_{[k]})$  define  $P_{\underline{\theta}}(i) = P_{\underline{\theta}}\{R \text{ select population } \pi_{(i)}\}$ , and recall the following definitions (see Santner [69]).

Definition 2.2.1. The rule  $R$  is strongly monotone in  $\pi_{(i)}$  means

$$P_{\underline{\theta}}(i) \text{ is } \begin{cases} \uparrow \text{ in } \theta_{[i]} \text{ when all other components of } \underline{\theta} \text{ are fixed} \\ \downarrow \text{ in } \theta_{[j]} \text{ (} j \neq i \text{) when all other components of } \underline{\theta} \text{ are fixed} \end{cases}$$

Definition 2.2.2.  $R$  is a monotone procedure means for every  $\underline{\theta} \in \Omega$  and  $1 \leq i < j \leq k$ ,  $P_{\underline{\theta}}(i) \leq P_{\underline{\theta}}(j)$ .

Definition 2.2.3.  $R$  is an unbiased procedure means for every  $\underline{\theta} \in \Omega$  and  $1 \leq j < k$ ,

$$P_{\underline{\theta}}\{R \text{ does not select } \pi_{(i)}\} \geq P_{\underline{\theta}}\{R \text{ does not select } \pi_{(k)}\}$$

Of course, if  $R$  is monotone it is also unbiased.

Theorem 2.2.2. For any  $i = 1, 2, \dots, k$ , the procedure  $R_5$  is strongly monotone in  $\pi_{(i)}$ .

Proof. The proof follows easily from the expression

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^k G_n(y + \theta_{[i]} - \theta_{[j]} + d) \right\} g_n(y) dy .$$

Corollary 2.2.1. The rule  $R_5$  is monotone and unbiased.

Proof. It is known and easy to see that if  $R$  is strongly monotone in

$\pi(i)$ , for all  $i = 1, 2, \dots, k$ , then it is monotone.

Now we consider some special configurations of  $\underline{\theta} \in \Omega$ .

$$\begin{cases} \theta_{[i]} = \theta & , \quad i = 1, 2, \dots, k-1 \\ \theta_{[k]} = \theta + \Delta & , \quad \Delta > 0 \end{cases} \quad (2.2.8)$$

$$\theta_{[i]} = \theta + (i-1)\Delta & , \quad \Delta > 0, \quad i = 1, 2, \dots, k. \quad (2.2.9)$$

Under (2.2.8),

$$p_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} [G_n(y+d)]^{k-2} G_n(y+d-\Delta) g_n(y) dy \quad \text{for } i=1, 2, \dots, k-1 \quad (2.2.10)$$

$$p_{\underline{\theta}}(k) = \int_{-\infty}^{\infty} [G_n(y+d+\Delta)]^{k-1} g_n(y) dy . \quad (2.2.11)$$

While under (2.2.9),

$$p_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^k G_n(y+d+(i-j)\Delta) \right\} g_n(y) dy, \quad i=1, 2, \dots, k.$$

From the above equations we can make the following remarks:

Remark 2.2.1. For fixed  $P^*$ ,  $k$ ,  $n$ ,  $i$  ( $i = 1, 2, \dots, k-1$ ), the probability of selecting population  $\pi(i)$  decreases from  $P^*$  to zero as  $\Delta$  increases from zero to infinity.

Remark 2.2.2. For fixed  $P^*$ ,  $k$  and  $n$ , the probability of selecting  $\pi(k)$  increases from  $P^*$  to one as  $\Delta$  increases from zero to infinity.

Remark 2.2.3. For fixed  $P^*$ ,  $k$ ,  $i$  ( $i=1, \dots, k-1$ ) and  $\Delta$ , the probability of selecting population  $\pi(i)$  tends to zero as  $n \rightarrow \infty$ . While the probability of selecting  $\pi(k)$  tends to one as  $n \rightarrow \infty$ .

Conclusion: Under either configuration (2.2.8), (2.2.9),

$E_{\underline{\theta}}(S|R_5) = \sum_{i=1}^k p_{\underline{\theta}}(i) \rightarrow 1$  as  $\Delta \rightarrow \infty$  for fixed  $n$  and  $E_{\underline{\theta}}(S|R_5) \rightarrow 1$  as  $n \rightarrow \infty$  for fixed  $\Delta$ .

(D) Asymptotic Results for the Procedure  $R_5$

It suffices to consider the parameter space  $\Omega_0$ . For  $n$  large, we discuss an asymptotic property of the procedure as follows. Let  $Y$  be the sample median from a sample of size  $(2n+1)$  with pdf  $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$ ,  $-\infty < x < \infty$ . Then it is known (see Chu [24]) that under  $\Omega_0$ ,  $\frac{Y-\theta}{\sigma_n}$  is asymptotically normally distributed (here  $\sigma_n^2 = \frac{1}{2n+1}$ ). Let  $Z$  denote a random variable which has a standard normal distribution, then  $\frac{Y-\theta}{\sigma_n}$  is asymptotically distributed as  $Z$ . Hence, under  $\Omega_0$ , the probability

$$Y_k \geq \max_{1 \leq j \leq k} Y_j - d$$

is asymptotically, the same as the probability that

$$Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1} d \quad (2.2.13)$$

where  $Z_i$ ,  $i = 1, 2, \dots, k$ , are iid standard normal variables. Hence,

$$\begin{aligned} \inf_{\underline{\theta} \in \Omega_0} P_{\underline{\theta}}(CS|R_5) &\doteq P_{\underline{\theta}}\{Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1} d\} \\ &= \int_{-\infty}^{\infty} \left[ \Phi(z + \sqrt{2n+1} d) \right]^{k-1} d\Phi(z) \end{aligned}$$

where  $\Phi(\cdot)$  is the cdf of the standard normal distribution.

(E) The Monotone Likelihood Ratio Property of the Sample Median

Suppose  $Y$  is the sample median of  $(2n+1)$  observations from the population with double exponential density function  $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$ .

The pdf  $g_n(y; \theta)$  and cdf  $G_n(y; \theta)$  of  $Y$  are given by equations (2.2.2) and (2.2.3).

After some algebraic computations, we see that  $G_n(\theta; \theta) = \frac{1}{2}$ ; also it is easy to show that  $g_n(y; \theta)$  is differentiable at  $y = \theta$ .

Let  $g_n(y; \theta) = \bar{g}_n(y - \theta)$ . It is shown in Lehmann [53, p.330] that a necessary and sufficient condition for  $\bar{g}_n(y - \theta)$  to have monotone likelihood ratio in  $y$  is that  $-\log \bar{g}_n$  is convex. Our main goal in this section is to prove this assertion. Now

$$\bar{g}_n(y) = c_n \left(\frac{1}{2} e^{-|y|}\right)^{n+1} \left(1 - \frac{1}{2} e^{-|y|}\right)^n \text{ where } c_n = \frac{(2n+1)!}{n!n!}, \text{ so,}$$

$$-\log \bar{g}_n(y) = -\log c_n + (n+1) \log 2 + (n+1)|y| - n \log \left(1 - \frac{1}{2} e^{-|y|}\right).$$

$$\text{Let } h(y) = (n+1)|y| - n \log \left(1 - \frac{1}{2} e^{-|y|}\right) = \begin{cases} h_1(y) & , y < 0 \\ h_2(y) & , y \geq 0 \end{cases} \text{ which is}$$

a continuous function. For  $y < 0$ ,

$$h(y) = h_1(y) = -(n+1)y - n \log \left(1 - \frac{1}{2} e^y\right), \text{ we have}$$

$$h_1'(y) = -(n+1) + \frac{\frac{n}{2} e^y}{1 - \frac{1}{2} e^y} < 0 \text{ since for } y < 0, \frac{\frac{1}{2} e^y}{1 - \frac{1}{2} e^y} < 1$$

$$\text{and } h_1''(y) = \frac{\frac{n}{2} e^y}{\left(1 - \frac{1}{2} e^y\right)^2} > 0.$$

Hence, for  $y < 0$ ,  $h_1(y)$  is a decreasing, convex function. Similarly,

for  $y \geq 0$ ,

$$h(y) = h_2(y) = (n+1)y - n \log \left(1 - \frac{1}{2} e^{-y}\right)$$

$$h_2'(y) = n+1 - \frac{\frac{n}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} > 0 \text{ since for } y \geq 0, \frac{\frac{1}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} < 1$$

$$h_2''(y) = \frac{\frac{n}{2} e^{-y}}{\left(1 - \frac{1}{2} e^{-y}\right)^2} > 0.$$

Hence, for  $y \geq 0$ ,  $h_2(y)$  is an increasing, convex function. Note that  $h(y)$  is continuous at  $y = 0$ , decreasing, convex for  $y < 0$  and increasing, convex for  $y \geq 0$ . Hence, this concludes that  $h(y)$  is a convex function, which implies  $-\log \bar{g}_n(y)$  is also a convex function.

Theorem 2.2.3.  $g_n(y; \theta)$  has monotone likelihood ratio in  $y$ .

(F) Expected Size of the Selected Subset

The procedure  $R$  satisfies the basic probability requirement (2.2.1) and is defined by (2.2.4). Consistent with the basic probability requirement, we would like the size of the selected subset to be small. Now  $S$ , the size of the selected subset is a random variable which takes integer values  $1, 2, \dots, k$ . Hence, one criterion of the efficiency of the procedure  $R$  is the expected value of the size of the subset. Now, we derive an expression for  $E(S|R_5)$ , the expected size of the selected subset using procedure  $R_5$ .

$$\begin{aligned} E(S|R_5) &= \sum_{i=1}^k P\{\text{Selecting the population with parameter } \theta_{[i]}\} \\ &= \sum_{i=1}^k P\{Y_{(i)} \geq \max_{1 \leq j \leq k} Y_{(j)} - d\} \\ &= \sum_{i=1}^k \int_{-\infty}^{\infty} \left[ \prod_{\substack{j=1 \\ j \neq i}}^k G_n(y + d + \theta_{[i]} - \theta_{[j]}) \right] g_n(y) dy \quad (2.2.16) \end{aligned}$$

If we set the  $m$  smallest parameters  $\theta_i$  ( $1 \leq m < k$ ) equal to a common value  $\theta$  (say) and define

$$Q = E(S \mid \theta_{[1]} = \dots = \theta_{[m]} = \theta) \quad (2.2.17)$$

then by an analogous argument as in Gupta [41] one can prove the following theorem.

Theorem 2.2.4. For given  $k$ ,  $P^*(\frac{1}{k} < P^* < 1)$ , the expected size of the selected subset  $E(S | \theta_{[1]} = \theta_{[2]} = \dots = \theta_{[m]} = \theta, m < k)$  in using the procedure  $R_5$  is strictly increasing in  $\theta$ .

Corollary 2.2.2.  $\sup_{\theta \in \Omega} E_{\theta}(S | R_5) = k \int_{-\infty}^{\infty} G_n^{k-1}(y+d) g_n(y) dy = k P^*$ .

Corollary 2.2.3. In the subset  $\Omega(\delta) = \{\theta: \theta_{[i]} \leq \theta_{[k]} - \delta, i = 1, 2, \dots, k-1\}$ , the function  $E_{\theta}(S | R_5)$  takes on its maximum value when  $\theta_{[i]} = \theta_{[k]} - \delta, i = 1, 2, \dots, k-1$ , and so

$$\begin{aligned} \sup_{\theta \in \Omega(\delta)} E_{\theta}(S | R_5) &= \int_{-\infty}^{\infty} G_n^{k-1}(y+d+\delta) g_n(y) dy \\ &+ (k-1) \int_{-\infty}^{\infty} G_n^{k-2}(y+d) G_n(y+d-\delta) g_n(y) dy . \end{aligned}$$

(G) Minimax Property of the Rule  $R_5$

Suppose that  $y_1, \dots, y_k$  are the sample medians from the  $k$  populations  $\pi_1, \dots, \pi_k$ , respectively, and with this set of observations, we select the  $i$ th population with probability  $\phi_i(y_1, \dots, y_k)$ . Then the selection rule  $R$  is said to be invariant or symmetric if

$$\phi_i(y_1, \dots, y_i, \dots, y_j, \dots, y_k) = \phi_j(y_1, \dots, y_j, \dots, y_i, \dots, y_k)$$

for all  $i$  and  $j$ , i.e. if  $y_j$  is observed from  $\pi_i$  and  $y_i$  from  $\pi_j$ , then we select the  $j$ th population with the same probability  $\phi_i(y_1, \dots, y_k)$ .

Notice that the rule  $R_5: Y_i \geq \max_{1 \leq j \leq k} Y_j - d$  satisfies the equations

$$\inf_{\theta \in \Omega} P_{\theta}(CS | R_5) = \inf_{\theta \in \Omega_0} P_{\theta}(CS | R_5) = P_{\theta_0}(CS | R_5) = P^* \quad (2.2.20)$$

$$\text{and } \sup_{\underline{\theta} \in \Omega} E_{\underline{\theta}}(S|R_5) = \sup_{\underline{\theta} \in \Omega_0} E_{\underline{\theta}}(S|R_5) = E_{\underline{\theta}_0}(S|R_5) = k P^* \quad (2.2.21)$$

where  $\underline{\theta}_0 = (\theta_0, \dots, \theta_0)$ .

For any invariant rule  $R'$ ,  $\underline{\theta}_0 \in \Omega$

$$\begin{aligned} E_{\underline{\theta}_0}(S|R') &= \sum_{i=1}^k P_{\underline{\theta}_0} \{\text{select population } \pi_i | R'\} \\ &= \sum_{i=1}^k \int \phi_i(y_1, \dots, y_k) \left[ \prod_{j=1}^k g_n(y_j) \right] dy_1 \dots dy_k \\ &= k P_{\underline{\theta}_0}(CS|R'). \end{aligned}$$

Hence for  $\underline{\theta}_0 \in \Omega_0$ ,

$$E_{\underline{\theta}_0}(S|R') - E_{\underline{\theta}_0}(S|R_5) = k [P_{\underline{\theta}_0}(CS|R') - P_{\underline{\theta}_0}(CS|R_5)] \quad (2.2.22)$$

If the rule  $R'$  satisfies the basic  $P^*$  condition, it follows from (2.2.20) that the right hand side of (2.2.22) is non-negative. Thus

$$E_{\underline{\theta}_0}(S|R') \geq E_{\underline{\theta}_0}(S|R_5) = \sup_{\underline{\theta} \in \Omega} E_{\underline{\theta}}(S|R_5).$$

So that  $\sup_{\underline{\theta} \in \Omega} E_{\underline{\theta}}(S|R') \geq \sup_{\underline{\theta} \in \Omega} E_{\underline{\theta}}(S|R_5)$

i.e. the rule  $R_5$  is minimax among all invariant rules satisfying the  $P^*$ -condition.

### 2.3. Selecting the Population with the Largest Location Parameter - Indifference Zone Approach

In this section, we would like to use the indifference zone approach of Bechhofer [11] to select one population which is guaranteed to be associated with the largest location parameter with a fixed probability  $P^*$  whenever the unknown parameters lie outside some subset,

or zone of indifference, of the entire parameter space. The goal is to define a sequence of rules  $\{R_6(n)\}$  each of which selects a single population  $\pi_{(k)}$  and find the smallest  $n$  so that

$$P_{\underline{\theta}}(CS|R_6(n)) \geq P^*, \forall \underline{\theta} \in \Omega(\delta^*) = \{\underline{\theta}: \theta_{[k]} - \theta_{[k-1]} \geq \delta^*\} \quad (2.3.1)$$

where  $P^*$  and  $\delta^*$  are preassigned numbers.

For the sake of clarity, we will use the notation  $Y_{[k]n}$  to denote the largest of the sample medians each based on  $(2n+1)$  observations.

$R_6(n)$ : Select the population corresponding to  $Y_{[k]n}$ .

Let  $\Omega_0(\delta^*) = \{\underline{\theta}: \theta_{[1]} = \dots = \theta_{[k-1]} = \theta_{[k]} - \delta^*\}$ . Then we have the following theorem.

**Theorem 2.3.1.**  $\inf_{\underline{\theta} \in \Omega(\delta^*)} P_{\underline{\theta}}(CS|R_6(n)) = \inf_{\underline{\theta} \in \Omega_0(\delta^*)} P_{\underline{\theta}}(CS|R_6(n))$

**Proof.** For  $\underline{\theta} \in \Omega(\delta^*)$ ,

$$\begin{aligned} P_{\underline{\theta}}(CS|R_6(n)) &= P_{\underline{\theta}}\{\max_{1 \leq j \leq k-1} Y_{(j)n} < Y_{(k)n}\} \\ &= P_{\underline{\theta}}\{Y_{(j)n} < Y_{(k)n}, j = 1, 2, \dots, k-1\} \\ &= P_{\underline{\theta}}\{Y_{(j)n}^{-\theta_{[j]}} < Y_{(k)n}^{-\theta_{[k]} + \theta_{[k]} - \theta_{[j]}}, j = 1, 2, \dots, k-1\} \\ &= \int_{-\infty}^{\infty} \left[ \prod_{j=1}^{k-1} G_n(y + \delta_{kj}) \right] d G_n(y) \end{aligned} \quad (2.3.2)$$

where  $G_n(y) = G_n(y; 0)$  is the cdf of the sample median of  $(2n+1)$  independent observation from the standard double exponential distribution with density function  $\frac{1}{2} e^{-|x|}$ ,  $-\infty < x < \infty$ , and  $\delta_{kj} = \theta_{[k]} - \theta_{[j]} \geq 0$ . Hence the infimum of the probability of a correct selection occurs when  $\theta_{[1]} = \theta_{[2]} = \dots = \theta_{[k-1]} = \theta_{[k]} - \delta^*$  provided  $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$ . This proves the theorem.

The minimum sample size required to achieve the  $P^*$  condition (2.3.1) is the smallest integer  $n$  such that

$$\int_{-\infty}^{\infty} [G_n(y + \delta^*)]^{k-1} dG_n(y) \geq P^* . \quad (2.3.3)$$

#### 2.4 Selecting the t-Best Populations - Indifference Zone Approach

Now, we consider the problem of selecting the best  $t$  populations, i.e., the populations with location parameters  $\theta_{[k-t+1]}, \theta_{[k-t+2]}, \dots, \theta_{[k]}$ , without regard to order. We are using the indifference zone approach based on the sample median  $Y_i$  of  $2n+1$  independent observations from population  $\pi_i$ ,  $i = 1, \dots, k$ . Define a sequence of procedures as follows:  $R_7(n)$ : Select the  $t$  populations associated with  $t$  largest values of  $Y_i$ .

Let  $\Omega^*(\delta^*) = \{\underline{\theta} : \theta_{[k-t+1]} - \theta_{[k-t]} \geq \delta\}$  and let

$$\Omega_0^*(\delta^*) = \{\underline{\theta} : \theta_{[1]} = \dots = \theta_{[k-t]} = \theta, \theta_{[k-t+1]} = \dots = \theta_{[k]} = \theta + \delta^*\}.$$

Theorem 2.4.1.  $\inf_{\underline{\theta} \in \Omega^*(\delta^*)} P_{\underline{\theta}}\{CS|R_7(n)\} = \inf_{\underline{\theta} \in \Omega_0^*(\delta^*)} P_{\underline{\theta}}\{CS|R_7(n)\}$

Proof. It was shown in Theorem 2.2.3 that the pdf  $g_n(y; \theta)$  of the sample median has monotone likelihood ratio in  $y$ , which implies that it is stochastically increasing in  $\theta$ . Using a theorem of Barr and Rizvi [8], it follows that, for  $\underline{\theta} \in \Omega^*(\delta^*)$

$$P_{\underline{\theta}}\{CS|R_7(n)\} = P_{\underline{\theta}}\left\{ \max_{1 \leq i \leq k-t} Y(i) < \min_{k-t+1 \leq j \leq k} Y(j) \right\}$$

is a non-increasing function of  $\theta_{[1]}, \dots, \theta_{[k-t]}$  and a non-decreasing function of  $\theta_{[k-t+1]}, \dots, \theta_{[k]}$ . Thus  $P_{\underline{\theta}}\{CS|R_7(n)\}$  attains its infimum when  $\theta_{[1]}, \dots, \theta_{[k-t]}$  attain their maximum possible values, while

$\theta_{[k-t+1]}, \dots, \theta_{[k]}$  attain their minimum possible values subject to  $\underline{\theta} \in \Omega^*(\delta^*)$ . The proof is thus completed.

Using the same notation as in Section 2.2, let  $G_n(y; \theta_i)$  denotes the cdf of the sample median  $Y_i$  with parameter  $\theta_i$ . Since  $\theta_i$  is the location parameter,  $G_n(y; \theta_i) = G_n(y - \theta_i; 0)$  and  $G_n$  is stochastically increasing continuous in both  $y$  and  $\theta_i$ . For  $\underline{\theta} \in \Omega^*(\delta^*)$

$$\begin{aligned} P_{\underline{\theta}}\{CS|R_7(n)\} &= P_{\underline{\theta}}\left\{\max_{1 \leq i \leq k-t} Y(i) < \min_{k-t+1 \leq \ell \leq k} Y(\ell)\right\} \\ &= P_{\underline{\theta}}\left\{\bigcup_{j=k-t+1}^k \{Y(j) = \min_{k-t+1 \leq \ell \leq k} Y(\ell) \text{ and } \max_{1 \leq \ell \leq k-t} Y(i) < Y(j)\}\right\} \\ &= \sum_{j=k-t+1}^k \int_{-\infty}^{\infty} \prod_{\substack{\beta=1 \\ \alpha \neq j}}^{k-t} G_n(y; \theta_{[\beta]}) \prod_{\substack{\alpha=k-t+1 \\ \alpha \neq j}}^k \{1 - G_n(y; \theta_{[\alpha]})\} dG_n(y; \theta_{[j]}) \end{aligned}$$

In particular, for  $\underline{\theta} \in \Omega_0^*(\delta^*) \subset \Omega^*(\delta^*)$ ,

$$\begin{aligned} P_{\underline{\theta}}\{CS|R_7(n)\} &= t \int_{-\infty}^{\infty} G_n^{k-t}(y; \theta) \{1 - G_n(y; \theta + \delta^*)\}^{t-1} dG_n(y, \theta + \delta^*) \\ &= t \int_{-\infty}^{\infty} G_n^{k-t}(y - \theta; 0) \{1 - G_n(y - \theta - \delta^*; 0)\}^{t-1} dG_n(y - \theta - \delta^*; 0) \\ &= t \int_{-\infty}^{\infty} G_n^{k-t}(y + \delta^*; 0) \{1 - G_n(y; 0)\}^{t-1} dG_n(y; 0) \end{aligned}$$

which is independent of the parameter  $\theta$ . Hence for specified values of  $\delta^*$  and  $P^*$  ( $\frac{1}{\binom{k}{t}} < P^* < 1$ ), we can solve the equation

$$t \int_{-\infty}^{\infty} G_n^{k-t}(y + \delta^*; 0) \{1 - G_n(y; 0)\}^{t-1} dG_n(y; 0) = P^*$$

for  $n$ .

### 2.5 Subset Selection with Respect to the Scale Parameter $\sigma$

Let  $X_i$ ,  $i = 1, 2, \dots, k$  be  $k$  independent random variables from double exponential population  $\pi_i$ ,  $i = 1, 2, \dots, k$ , respectively, with  $\pi_i$  having the probability density function

$$f(x; \theta_i, \sigma_i) = \frac{1}{2\sigma_i} \exp[-|x - \theta_i|/\sigma_i], \quad -\infty < x < \infty, \quad -\infty < \theta_i < \infty, \quad \sigma_i > 0.$$

Take  $n$  independent observations from  $\pi_i$ ,  $i = 1, 2, \dots, k$ . From these data one wishes to select a subset contains the population with the largest  $\sigma_i$ . Let  $\sigma_{[1]} \leq \dots \leq \sigma_{[k]}$  be the ordered parameters. We consider two different cases.

Case (i):  $\theta_1, \theta_2, \dots, \theta_k$  known.

In this case, the maximum likelihood estimator of  $\sigma_i$  is  $Y_i = \frac{1}{n} \sum_{j=1}^n |X_{ij} - \theta_i|$  which is distributed as a gamma variable with parameters  $n$  and  $\frac{\sigma_i}{n}$ , i.e.  $Y_i$  has density  $\frac{n}{\sigma_i \Gamma(n)} \left(\frac{ny}{\sigma_i}\right)^{n-1} e^{-\frac{ny}{\sigma_i}}$ ,  $y > 0$ . Thus the problem reduces to the one considered by Gupta [40]. The selection procedure is

R: Select the population  $\pi_i$  in the subset if and only if

$$Y_i \geq c \max_{1 \leq j \leq k} Y_j.$$

Case (ii):  $\theta_i$ 's are unknown.

When  $\theta_i$  is unknown, it is well known that the maximum likelihood estimate of  $\sigma_i$  is given by  $\hat{\sigma}_i = \frac{1}{n} \sum_{j=1}^n |X_{ij} - \tilde{X}_i|$ , where  $\tilde{X}_i$  denotes the sample median from population  $\pi_i$ . For this problem, we propose the following selection procedure.

R<sub>g</sub>: Select the population  $\pi_i$  in the subset if and only if

$$\hat{\sigma}_i \geq c_g \max_{1 \leq j \leq k} \hat{\sigma}_j$$

where  $0 < c_8 < 1$  is so determined as to satisfy the basic probability requirement regardless of what the unknown  $\sigma_i$ 's may be.

Let  $V_i = \frac{n\hat{\sigma}_i}{\sigma_i}$ ,  $i = 1, 2, \dots, k$ . Then

$$\begin{aligned} P(CS|R_8) &= P\{\hat{\sigma}_{(k)} \geq c_8 \max_{1 \leq j \leq k-1} \hat{\sigma}_{(j)}\} \\ &= \int_0^\infty \left[ \prod_{j=1}^{k-1} F_{V(j)} \left( \frac{1}{c_8} \frac{\sigma_{[k]}}{\sigma_{[j]}} x \right) \right] dF_{V(k)}(x). \end{aligned}$$

So

$$\inf_{\underline{\sigma} \in \Omega'} P(CS|R_8) = \inf_{\underline{\sigma} \in \Omega'_0} P(CS|R_8) = \int_0^\infty F_V^{k-1} \left( \frac{x}{c_8} \right) dF_V(x),$$

where  $\Omega' = \{\underline{\sigma} = (\sigma_1, \dots, \sigma_k), \sigma_i > 0, i = 1, \dots, k\}$ ,

$\Omega'_0 = \{\underline{\sigma} = (\sigma, \dots, \sigma), \sigma > 0\}$  and  $F_V(\cdot)$ ,  $F_{V(j)}(\cdot)$ ,  $j = 1, \dots, k$  are the cdf's of  $V = \frac{n\hat{\sigma}}{\sigma}$ ,  $V(j) = \frac{n\hat{\sigma}_{(j)}}{\sigma_{[j]}}$ ,  $j = 1, \dots, k$ , respectively.

Hence if the distribution  $F_V(\cdot)$  is known, then the constant  $c_8$  can be determined by the equation

$$\int_0^\infty F_V^{k-1} \left( \frac{x}{c_8} \right) dF_V(x) = P^*.$$

The exact distribution  $F$  of  $V$  is worked out for  $n = 3$  by Bain and Engelhardt [4], and a chi-square approximation is also given by

them which is quite good even for small  $n$ . However, it follows from Chernoff, Gastwirth and Johns [22], that  $\frac{1}{\sqrt{n}}(V-n) = \sqrt{n} \left[ \frac{\hat{\sigma}}{\sigma} - 1 \right]$

is asymptotically a standard normal variable. When all  $\sigma_i$  are identical

$$\begin{aligned}
P(\text{CS} | R_8) &= P\{\hat{\sigma}_k \geq c_8 \hat{\sigma}_j, j = 1, \dots, k-1\} \\
&= P\{\sqrt{n}(\frac{\hat{\sigma}_k}{\sigma} - 1) \geq c_8 \sqrt{n}(\frac{\hat{\sigma}_j}{\sigma} - 1) + \sqrt{n}(c_8 - 1), j = 1, \dots, k-1\} \\
&\approx \int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x - \sqrt{n}(c_8 - 1)}{c_8} \right) d\phi(x).
\end{aligned}$$

## 2.6 A Test of Homogeneity Based on the Sample Median Range

Let  $\pi_1, \pi_2, \dots, \pi_k$  be  $k$  independent double exponential populations such that the observations  $X_{i1}, \dots, X_{i,2n+1}$  from  $\pi_i$  has density  $\frac{1}{2} e^{-|x-\theta_i|}$ , for  $i = 1, 2, \dots, k$ . As before, let the sample median of these  $(2n+1)$  observations be denoted as  $Y_i$ ,  $i = 1, \dots, k$ . In some practical situations one wishes to know whether  $\theta_i$  are significantly different or not. This problem is to test the homogeneity of the double exponential populations. We are interested in using a test based on the sample range of  $Y$ 's and hence we wish to derive the distribution of the sample median range  $R = \max_{1 \leq j \leq k} Y_j - \min_{1 \leq j \leq k} Y_j$ , considering all  $\theta_i$  to be equal to a common unknown  $\theta$ . When the value of  $R$  is large, the hypothesis of homogeneity is rejected. We wish to find a constant  $r$ , such that  $P(R > r) \leq \alpha$  under the hypothesis  $H_0: \theta_1 = \dots = \theta_k = \theta$ . This will provide an  $\alpha$ -level test.

Theorem 2.5.1. For  $\alpha$ ,  $0 < \alpha < 1$ , let  $r$  be a constant such that

$$P_{\Omega_0} \{Y_k \geq \max_{1 \leq j \leq k-1} Y_j - r\} \geq 1 - \frac{\alpha}{k}.$$

Then  $P_{\Omega_0} (R > r) \leq \alpha$ .

Proof. When  $H_0$  is true, i.e., under  $\Omega_0$ ,

$$\begin{aligned}
P(R > r) &= P\left(\max_{1 \leq j \leq k} Y_j - \min_{1 \leq j \leq k} Y_j > r\right) \\
&\leq k - \sum_{i=1}^k P\{Y_i \geq \max_{1 \leq j \leq k} Y_j - r\} \\
&= k - k P\{Y_k \geq \max_{1 \leq j \leq k-1} Y_j - r\} \\
&\leq k - k \cdot \left(1 - \frac{\alpha}{k}\right) \\
&= \alpha.
\end{aligned}$$

The above theorem establishes a connection between the selection rule  $R_5$  and the above test for equality of  $\theta$ 's.

### 2.7 On the Distribution of the Statistic Associated with $R_5$

Let  $X_i$  ( $i = 0, 1, \dots, p$ ) be  $(p+1)$  independent and identically distributed random variables each representing the median in a random sample of size  $(2n+1)$  from a population with standard double exponential density function  $f(x) = \frac{1}{2} e^{-|x|}$ . Consider the differences  $Y_i = X_i - X_0$  ( $i = 1, 2, \dots, p$ ). The random variables  $Y_i$  ( $i = 1, 2, \dots, p$ ) are correlated and the distribution of the maximum of  $Y_i$  is of interest in problems of selection and ranking for double exponential distribution as explained earlier when discussing  $R_5$ . In this section, we give a closed form of the distribution of  $Y = \max_{1 \leq i \leq p} Y_i$  for some special cases. We have also computed tables of the upper percentage points of  $Y = \max_{1 \leq i \leq p} Y_i$  corresponding to the probability levels  $\alpha = P^* = 0.75, 0.90, 0.95, 0.99$  for  $p = 1(1) 9, n = 1(1) 10$ .

For the special case  $P = 1$  ( $k=2$ ),  $n = 1$  (sample size = 3), straight forward integration gives the cdf of  $Y$  (see formulae (2.2.2), (2.2.3)) as

$$\begin{aligned}
 P(Y \leq y) &= \int_{-\infty}^{\infty} G(x + y) g(x) dx \\
 &= 1 - \frac{9}{8} ye^{-2y} - \frac{3}{16} ye^{-3y} + \frac{9}{40} e^{-2y} - \frac{29}{40} e^{-3y}
 \end{aligned}$$

Again, for  $p = 1$  ( $k=2$ ),  $n = 2$  (sample size = 5),

$$\begin{aligned}
 P(Y \leq y) &= 1 - \frac{75}{16} ye^{-3y} - \frac{225}{64} ye^{-4y} - \frac{45}{256} ye^{-5y} + \frac{10975}{1792} e^{-3y} \\
 &\quad - \frac{5225}{896} e^{-4y} - \frac{203}{256} e^{-5y}
 \end{aligned}$$

All computations related to and given at the end of this chapter were made on a CDC 6500 using Gauss Laguerre quadrature based on fifteen nodes to perform the numerical integration. Checks on the accuracy of the program for  $p = 1$ ,  $n = 1$  showed that these values seem to be correct to three decimal places.

Table VI

Upper  $100(1-P^*)$  percentage points of  $Y = \max_{1 \leq i \leq p} (X_i - X_0)$  where  $X_0, X_1, \dots, X_p$  are iid sample median random variables in samples of sizes  $(2n+1)$  from the standard double exponential (Laplace) distribution.

$p \backslash n$	1	2	3	4	5	6	7	8	9	10
1	.6781	.5145	.4274	.3721	.3334	.3049	.2828	.2651	.2504	.2378
	1.3777	1.0311	.8496	.7357	.6568	.5987	.5541	.5184	.4888	.4634
	1.8508	1.3735	1.1261	.9718	.8658	.7885	.7294	.6825	.6436	.6102
	2.8631	2.0931	1.7002	1.4598	1.2994	1.1876	1.1070	1.0464	.9974	.9533
2	1.0434	.7875	.6522	.5667	.5072	.4631	.4289	.4014	.3786	.3591
	1.7380	1.2948	1.0640	.9195	.8195	.7459	.6892	.6437	.6060	.5738
	2.2092	1.6330	1.3354	1.1503	1.0231	.9302	.8590	.8024	.7555	.7153
	3.2186	2.3459	1.9015	1.6297	1.4479	1.3203	1.2274	1.1569	1.0998	1.0489
3	1.2507	.9401	.7767	.6737	.6021	.5490	.5077	.4746	.4470	.4234
	1.9451	1.4445	1.1847	1.0224	.9102	.8275	.7637	.7126	.6703	.6342
	2.4159	1.7811	1.4540	1.2509	1.1114	1.0094	.9313	.8690	.8176	.7735
	3.4247	2.4913	2.0166	1.7265	1.5322	1.3955	1.2956	1.2194	1.1576	1.1029
4	1.3972	1.0468	.8632	.7476	.6673	.6078	.5616	.5244	.4935	.4672
	2.0917	1.5496	1.2689	1.0938	.9729	.8838	.8151	.7601	.7146	.6758
	2.5624	1.8852	1.5369	1.3210	1.1728	1.0644	.9813	.9151	.8605	.8138
	3.5706	2.5936	2.0974	1.7942	1.5911	1.4479	1.3430	1.2629	1.1979	1.1405
5	1.5109	1.1290	.9294	.8040	.7170	.6525	.6024	.5622	.5288	.5003
	2.2053	1.6306	1.3336	1.1485	1.0208	.9268	.8543	.7962	.7482	.7074
	2.6759	1.9655	1.6007	1.3747	1.2197	1.1064	1.0195	.9503	.8932	.8444
	3.6838	2.6727	2.1596	1.8463	1.6363	1.4881	1.3794	1.2962	1.2287	1.1693
6	1.6039	1.1958	.9830	.8496	.7570	.6885	.6353	.5925	.5571	.5269
	2.2982	1.6965	1.3860	1.1928	1.0595	.9615	.8859	.8253	.7753	.7328
	2.7686	2.0308	1.6524	1.4183	1.2577	1.1403	1.0503	.9787	.9195	.8691
	3.7762	2.7371	2.2101	1.8855	1.6729	1.5207	1.4088	1.3232	1.2537	1.1925

Table VI (continued)

$p \backslash n$	1	2	3	4	5	6	7	8	9	10
7	1.6826	1.2521	1.0281	.8878	.7905	.7186	.6628	.6179	.5807	.5491
	2.3767	1.7520	1.4301	1.2299	1.0920	.9905	.9134	.8497	.7979	.7540
	2.8470	2.0859	1.6959	1.4548	1.2895	1.1687	1.0762	1.0024	.9416	.8898
	3.8543	2.7913	2.2526	1.9240	1.7037	1.5480	1.4336	1.3458	1.2745	1.2120
8	1.7507	1.3007	1.0669	.9207	.8193	.7444	.6863	.6396	.6009	.5681
	2.4447	1.7999	1.4681	1.2619	1.1199	1.0155	.9350	.8706	.8174	.7722
	2.9148	2.1334	1.7335	1.4863	1.3169	1.1932	1.0984	1.0228	.9606	.9075
	3.9219	2.8383	2.2894	1.9546	1.7302	1.5716	1.4548	1.3653	1.2925	1.2288
9	1.8109	1.3435	1.1010	.9495	.8446	.7670	.7069	.6586	.6186	.5847
	2.5047	1.8421	1.5014	1.2900	1.1444	1.0373	.9548	.8888	.8344	.7881
	2.9747	2.1753	1.7665	1.5139	1.3410	1.2146	1.1178	1.0407	.9772	.9231
	3.9816	2.8796	2.3217	1.9815	1.7535	1.5922	1.4735	1.3823	1.3083	1.2435

For given  $p, n$  and  $P^* = .75$  (top),  $.90$  (second),  $.95$  (third),  $.99$  (bottom), the entries in this table are the values of  $d$  for which  $\int_{-\infty}^{\infty} G_n^P(x+d) dG_n(x) = P^*$  where  $G_n(\cdot)$  is the cdf of the median in a sample of size  $(2n+1)$  from the standard double exponential distribution.

## CHAPTER III

SOME CLASSIFICATION RULES FOR  $k$  UNIVARIATE NORMAL POPULATIONS3.1 Introduction

In problems of classification, one usually assumes that an individual belongs to one of the  $k$  populations. Based on the observations from these populations one wishes to assign it to the correct population. Such problems of classification often arise in several branches of science. About forty years ago Fisher was consulted by *Barnard* [6] as to the best method of classifying skeletal remains unearthed by archaeological excavations. Fisher [29] suggested the use of the now well-known discriminant function. A general mathematical theory of statistical taxonomy was built by *Welch* [79] on foundations laid by *Neyman* and *Pearson's* theory of tests of hypotheses. The technique of discriminant functions which was devised by *Fisher* [29] has proved to be invaluable in tackling classification problems. But the construction of the discriminant function is possible only when we know the values of the parameters characterizing the populations to be discriminated between. This raises the question as to what is to be done when such knowledge is absent. In this chapter we describe some classification procedures suited to such situations.

The problem of classifying an individual into one of two categories, discriminant function analysis as some prefer to call it in the parametric case, has been considered by many authors in the statistical literature.

For an extensive bibliography, the reader is referred to Anderson, et.al. [1]. The probability of misclassification, inherent in such classification procedures is not necessarily known to the experimenter. Several authors have treated the problems dealing with the misclassification (e.g., see Smith [74], John [51], [52], Okamoto [59], Sedransk [73], Hills [48], and Sorum [76]). It should be stressed that the above papers deal with the case of two populations only.

In this chapter, we use the subset selection approach to the problem of classification where the probability of correct classification  $P(CC)$  is guaranteed to be at least a preassigned number  $P^*$  ( $\frac{1}{k} < P^* < 1$ ) regardless of what the unknown state of nature might be. The classification rules proposed here are different from those considered by Gupta and Govindarajulu [37].

Let  $\pi_i$  denote a normal population with an unknown mean  $\theta_i$  and variance  $\sigma_i^2$  ( $i = 0, 1, \dots, k$ ). From population  $\pi_i$  one observes a random sample  $X_{ij}$ ,  $j = 1, \dots, n_i$ ,  $i = 0, 1, \dots, k$ . Based on the above data we allocate  $\pi_0$  to one of the  $k$  populations with respect to the mean, variance, and the reciprocal of the coefficient of variation. In each case it is assumed that the parameter, for example, the mean  $\theta_0$  of  $\pi_0$  is equal to one of the  $\theta_i$ ,  $i = 1, \dots, k$ .

### 3.2 Classification Rules with Respect to the Mean

Case (i). Common known variance  $\sigma^2$ .

Without loss of generality  $\sigma^2$  will be taken to be unity. Let

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \quad i = 0, 1, \dots, k. \quad (3.2.1)$$

Then we propose the following classification rule,

$R_g$ : Classify  $\pi_0$  as  $\pi_i$  if and only if

$$|\bar{X}_i - \bar{X}_0| \leq c_g \sqrt{\frac{1}{n_i} + \frac{1}{n_0}} \quad (3.2.2)$$

where  $c_g$  is chosen such that  $P(CC|R_g)$  is at least  $P^*$  ( $\frac{1}{k} < P^* < 1$ ) which is a specified number. Then  $P(CC|R_g)$  is given by

$$\begin{aligned} P(CC|R_g) &= \sum_{i=1}^k P(|\bar{X}_i - \bar{X}_0| \leq c_g \sqrt{\frac{1}{n_i} + \frac{1}{n_0}} \mid \theta_i = \theta_0) P(\theta_i = \theta_0) \\ &= \sum_{i=1}^k [2\Phi(c_g) - 1] q_i \\ &= 2\Phi(c_g) - 1 \end{aligned} \quad (3.2.3)$$

where  $\Phi(\cdot)$  denotes the distribution function of the standard normal variate and  $q_i$  is the a priori probability of  $\theta_i = \theta_0$ ,  $i = 1, \dots, k$ . Since  $P(CC|R_g) \geq P^*$ , we obtain  $c_g$  as the smallest non-negative number which satisfies the equation

$$2\Phi(c_g) - 1 = P^*$$

or equivalently

$$c_g = \Phi^{-1} \left( \frac{1+P^*}{2} \right) \quad (3.2.4)$$

**Theorem 3.2.1.** Let  $A_i$  be the event "classify  $\pi_0$  as  $\pi_i$  when  $\theta_i \neq \theta_0$ ".

Then  $P(A_i|R_g) \rightarrow 0$  as each of  $n_0, n_i \rightarrow \infty$ .

**Proof.**  $P(A_i|R_g) = P(|\bar{X}_i - \bar{X}_0| \leq c_g \sqrt{\frac{1}{n_i} + \frac{1}{n_0}} \mid \theta_i \neq \theta_0)$

$$= P \left[ -c_g + \frac{\theta_0 - \theta_i}{\sqrt{\frac{1}{n_i} + \frac{1}{n_0}}} \leq \frac{\bar{X}_i - \bar{X}_0 - (\theta_i - \theta_0)}{\sqrt{\frac{1}{n_i} + \frac{1}{n_0}}} \leq c_g + \frac{\theta_0 - \theta_i}{\sqrt{\frac{1}{n_i} + \frac{1}{n_0}}} \mid \theta_i \neq \theta_0 \right]$$

$$\rightarrow 0 \quad \text{as } n_0, n_i \rightarrow \infty. \quad (3.2.5)$$

Case (ii). Common unknown variance  $\sigma^2$ .

$$\text{Let } \bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \quad s^2 = \frac{1}{v} \sum_{i=0}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2.$$

where  $v = \sum_{i=0}^k (n_i - 1)$ . In this case, we propose the classification rule as follows:

$R_{10}$ : Classify  $\pi_0$  as  $\pi_i$  if and only if

$$|\bar{X}_i - \bar{X}_0| \leq c_{10} s \sqrt{\frac{1}{n_i} + \frac{1}{n_0}} \quad (3.2.7)$$

where, as before,  $c_{10}$  is determined as the smallest non-negative number which satisfies  $P(CC|R_{10}) \geq P^*$ . Here, we have

$$\begin{aligned} P(CC|R_{10}) &= \sum_{i=1}^k P(|\bar{X}_i - \bar{X}_0| \leq c_{10} s \sqrt{\frac{1}{n_i} + \frac{1}{n_0}} \mid \theta_i = \theta_0) P(\theta_i = \theta_0) \\ &= \sum_{i=1}^k P(|T_i| \leq c_{10}) q_i \end{aligned} \quad (3.2.8)$$

where  $T_i$ ,  $i = 1, \dots, k$  are identically distributed (not independent) as Student's-t with  $v$  degrees of freedom. Hence

$$P(CC|R_{10}) = P(|T| \leq c_{10})$$

where  $T$  is distributed as Student's-t with  $v$  degrees of freedom. It should be pointed out that the joint distribution of  $T_i$ ,  $i = 1, \dots, k$  is a multivariate t as studied, for example, in Gupta [39]. The covariance matrix is  $\Sigma = (\sigma_{ij})$  where  $\sigma_{ij} = \sigma^2$  for  $i = j$  and

$$\sigma_{ij} = \sigma^2 \left[ \left(1 + \frac{n_0}{n_i}\right) \left(1 + \frac{n_0}{n_j}\right) \right]^{-\frac{1}{2}} \quad \text{for } i \neq j.$$

$$P(CC|R_{10}) = F_T(c_{10}) - F_T(-c_{10}) = 2F_T(c_{10}) - 1 \quad (3.2.9)$$

where  $F_T(\cdot)$  is the cumulative distribution function of  $T$  and  $c_{10}$  is determined to be the smallest non-negative number which satisfies the equation

$$c_{10} = F_T^{-1} \left( \frac{1+P^*}{2} \right). \quad (3.2.10)$$

If  $A_i$  is the same event as defined in Theorem 3.2.1., then we have a similar result for the rule  $R_{10}$ .

Theorem 3.2.2.  $P(A_i | R_{10}) \rightarrow 0$  as each of  $n_0, n_1, \dots, n_k \rightarrow \infty$ .

Proof. The proof is straightforward and hence omitted.

### 3.3 Classification Rule with Respect to the Variance

It is assumed that the population  $\pi_i$  has unknown mean  $\theta_i$  and variance  $\sigma_i^2$ ,  $i = 1, \dots, k$ . As before we assume that one of the variances  $\sigma_i^2$ ,  $i = 1, 2, \dots, k$  is equal to  $\sigma_0^2$ . We, then, wish to allocate  $\pi_0$  to one of the  $k$  populations  $\pi_1, \dots, \pi_k$  with respect to the variance. Assume we take  $n_i \geq 4$  observations from  $\pi_i$ ,  $i = 0, 1, \dots, k$ . We propose the classification rule:

$R_{11}$ : Classify  $\pi_0$  as  $\pi_i$  if and only if

$$\left| \frac{n_0(n_i - 3)}{n_i(n_0 - 1)} \frac{s_0^2}{s_i^2} - 1 \right| \leq c_{11}$$

where  $s_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$ ,  $i = 0, 1, \dots, k$  and  $c_{11}$  is the smallest non-negative number which satisfies the inequality  $P(CC | R_{11}) \geq P^*$ . We have

$$\begin{aligned}
P(CC|R_{11}) &= \sum_{i=1}^k P\left\{ \left| \frac{n_o(n_i-3)}{n_i(n_o-1)} \frac{s_o^2}{s_i^2} - 1 \right| \leq c_{11} \mid \sigma_i = \sigma_o \right\} P\{\sigma_i = \sigma_o\} \\
&= \sum_{i=1}^k P\left\{ \left| \frac{n_o s_o^2 / [\sigma_o^2(n_o-1)]}{n_i s_i^2 / [\sigma_i^2(n_i-1)]} - \frac{n_i-1}{n_i-3} \right| \leq \frac{n_i-1}{n_i-3} c_{11} \mid \sigma_i = \sigma_o \right\} P\{\sigma_i = \sigma_o\} \\
&= \sum_{i=1}^k q_i \left\{ F_{n_o-1, n_i-1} \left[ \frac{n_i-1}{n_i-3} (1+c_{11}) \right] - F_{n_o-1, n_i-1} \left[ \frac{n_i-1}{n_i-3} (1-c_{11}) \right] \right\}
\end{aligned} \tag{3.3.2}$$

where  $F_{v_1, v_2}$  is the cdf of an F random variable with  $v_1$  and  $v_2$  degrees of freedom, and  $q_i$  is the a priori probability of  $\sigma_i = \sigma_o$ . In the special case when  $n_1 = n_2 = \dots = n_k = n$ , (3.3.2) becomes

$$P(CC|R_{11}) = F_{n_o-1, n-1} \left[ \frac{n-1}{n-3} (1+c_{11}) \right] - F_{n_o-1, n-1} \left[ \frac{n-1}{n-3} (1-c_{11}) \right] \tag{3.3.3}$$

and  $c_{11}$  is the smallest non-negative constant determined from the equation

$$F_{n_o-1, n-1} \left[ \frac{n-1}{n-3} (1+c_{11}) \right] - F_{n_o-1, n-1} \left[ \frac{n-1}{n-3} (1-c_{11}) \right] = P^*.$$

In using the rules  $R_9$ ,  $R_{10}$ ,  $R_{11}$ , we might classify  $\pi_o$  as none, one, two or k of the k populations. In the following sections, we use the subset (non-empty) selection approach to propose rules that will classify  $\pi_o$  to be at least one of the k populations. This overcomes the objection that  $\pi_o$  may not be classified as any one of the k populations.

3.4 A Subset Selection Approach to Classification Rule  
with Respect to the Mean

Without loss of generality, we may assume that  $\sigma^2 = 1$ . Again we assume that there is exactly one population with  $\theta_i = \theta_0$ . In this case, we propose the following classification rule

$R_{12}$ : Classify  $\pi_0$  as  $\pi_i$  if and only if

$$|\bar{X}_i - \bar{X}_0| \leq c_{12} \min_{1 \leq j \leq k} |\bar{X}_j - \bar{X}_0| \quad (3.4.1)$$

where  $c_{12} (\geq 1)$  is the smallest number which satisfies the inequality  $P(CC|R_{12}) \geq P^*$ .

The classification rule  $R_{12}$  has the following desirable asymptotic property, i.e., the probability of misclassification approaches zero as the sample sizes  $n_0, n_1, \dots, n_k$  become large. Before we prove this we need the following lemma.

Lemma 3.4.1.  $P(MC|R_{12}) \leq \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k P(|\bar{X}_i - \bar{X}_0| > c_{12} |\bar{X}_j - \bar{X}_0| | \theta_i = \theta_0)$

where MC denotes misclassification.

Proof. Since

$$\begin{aligned} P(MC|R_{12}) &= \sum_{i=1}^k P(\pi_0 \text{ is not classified as } \pi_i | \theta_i = \theta_0) P(\theta_i = \theta_0) \\ &= \sum_{i=1}^k P\{|\bar{X}_i - \bar{X}_0| > c_{12} \min_{1 \leq j \leq k} |\bar{X}_j - \bar{X}_0| | \theta_i = \theta_0\} \cdot P\{\theta_i = \theta_0\} \\ &\leq \sum_{i=1}^k P\{|\bar{X}_i - \bar{X}_0| > c_{12} |\bar{X}_j - \bar{X}_0| \text{ for some } j \neq i | \theta_i = \theta_0\} \\ &\leq \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k P\{|\bar{X}_i - \bar{X}_0| > c_{12} |\bar{X}_j - \bar{X}_0| | \theta_i = \theta_0\} \quad (3.4.2) \end{aligned}$$

thus proved the lemma.

Theorem 3.4.1. With the above notation, if  $\frac{n_j}{N} \rightarrow \lambda_j$ ,  $j = 0, 1, \dots, k$ , then

$$P(\text{MC} | R_{12}) \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

where 
$$N = \sum_{i=0}^k n_i.$$

Proof. By Lemma 3.4.1., we have

$$\begin{aligned} P(\text{MC} | R_{12}) &\leq \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k P\{|\bar{X}_j - \bar{X}_0| \leq \frac{1}{c_{12}} |\bar{X}_i - \bar{X}_0| \mid \theta_i = \theta_0\} \\ &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{P[-\frac{1}{c_{12}} (\bar{X}_i - \bar{X}_0) \leq \bar{X}_j - \bar{X}_0 \leq \frac{1}{c_{12}} (\bar{X}_i - \bar{X}_0), \bar{X}_i > \bar{X}_0 \mid \theta_i = \theta_0] \\ &\quad + P[-\frac{1}{c_{12}} (\bar{X}_0 - \bar{X}_i) \leq \bar{X}_j - \bar{X}_0 \leq \frac{1}{c_{12}} (\bar{X}_0 - \bar{X}_i), \bar{X}_i < \bar{X}_0 \mid \theta_i = \theta_0]\} \\ &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{P[\bar{X}_0 - \frac{1}{c_{12}} (\bar{X}_i - \bar{X}_0) \leq \bar{X}_j - \bar{X}_0 + \frac{1}{c_{12}} (\bar{X}_i - \bar{X}_0), \bar{X}_i > \bar{X}_0 \mid \theta_i = \theta_0] \\ &\quad + P[\bar{X}_0 - \frac{1}{c_{12}} (\bar{X}_0 - \bar{X}_i) \leq \bar{X}_j - \bar{X}_0 + \frac{1}{c_{12}} (\bar{X}_0 - \bar{X}_i), \bar{X}_i < \bar{X}_0 \mid \theta_i = \theta_0]\} \\ &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{P[\frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 - \frac{1}{c_{12}} (\frac{\sqrt{n_j}}{\sqrt{n_i}} Z_i - \frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0) + \sqrt{n_j} (\theta_0 - \theta_j) \leq Z_j \\ &\quad \leq \frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 + \frac{1}{c_{12}} (\frac{\sqrt{n_j}}{\sqrt{n_i}} Z_i - \frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0) + \sqrt{n_j} (\theta_0 - \theta_j), \\ &\quad Z_i > \frac{\sqrt{n_i}}{\sqrt{n_0}} Z_0 \mid \theta_i = \theta_0] \\ &\quad + P[\frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 - \frac{1}{c_{12}} (\frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 - \frac{\sqrt{n_j}}{\sqrt{n_i}} Z_i) + \sqrt{n_j} (\theta_0 - \theta_j) \leq Z_j \\ &\quad \leq \frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 + \frac{1}{c_{12}} (\frac{\sqrt{n_j}}{\sqrt{n_0}} Z_0 - \frac{\sqrt{n_j}}{\sqrt{n_i}} Z_i) + \sqrt{n_j} (\theta_0 - \theta_j), Z_i \leq \frac{\sqrt{n_i}}{\sqrt{n_0}} Z_0 \mid \theta_i = \theta_0]\} \end{aligned}$$

where  $Z_i$ ,  $i = 0, 1, \dots, k$ , are iid standard normal variates.

$$\begin{aligned}
 &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \left\{ \int_{-\infty}^{\infty} \int_{\sqrt{\frac{n_i}{n_0}} x}^{\infty} \left[ \Phi\left(\sqrt{\frac{n_j}{n_0}} x + \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_i}} y - \sqrt{\frac{n_j}{n_0}} x\right) + \sqrt{n_j}(\theta_0 - \theta_j)\right) \right. \right. \\
 &\quad \left. \left. - \Phi\left(\sqrt{\frac{n_j}{n_0}} x - \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_i}} y - \sqrt{\frac{n_j}{n_0}} x\right) + \sqrt{n_j}(\theta_0 - \theta_j)\right) \right] d\Phi(y) d\Phi(x) \right. \\
 &\quad \left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\sqrt{\frac{n_i}{n_0}} x} \left[ \Phi\left(\sqrt{\frac{n_j}{n_0}} x + \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_0}} x - \sqrt{\frac{n_j}{n_i}} y\right) + \sqrt{n_j}(\theta_0 - \theta_j)\right) \right. \right. \\
 &\quad \left. \left. - \Phi\left(\sqrt{\frac{n_j}{n_0}} x - \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_0}} x - \sqrt{\frac{n_j}{n_i}} y\right) + \sqrt{n_j}(\theta_0 - \theta_j)\right) \right] d\Phi(y) d\Phi(x) \right\} \\
 &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{I_1(i, j) + I_2(i, j)\}
 \end{aligned}$$

where  $I_1(i, j)$ ,  $I_2(i, j)$  are the first and second double integrals inside the double summation signs.

Now, for every  $\epsilon > 0$ , there exists  $\delta(\epsilon)$  such that

$$\int_{|x| > \delta(\epsilon)} d\Phi(x) \leq \epsilon \tag{3.4.7}$$

Hence, we get

$$\begin{aligned}
I_1(i, j) &\leq \int_{-\delta(\epsilon)}^{\delta(\epsilon)} \int_{-\delta(\epsilon)}^{\delta(\epsilon)} [\Phi((1 - \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} x + \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} y + \sqrt{n_j}(\theta_0 - \theta_j)) \\
&\quad - \Phi((1 + \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} x - \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} y + \sqrt{n_j}(\theta_0 - \theta_j))] \\
&\quad d\Phi(y) d\Phi(x) + \epsilon^2 \\
&\leq \int_{-\delta(\epsilon)}^{\delta(\epsilon)} [\Phi((1 - \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} x + \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} \delta(\epsilon) + \sqrt{n_j}(\theta_0 - \theta_j)) \\
&\quad - \Phi((1 + \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} x - \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} \delta(\epsilon) + \sqrt{n_j}(\theta_0 - \theta_j))] d\Phi(x) + \epsilon^2 \\
&\leq \Phi((1 - \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} \delta(\epsilon) + \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} \delta(\epsilon) + \sqrt{n_j}(\theta_0 - \theta_j)) \\
&\quad - \Phi(-(1 + \frac{1}{c_{12}}) \sqrt{\frac{n_j}{n_0}} \delta(\epsilon) - \frac{1}{c_{12}} \sqrt{\frac{n_j}{n_1}} \delta(\epsilon) + \sqrt{n_j}(\theta_0 - \theta_j)) + \epsilon^2.
\end{aligned}
\tag{3.4.7}$$

For every  $\eta > 0$ , there exists an  $N(\epsilon, \eta)$  such that for  $N > N(\epsilon, \eta)$ ,

$$\begin{aligned}
&\Phi((1 - \frac{1}{c_{12}}) \sqrt{\frac{\lambda_j}{\lambda_0}} \delta(\epsilon) + \frac{1}{c_{12}} \sqrt{\frac{\lambda_j}{\lambda_1}} \delta(\epsilon) + \sqrt{N\lambda_j}(\theta_0 - \theta_j)) \\
&\quad - \Phi(-(1 + \frac{1}{c_{12}}) \sqrt{\frac{\lambda_j}{\lambda_0}} \delta(\epsilon) - \frac{1}{c_{12}} \sqrt{\frac{\lambda_j}{\lambda_1}} \delta(\epsilon) + \sqrt{N\lambda_j}(\theta_0 - \theta_j)) \leq \eta
\end{aligned}
\tag{3.4.8}$$

Now since  $\epsilon$  and  $\eta$  are arbitrary,  $I_1(i, j) \rightarrow 0$  as  $N \rightarrow \infty$ . Using an identical argument one can show that  $I_2(i, j) \rightarrow 0$  as  $N \rightarrow \infty$ . Thus,  $P(MC | R_{12}) \rightarrow 0$  as  $N \rightarrow \infty$ . This completes the proof of the theorem.

Suppose  $n_0 = n_1 = \dots = n_k = n$ , we have the following.

**Theorem 3.4.2.** If  $c_{12}$  is chosen to be the smallest constant such that

$$\int_0^\infty \Phi\left(\frac{\sqrt{2}x}{c_{12}}\right) d\Phi(x) \leq \frac{1}{4} + \frac{1 - P^*}{4k(k-1)}
\tag{3.4.10}$$

then

$$P(CC|R_{12}) \geq P^* \quad (3.4.11)$$

Proof. Since

$$\begin{aligned} P(CC|R_{12}) &= 1 - P(MC|R_{12}) \\ &\geq 1 - \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{P[\bar{X}_0 - \frac{1}{c_{12}}(\bar{X}_i - \bar{X}_0) < \bar{X}_j < \bar{X}_0 + \frac{1}{c_{12}}(\bar{X}_i - \bar{X}_0), \bar{X}_i > \bar{X}_0 \mid \theta_i = \theta_0] \\ &\quad + P[\bar{X}_0 - \frac{1}{c_{12}}(\bar{X}_0 - \bar{X}_i) < \bar{X}_j < \bar{X}_0 + \frac{1}{c_{12}}(\bar{X}_0 - \bar{X}_i), \bar{X}_i < \bar{X}_0 \mid \theta_i = \theta_0]\} \\ &= 1 - \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \{P[Z_0 - \frac{1}{c_{12}}(Z_i - Z_0) + \sqrt{n}(\theta_0 - \theta_j) < Z_j < Z_0 + \frac{1}{c_{12}}(Z_i - Z_0) \\ &\quad + \sqrt{n}(\theta_0 - \theta_j), Z_i > Z_0] \\ &\quad + P[Z_0 + \frac{1}{c_{12}}(Z_i - Z_0) + \sqrt{n}(\theta_0 - \theta_j) < Z_j < Z_0 - \frac{1}{c_{12}}(Z_i - Z_0) \\ &\quad + \sqrt{n}(\theta_0 - \theta_j), Z_i \leq Z_0]\} \quad (3.4.12) \end{aligned}$$

where  $Z_i$ ,  $i = 0, 1, \dots, k$ , are iid standard normal variables. But

$$\begin{aligned} &P[Z_0 + \sqrt{n}(\theta_0 - \theta_j) - \frac{1}{c_{12}}(Z_i - Z_0) < Z_j < Z_0 + \sqrt{n}(\theta_0 - \theta_j) + \frac{1}{c_{12}}(Z_i - Z_0), Z_i - Z_0 > 0] \\ &\leq P\left[-\frac{\sqrt{2}}{c_{12}} \frac{(Z_i - Z_0)}{\sqrt{2}} < Z_j < \frac{\sqrt{2}}{c_{12}} \frac{(Z_i - Z_0)}{\sqrt{2}}, \frac{Z_i - Z_0}{\sqrt{2}} > 0\right] \\ &= \int_0^\infty \left[\Phi\left(\frac{\sqrt{2}}{c_{12}} x\right) - \Phi\left(-\frac{\sqrt{2}}{c_{12}} x\right)\right] d\Phi(x) \\ &= \int_0^\infty \left[2\Phi\left(\frac{\sqrt{2}}{c_{12}} x\right) - 1\right] d\Phi(x) \quad (3.4.13) \end{aligned}$$

Similarly,

$$\begin{aligned}
& P\left\{Z_0 + \frac{1}{c_{12}}(Z_i - Z_0) + \sqrt{n}(\theta_0 - \theta_j) < Z_j < Z_0 - \frac{1}{c_{12}}(Z_i - Z_0) + \sqrt{n}(\theta_0 - \theta_j), Z_i - Z_0 \leq 0\right\} \\
& \leq \int_{-\infty}^0 \left[2\Phi\left(-\frac{\sqrt{2}}{c_{12}}x\right) - 1\right] d\Phi(x) \\
& = \int_0^{\infty} \left[2\Phi\left(\frac{\sqrt{2}}{c_{12}}x\right) - 1\right] d\Phi(x). \tag{3.4.14}
\end{aligned}$$

So that

$$\begin{aligned}
P(CC|R_{12}) & \geq 1 - 2 \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \int_0^{\infty} \left[2\Phi\left(\frac{\sqrt{2}}{c_{12}}x\right) - 1\right] d\Phi(x) \\
& = 1 - 2k(k-1) \int_0^{\infty} \left[2\Phi\left(\frac{\sqrt{2}}{c_{12}}x\right) - 1\right] d\Phi(x) \tag{3.4.15}
\end{aligned}$$

Hence, for any  $P^*$ , let  $c_{12}$  be the smallest non-negative number such that

$$\int_0^{\infty} \Phi\left(\frac{\sqrt{2}}{c_{12}}x\right) d\Phi(x) \leq \frac{1}{4} + \frac{1 - P^*}{4k(k-1)},$$

then  $P(CC|R_{12}) \geq P^*$ .

### 3.5 A Subset Selection Approach to Classification Rule

with Respect to the Reciprocal of the Coefficient of Variation

$$\text{Let } \bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \quad i = 0, 1, \dots, k \tag{3.5.1}$$

$$s_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2, \quad \frac{n_i s_i^2}{\sigma_i^2} \sim \chi_{n_i-1}^2 \tag{3.5.2}$$

$$\alpha_i = \frac{\theta_i}{\sigma_i} = \frac{1}{\text{coefficient of variation}}, \quad i = 0, 1, \dots, k \tag{3.5.3}$$

We assume  $\alpha_0 = \frac{\theta_0}{\sigma_0}$  to be known and further that there exists only one population with  $\alpha_i = \alpha_0$ . The classification rule proposed in this case is

$R_{13}$ : Classify  $\pi_0$  as  $\pi_i$  if and only if

$$\left| \frac{\bar{X}_i}{s_i} - \alpha_0 \right| \leq c_{13} \min_{1 \leq j \leq k} \left| \frac{\bar{X}_j}{s_j} - \alpha_0 \right| \quad (3.5.4)$$

where  $c_{13} (\geq 1)$  is chosen to be the smallest nonnegative number such that  $P(CC|R_{13}) \geq P^*$ .

The classification rule  $R_{13}$  has the asymptotic optimum property that  $P(CC|R_{13})$  approaches unity as the sample sizes  $n_i$ 's,  $i = 1, \dots, k$ , become large. This result is proved in the following theorem where as before we write  $N = \sum_{i=1}^k n_i$  and  $\frac{n_i}{N} \rightarrow \lambda_i$ ,  $i = 1, \dots, k$ .

**Theorem 3.5.1.**  $P(MC|R_{13}) \rightarrow 0$  as  $N \rightarrow \infty$ , where  $N = \sum_{i=1}^k n_i$ .

**Proof.** Using a similar argument as in Lemma 3.4.1, we have

$$\begin{aligned} P(MC|R_{13}) &\leq \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k P\left\{ \left| \frac{\bar{X}_i}{s_i} - \alpha_0 \right| > c_{13} \left| \frac{\bar{X}_j}{s_j} - \alpha_0 \right| \mid \alpha_i = \alpha_0 \right\} \\ &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k [P\{\alpha_0 - \frac{1}{c_{13}} (\frac{\bar{X}_i}{s_i} - \alpha_0) \leq \frac{\bar{X}_j}{s_j} \leq \alpha_0 + \frac{1}{c_{13}} (\frac{\bar{X}_i}{s_i} - \alpha_0), \\ &\quad \frac{\bar{X}_i}{s_i} \geq \alpha_0 \mid \alpha_i = \alpha_0\} \\ &\quad + P\{\alpha_0 + \frac{1}{c_{13}} (\frac{\bar{X}_i}{s_i} - \alpha_0) \leq \frac{\bar{X}_j}{s_j} \leq \alpha_0 - \frac{1}{c_{13}} (\frac{\bar{X}_i}{s_i} - \alpha_0), \\ &\quad \frac{\bar{X}_i}{s_i} < \alpha_0 \mid \alpha_i = \alpha_0\}] \\ &= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k [P\{(1 + \frac{1}{c_{13}})\alpha_0 - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n_i}\alpha_0}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j} \leq \frac{Z_j}{U_j} \leq (1 - \frac{1}{c_{13}})\alpha_0 \\ &\quad + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n_i}\alpha_0}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j}, \\ &\quad \frac{Z_i}{U_i} \geq \alpha_0 - \frac{\sqrt{n_i}\alpha_0}{U_i} \mid \alpha_i = \alpha_0\}] \end{aligned}$$

$$+ P\left\{\left(1 - \frac{1}{c_{13}}\right)\alpha_0 + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n_i}\alpha_0}{c_{13}U_i} - \frac{\sqrt{n_i}\alpha_i}{U_j} \leq \frac{Z_j}{U_j} \leq \left(1 + \frac{1}{c_{13}}\right)\alpha_0 - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n_i}\alpha_0}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j}, \frac{Z_i}{U_i} < \alpha_0 - \frac{\sqrt{n_i}\alpha_0}{U_i} \mid \alpha_i = \alpha_0\right\}$$

where

$$Z_i = \frac{\sqrt{n_i}(\bar{X}_i - \theta_i)}{\sigma_i}, \quad U_i = \frac{\sqrt{n_i}s_i}{\sigma_i}, \quad i = 1, \dots, k.$$

$$= S_1 + S_2 \text{ (say)}$$

Now,

$$S_1 = \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \int_0^\infty \int_0^\infty \int_{\alpha_0 - \sqrt{n_i}\alpha_0}^\infty [\Phi\left(\left(1 - \frac{1}{c_{13}}\right)\alpha_0 t + \frac{xt}{c_{13}v} + \frac{\sqrt{n_i}\alpha_0 t}{c_{13}v} - \sqrt{n_j}\alpha_j\right) - \Phi\left(\left(1 + \frac{1}{c_{13}}\right)\alpha_0 t - \frac{xt}{c_{13}v} - \frac{\sqrt{n_i}\alpha_0 t}{c_{13}v} - \sqrt{n_j}\alpha_j\right)] d\Phi(x) dF_i(v) dF_j(t)$$

where  $\Phi(\cdot)$  is the cdf of unite normal variate and  $F_i(\cdot)$  represents the cdf of the square root of a  $\chi^2$  random variable with  $(n_i-1)$  degrees of freedom.

For any  $\epsilon > 0$ , there exists  $\delta(\epsilon)$  such that

$$\int_{|x| \geq \delta(\epsilon)} d\Phi(x) \leq \epsilon$$

and

$$\int_{|x| \geq \delta(\epsilon)} dF_r(x) \leq \epsilon \quad \text{for } r = 1, 2, \dots, k.$$

Hence, we get

$$S_1 \leq \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \int_{|t| < \delta(\epsilon)} \int_{|v| < \delta(\epsilon)} \int_{|x| < \delta(\epsilon)} [\Phi((1 - \frac{1}{c_{13}})\alpha_0 t + \frac{xt}{c_{13}^v} + \frac{\sqrt{n_i}\alpha_0 t}{c_{13}^v} - \sqrt{n_j}\alpha_j) - \Phi((1 + \frac{1}{c_{13}})\alpha_0 t - \frac{xt}{c_{13}^v} - \frac{\sqrt{n_i}\alpha_0 t}{c_{13}^v} - \sqrt{n_j}\alpha_j)] d\Phi(x) dF_i(v) dF_j(t) + \epsilon^3.$$

$$= \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \int_{|t| < \delta(\epsilon)} \int_{|v| < \delta(\epsilon)} \int_{|x| < \delta(\epsilon)} [\Phi((1 - \frac{1}{c_{13}})\alpha_0 t + \frac{xt}{c_{13}^v} - (\alpha_j \sqrt{\lambda_j} - \frac{\sqrt{\lambda_i}\alpha_0 t}{c_{13}^v})\sqrt{N}) - \Phi((1 + \frac{1}{c_{13}})\alpha_0 t - \frac{xt}{c_{13}^v} - (\alpha_j \sqrt{\lambda_j} + \frac{\sqrt{\lambda_i}\alpha_0 t}{c_{13}^v})\sqrt{N})] d\Phi(x) dF_i(x) dF_j(t) + \epsilon^3.$$

If  $t$ ,  $v$  and  $x$  are bounded, then for any  $\epsilon > 0$ , there exists an  $N(\epsilon)$  such that whenever  $N > N(\epsilon)$ ,

$$\Phi((1 - \frac{1}{c_{13}})\alpha_0 t + \frac{xt}{c_{13}^v} - (\alpha_j \sqrt{\lambda_j} - \frac{\sqrt{\lambda_i}\alpha_0 t}{c_{13}^v})\sqrt{N}) - \Phi((1 + \frac{1}{c_{13}})\alpha_0 t - \frac{xt}{c_{13}^v} - (\alpha_j \sqrt{\lambda_j} + \frac{\sqrt{\lambda_i}\alpha_0 t}{c_{13}^v})\sqrt{N}) < \epsilon.$$

Since  $\epsilon > 0$  is arbitrarily, hence  $S_1 \rightarrow 0$  as  $N \rightarrow \infty$ . Similarly, one can show that  $S_2 \rightarrow 0$  as  $N \rightarrow \infty$ . Then  $P(MC|R_{13}) \rightarrow 0$  as  $N \rightarrow \infty$ .

**Theorem 3.5.2.** Suppose  $n_1 = \dots = n_k = n$ . If  $c_{13}$  is the smallest non-negative number such that

$$\int_0^{\infty} \int_0^{\infty} \left\{ \int_{\alpha_0 v - \sqrt{n}\alpha_0}^{\infty} 2\phi\left(\frac{1}{c_{13}}(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v})t\right) d\phi(x) \right. \\ \left. - \int_{-\infty}^{\alpha_0 v - \sqrt{n}\alpha_0} 2\phi\left(\frac{1}{c_{13}}(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v})t\right) d\phi(x) + 2\phi(\alpha_0 v - \sqrt{n}\alpha_0) \right\} \\ dF_n(v) dF_n(t) \leq 1 + \frac{1 - P^*}{k(k-1)}$$

Then  $P(CC|R_{13}) \geq P^*$ .

Proof.

$$P(CC|R_{13}) = 1 - P(MC|R_{13})$$

$$\geq 1 - \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \left[ P\left\{ \frac{\alpha_0}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_0}{c_{13}U_i} + \alpha_0 - \frac{\sqrt{n}\alpha_j}{U_j} \leq \frac{Z_j}{U_j} \leq -\frac{\alpha_0}{c_{13}} \right. \right. \\ \left. \left. + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_0}{c_{13}U_i} + \alpha_0 - \frac{\sqrt{n}\alpha_j}{U_j}, \frac{Z_i}{U_i} \geq \alpha_0 - \frac{\sqrt{n}\alpha_0}{U_i} \right\} \right. \\ \left. + P\left\{ -\frac{\alpha_0}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_0}{c_{13}U_i} + \alpha_0 - \frac{\sqrt{n}\alpha_j}{U_j} \leq \frac{Z_j}{U_j} \leq \frac{\alpha_0}{c_{13}} \right. \right. \\ \left. \left. - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_0}{c_{13}U_i} + \alpha_0 - \frac{\sqrt{n}\alpha_j}{U_j}, \frac{Z_i}{U_i} < \alpha_0 - \frac{\sqrt{n}\alpha_0}{U_i} \right\} \right] \\ \geq 1 - \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \left[ P\left\{ \frac{\alpha_0}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_0}{c_{13}U_i} \leq \frac{Z_j}{U_j} \leq -\frac{\alpha_0}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_0}{c_{13}U_i}, \right. \right. \\ \left. \left. \frac{Z_i}{U_i} \geq \alpha_0 - \frac{\sqrt{n}\alpha_0}{U_i} \right\} \right. \\ \left. + P\left\{ -\frac{\alpha_0}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_0}{c_{13}U_i} \leq \frac{Z_j}{U_j} \leq \frac{\alpha_0}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_0}{c_{13}U_i}, \right. \right. \\ \left. \left. \frac{Z_i}{U_i} < \alpha_0 - \frac{\sqrt{n}\alpha_0}{U_i} \right\} \right]$$

$$\begin{aligned}
&= 1 - \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \int_0^{\infty} \int_0^{\infty} \int_{\alpha_0 v - \sqrt{n}\alpha_0}^{\infty} \left\{ \Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v}\right)t\right) \right. \\
&\quad \left. - \Phi\left(\frac{1}{c_{13}}\left(\alpha_0 - \frac{x}{v} - \frac{\sqrt{n}\alpha_0}{v}\right)t\right) \right\} d\Phi(x) dF_n(v) dF_n(t) \\
&\quad + \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\alpha_0 v - \sqrt{n}\alpha_0} \left\{ \Phi\left(\frac{1}{c_{13}}\left(\alpha_0 - \frac{x}{v} - \frac{\sqrt{n}\alpha_0}{v}\right)t\right) \right. \\
&\quad \left. - \Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v}\right)t\right) \right\} d\Phi(x) dF_n(v) dF_n(t)
\end{aligned}$$

where  $F_n(\cdot)$  is the cdf of the square root of a  $x^2$  random variable with  $(n-1)$  degrees of freedom.

$$\begin{aligned}
&= 1 - k(k-1) \int_0^{\infty} \int_0^{\infty} \int_{\alpha_0 v - \sqrt{n}\alpha_0}^{\infty} \{2\Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v}\right)t\right) - 1\} d\Phi(x) dF_n(v) dF_n(t) \\
&\quad + \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\alpha_0 v - \sqrt{n}\alpha_0} \{1 - 2\Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v}\right)t\right)\} d\Phi(x) dF_n(v) dF_n(t)
\end{aligned}$$

So if  $c_{13}$  is chosen to be the smallest nonnegative number such that

$$\begin{aligned}
&\int_0^{\infty} \int_0^{\infty} \int_{\alpha_0 v - \sqrt{n}\alpha_0}^{\infty} 2\Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} + \frac{\sqrt{n}\alpha_0}{v}\right)t\right) d\Phi(x) - \int_{-\infty}^{\alpha_0 v - \sqrt{n}\alpha_0} 2\Phi\left(\frac{1}{c_{13}}\left(-\alpha_0 + \frac{x}{v} \right. \right. \\
&\quad \left. \left. + \frac{\sqrt{n}\alpha_0}{v}\right)t\right) d\Phi(x) \\
&\quad + 2\Phi(\alpha_0 v - \sqrt{n}\alpha_0) \} dF_n(v) dF_n(t) \leq 1 + \frac{1 - P^*}{k(k-1)}
\end{aligned}$$

then

$$P(CC | R_{13}) \geq P^*.$$

## CHAPTER IV

## SELECTION PROCEDURES FOR NEGATIVE BINOMIAL POPULATIONS

4.1 Introduction

Let  $X_i$ ,  $i = 1, 2, \dots, k$  be  $k$  independent random observations from population  $\pi_i$ ,  $i = 1, 2, \dots, k$ , respectively, which has a negative binomial distribution with parameters  $r_i$ ,  $p_i$ . To select a subset of populations which contains the population associated with the largest  $p_i$  when  $r_1 = \dots = r_k = r$ , Bartlett and Govindarajulu [10] proposed a rule which is based on the statistic  $b \min_{1 \leq j \leq k} X_j - X_i$ , where  $b$  is a constant. However, in many situations one may be interested in small values of  $p_i$ . In this chapter, our aim is to select a subset of  $k$  negative binomial populations which contains the population with the smallest  $p_i$  based on a statistic of the type  $c \max_{1 \leq j \leq k} X_j - X_i$ . It should be pointed out that although the two problems seem similar, they are not equivalent, i.e. the procedures proposed here cannot be obtained from the above paper [10]. In Section 4.2, we present a result which gives a conservative constant for the unconditional rule we propose, which is based on an exact computation of the conditional distribution of the statistic  $c \max_{1 \leq j \leq k} X_j - X_i$ . In Section 4.3, we propose a similar rule as in Section 4.2, except that the rule is conditioned on the total number of observations  $T = \sum_{i=1}^k X_i$ . We obtain a lower bound for the infimum of the

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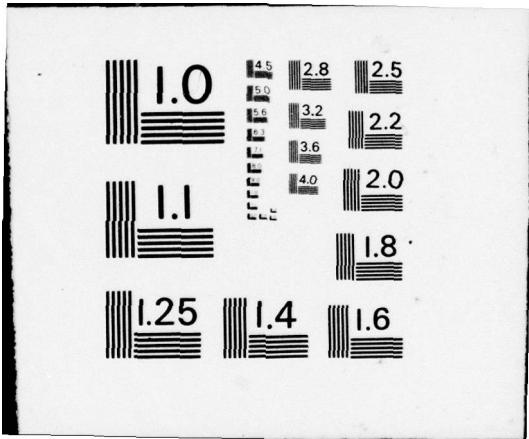
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probability of a correct selection. It is shown that when  $k = 2$ , the infimum of this procedure is attained when  $p_1 = p_2 = p$ , and it is independent of  $p$ . A method leading to a conservative solution for the constant  $c(t)$  depending on  $T = t$  is also given. An upper bound for the expected subset size is derived which holds for all values of the parameters. The problem of selecting all populations better than a standard is also considered in Section 4.4.

#### 4.2 An Unconditional Subset Selection Procedure

Let  $X$  be a random variable which has the negative binomial distribution with parameters  $r, p$ , i.e.  $X$  denotes the number of failures before the  $r$ th success is observed,  $p$  being the probability of a success in an independent trial. Then  $X$  is distributed with the probability mass function

$$P(X = x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, 2, \dots \quad (4.2.1)$$

It is known that the sum of  $n$  independent and identically distributed negative binomial random variables with parameters  $r, p$  is again a negative binomial random variable with parameters  $nr, p$ . We may, therefore, think of the selection problem as being one of picking a subset containing the negative binomial populations with the smallest  $p$  value, based on a single observation on  $X$  from each of the  $k$  populations.

Suppose then that  $X_i, i = 1, 2, \dots, k$  are independent observations from populations  $\pi_i, i = 1, 2, \dots, k$ , respectively, which have negative binomial distributions with parameters  $r, p_i$ . Without loss of generality and in order to avoid a more complicated notation, we assume that  $p_1 \leq p_2 \leq \dots \leq p_k$ . Any population whose parameter value equals  $p_1$  will

be defined as a best population. A correct selection (CS) is defined as a selection of any subset of the  $k$  given populations which contains at least one best population. Let  $P\{CS; k, \underline{p}, R\}$  denote the probability of a correct selection when the procedure  $R$  is used with the given  $k$  and when the true configuration of parameter values in  $\underline{p} = (p_1, \dots, p_k)$ . Let  $\Omega$  be the space of all configuration  $\underline{p} = (p_1, \dots, p_k)$  such that  $0 < p_i < 1, i = 1, \dots, k$ .

(A) The Rule  $R_{14}$  and its Associated Probability of a Correct Selection

We propose the following selection rule:

$R_{14}$ : Retain population  $\pi_i$  in the selected subset if and only if

$$\frac{r}{r + X_i} \leq d \min_{1 \leq j \leq k} \frac{r}{r + X_j} \quad (4.2.2)$$

where  $d \geq 1$  is the smallest number such that the basic probability requirement

$$\inf_{\underline{p} \in \Omega} P_{\underline{p}}(CS | R_{14}) \geq P^* \quad (4.2.3)$$

is satisfied.

Letting  $c_{14} = \frac{1}{d}$ , the rule in (4.2.2) becomes

$R_{14}$ : Retain population  $\pi_i$  in the selected subset if and only if

$$X_i \geq c_{14} \max_{1 \leq j \leq k} X_j - (1 - c_{14})r \quad (4.2.4)$$

where  $0 \leq c_{14} \leq 1$  is the largest number for which the basic probability requirement (4.2.3) is satisfied for all parameter points  $\underline{p} = (p_1, \dots, p_k)$  in  $\Omega$ .

Now for any  $\underline{p} \in \Omega$ ,

$$\begin{aligned}
 P_{\underline{p}}(CS|R_{14}) &= P_{\underline{p}}(X_{(1)} \geq c_{14} X_{(j)} - (1-c_{14})r, \quad j = 2, \dots, k) \\
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_1^r q_1^x \prod_{j=2}^k \left[ \frac{x+(1-c_{14})r}{c_{14}} \sum_{y=0}^{\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor} \binom{r+y-1}{y} p_j^r q_j^y \right]
 \end{aligned} \tag{4.2.5}$$

where  $q_i = 1 - p_i$ ,  $i = 1, 2, \dots, k$  and  $\lfloor a \rfloor$  denotes the greatest integer less than or equal to  $a$ .

Using the well-known identity for the incomplete beta function

$$q_1^r \sum_{j=0}^{s-1} \binom{r+j-1}{j} p_1^j = I_q(r, s) = p_1^s \sum_{j=r}^{\infty} \binom{s+j-1}{j} q_1^j \tag{4.2.6}$$

where  $I_q(r, s) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^q t^{r-1} (1-t)^{s-1} dt$ , it follows that

$$\begin{aligned}
 P_{\underline{p}}(CS|R_{14}) &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_1^r q_1^x \prod_{j=2}^k \\
 &\left[ \frac{\Gamma(r + \lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor + 1)}{\Gamma(r) \Gamma(\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor + 1)} p_j \int_0^{\frac{x+(1-c_{14})r}{c_{14}}} t^{r-1} (1-t)^{\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor} dt \right].
 \end{aligned}$$

It is immediately evident from the above that  $P_{\underline{p}}(CS|R_{14})$  is an increasing function of  $p_j$ , for  $j = 2, \dots, k$  and consequently  $P_{\underline{p}}(CS|R_{14})$  is partially minimized as we let each  $p_j$  approach  $p_1$  from above. Hence we have the following result.

Theorem 4.2.1.  $\inf_{p \in \Omega} P_p(CS|R_{14}) = \inf_{p \in \Omega_0} P_p(CS|R_{14})$

$$= \inf_{0 < p < 1} \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \left[ \frac{x+(1-c_{14})r}{c_{14}} \right] \sum_{y=0}^{\infty} \binom{r+y-1}{y} p^r q^y \}^{k-1} \quad (4.2.7)$$

where  $\Omega_0 = \{p = (p, \dots, p), 0 < p < 1\}$ .

It is difficult to determine analytically and also numerically where the infimum of the above expression with respect to the common  $p$  ( $0 < p < 1$ ) takes place. If we could find the infimum then we can solve for the constant  $c_{14}$  to satisfy (4.2.3). In order to overcome this difficulty, we give a method leading a conservative solution for the constant.

For any fixed non-negative integer  $t$ , let

$$N(t, c(t), r) = \sum_{x=0}^{\lfloor \frac{t+(1-c(t))r}{1+c(t)} \rfloor} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \quad (4.2.8)$$

where  $[a]$  is defined as before and  $a \wedge b = \min(a, b)$ .

Theorem 4.2.2. For given  $P^*$ , let  $P_1^* = (P^*)^{\frac{1}{k-1}}$ , and  $t \geq 0$ , let  $c_{14}(t)$  be the largest value such that

$$N(t, c_{14}(t), r) \geq \binom{2r+t-1}{t} P_1^* . \quad (4.2.9)$$

If  $c_{14} = \inf \{c_{14}(t) : t \geq 0\}$ , then

$$\inf_{p \in \Omega} P_p(CS|R_{14}) \geq P^* .$$

Proof. For  $\underline{p} \in \Omega_0$

$$\begin{aligned}
 P_{\underline{p}}(CS|R_{14}) &= P_{\underline{p}}(X_1 \geq c_{14} \max_{2 \leq j \leq k} X_j - (1-c_{14})r) \\
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \left[ \begin{array}{c} \frac{x+(1-c_{14})r}{c_{14}} \\ \sum_{y=0} \end{array} \right] \binom{r+y-1}{y} p^r q^y \Big]^{k-1} \\
 &\geq \left\{ \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \left[ \begin{array}{c} \frac{x+(1-c_{14})r}{c_{14}} \\ \sum_{y=0} \end{array} \right] \binom{r+y-1}{y} p^r q^y \right\}^{k-1} \\
 &= \{P_{\underline{p}}(X_1 \geq c_{14} X_2 - (1-c_{14})r)\}^{k-1} \\
 &= \left\{ \sum_{t=0}^{\infty} P_{\underline{p}}(X_1 \geq c_{14} X_2 - (1-c_{14})r \mid X_1 + X_2 = t) \cdot P_{\underline{p}}(X_1 + X_2 = t) \right\}^{k-1} \\
 &\geq \left\{ \sum_{t=0}^{\infty} P_{\underline{p}}(X_1 \geq c_{14}(t) X_2 - (1-c_{14}(t))r \mid X_1 + X_2 = t) \cdot P_{\underline{p}}(X_1 + X_2 = t) \right\}^{k-1} \\
 &= \left\{ \sum_{t=0}^{\infty} P_{\underline{p}}\left(X_2 \leq \frac{t+(1-c_{14}(t))r}{1+c_{14}(t)} \mid X_1 + X_2 = t\right) \cdot P_{\underline{p}}(X_1 + X_2 = t) \right\}^{k-1} \\
 &= \left\{ \sum_{t=0}^{\infty} \frac{N(t, c_{14}(t), r)}{\binom{2r+t-1}{t}} \cdot P_{\underline{p}}(X_1 + X_2 = t) \right\}^{k-1} \\
 &\geq (P_1^*)^{k-1} = P^*.
 \end{aligned}$$

From Theorem 4.2.1, the proof is completed.

(B) Some Properties of the Procedure  $R_{14}$

We now discuss some properties of the rule  $R_{14}$ . For  $\underline{p} \in \Omega$ , define

$$P_{\underline{p}}(i) = P_{\underline{p}}\{R_{14} \text{ selects } \pi(i)\}$$

Theorem 4.2.3.  $p_{\underline{p}}(i)$  is { + in  $p_i$  when all other components of  $\underline{p}$  are fixed.

† in  $p_j$  ( $j \neq i$ ) when all other components of  $\underline{p}$  are fixed.

Proof.  $p_{\underline{p}}(i) = P(R_{14} \text{ selects } \pi(i))$

$$\begin{aligned}
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_i^r q_i^x \left[ \prod_{\substack{j=1 \\ j \neq i}}^k \sum_{y=0}^{\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor} \binom{r+y-1}{y} p_j^r q_j^y \right] \\
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_i^r q_i^x \left[ \prod_{\substack{j=1 \\ j \neq i}}^k \frac{\Gamma(r + \lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor + 1)}{\Gamma(r) \Gamma(\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor + 1)} \right. \\
 &\quad \left. \cdot \int_0^1 p_j t^{r-1} (1-t)^{\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor} dt \right].
 \end{aligned}$$

It is obvious that  $p_{\underline{p}}(i)$  is increasing in  $p_j$  ( $j \neq i$ ) when all other components of  $\underline{p}$  are fixed. On the other hand,  $p_{\underline{p}}(i)$  can be written as

$$p_{\underline{p}}(i) = E_{p_i} f(X; p_1, \dots, \hat{p}_i, \dots, p_k)$$

where  $f(x; p_1, \dots, \hat{p}_i, \dots, p_k) = \prod_{\substack{j=1 \\ j \neq i}}^k \sum_{y=0}^{\lfloor \frac{x+(1-c_{14})r}{c_{14}} \rfloor} \binom{r+y-1}{y} p_j^r q_j^y$  and  $\hat{p}_i$

denotes that  $p_i$  is deleted. Note that  $f(x; p_1, \dots, \hat{p}_i, \dots, p_k)$  is a non-decreasing function in  $x$  and the cdf  $F_p(x)$  of a negative binomial random variable is stochastically increasing in  $1-p$ . Then for

$$p_i \leq \hat{p}_i,$$

$$\begin{aligned}
E_{p_i} f(X; p_1, \dots, \hat{p}_i, \dots, p_k) &= \sum_{x=0}^{\infty} [f(x; p_1, \dots, \hat{p}_i, \dots, p_k) \\
&\quad - f(x+1; p_1, \dots, \hat{p}_i, \dots, p_k)] F_{p_i}(x) \\
&\geq \sum_{x=0}^{\infty} [f(x; p_1, \dots, \hat{p}_i, \dots, p_k) \\
&\quad - f(x+1; p_1, \dots, \hat{p}_i, \dots, p_k)] F_{p_i^-}(x) \\
&= E_{p_i^-} f(X; p_1, \dots, \hat{p}_i, \dots, p_k).
\end{aligned}$$

Hence  $p_{\underline{p}}(i)$  is non-increasing in  $p_i$  when all other components of  $\underline{p}$  are fixed.

Corollary 4.2.1. For every  $\underline{p} \in \Omega$  and  $1 \leq i < j \leq k$ ,  $p_{\underline{p}}(i) \geq p_{\underline{p}}(j)$ .

Proof. The proof follows easily from the above theorem hence is omitted. We also have the following corollary.

Corollary 4.2.2. For every  $\underline{p} \in \Omega$  and  $1 < j \leq k$ ,

$$P_{\underline{p}}(R_{14} \text{ does not select } \pi_{(1)}) \leq P_{\underline{p}}(R_{14} \text{ does not select } \pi_{(j)}).$$

Remark 4.2.1. It also follows from Theorem 4.2.3. that  $p_{\underline{p}}(1)$ , the probability of a correct selection, attains its minimum when  $p_2, \dots, p_k$  tend to  $p_1$  from above.

### 4.3 A Conditional Subset Selection Procedure

In this section, we use the same notation as in Section 4.2.

We propose a rule  $R_{15}$ , similar to the rule  $R_{14}$  except that this rule is based on the total number of observations  $T = \sum_{i=1}^k X_i$ , as follows:

$R_{15}$ : Selection  $\pi_i$  if and only if

$$X_i \geq c_{15}(t) \max_{1 \leq j \leq k} X_j - (1 - c_{15}(t))r, \text{ given } \sum_{i=1}^k X_i = t,$$

where  $t \geq 0$  and  $0 \leq c_{15}(t) \leq 1$  is chosen to satisfy the basic probability requirement. For this rule  $R_{15}$ , we obtain an exact result for  $k = 2$  in Theorem 4.3.1.. For  $k \geq 3$ , we have a lower bound for the probability of a correct selection in Theorem 4.3.2.

Theorem 4.3.1. For a given  $P^*$  ( $\frac{1}{k} < P^* < 1$ ),  $k = 2$ , and any  $t \geq 0$ , let  $c_{15}(t)$  be the largest value such that

$$N(t, c_{15}(t), r) \geq P^*(2^{r+t-1}). \quad (4.3.1)$$

Then,  $\inf_{p \in \Omega} P_p(\text{CS} | R_{15}) = \inf_{p \in \Omega_0} P_p(\text{CS} | R_{15}) \geq P^*$ .

Proof. Let  $D(t) = \left\langle \frac{tc_{15}(t) - (1-c_{15}(t))r}{1 + c_{15}(t)} \right\rangle$ , where  $\langle \alpha \rangle$  denote the smallest integer  $\geq \alpha$ , then for any  $p \in \Omega$ ,

$$P_p(\text{CS} | R_{15}) = P_p(X_{(1)} \geq c_{15}(t) X_{(2)} - (1-c_{15}(t))r \mid X_{(1)} + X_{(2)} = t)$$

$$= P_p(X_{(1)} \geq \frac{tc_{15}(t) - (1-c_{15}(t))r}{1 + c_{15}(t)} \mid X_{(1)} + X_{(2)} = t)$$

$$= \frac{\sum_{x=D(t)}^t P_p(X_{(1)} = x, X_{(2)} = t-x)}{\sum_{x=0}^t P_p(X_{(1)} = x, X_{(2)} = t-x)}$$

$$= \frac{\sum_{x=D(t)}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}{\sum_{x=0}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}$$

$$= \frac{\sum_{x=D(t)}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=0}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x} \quad \text{where } \lambda = \frac{q_1}{q_2}$$

$$\begin{aligned}
&= \left\{ 1 + \frac{\sum_{x=0}^{D(t)-1} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=D(t)}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x} \right\}^{-1} \\
&= \{1 + \phi(\lambda)\}^{-1} \tag{4.3.2}
\end{aligned}$$

By differentiating  $\phi(\lambda)$  with respect to  $\lambda$ , we get

$$\begin{aligned}
\phi'(\lambda) &= \frac{1}{\left\{ \sum_{x=D(t)}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x \right\}^2} \left[ \begin{aligned} &\left\{ \sum_{x=0}^{D(t)-1} x \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^{x-1} \right\} \\ &\left\{ \sum_{x=D(t)}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x \right\} \\ &- \left\{ \sum_{x=D(t)}^t x \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^{x-1} \right\} \\ &\left\{ \sum_{x=0}^{D(t)-1} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x \right\} \end{aligned} \right]
\end{aligned}$$

$$\leq 0.$$

Hence  $\phi(\lambda)$  is non-increasing in  $\lambda$  and the right hand side of (4.3.2) is non-decreasing in  $\lambda$ . Since  $\lambda \geq 1$ , the infimum of  $P_{\underline{p}}(CS|R_{15})$  occurs at  $\lambda = 1$  i.e. when  $p_1 = p_2$ . Thus  $\inf_{\underline{p} \in \Omega} P_{\underline{p}}(CS|R_{15}) = \inf_{\underline{p} \in \Omega_0} P_{\underline{p}}(CS|R_{15})$ . Note that this infimum probability does not depend on the common value of  $p_1 = p_2 = p$ .

For  $k = 2$  and selected values of  $r$  and  $t$ , tables of the constants  $c_{15}(t)$  satisfying (4.3.1) are given at the end of the chapter.

**Theorem 4.3.2.** For  $\underline{p} \in \Omega$ ,

$$P_{\underline{p}}(CS|R_{15}) \geq 2^{-k} + \sum_{j=2}^k \sum_{\ell=0}^t \{ \sum_{\underline{p}} P(X_{(1)} \geq c_{15}(\ell) X_{(j)} - (1-c_{15}(\ell))r | X_{(1)} + X_{(j)} = \ell) \\ \cdot P_{\underline{p}}(X_{(1)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t) / P_{\underline{p}}(\sum_{i=1}^k X_i = t) \}$$

Proof. The proof is straightforward and hence omitted.

Theorem 4.3.3. For given  $P^*$ ,  $\frac{1}{k} < P^* < 1$ , let  $P_2^* = 1 - \frac{1-P^*}{k-1}$ ,  $0 \leq \ell \leq t$ , let  $c_{15}(\ell)$  be the largest value such that

$$N(\ell, c_{15}(\ell), r) \geq P_2^* \binom{2r+\ell-1}{\ell}$$

If,  $c_{15}(t) = \min \{c_{15}(\ell) : 0 \leq \ell \leq t\}$ , then

$$\inf_{\underline{p} \in \Omega} P_{\underline{p}}(CS|R_{15}) \geq P^*.$$

Proof. It follows from Theorem 4.3.2. that for  $\underline{p} \in \Omega$ ,

$$P_{\underline{p}}(CS|R_{15}) \geq 2^{-k} + \sum_{j=2}^k \sum_{\ell=0}^t \{ \sum_{\underline{p}} P(X_{(1)} \geq c_{15}(\ell) X_{(j)} - (1-c_{15}(\ell))r | X_{(1)} + X_{(j)} = \ell) \\ \cdot P_{\underline{p}}(X_{(1)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t) / P_{\underline{p}}(\sum_{i=1}^k X_i = t) \} \\ \geq 2^{-k} + \sum_{j=2}^k \sum_{\ell=0}^t \{ \sum_{\underline{p}} P(X_{(1)} \geq c_{15}(\ell) X_{(j)} - (1-c_{15}(\ell))r | X_{(1)} + X_{(j)} = \ell) \\ \cdot P_{\underline{p}}(X_{(1)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t) / P_{\underline{p}}(\sum_{i=1}^k X_i = t) \}$$

and from Theorem 4.3.1., for any  $j$ ,  $j = 2, \dots, k$ ,

$$\inf_{\underline{p} \in \Omega} P_{\underline{p}}(X_{(1)} \geq c_{15}(\ell) X_{(j)} - (1-c_{15}(\ell))r | X_{(1)} + X_{(j)} = \ell) \\ = \inf_{\underline{p} \in \Omega_0} P_{\underline{p}}(X_1 \geq c_{15}(\ell) X_j - (1-c_{15}(\ell))r | X_1 + X_j = \ell) \\ = \frac{N(\ell, c_{15}(\ell), r)}{\binom{2r+\ell-1}{\ell}} \geq P_2^* ,$$

we have

$$\begin{aligned} P_{\underline{p}}(CS|R_{15}) &\geq 2^{-k} \sum_{j=2}^k \sum_{\ell=0}^t P_2^* \cdot P_{\underline{p}}(X_{(1)}+X_{(j)}=\ell, \sum_{i=1}^k X_i=t) / P_{\underline{p}}(\sum_{i=1}^k X_i=t) \\ &= 2^{-k+(k-1)} P_2^* = P^*. \end{aligned}$$

Thus the proof is completed.

Hence, for each  $k$ ,  $P^*$ , Theorem 4.3.3. guarantees the existence of  $c_{15}(t)$  for the rule  $R_{15}$  and gives a method to find  $c_{15}(t)$  for given  $\sum_{i=1}^k X_i = t$  such that  $P_{\underline{p}}(CS|R_{15}) \geq P^*$  for any  $\underline{p} \in \Omega$ .

#### An Upper Bound on the Expected Subset Size for $R_{15}$

For the procedure  $R_{15}$ , the subset size  $S$  of the selected subset is a random variable which takes on only integer values from 1 to  $k$ , inclusively. For any fixed values of  $k$ , and  $P^*$ , then expected size of the selected subset is a function of the true configuration  $\underline{p} = (p_1, \dots, p_k)$ .

Lemma 3.3.1. For  $k = 2$ ,  $a = \left\langle \frac{t c_{15}(t) - (1 - c_{15}(t))r}{1 + c_{15}(t)} \right\rangle$ ,  $b = \left\lfloor \frac{t + (1 - c_{15}(t))r}{1 + c_{15}(t)} \right\rfloor$ .

$$\sup_{\underline{p} \in \Omega} E_{\underline{p}}(S|R_{15}) \leq 1 + \frac{1}{1 + \frac{\sum_{x=b+1}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x}}{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x}}}$$

Proof. For  $\underline{p} \in \Omega$ , let  $\lambda = \frac{q_1}{q_2}$ , then  $\lambda \geq 1$

$$\begin{aligned}
E_{\underline{P}}(S|R_{15}) &= P_{\underline{P}}(X_{(1)} \geq c_{15}(t)X_{(2)} - (1-c_{15}(t))r | X_{(1)}+X_{(2)}=t) \\
&\quad + P_{\underline{P}}(X_{(2)} \geq c_{15}(t)X_{(1)} - (1-c_{15}(t))r | X_{(1)}+X_{(2)}=t) \\
&= P_{\underline{P}}\left(X_{(1)} \geq \frac{tc_{15}(t) - (1-c_{15}(t))r}{1+c_{15}(t)} \mid X_{(1)}+X_{(2)}=t\right) \\
&\quad + P_{\underline{P}}\left(X_{(1)} \leq \frac{t+(1-c_{15}(t))r}{1+c_{15}(t)} \mid X_{(1)}+X_{(2)}=t\right) \\
&= 1 + P_{\underline{P}}\left(\frac{tc_{15}(t) - (1-c_{15}(t))r}{1+c_{15}(t)} \leq X_{(1)} \leq \frac{t+(1-c_{15}(t))r}{1+c_{15}(t)} \mid X_{(1)}+X_{(2)}=t\right) \\
&= 1 + P_{\underline{P}}(a \leq X_{(1)} \leq b \mid X_{(1)}+X_{(2)}=t) \\
&= 1 + \frac{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}{\sum_{x=0}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}} \\
&= 1 + \frac{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=0}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x} \\
&= 1 + \frac{1}{1 + \frac{\sum_{x=0}^{a-1} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x} + \frac{\sum_{x=b+1}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}}
\end{aligned}$$

(4.3.2)

$$\text{Let } g(\lambda) = \frac{\sum_{x=0}^{a-1} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x} \text{ and } h(\lambda) = \frac{\sum_{x=b+1}^t \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^b \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \lambda^x}$$

Then, by differentiating  $g(\lambda)$  and  $h(\lambda)$  with respect to  $\lambda$ , it is easily seen that  $g(\lambda)$  is a non-increasing function of  $\lambda$ ,  $h(\lambda)$  is a non-decreasing function of  $\lambda$ . Hence, from (4.3.2) the Lemma follows.

**Theorem 4.3.4.** For  $k \geq 2$ ,  $0 \leq \ell \leq t$ , let  $b(\ell) = \left[ \frac{\ell + (1 - c_{15}(\ell))r}{1 + c_{15}(\ell)} \right]$

$$a(\ell) = \left\langle \frac{\ell c_{15}(\ell) - (1 - c_{15}(\ell))r}{1 + c_{15}(\ell)} \right\rangle$$

$$\sup_{p \in \Omega} E_p(S | R_{15}) \leq \frac{k}{2} \max_{0 \leq \ell \leq t} \left\{ 1 + \frac{1}{1 + \frac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}} \right\}$$

**Proof.** For  $p \in \Omega$ ,

$$\begin{aligned} E_p(S | R_{15}) &= \sum_{i=1}^k P_p(X(i) \geq c_{15}(t) \max_{j \neq i} X(j) - (1 - c_{15}(t))r \mid \sum_{i=1}^k X_i = t) \\ &\leq \frac{1}{k-1} \sum_{i=1}^k \sum_{j \neq i} P_p(X(i) \geq c_{15}(t) X(j) - (1 - c_{15}(t))r \mid \sum_{i=1}^k X_i = t) \\ &= \frac{1}{(k-1) P_p(\sum_{i=1}^k X_i = t)} \sum_{i=1}^k \sum_{j \neq i} \sum_{\ell=0}^t P_p(X(i) \geq c_{15}(t) X(j) - (1 - c_{15}(t))r, \\ &\quad X(i) + X(j) = \ell, \sum_{s \neq i, j} X(s) = t - \ell) \\ &= \frac{1}{(k-1) P_p(\sum_{i=1}^k X_i = t)} \sum_{i=1}^k \sum_{j \neq i} \sum_{\ell=0}^t P_p(X(i) \geq c_{15}(t) X(j) - (1 - c_{15}(t))r \mid \\ &\quad X(i) + X(j) = \ell) \cdot P_p(X(i) + X(j) = \ell, \sum_{s \neq i, j} X(s) = t - \ell) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(k-1)P_{\underline{P}}\left(\sum_{i=1}^k X_i = t\right)} \sum_{i < j} \sum_{\ell=0}^t \{P_{\underline{P}}(X_{(i)} \geq c_{15}(t)X_{(j)} - (1-c_{15}(t))r | \\
&\quad X_{(i)} + X_{(j)} = \ell) \\
&\quad + P_{\underline{P}}(X_{(j)} \geq c_{15}(t)X_{(i)} - (1-c_{15}(t))r | X_{(i)} + X_{(j)} = \ell)\} \\
&\quad \cdot P_{\underline{P}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t) \\
&\leq \frac{1}{(k-1)P_{\underline{P}}\left(\sum_{i=1}^k X_i = t\right)} \sum_{i < j} \sum_{\ell=0}^t \{ \sup_{\Omega} [P_{\underline{P}}(X_{(i)} \geq c_{15}(t)X_{(j)} \\
&\quad - (1-c_{15}(t))r | X_{(i)} + X_{(j)} = \ell) \\
&\quad + P_{\underline{P}}(X_{(j)} \geq c_{15}(t)X_{(i)} - (1-c_{15}(t))r | X_{(i)} + X_{(j)} = \ell)]\} \\
&\quad \cdot P_{\underline{P}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t) \\
&\leq \frac{1}{(k-1)P_{\underline{P}}\left(\sum_{i=1}^k X_i = t\right)} \sum_{i < j} \sum_{\ell=0}^t \left\{ 1 + \frac{1}{1 + \frac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}} \right\} \\
&\quad \cdot P_{\underline{P}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^k X_i = t)
\end{aligned}$$

by Lemma 4.3.1.,

$$\leq \frac{1}{k-1} \sum_{i < j} \max_{0 \leq \ell \leq t} \left\{ 1 + \frac{1}{1 + \frac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x} \binom{r+\ell-x-1}{\ell-x}}} \right\}$$

The proof is thus completed.

#### 4.4 Selection of all Populations Better Than a Standard

##### Case 1. Known Standard.

We consider the problem where  $k$  negative binomial populations  $\pi_i$  with parameters  $r_i, p_i$  ( $i = 1, \dots, k$ ) are to be compared with a standard  $\pi_0$  with parameters  $r_0, p_0$  where  $p_0$  is the known value of the probability of a success in the standard population. Since  $p_i$  is the probability of a success, we define  $\pi_i$  to be better than  $\pi_0$  when  $p_i > p_0$ . Let  $X_i$  denote the number of failures before the  $r_i$ th success is observed in population  $\pi_i$ . Then for selecting all populations better than the standard we define the following procedure:

$R_{D_1}$ : Retain in the selected subset those and only those populations  $\pi_i$  ( $i = 1, \dots, k$ ) for which

$$X_i \leq p_0 + D_1. \quad (4.4.1)$$

Let  $l_1$  and  $l_2$  denote the number of populations with  $p_i \geq p_0$  and  $p_i < p_0$ , respectively, so that  $l_1 + l_2 = k$ . In general,  $l_1$  and  $l_2$  are unknown.

Then the probability  $P_1$  of a correct decision is given by

$$\begin{aligned} P_1 &= \prod_{i=1}^{l_1} P\{X_i' \leq p_0 + D_1\} \\ &= \prod_{i=1}^{l_1} \left\{ \sum_{x=0}^{[p_0 + D_1]} \binom{r_i + x - 1}{x} p_i'^{r_i} q_i'^x \right\} \end{aligned}$$

where primes refer to the  $l_1$  populations with  $p_i \geq p_0$ . Let  $m = [p_0 + D_1]$ .

Then

$$P_1 = \prod_{i=1}^{l_1} \frac{\Gamma(r_i + m + 1)}{\Gamma(r_i)\Gamma(m + 1)} \int_0^{p_i'} t^{r_i - 1} (1 - t)^m dt$$

A lower bound on the above probability is obtained by setting  $p_i = p_0$  ( $i = 1, \dots, k$ ) and  $\ell_1 = k$ . Hence the inequality determining  $D_1$  becomes

$$\prod_{i=1}^k \left\{ \sum_{x=0}^{[p_0 + D_1]} \binom{r_i + x - 1}{x} p_0^r q_0^x \right\} \geq P^* \quad (4.4.2)$$

and the solution is the smallest value of  $D_1$  satisfying (4.4.2). If  $r_i = r$  ( $i = 1, \dots, k$ ), then (4.4.2) reduces to

$$\sum_{x=0}^{[p_0 + D_1]} \binom{r+x-1}{x} p_0^r q_0^x \geq (P^*)^{\frac{1}{k}}.$$

This is easily solved by consulting a table of cumulative negative binomial probabilities.

#### Case 2. Unknown Standard.

The assumptions are the same as in case 1 except that  $p_0$  is not known. Let  $X_0$  be the number of failures before the  $r_0$ th success is observed in population  $\pi_0$ . Consider the following procedure:

$R_{D_2}$ : Retain in the selected subset those and only those populations  $\pi_i$  ( $i = 1, \dots, k$ ) for which

$$X_i \leq \frac{1}{D_2} X_0 + \left( \frac{1}{D_2} - 1 \right) r_i. \quad (4.4.3)$$

The probability  $P_2$  of retaining all the  $\ell_1$  populations with  $p_i \geq p_0$  attains a minimum where  $p_i = p_0$  ( $i = 1, \dots, k$ ) and  $\ell_1 = k$  and is given by

$$P_2(p_0, D_2) = \sum_{x=0}^{\infty} \prod_{i=1}^k \left\{ \sum_{y=0}^{[\frac{1}{D_2} x + (\frac{1}{D_2} - 1) r_i]} \binom{r_i + y - 1}{y} p_0^{r_i} q_0^y \right\} \left( \sum_{x=0}^{r_0 + x - 1} \binom{r_0 + x - 1}{x} p_0^{r_0} q_0^x \right).$$

Then the desired value of  $D_2$  for the rule defined in (4.4.3) is the largest number for which

$$\min_{0 < p_0 < 1} P_2(p_0, D_2) \geq P^* . \quad (4.4.4)$$

For  $r_i = r$  ( $i = 0, 1, \dots, k$ ),

$$P_2 \geq \left\{ \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_0^r q_0^x \left[ \sum_{y=0}^{\lfloor \frac{1}{D_2} x + (\frac{1}{D_2} - 1)r \rfloor} \binom{r+y-1}{y} p_0^r q_0^y \right] \right\}^k$$

$$= \{P(X_1 \leq \frac{1}{D_2} X_0 + (\frac{1}{D_2} - 1)r)\}^k$$

For a given  $P^*$ , let  $P_1^* = (P^*)^{\frac{1}{k}}$  and for any  $t \geq 0$ , let  $D_2(t)$  be the largest value such that

$$N(t, D_2(t), r) \geq \binom{2r+t-1}{t} P_1^* ,$$

where  $N(t, D_2(t), r)$  is defined in (4.2.8). Let  $D_2 = \inf \{D_2(t) : t \geq 0\}$ .

Then a conservative value of  $D_2$  is obtained such that  $P_2 \geq P^*$ . The arguments for the above statement are essentially the same as those in Section 4.2.

#### 4.5 Application

If a structure consists of  $n$  components it will be called a structure of order  $n$ . The state of all components of such a system will be described by a vector  $\underline{x} = (x_1, \dots, x_n)$  where  $x_i = 1$  means "ith component performs" and  $x_i = 0$  means "ith component fails". In reliability theory, structures satisfying the following conditions

- (i)  $\emptyset(\underline{1}) = 1$  where  $\underline{1} = (1, \dots, 1)$
- (ii)  $\emptyset(\underline{0}) = 0$  where  $\underline{0} = (0, \dots, 0)$
- (iii)  $\emptyset(\underline{x}) \geq \emptyset(\underline{y})$  whenever  $x_i \geq y_i$ ,  $i = 1, \dots, n$

are called coherent (see Birnbaum, Esary and Saunders [17]) or monotonic (see Barlow and Proschan [5]).

Now consider  $k$  independent structures each of which consists of  $n$  components in parallel. We put each of the components of the structure on test. Suppose a component might not perform at the first trial. Let  $Y_{ij}$  denote the number of failures of the  $j$ th component of the structure  $\pi_i$  before it performs,  $j = 1, 2, \dots, n$ ,  $i = 1, 2, \dots, k$ . It is easy to see that  $Z_i = \min(Y_{i1}, \dots, Y_{in})$  is a geometric random variable with parameter  $p_i = 1 - \prod_{j=1}^n (1 - p_{ij})$  where  $p_{ij}$  is the probability that the  $j$ th component of the  $i$ th structure will perform. Hence, the problem of selecting the structure with the smallest reliability is the same as that of selecting the geometric population with the smallest parameter as discussed in this chapter when  $r=1$ .

Table VII

For  $k=2$ ,  $r=5$ , this table gives the value of  $c_{15}(t)$  for various  $t$  and  $P^*$  value.

$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.8333	.8333	.8333	.8333	26	.7142	.5000	.3846	.2857
2	.7142	.7142	.7142	.7142	27	.6817	.4800	.4230	.2758
3	.8571	.6250	.6250	.6250	28	.6521	.5199	.4073	.2666
4	.7500	.7500	.5555	.5555	29	.6956	.5000	.3928	.2580
5	.8750	.6666	.5000	.5000	30	.6666	.4814	.3793	.2903
6	.7778	.6000	.6000	.4545	31	.7083	.4642	.4137	.2812
7	.6999	.5454	.5454	.4166	32	.6800	.5000	.3999	.2727
8	.8000	.6363	.5000	.3846	33	.6538	.4827	.3871	.2646
9	.7272	.5833	.4615	.3571	34	.6922	.4666	.3750	.2571
10	.6666	.5384	.5384	.3333	35	.6666	.5000	.4062	.2500
11	.7500	.5000	.5000	.3999	36	.7037	.4838	.3939	.2778
12	.6922	.5714	.4666	.3750	37	.6785	.4687	.3823	.2702
13	.7692	.5333	.4375	.3529	38	.6551	.5000	.3714	.2631
14	.7142	.5000	.4117	.3333	39	.6896	.4848	.3999	.2564
15	.6666	.5625	.4706	.3157	40	.6666	.4706	.3888	.2500
16	.7333	.5293	.4444	.3000	41	.6999	.4571	.3783	.2439
17	.6875	.5000	.4210	.3500	42	.6773	.4856	.3684	.2682
18	.7500	.5555	.3999	.3333	43	.6562	.4721	.3947	.2619
19	.7059	.5263	.4499	.3182	44	.6875	.4594	.3846	.2558
20	.6666	.5000	.4285	.3043	45	.6666	.4864	.3750	.2500
21	.7221	.4761	.4090	.2916	46	.6969	.4736	.3658	.2444
22	.6842	.5238	.3913	.2799	47	.6764	.4615	.3902	.2666
23	.7368	.5000	.4347	.2692	48	.6571	.4499	.3809	.2608
24	.6999	.4782	.4166	.3077	49	.6856	.4750	.3721	.2553
25	.6666	.5217	.3999	.2962	50	.6666	.4634	.3636	.2500

Table VIII

For  $k=2$ ,  $r=6$ , this table gives the value of  $c_{15}(t)$  for various  $t$  and  $P^*$  value.

$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.8571	.8571	.8571	.8571	26	.7272	.5199	.4615	.3103
2	.7500	.7500	.7500	.7500	27	.6956	.5600	.4444	.3448
3	.8750	.6666	.6666	.6666	28	.7391	.5384	.4285	.3333
4	.7778	.7778	.6000	.6000	29	.7083	.5185	.4642	.3226
5	.8888	.6999	.5454	.5454	30	.7500	.5555	.4482	.3125
6	.8000	.6363	.6363	.5000	31	.7200	.5357	.4333	.3030
7	.7272	.5833	.5833	.4615	32	.6922	.5172	.4193	.3333
8	.8182	.6666	.5384	.4285	33	.7307	.5000	.4516	.3235
9	.7500	.6153	.5000	.3999	34	.7037	.5333	.4375	.3143
10	.8333	.5714	.5714	.4666	35	.7407	.5161	.4242	.3055
11	.7692	.6428	.5333	.4375	36	.7142	.5000	.4545	.2972
12	.7142	.6000	.5000	.4117	37	.6896	.5312	.4411	.3243
13	.7857	.5625	.4706	.3888	38	.7241	.5151	.4285	.3157
14	.7333	.6500	.5293	.3684	39	.6999	.5000	.4166	.3077
15	.8000	.5882	.5000	.3500	40	.7333	.5293	.4444	.3000
16	.7500	.5555	.4736	.3999	41	.7096	.5143	.4324	.2926
17	.7059	.5263	.4499	.3809	42	.6875	.5000	.4210	.2857
18	.7646	.5789	.5000	.3636	43	.7187	.5278	.4102	.3095
19	.7221	.5500	.4761	.3478	44	.6969	.5135	.5358	.3023
20	.7778	.5238	.4545	.3333	45	.7272	.5000	.4250	.2954
21	.7368	.5714	.4347	.3750	46	.7059	.5263	.4146	.2889
22	.6999	.5454	.4782	.3600	47	.6856	.5128	.4047	.2826
23	.7500	.5217	.4583	.3461	48	.7142	.5000	.4285	.3043
24	.7142	.5652	.4399	.3333	49	.6944	.5249	.4186	.2978
25	.7619	.5416	.4800	.3214	50	.7221	.5121	.4090	.2916

Table IX

For  $k=2$ ,  $r=10$ , this table gives the value of  $c_{15}(t)$  for various  $t$  and  $P^*$  value.

$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	$t$	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.9090	.9090	.9090	.9090	26	.7692	.6428	.5862	.4838
2	.8333	.8333	.8333	.8333	27	.8077	.6206	.5666	.4687
3	.9166	.7692	.7692	.7692	28	.7778	.6551	.5483	.4545
4	.8461	.8461	.7142	.7142	29	.8147	.6333	.5806	.4411
5	.9230	.7857	.7857	.6666	30	.7857	.6666	.5625	.4706
6	.8571	.7333	.7333	.6250	31	.7586	.6451	.5454	.4571
7	.8000	.8000	.6875	.5882	32	.7931	.6250	.5757	.4444
8	.8666	.7500	.6470	.5555	33	.7666	.6562	.5588	.4324
9	.8125	.7059	.7059	.6111	34	.8000	.6363	.5428	.4594
10	.8750	.7646	.6666	.5789	35	.7742	.6176	.5714	.4473
11	.8235	.7221	.6315	.5500	36	.8064	.6470	.5555	.4358
12	.8823	.6842	.6842	.5238	37	.7812	.6285	.5405	.4250
13	.8333	.7368	.6499	.5714	38	.7575	.6111	.5675	.4146
14	.7894	.6999	.6190	.5454	39	.7878	.6388	.5526	.4389
15	.8421	.6666	.5909	.5217	40	.7646	.6216	.5384	.4285
16	.8000	.7142	.6363	.5000	41	.7940	.6052	.5641	.4186
17	.8500	.6817	.6086	.4800	42	.7714	.6315	.5500	.4090
18	.8095	.6521	.5833	.5199	43	.8000	.6151	.5365	.4317
19	.7727	.6956	.6250	.5000	44	.7778	.6000	.5238	.4222
20	.8182	.6666	.6000	.4814	45	.7567	.6250	.5476	.4130
21	.7826	.6399	.5769	.4642	46	.7837	.6097	.5348	.4042
22	.8260	.6800	.6153	.5000	47	.7631	.6341	.5227	.4255
23	.7916	.6538	.5926	.4827	48	.7894	.6190	.5454	.4166
24	.8333	.6296	.5714	.4666	49	.7692	.6046	.5186	.4081
25	.8000	.6666	.5517	.4516	50	.7948	.6278	.4090	.3999

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selection for the scale parameters is also investigated. A test of homogeneity is proposed which is based on the range of sample medians. An indifference zone approach to the problem of selecting the populations with the  $t$ -largest unknown means is also studied. Chapter III discusses some classification rules for  $k$  univariate normal populations using the subset selection approach. The classification problem is studied in terms of (i) the mean (ii) the variance and (iii) the reciprocal of the coefficient of variation. Chapter IV deals with a conditional and an unconditional procedure for selecting a subset which contains the negative binomial population with the smallest unknown probability of a success. Selection of populations better than a standard is also investigated. An application of the procedure to reliability theory is described.

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