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CLASSIFICATION AND ENUMERATION OF MINIMUM (D,3,3)-GRAPHS FOR OD--ETC(U)  
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13. ABSTRACT

A (d,c,v)-graph is a c-connected graph of diameter d in which each node is of valence v. A minimum (d,c,v)-graph is one with the minimum number of nodes. Each minimum (d,c,v)-graph corresponds to an efficient way of arranging the stations of a communication network so that if any c - 1 stations are incapacitated, the rest of the network is still connected, and so that in case of breakdown or other difficulty each station can rely for assistance on precisely v others. Here the minimum (d,3,3)-graphs are classified and counted for odd d.

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#### KEY WORDS

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CLASSIFICATION AND ENUMERATION OF MINIMUM  $(d,3,3)$ -GRAPHS FOR ODD  $d$

by

Victor Klee

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Classification and Enumeration of Minimum  $(d, 3, 3)$ -Graphs for Odd  $d$ 

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A  $(d, c, v)$ -graph is a  $c$ -connected graph of diameter  $d$  in which each node is of valence  $v$ . A minimum  $(d, c, v)$ -graph is one with the minimum number of nodes. In [4], minimum  $(d, c, v)$ -graphs are constructed for all  $d, c$  and  $v$ . In [5] the minimum  $(d, 1, 3)$ -graphs and minimum  $(d, 2, 3)$ -graphs are classified and counted. These tasks are performed here for the minimum  $(d, 3, 3)$ -graphs when  $d$  is odd. The main result is that if  $\xi = (13 - \sqrt{129})/2$ ,  $\eta = (13 + \sqrt{129})/2$ , and

$$\zeta_j = \frac{\eta^j - \xi^j}{\eta - \xi} = 2^{1-j} \sum_{1 \leq \text{odd } i \leq j} \binom{j}{i} 13^{j-i} 129^{(i-1)/2}$$

then the number of (isomorphism types of) minimum  $(d, 3, 3)$ -graphs is equal to

$$\frac{3}{4}\zeta_{2j} - \frac{5}{2}\zeta_{2j-2} + \frac{21}{4}\zeta_j - 22\zeta_{j-1} + 15\zeta_{j-2} - 4^{j-2}$$

when  $d = 4j + 1 \geq 9$  and to

$$\frac{3}{4}\zeta_{2j-1} - \frac{5}{2}\zeta_{2j} + \frac{3}{4}\zeta_{j+1} - \frac{13}{4}\zeta_j + \frac{5}{2}\zeta_{j-1} - 4^{j-1} - 2^{j-1} + \frac{1}{4}$$

when  $d = 4j + 3 \geq 7$ .

The methods used here can be adapted to yield information about the number of minimum  $(d, 3, 3)$ -graphs when  $d$  is even, but the precise determination of that number would seem to require additional ideas not included here. In any case, even values of  $d$  are ignored except in the first section of the paper.

## 1. SEINES AND ROOTED SEINES

In a connected graph  $G$ , two nodes  $x$  and  $y$  form a diametral pair if the  $G$ -distance  $\delta_G(x,y)$  is equal to the diameter  $\delta(G)$ . Diametral pairs are unordered unless otherwise specified. A diametral node is one that belongs to a diametral pair. Two or more paths are independent if any node common to two of the paths is an end of both.

When  $n$  is even, a simple  $n$ -seine is a 3-valent graph of diameter  $d = n + 1$  formed from three independent paths  $A$ ,  $B$  and  $C$  of length  $d$  joining the two nodes of a diametral pair, together with the  $3n/2$  edges of a matching that covers all intermediate nodes of the paths. The same definition applies when  $n$  is odd, except that then two paths are of length  $d$ , one is of length  $d + 1$ , and there are  $(3n + 1)/2$  edges in the matching. When  $n$  is odd, a split  $n$ -seine is a 3-valent graph of diameter  $d = n + 1$  formed from three independent paths of length  $d$  joining the two nodes of a diametral pair, together with an additional node  $z$  and the  $(3n + 1)/2$  edges of a matching that covers  $z$  and all intermediate nodes of the paths. It is easy to verify that simple  $n$ -seines exist for all  $n \geq 1$  and split  $n$ -seines exist for all odd  $n \geq 1$ . The numbers of (isomorphism types of) split 1-seines, simple 1-seines and simple 2-seines are respectively 1, 1 and 2. See Figures 1 and 2.

(Figure 1 is to be inserted here)

Fig. 1: The split 1-seine and the simple 1-seine

(Figure 2 is to be inserted here)

Fig. 2: The two simple 2-seines

1.1 THEOREM When  $d$  is odd, a graph is a minimum  $(d,3,3)$ -graph if and only if it is a simple  $(d-1)$ -seine. When  $d$  is even, a graph is a minimum  $(d,3,3)$ -graph if and only if it is a split  $(d-1)$ -seine or a simple  $(d-1)$ -seine.

Proof. Note first that all simple  $n$ -seines and all split  $n$ -seines are 3-connected. That is, for any two nodes  $p$  and  $q$  of such a graph  $S$ ,  $p$  and  $q$  are not separated by any pair of nodes of  $S$ . To prove this, consider separately the cases in which  $\{p,q\}$  intersects the diametral pair  $\{x,y\}$ , in which  $p$  and  $q$  are intermediate nodes of the same path  $A$ ,  $B$  or  $C$ , and in which  $p$  and  $q$  are intermediate nodes of different paths. Consider also, for split  $n$ -seines, the case in which the extra node  $z$  is  $p$  or  $q$ . In each case the argument is straightforward.

To complete the proof, consider an arbitrary diametral pair  $\{x,y\}$  in a 3-connected graph  $G$  of diameter  $d$ . Each path joining  $x$  and  $y$  has at least  $d-1$  intermediate nodes, and since  $x$  and  $y$  are joined by three independent paths there are at least  $3d-1$  nodes in all. If, in addition,  $d$  is even and  $G$  is 3-valent then there are at least  $3d$  nodes because odd-valent graphs are of even order. But in view of the preceding paragraph, a minimum  $(d,3,3)$ -graph has at most  $3d-1$  nodes when  $d$  is odd and at most  $3d$  nodes when  $d$  is even. The desired conclusion follows readily.  $\square$

Because of 1.1, the minimum  $(d,3,3)$ -graphs are henceforth called  $(d-1)$ -seines. Thus all  $(d-1)$ -seines are simple when  $d$  is odd, and are simple or split when  $d$  is even. A rooted  $n$ -seine is an ordered triple  $(S,x,y)$ , where  $S$  is an  $n$ -seine and  $(x,y)$  is an ordered diametral pair of nodes of  $S$ . Two rooted  $n$ -seines  $(S,x,y)$  and  $(S^*,x^*,y^*)$  are isomorphic if there is a graph isomorphism of  $S$  onto  $S^*$  that carries  $x$  and  $y$  onto  $x^*$  and  $y^*$

respectively. (When  $S$  is a 1-seine or 2-seine,  $S$  has at least two diametral pairs and hence yields at least four distinct rooted seines, but all rooted seines associated with  $S$  are isomorphic.)

In all that follows, the distinction between rooted seines and unrooted seines is essential. A seine not specifically described as rooted is assumed to be unrooted.

2. THE NUMBER OF ROOTED  $(d - 1)$ -SEINES FOR ODD  $d$ 

Henceforth there is the

STANDING HYPOTHESIS:  $d = n + 1$ , where  $n$  is an even integer  $\geq 2$ .

Let  $Q$  denote the infinite graph formed from three paths  $(x, p_{11}, p_{12}, \dots)$ ,  $(x, p_{21}, p_{22}, \dots)$  and  $(x, p_{31}, p_{32}, \dots)$  which are pairwise node-disjoint except for having the same initial node  $x$ . For each positive integer  $k$  let  $Q_k$  denote the subgraph of  $Q$  spanned by  $\{x\} \cup (V_1^k P_i)$ , where  $P_i = \{p_{1i}, p_{2i}, p_{3i}\}$ . A  $k$ -start is a graph  $T$  formed by adding to  $Q_k$  the edges of a matching  $M$  such that

(a) each member of  $M$  joins two points of  $V_1^k P_i$  not joined in  $Q_k$ ,

(b) each point of  $V_1^{k-1} P_i$  is covered by  $M$ , and

(c)  $\delta_T(x, p_{1k}) = \delta_T(x, p_{2k}) = \delta_T(x, p_{3k}) = k$ , where  $\delta$  is the usual graph-theoretic distance.

A consequence of (a) and (c) is that if  $\{p_{\ell i}, p_{mj}\} \in M$  then  $\ell \neq m$  and  $|i - j| \leq 1$ . This and the following fact are used frequently without explicit reference.

2.1 PROPOSITION Every automorphism of a  $k$ -start  $T$  carries  $x$  onto  $x$  and carries  $P_i$  onto  $P_i$  for  $1 \leq i \leq k$ .

Proof. Note that  $P_k$  is the set of all nodes of  $T$ -valence  $\leq 2$ , and that for  $1 \leq i < k$ ;  $P_i$  is the set of all nodes at  $T$ -distance  $k - i$  from  $P_k$ .  $\square$

When  $T$  is a  $k$ -start, a node  $p \in P_k$  is matched or unmatched in  $T$  according as its  $T$ -valence is 2 or 1. The end set of  $T$  is the set of all  $r \in \{1, 2, 3\}$  such that  $p_{kr}$  is matched in  $T$ . The cardinality  $\rho(T)$  of  $T$ 's end set is called the reach of  $T$ . The

end group of  $T$  is the group of all permutations  $\pi$  of  $\{1,2,3\}$  such that  $T$  admits an automorphism (a graph isomorphism of  $T$  onto  $T$ ) carrying  $p_{kr}$  onto  $p_{k\pi(r)}$  for  $r \in \{1,2,3\}$ . The cardinality  $\sigma(T)$  of  $T$ 's end group is called the style of  $T$ . Plainly  $\sigma(T) \in \{1,2,3,6\}$ , for the end group is a subgroup of the symmetric group  $S_3$ . Figure 3 shows all isomorphism types of  $k$ -starts for  $k \leq 3$ . Under each example the pair  $\rho(T), \sigma(T)$  is shown.

(Figure 3 is to be inserted here)

Fig. 3: Examples of all isomorphism types of  $k$ -starts for  $k \leq 3$

For each pair of integers  $r \in \{0,1,2,3\}$  and  $s \in \{1,2,3,6\}$ , let  $g_{rs}(k)$  denote the number of isomorphism types of  $k$ -starts of reach  $r$  and style  $s$ . Note that the number of isomorphism types of rooted  $n$ -seines is

$$(g_{31} + g_{32} + g_{33} + g_{36})(n).$$

We now proceed by recursion to determine the various functions  $g_{rs}$ .

2.2 PROPOSITION Of the sixteen functions  $g_{rs}$  for  $r \in \{0,1,2,3\}$  and  $s \in \{1,2,3,6\}$ , all but  $g_{01}, g_{06}, g_{11}, g_{21}, g_{22}, g_{31}$  and  $g_{36}$  are identically zero.

Proof. To see that  $g_{23} = g_{26} = 0$ , note that if  $\{1,2\}$  is the end set of a  $k$ -start  $T$  then each automorphism of  $T$  is the identity on  $P_k$  or interchanges  $p_{k1}$  and  $p_{k2}$ .

That  $g_{12} = g_{13} = g_{16} = 0$  follows from valency and distance considerations. Suppose, for example, that  $\{1\}$  is the end set of a  $k$ -start  $T$  and  $\{p_{(k-1)2}, p_{k1}\}$  is an edge of  $T$ . Then the

T-valence of  $p_{k1}$  is 2 while that of  $p_{k2}$  and  $p_{k3}$  is 1. Also, the T-distance from  $p_{k1}$  is 2 for  $p_{k2}$  and  $> 2$  for  $p_{k3}$ . Hence each automorphism of  $T$  is the identity on  $P_k$ .

That  $g_{02} = g_{03} = g_{32} = g_{33} = 0$  follows from an induction that is based on the initial conditions

(a)  $g_{06}(1) = g_{22}(1) = 1$ ,  $g_{rs}(1) = 0$  for all  $(r,s) \notin \{(0,6), (2,2)\}$  in conjunction with the observation that for all  $k \geq 1$ ,

(b)  $g_{0s}(k+1) = g_{3s}(k)$  for all  $s$ , and

(c)  $g_{3s}(k+1) = g_{0s}(k)$  for all  $s \neq 1$ .

To establish (b), use 2.1 to verify that if a  $k$ -start of reach 3 and a  $(k+1)$ -start of reach 0 are associated with the same matching  $M$  then they have the same end group. For (c), consider an arbitrary  $(k+1)$ -start  $T$  of reach 3 and style  $s \neq 1$ , and let  $\phi$  be an automorphism of  $T$  that is not the identity on  $P_{k+1}$ . If  $\{p_{(k+1)1}, p_{(k+1)2}\}$  is an edge of  $T$  then  $\phi$  carries  $p_{(k+1)3}$  onto itself and interchanges  $p_{(k+1)1}$  and  $p_{(k+1)2}$ , whence  $\phi$  interchanges  $p_{k1}$  and  $p_{k2}$ . A contradiction then arises from the fact that precisely one of  $p_{k1}$  and  $p_{k2}$  is joined to  $p_{(k+1)3}$ . It follows that no edge of  $T$  joins two points of  $P_{k+1}$ , whence  $T$  is formed from a  $k$ -start  $T'$  of reach 0 by adding, for each  $p \in P_k$ , a node  $q_p$  that is not joined to  $p$  but is joined to both members of  $P_k \sim \{p\}$ . Then  $T$  and  $T'$  have the same end group, and (c) follows.  $\square$

The order of the  $g_{rs}$ 's in the next result is chosen so as to simplify a later computation.

2.3 PROPOSITION For each positive integer  $k$  let the column vector  $z_k$  be the transpose of

$$(g_{06}(k), g_{22}(k), g_{01}(k), g_{11}(k), g_{31}(k), g_{36}(k), g_{21}(k)).$$

Then  $z_{k+1} = Az_k$  where  $A$  is the matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & 1 & 2 & \cdot & \cdot & \cdot & 2 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 3 & \cdot & \cdot \end{bmatrix}$$

Proof. The first and third rows are justified by (b) above, the sixth by (c). For the fourth row, note that each  $(k+1)$ -start  $T$  of reach 1 is also of style 1 by 2.2, and  $T$  has as subgraph a  $k$ -start of reach 2. Each  $k$ -start of reach 2 and style 1 <resp. 2> yields 2 <resp. 1> isomorphism types of  $(k+1)$ -starts of reach 1.

To justify the fifth row of  $A$ , note that  $g_{11}(k+1)$  <resp.  $2g_{01}(k)$ > is the number of isomorphism types of  $(k+1)$ -starts of reach 3 and style 1 in which there is <resp. is not> an edge joining two nodes of  $P_{k+1}$ . Then use the fact, provided by  $A$ 's fourth row, that  $g_{11}(k+1) = 2g_{21}(k) + g_{22}(k)$ .

For the second and seventh rows of  $A$ , consider an arbitrary  $(k+1)$ -start  $T$  of reach 2. If two nodes of  $P_{k+1}$  are joined in  $T$  then  $T$  has as subgraph a  $k$ -start  $T'$  of reach 3 and it follows with the aid of 2.2 that the style of  $T'$  is 1 <resp. 6> when that of  $T$  is 1 <resp. 2>. Further, each  $k$ -start of reach 3 and style 1 <resp. 6> yields 3 <resp. 1> isomorphism types of  $(k+1)$ -starts of reach 2 and style 1 <resp. 2>. If no edge of  $T$  joins two nodes of  $P_{k+1}$  then  $T$  has as subgraph a  $k$ -start  $T'$  of reach 1 whose style is also 1 by 2.1. Each such  $T'$  yields 2 <resp. 1> isomorphism types of  $(k+1)$ -starts of reach 2 and style 1 <resp. 2>.  $\square$

2.4 THEOREM Let  $\xi = (13 - \sqrt{129})/2$ ,  $\eta = (13 + \sqrt{129})/2$ , and for  
each positive integer  $j$  let

$$\zeta_j = \frac{\eta^j - \xi^j}{\eta - \xi} = 2^{1-j} \sum_{1 \leq \text{odd } i \leq j} \binom{j}{i} 13^{j-i} 129^{(i-1)/2}$$

Then the functions  $g_{rs}$  that are not identically 0 are as tabulated  
below.

|             | <u><math>k = 2_j</math></u>                                | <u><math>k = 2_j + 1</math></u>                                            |
|-------------|------------------------------------------------------------|----------------------------------------------------------------------------|
| $g_{01}(k)$ | 0                                                          | $\frac{1}{2}(-1 + 3\zeta_j) - 5\zeta_{j-1}$                                |
| $g_{06}(k)$ | 0                                                          | 1                                                                          |
| $g_{11}(k)$ | $\frac{1}{2}(1 + 3\zeta_j) - 8\zeta_{j-1} + 10\zeta_{j-2}$ | 0                                                                          |
| $g_{21}(k)$ | 0                                                          | $\frac{1}{2}(-1 + 15\zeta_j) - 31\zeta_{j-1} + 20\zeta_{j-2}$              |
| $g_{22}(k)$ | 0                                                          | $\frac{3}{2}(1 + \zeta_{j+1}) - 23\zeta_j + 72\zeta_{j-1} - 40\zeta_{j-2}$ |
| $g_{31}(k)$ | $\frac{1}{2}(-1 + 3\zeta_j) - 5\zeta_{j-1}$                | 0                                                                          |
| $g_{36}(k)$ | 1                                                          | 0                                                                          |

Proof. Let  $A$  be the matrix of 2.3. A first inspection shows that 0 is an eigenvalue of  $A$  with associated eigenvector  $(0, 2, 0, 0, 0, 0, -1)^t$ . When  $\lambda \neq 0$  the matrix  $A - \lambda I$  is row-equivalent to the matrix

$$\begin{bmatrix} -\lambda & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & -\lambda & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & -\lambda & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1-\lambda^2 & \cdot & 1 & 2\lambda \\ \cdot & \cdot & \cdot & 1 & 2-\lambda^2 & 1 & 2\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1-\lambda^2 & \cdot \\ \cdot & \cdot & \cdot & 2 & 3 & \cdot & -\lambda \end{bmatrix}.$$

Hence 1 and -1 are eigenvalues of  $A$ , and associated eigenvectors turn out to be  $(2, 3, -1, 1, -1, 2, -1)^t$  and  $(2, 3, -1, -1, 1, -2, -1)^t$  respectively. When  $\lambda^2 \notin \{0, 1, 2\}$  the matrix  $M$  is row-equivalent to the upper triangular matrix whose first four rows are those of  $M$  and whose lower right  $3 \times 3$  minor is

$$\begin{bmatrix} (1-\lambda^2)(2-\lambda^2) & -\lambda^2 & -2\lambda^3 \\ \cdot & 1-\lambda^2 & \cdot \\ \cdot & \cdot & \lambda^4 - 13\lambda^2 + 10 \end{bmatrix}$$

It follows that when  $\lambda \in \{-\xi^{\frac{1}{2}}, \xi^{\frac{1}{2}}, -\eta^{\frac{1}{2}}, \eta^{\frac{1}{2}}\}$ ,  $\lambda$  is an eigenvalue of  $A$  with associated eigenvector

$$(0, \psi_{\lambda^2}, \lambda^2, \lambda^3 - 2\lambda, \lambda^3, 0, 5\lambda^2 - 4),$$

where  $\psi_\tau = \tau^2 - 12\tau + 8$ .

There is now enough information to diagonalize  $A$  by means of a similarity transformation, but it is computationally more convenient to work with  $A^2$  instead.

The eigenvalues of  $A^2$  are  $0, 1, \xi$  and  $\eta$ , and for all but the first of these an independent pair of eigenvectors is available. Replacing each of these pairs with half of its sum and half of its difference, thereby obtaining other independent eigenvectors for the eigenvalue in question, and arranging the seven eigenvectors in an order that is convenient for subsequent computation, yields the matrix

$$B = \begin{bmatrix} 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 2 & \psi_\xi & \cdot & \cdot & \cdot & \psi_\eta \\ -1 & \cdot & \xi & \cdot & \cdot & \cdot & \eta \\ \cdot & \cdot & \cdot & \xi^{3/2} - 2\xi^{1/2} & \eta^{3/2} - 2\eta^{1/2} & 1 & \cdot \\ \cdot & \cdot & \cdot & \xi^{3/2} & \eta^{3/2} & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\ -1 & -1 & 5\xi - 4 & \cdot & \cdot & \cdot & 5\eta - 4 \end{bmatrix}.$$

The columns of  $B$  are eigenvectors of  $A^2$  associated with the respective eigenvalues  $1, 0, \xi, \xi, \eta, 1, \eta$ . By a standard result on diagonalization of matrices,

$$A^2 = BDB^{-1}$$

where

$$D = \text{diag} (1, 0, \xi, \xi, \eta, 1, \eta).$$

For each integer  $j > 0$ ,

$$z_{2j+1} = (A^2)^j z_1 = BD^j B^{-1} z_1.$$

Let  $x = B^{-1} z_1$ , so that  $Bx = z_1$ . To compute  $x$ , recall that  $z_1 = (1, 1, 0, 0, 0, 0, 0)^t$  and use row operations to transform the matrix  $(B, z_1)$  into the following:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \\ \cdot & 1 & \frac{1}{2}\psi_\xi & \cdot & \cdot & \cdot & \frac{1}{2}\psi_\eta & -\frac{1}{4} \\ \cdot & \cdot & \xi & \cdot & \cdot & \cdot & \eta & \frac{1}{2} \\ \cdot & \cdot & \cdot & \xi^{\frac{1}{2}} & \eta^{\frac{1}{2}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \eta^{\frac{1}{2}}(\eta - \xi) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta(\eta - \xi) & \frac{3 - \xi}{2} \end{bmatrix}.$$

Back substitution then shows that

$$x = \left( \frac{1}{2}, \mu, \frac{\eta - 3}{2\xi(\eta - \xi)}, 0, 0, 0, \frac{\xi - 3}{2\eta(\xi - \eta)} \right)^t$$

where the value of the second entry  $\mu$  is immaterial for our purpose. It follows that

$$D^j x = \frac{1}{2} \left( 1, 0, \xi^{j-1} \frac{\eta - 3}{\eta - \xi}, 0, 0, 0, \eta^{j-1} \frac{\xi - 3}{\xi - \eta} \right),$$

whence the successive entries of  $z_{2j+1} = BD^j x$  are

$$g_{06}(2j + 1) = 1,$$

$$g_{22}(2j + 1) = \frac{3}{2} + \frac{1}{2(\eta - \xi)} [\psi_\xi \xi^{j-1}(\eta - 3) - \psi_\eta \eta^{j-1}(\xi - 3)]$$

$$g_{01}(2j + 1) = -\frac{1}{2} + \frac{1}{2(\eta - \xi)} [\xi^j(\eta - 3) - \eta^j(\xi - 3)],$$

$$g_{11}(2j + 1) = g_{31}(2j + 1) = g_{36}(2j + 1) = 0, \quad \text{and}$$

$$g_{21}(2j+1) = -\frac{1}{2} + \frac{1}{2(\eta - \xi)}[(5\xi - 4)\xi^{j-1}(\eta - 3) - (5\eta - 4)\eta^{j-1}(\xi - 3)].$$

By simple manipulation and use of the fact that  $\xi\eta = 10$ , the formulae for  $g_{rs}(2j+1)$  stated in 2.4 are derived from the ones just obtained. Note, for example, that

$$\begin{aligned} & \frac{1}{\eta - \xi}[\psi_\xi \xi^{j-1}(\eta - 3) - \psi_\eta \eta^{j-1}(\xi - 3)] = \\ & \frac{3}{\eta - \xi}[(\eta^2 - 12\eta + 8)\eta^{j-1} - (\xi^2 - 12\xi + 8)\xi^{j-1}] - \frac{\xi\eta}{\eta - \xi}[(\eta^2 - 12\eta + 8)\eta^{j-2} - (\xi^2 - 12\xi + 8)\xi^{j-2}] \\ & = 3(\zeta_{j+1} - 12\zeta_j + 8\zeta_{j-1}) - 10(\zeta_j - 12\zeta_{j-1} + 8\zeta_{j-2}) = 3\zeta_{j+1} - 46\zeta_j + 144\zeta_{j-1} - 80\zeta_{j-2}. \end{aligned}$$

The formulae for  $g_{rs}(2j)$  follow from the formulae for  $g_{rs}(2j-1)$  and the fact that  $z_{2j} = Az_{2j-1}$ .  $\square$

2.5 THEOREM. For even  $n \geq 2$  the number of isomorphism types of rooted  $n$ -seines is  $\frac{1}{2}(-1 + 3\xi_{n/2}) - 5\xi_{(n-2)/2}$ .

Proof. Since  $g_{32} = g_{33} = 0$ , the number in question is  $g_{31}(n) + g_{36}(n)$ . Use 2.4.  $\square$

By 2.5 there are 14 types of rooted 4-seines. The 8 types  $(S, x, y)$  shown in Figure 4 are reversible, meaning that  $(S, x, y)$  is isomorphic with  $(S, y, x)$ . The 3 types shown in Figure 5 are nonreversible. The remaining 3 nonreversible types are of the form  $(S, y, x)$  where  $(S, x, y)$  is as in Figure 5.

*(Figure 4 is to be inserted here)*

Fig. 4: The eight types of reversible rooted 4-seine  $(S, x, y)$

*(Figure 5 is to be inserted here)*

Fig. 5: Three of the six types of nonreversible rooted 4-seine  $(S, x, y)$

A link is any of the five graphs shown in Figure 6. In each link  $L$ , the six nodes are partitioned into a set  $L_W$  of three white nodes and a set  $L_b$  of three black nodes. When  $G$  is a link and  $p$  a node of  $G$ , or  $G$  is a  $k$ -start and  $p \in P_k$ , the node  $p$  is matched if its  $G$ -valence is 2; otherwise the  $G$ -valence of  $p$  is 1 and  $p$  is unmatched. Each reversible rooted  $n$ -seine can be formed, for some  $r$  between 0 and 3, by amalgamating a link  $L$  having  $r$  unmatched white nodes with two copies,  $T_W$  and  $T_b$ , of an  $(n/2)$ -start of reach  $r$ . In the amalgamation,  $L$ 's unmatched white <resp. black> nodes are identified with the matched nodes of  $T_W$  <resp.  $T_b$ > and  $L$ 's matched white <resp. black> nodes are identified with the unmatched nodes of  $T_W$  <resp.  $T_b$ >. In Figure 4, five of the examples use the link  $L_5$ ; the last examples on the successive rows use  $L_3$ ,  $L_3$  and  $L_2$  respectively.

(Figure 6 is to be inserted here)

$L_1$              $L_2$              $L_3$              $L_4$              $L_5$

Fig. 6: The five links

2.6 THEOREM. For even  $n \geq 4$  the number of isomorphism types of reversible rooted  $n$ -seines is

$$\begin{aligned} & \frac{1}{2}(1 + 21\zeta_j) - 44\zeta_{j-1} + 30\zeta_{j-2} && \text{when } n = 4_j \text{ and} \\ & 1 + \frac{1}{2}(3\zeta_{j+1} - 13\zeta_j) + 5\zeta_{j-1} && \text{when } n = 4_j + 2. \end{aligned}$$

Proof. For each link  $L$  having  $r$  unmatched white nodes, and for each  $(n/2)$ -start  $T$  of reach  $r$  and specified style, the number of isomorphism types of reversible rooted  $n$ -seines that can be formed by amalgamating  $L$  with two copies of  $T$  is shown in the table of Figure 7.

| <u>Link number</u> | <u>Reach of start</u> | <u>Style of start</u> | <u>Number of types</u> |
|--------------------|-----------------------|-----------------------|------------------------|
| 1                  | 0                     | 1                     | 1                      |
| 1                  | 0                     | 6                     | 1                      |
| 2                  | 1                     | 1                     | 1                      |
| 3                  | 1                     | 1                     | 2                      |
| 4                  | 2                     | 1                     | 2                      |
| 4                  | 2                     | 2                     | 1                      |
| 5                  | 3                     | 1                     | 4                      |
| 5                  | 3                     | 6                     | 1                      |

Fig. 7: Number of types of reversible rooted  $n$ -seines yielded by certain amalgamations

Justification of the table is facilitated by a more formal description of the amalgamation process for forming reversible rooted  $n$ -seines. For each link  $L$ , let  $A(L)$  denote the set of all automorphisms  $\alpha$  of  $L$  such that  $\alpha L_w = L_b$ . For each  $k$ -start  $T$  and each  $\alpha \in A(L)$ , let  $W(T, \alpha)$  denote the set of all one-to-one mappings  $\omega: P_k \rightarrow L_w$  such that

- (a)  $\omega$  carries matched <resp. unmatched> nodes of  $P_k$  onto unmatched <resp. matched> nodes of  $L_w$ , and
- (b) the mapping  $\omega^{-1} \alpha^2 \omega: P_k \rightarrow P_k$  belongs to the end group of  $T$ .

Note that if  $W(T, \alpha) \neq \emptyset$  then by (a) the reach of  $T$  is equal to the number of unmatched white nodes of  $L$ . For  $T, L, \alpha$  and  $\omega$  as described, with  $2k = n$ , the reversible rooted  $(2k)$ -seine  $R = R(T, L, \alpha, \omega)$  is constructed in the manner set forth in the next paragraph.

Let  $\phi$  be an isomorphism of  $T$  onto a graph  $T'$  that is node-disjoint from both  $T$  and  $L$ . Let  $\mu$  be an automorphism of  $T$  whose restriction to  $P_k$  is  $\omega^{-1}\alpha^2\omega$ . Let  $x' = \phi(x)$ ,  $P'_k = \phi P_k$ , and define the mapping  $\beta: P'_k \rightarrow L_b$  by  $\beta = \alpha\omega\mu^{-1}\phi^{-1}$ . The graphs  $T, L$  and  $T'$  are amalgamated to form an  $n$ -seine  $S$  by identifying, as an individual node of  $S$ , each of the pairs  $\{p, \omega(p)\}$  for  $p \in P_k$  and each of the pairs  $\{p', \beta(p')\}$  for  $p' \in P'_k$ . To see that the rooted  $n$ -seine  $R = (S, x, x')$  is reversible, let  $\psi(q) = \phi(\mu(q))$  for each node  $q$  of  $T$  and  $\psi(q') = \phi^{-1}(q')$  for each node  $q'$  of  $T'$ . Plainly  $\psi$  is an automorphism of  $T \cup T'$  that interchanges  $x$  and  $x'$ . By means of the identifications,  $\psi$  is defined also on the nodes of  $L$ . On  $L_w$ ,  $\psi = \beta\phi\mu\omega^{-1}$ , which is equal to  $\alpha$  because  $\beta = \alpha\omega\mu^{-1}\phi^{-1}$  by definition. And since  $\omega\mu\omega^{-1} = \alpha^2$ , on  $L_b$  it is true that

$$\psi = \omega\phi^{-1}\beta^{-1} = \omega\phi^{-1}\phi\mu\omega^{-1}\alpha^{-1} = \omega\mu\omega^{-1}\alpha^{-1} = \alpha^2\alpha^{-1} = \alpha.$$

Thus  $\psi$  is an automorphism of  $S$  and an isomorphism of  $(S, x, x')$  onto  $(S, x', x)$ . A straightforward argument shows that every reversible rooted  $n$ -seine can be constructed in this way.

In justifying the rows of the table in Figure 7, let the nodes of the  $k$ -start  $T$  that belong to  $P_k$  be denoted by 1, 2 and 3. It may be assumed without loss of generality that the matched members of  $P_k$  have smaller indices than the unmatched members, and then because of the symmetries of  $L$  it may be assumed that the identification of  $P_k$  with those of  $L_w$  is as shown in Figure 8.

(Figure 8 is to be inserted here)

Fig. 8: Identification of nodes of  $P_k$  with white nodes of link  $L_1$

Now recall that  $\alpha$  is an automorphism of  $L$  for which  $L_w = L_b$ , whence  $\alpha$  interchanges the sets  $L_w$  and  $L_b$ . Consider first the case in which  $T$  is of style 1, whence  $\alpha^2$  is the identity. In the case of  $L_1$  this fixes  $\alpha$  and hence (for a given  $\omega$  and  $\phi$ ) also fixes  $\beta$ , so that only one type of  $R(T, L, \alpha, \omega)$  emerges. In the case of  $L_2$  there are two possibilities for  $\alpha$  and hence for  $\beta$ , but the two pairs of identifications  $(\omega, \beta)$  are equivalent under the automorphism group of  $L_2$  and thus only one type of  $R$  emerges. In the case of  $L_3$  or  $L_4$ ,  $\alpha$  is fixed and hence  $\beta$  is fixed but (as shown in Figure 8) there are two essentially different possibilities for  $\omega$ ; thus in each case two types of  $R$  emerge. In the case of  $L_5$  the four possibilities for  $\alpha$  lead to the four possibilities for  $(\omega, \beta)$  indicated in Figure 9 and hence to four different types of  $R$ .

(Figure 9 is to be inserted here)

Fig. 9: Different identification pairs  $(\omega, \beta)$  when  $L$  is  $L_5$  and  $T$  is of style 1

Only the second, sixth and eighth rows of the table remain. The number of possibilities for  $\alpha$  is 3, 1 and 6 respectively. However, because of the nature of  $T$ 's and groups, in each case only a single type of  $R$  emerges.

From the table it follows that the total number of reversible rooted  $n$ -seines is

$$(g_{01} + g_{06} + 3g_{11} + 2g_{21} + g_{22} + 4g_{31} + g_{36})(n/2),$$

whence the stated formulae follow with the aid of 2.4.  $\square$

3. THE NUMBER OF UNROOTED  $(d - 1)$ -SEINES FOR ODD  $d$ 

This section also comes under the

STANDING HYPOTHESIS:  $d = n + 1$ , where  $n$  is an even integer  $\geq 2$ .

Each diametral pair  $\{x, y\}$  of nodes of an  $n$ -seine  $S$  is joined by at least one set of three independent paths of length  $d$ , and for a given  $\{x, y\}$  there may be more than one such set of paths. It is often convenient to pick a particular set  $A = (x, a_1, \dots, a_n, y)$ ,  $B = (x, b_1, \dots, b_n, y)$ ,  $C = (x, c_1, \dots, c_n, y)$  and carry on the discussion with respect to it. Note that, in the matching involved in the definition of a simple  $n$ -seine,  $a_i$  is matched with one of the six points  $b_{i-1}, b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}$ , there being two exclusions when  $i$  is 1 or  $n$ . This fact is used frequently without explicit reference.

Each diametral node  $x$  of an  $n$ -seine belongs to a unique diametral pair  $\{x, y\}$ , but there may be diametral pairs disjoint from  $\{x, y\}$ . An  $n$ -seine with more than one diametral pair is called ambiguous. The first 2-seine in Figure 2 has four diametral pairs. It is in that respect an anomaly, as the following result shows.

3.1 THEOREM If  $S$  is an ambiguous  $n$ -seine for even  $n \geq 4$ , there are cliques  $\{x, x', w\}$  and  $\{y, y', z\}$  in  $S$  such that

$$\delta_S(x, y) = \delta_S(x', y') = n+1, \quad \delta_S(w, z) = n - 1,$$

and  $\{x, y\}$  and  $\{x', y'\}$  are the only diametral pairs of  $S$ .

Proof. Let  $\{x, y\}$  and  $\{x', y'\}$  be distinct diametral pairs of  $S$  and let  $A = (x, a_1, \dots, a_n, y)$ ,  $B = (x, b_1, \dots, b_n, y)$  and  $C = (x, c_1, \dots, c_n, y)$  be three independent paths from  $x$  to  $y$ . Since

$\delta_S(x', y') = d$  and each pair of the paths forms a circuit of length  $2d$ , it may be assumed without loss of generality that  $x' = a_j$  and  $y' = c_k$  with  $j + k = d$  and  $1 \leq j \leq k \leq n$ . If  $a_j$  is matched with  $c_r$  the sequence  $(a_j, c_r, c_{r+1}, \dots, c_k)$  is a path of length  $< d$ , and the situation is similar if  $c_k$  is matched with  $a_s$ . Hence  $a_j$  is matched with  $b_r$  and  $c_k$  with  $b_s$ , where

$$|r - s| + 2 \geq d - 2.$$

It follows that  $r = 1$  and  $s = n$ , whence  $j$  is 1 or 2.

If  $j = 2$ ,  $\{a_2, b_1\}$  is an edge, whence  $a_1$  is not matched with  $b_2$  for if it were there would be no admissible mate for  $c_1$ . Thus  $a_1$  is matched with  $c_t$  for  $t \in \{1, 2\}$ , whence (with  $n \geq 4$ )  $t < d - 2$  and  $(a_2, a_1, c_t, \dots, c_{d-2})$  is a path of length  $< d$  from  $x'$  to  $y'$ . Since this is contradictory, it follows that  $j = 1$ . That is,  $x' = a_1$ ,  $y' = c_n$ , and  $\{a_1, b_1\}, \{c_n, b_n\}$  are edges of  $S$ . With  $w = b_1$  and  $z = b_n$ ,  $\{x, x', w\}$  and  $\{y, y', z\}$  are the desired cliques. Note also, for future reference, that  $a_2$  is matched with  $c_1$ , for otherwise  $a_2$  is adjacent to  $\{b_2, b_3, c_2, c_3\}$  and a path of length  $< d$  from  $x'$  to  $y'$  is created. Similarly,  $c_{n-1}$  is matched with  $a_n$ . Thus  $S$  is as shown in Figure 10. Further, all additional edges joining  $C$  to  $A$  have positive slope (in terms of the geometric representation chosen), while those joining  $B$  to  $C$  or  $A$  have positive slope or are vertical.

(Figure 10 is to be inserted here)

Fig. 10: Labeled representation of an ambiguous  $n$ -seine for  $n \geq 4$

At this point a certain relationship of  $\{x', y'\}$  to  $\{x, y\}$  has been established. A third diametral pair  $\{x'', y''\}$  would bear this

same relationship to both  $\{x,y\}$  and  $\{x',y'\}$ , whence  $\{x'',y''\} = \{w,z\}$  for (with  $n \geq 4$ )  $w$  and  $z$  are the only two nodes adjacent to both  $\{x,y\}$  and  $\{x',y'\}$ . Since  $\{w,z\}$  is not diametral it follows that  $\{x,y\}$  and  $\{x',y'\}$  are the only diametral pairs.  $\square$

It follows from 3.1 that the number of isomorphism types of rooted  $n$ -seine that arise from a given ambiguous  $n$ -seine is at most 4. It is 1 for each of the 2-seines, 2 for the upper right 4-seine of Figure 4 (which is ambiguous and is isomorphic to the one on its left), and 4 for the ambiguous 8-seine of Figure 11. In order to use 2.5 and 2.6 in counting the total number of  $n$ -seines we must determine, for each  $t \in \{1,2,3,4\}$ , how many types of ambiguous  $n$ -seine yield precisely  $t$  types of rooted  $n$ -seine. That is done by first counting the number of types of rooted ambiguous  $n$ -seines and then taking symmetries into account.

(Figure 11 is to be inserted here)

Fig. 11: An ambiguous 8-seine that yields 4 different types of rooted 8-seines

For a rooted  $n$ -seine  $R = (S,x,y)$  and for  $1 \leq i \leq n$ , let  $R_i$  denote the set of all nodes  $v$  of  $S$  such that  $\delta_S(x,y) = i$ ; the members of  $R_i$  are said to be of level  $i$ . Note that  $|R_i| = 3$  for each  $i$ , and each  $R_i$  is carried onto itself by each automorphism of  $R$ . For each interval  $I$  of integers in  $[1,n]$ , let  $R_I = \bigcup_{i \in I} R_i$ . An  $\ell$ -block of  $R$ , or block of length  $\ell$ , is an interval  $I \subset [1,n]$  such that

$$(a) \quad |I| = \ell,$$

- (b) for each  $p \in R_I$ , at least two of the three S-neighbors of  $p$  also belong to  $R_I$ , and
- (c)  $I$  is minimal with respect to (b).

Plainly each block-length is even. The specification of  $R$  is the sequence  $H = (h_1, \dots, h_b)$  of half-lengths of the successive blocks. Thus  $\sum_{j=1}^b h_j = n/2$ . For the 8-seine  $S$  of Figure 10, the specification of  $(S, x, y)$  is  $(2, 1, 1)$  and of  $(S, x', y')$  is  $(1, 3)$ .

A rooted  $n$ -seine  $R = (S, x, y)$  is rigid if  $S$  admits only one set of three independent paths from  $x$  to  $y$ . A switch is a 2-block  $I$  such that for each  $i \in I$ , no edge of  $S$  joins two points of  $R_I$ ; equivalently, each node in either of the two levels represented in  $R_I$  has two neighbors in the other level. A cross is a set of four nodes that span a circuit whose nodes alternate between two levels. The nonrigid rooted 10-seine shown below has a switch in levels 3-4 and a cross in levels 8-9. The 4-circuit in levels 6-7 does not correspond to a cross because its nodes do not alternate between two levels.

(Figure 12 is to be inserted here)

Fig. 12: A nonrigid rooted 10-seine having one switch and one cross

Let us say that a rooted  $n$ -seine  $R = (S, x, y)$  is canonically labeled when each node other than  $x$  or  $y$  has been assigned a unique label from the set  $\bigcup_{i=1}^n \{a_i, b_i, c_i\}$  in such a way that  $A = (x, a_1, \dots, a_n, y)$ ,  $B = (x, b_1, \dots, b_n, y)$  and  $C = (x, c_1, \dots, c_n, y)$  are independent paths from  $x$  to  $y$  and  $\{a_1, b_1\}$ ,  $\{c_1, a_2\}$  are edges of  $S$ .

3.2 PROPOSITION Suppose that  $R$  is a rooted  $n$ -seine  $(S, x, y)$ . Then

- (a)  $R$  admits a canonical labeling if and only if  $R$  does not have a switch in levels 1-2;
- (b)  $R$  is rigid if and only if  $R$  has no switches and no crosses;
- (c) if  $R$  is rigid then  $R$  admits a unique canonical labeling;
- (d) if  $R$  is rigid then the identity is the only automorphism of  $R$ .

Proof. Assertions (a) - (c) are almost obvious and are left to the reader. For (d), let  $R$  be canonically labeled and note that  $\{x, a_1, b_1\}$  is the unique 3-clique containing  $x$ , whence each automorphism of  $R$  carries the set  $\{a_1, b_1\}$  onto itself. But then  $c_1$  is carried onto itself because levels are preserved, and  $a_2$  is carried onto itself because it is the only node in level 2 that has two neighbors in level 1. When  $R$  is rigid this implies  $A$  is carried onto  $A$  and  $C$  onto  $C$ , whence  $B$  is carried onto  $B$  and the automorphism is the identity.  $\square$

3.3 PROPOSITION If  $h_1, \dots, h_b$  are positive integers whose sum is  $n/2$ , the number of isomorphism types of rigid rooted  $n$ -seines with specification  $(h_1, \dots, h_b)$  is  $3^{b-1} 2^{n-b-1}$ .

Proof. It follows from 3.2 that two rigid rooted  $n$ -seines are isomorphic if and only if the correspondence of their node-sets given by the canonical labelings is an isomorphism. Thus it remains only to determine the number of matchings  $M$  which are of the sort involved in the definition of an  $n$ -seine, have neither switch nor cross, match  $a_1$  with  $b_1$  and  $c_1$  with  $a_2$ , and generate blocks according to the specification  $(h_1, \dots, h_b)$ .

If  $h_1 = 1$  the part of the  $n$ -seine corresponding to the first block is already determined. If  $h_1 > 1$  there are 2 ways of choosing the two edges of  $M$  that join  $R_2$  to  $R_3$ , then 2 ways of choosing the single edge of  $M$  that joins  $R_3$  to  $R_4, \dots$ , and hence a total of  $2^{2h_1-2}$  ways of completing the first block. For each subsequent block of length  $2h_i$ , there are 6 ways of choosing the single edge of  $M$  that joins the first two levels of the block and then (as when  $i = 1$ )  $2^{2h_i-1}$  ways of completing the block. Thus the total number of rigid rooted  $n$ -seines with specification  $(h_1, \dots, h_b)$  is

$$6^{b-1} 2^{2\sum_{i=1}^b (h_i-1)} = 3^{b-1} 2^{n-b-1}. \quad \square$$

**3.4 PROPOSITION** Except for the first 2-seine, each rooted ambiguous  $n$ -seine is rigid. If  $n \geq 4$  and  $h_1, \dots, h_b$  are positive integers whose sum is  $n/2$  there is a unique isomorphism type of rooted ambiguous  $n$ -seine with specification  $(h_1, \dots, h_b)$ .

**Proof.** Consider a rooted ambiguous  $n$ -seine  $R = (S, x, y)$  with  $n \geq 4$ , let  $(x', y')$  be as in 3.1, and note that by the proof of 3.1  $R$  admits a canonical labeling in which  $x' = a_1$  and  $y' = c_n$ . To show  $R$  is rigid, refer to Figure 9 and note that if there is a switch or cross then  $\delta_S(x', y') < d$ .

The proof of 3.4 is completed by working through the successive levels to show that the remaining edges of  $M$  are uniquely determined by the specification  $(h_1, \dots, h_b)$  in conjunction with the fact that  $\delta_S(x', y') = d$ . Indeed, the following properties of  $R$  can be verified by induction on  $i$ :

a block begins at level  $i$  if and only if  $\{a_i, b_i\}$  is an edge;  
 a block ends at level  $i$  if and only if  $\{b_i, c_i\}$  is an edge;  
 for each odd  $i$ ,  $\{c_i, a_{i+1}\}$  is an edge;  
 for each even  $i$  that does not end a block,  $\{b_i, a_{i+1}\}$  and  
 $\{c_i, b_{i+1}\}$  are edges.  $\square$

The next result could also have been proved by a recursive procedure similar to the one used in proving 2.5.

3.5 THEOREM The total number of isomorphism types of rigid rooted n-seines is  $10^{(n-2)/2}$ ; of these,  $2^{(n-2)/2}$  are ambiguous.

Proof. Let  $m = n/2$  and recall that  $\binom{m-1}{b-1}$  is the number of ordered partitions of  $m$  into  $b$  positive integers. By 3.3 the number of rigid rooted  $n$ -seines is

$$2^{2m-2} \sum_{b=1}^m \binom{m-1}{b-1} \left(\frac{3}{2}\right)^{b-1} = 2^{2m-2} \left(1 + \frac{3}{2}\right)^{m-1} = 10^{m-1}$$

and by 3.4 the number of rooted ambiguous  $n$ -seines is

$$\sum_{b=1}^m \binom{m-1}{b-1} = 2^{m-1}. \quad \square$$

For each finite sequence  $H = (h_1, \dots, h_b)$  let  $H^0$  denote the reverse sequence  $(h_b, \dots, h_1)$ . If  $H$  is the specification of  $(S, x, y)$  then  $H^0$  is the specification of  $(S, y, x)$ . An unrooted  $n$ -seine  $S$  is called symmetric if it admits an automorphism interchanging the two nodes of a diametral pair. If  $\{x, y\}$  is the pair in question and  $H$  is the specification of  $(S, x, y)$  then  $H^0 = H$ . Each unambiguous  $n$ -seine  $S$  yields 1 or 2 isomorphism types of rooted  $n$ -seines according as  $S$  is or is not symmetric. However, for

ambiguous  $n$ -seines the situation is more complicated and is elucidated with the aid of some additional operations on sequences.

For each finite sequence  $H$  of positive integers let  $H^\alpha$  denote the sequence obtained from  $H$  by doing the following:

replace each maximal segment of 1's in  $H$  by  $-r$ , where  $r$  is the number of 1's in the segment;

insert a 0 between each pair of consecutive entries  $> 1$ .

For example, if

$$H = (1, 1, 1, 7, 2, 1, 3, 1, 1, 4, 5)$$

then  $H^\alpha = (-3, 7, 0, 2, -1, 3, -2, 4, 0, 5).$

For each finite sequence  $K$  of integers let  $K^\beta$  be obtained from  $K$  by the simultaneous replacement of each entry  $k$  of  $K$  as follows, where  $s$  is the number of neighbors of the entry in  $K$  -- that is,  $s = 0$  if there is only one entry, else  $s = 1$  for the first and last entries and  $s = 2$  for all intermediate entries:

if  $k \leq 0$  replace  $k$  by  $s - k$ ;

if  $k \geq 2$  replace  $k$  by a segment of  $k - s$  1's.

For example, if  $H$  and  $H^\alpha$  are as above then

$$H^{\alpha\beta} = (4, 1, 1, 1, 1, 1, 2, 3, 1, 4, 1, 1, 2, 1, 1, 1, 1),$$

$$H^{\alpha\beta\alpha} = (4, -5, 2, 0, 3, -1, 4, -2, 2, -4),$$

and  $H^{\alpha\beta\alpha\beta} = (1, 1, 1, 7, 2, 1, 3, 1, 1, 4, 5) = H.$

It can be proved directly, or as a corollary of 3.6 below, that  $H^{\alpha\beta\alpha\beta} = H$  for every finite sequence  $H$  of positive integers.

3.6 PROPOSITION If  $S, x, y, x'$  and  $y'$  are as in 3.1 and  $H$  is the specification of  $(S, x, y)$  then  $H^{\alpha\beta}$  is the specification of

$(S, x', y)$ . In particular, the specification of  $(S, x, y)$  starts with 1 if and only if the specification of  $(S, x', y')$  does not start with 1.

Proof. Let  $(S, x, y)$  and  $(S, x', y')$  be canonically labeled using labels  $a_i, b_i$  and  $c_i$  in the first case and labels  $a'_i, b'_i$  and  $c'_i$  in the second case. Then the pairs of labels attached to the various nodes of  $S$  are those appearing in corresponding positions in the following two lists:

$$x'a_1a'_2 \cdots a'_ny', x'b_1 \cdots b'_ny', x'c'_1 \cdots c'_{n-1}c'_ny'$$

$$a_1xc_1 \cdots c_{n-1}c_n, a_1b_1 \cdots b_nc_n, a_1a_2 \cdots a_nyc_n.$$

The desired conclusion is a consequence of this correspondence in conjunction with the properties of  $(S, x, y)$  listed at the end of the proof of 3.4 and the analogous properties of  $(S, x', y')$ .  $\square$

3.7 PROPOSITION If  $S, x, y, x'$  and  $y'$  are as in 3.1 and  $H$  is the specification of  $(S, x, y)$  then exactly one of the following statements is true:

- (a)  $H \neq H^0 \neq H^{\alpha\beta}$ ,  $S$  admits no nontrivial automorphism;
- (b)  $H^0 = H$ ,  $S$  admits a unique nontrivial automorphism, and it interchanges  $x$  with  $y$  and  $x'$  with  $y'$ ;
- (c)  $H^0 = H^{\alpha\beta}$ ,  $S$  admits a unique nontrivial automorphism, and it interchanges  $x$  with  $y'$  and  $x'$  with  $y$ .

Proof. If  $\{p, q\}$  and  $\{u, v\}$  are diametral pairs in  $S$  and  $\xi$  and  $\eta$  are automorphisms of  $S$  that carry  $p$  onto  $u$ , then  $\xi\eta^{-1}$  is an automorphism of the rooted  $n$ -seine  $(S, p, q)$ , whence  $\xi = \eta$

by 3.4 and 3.2c. It follows from 3.6 that no automorphism of  $S$  carries  $x$  onto  $x'$ . Thus perhaps

there is an automorphism  $\phi$  of  $S$  that carries  $x$  onto  $y$ , or  
 there is an automorphism  $\psi$  of  $S$  that carries  $x$  onto  $y'$ ,  
 but there is at most one such  $\phi$ , at most one such  $\psi$ , and there are  
 no other automorphisms of  $S$  except the identity. With the aid of  
 3.4 and 3.6 it follows that:

$\phi$  exists if and only if  $H^p = H$ , and  $\phi$  actually interchanges  $x$   
 with  $y$  and  $x'$  with  $y'$ ;

$\psi$  exists if and only if  $H^p = H^{\alpha\beta}$ , and  $\psi$  actually interchanges  
 $x$  with  $y'$  and  $x'$  with  $y$ ;

$\phi$  and  $\psi$  do not both exist, because  $H^{\alpha\beta} \neq H$ .  $\square$

An unrooted  $n$ -seine  $S$  is called skew-symmetric if  $n \geq 4$  and  
 condition (c) is satisfied. (See Figures 13 and 14.) By 3.7, no  
 $n$ -seine is both symmetric and skew-symmetric.

(Figure 13 is to be inserted here)

$(S, x, y)$  is of specification (2)       $(S, x', y')$  is of specification (1,1)

Fig. 13: Two views of a symmetric ambiguous 4-seine

(Figure 14 is to be inserted here)

$(S, x, y)$  is of specification (2,1)       $(S, x', y')$  is of specification (1,2)

Fig. 14: Two views of a skew-symmetric ambiguous 6-seine

The following result sharpens the second part of 3.5.

3.8 PROPOSITION If  $\gamma_n$  <resp.  $\gamma'_n, \gamma''_n$ > is the number of rooted ambiguous n-seines for which the specification starts and ends with a 1 <resp. starts but does not end with a 1, neither starts nor ends with a 1>, then  $\gamma_2 = \gamma_4 = \gamma''_4 = 1, \gamma'_2 = \gamma''_2 = \gamma'_4 = 0$ , and for all  $n \geq 6$ ,

$$\gamma_n = \gamma'_n = \gamma''_n = 2^{(n-6)/2}.$$

Proof. The initial conditions are obvious. From the second part of 3.6, applied to ends as well as starts, it follows that  $\gamma_n = \gamma''_n$  when  $n \geq 4$ . In view of the second part of 3.4, the following recursions are obvious:  $\gamma'_{n+2} = \gamma'_n + \gamma''_n$ ;  $\gamma_{n+4} = \gamma_n + 2\gamma'_n + \gamma''_n$ . The stated conclusions follow by induction.  $\square$

3.9 PROPOSITION If  $\delta_n$  <resp.  $\delta'_n$ > is the number of rooted symmetric n-seines for which the specification starts and ends <resp. neither starts nor ends> with 1, then  $\delta_2 = 1, \delta'_2 = 0$ , and for all  $n \geq 4$ ,

$$\delta_n = \delta'_n = 2^{\lfloor n/4 \rfloor - 1}.$$

Proof. By 3.7 and the second part of 3.6,  $\delta_n = \delta'_n$  for all  $n \geq 4$ . In view of 3.4,  $\delta_{n+4} = \delta_n + \delta'_n$ . Use induction.  $\square$

3.10 PROPOSITION If  $\epsilon_n$  <resp.  $\epsilon'_n$ > is the number of rooted skew symmetric n-seines for which the specification starts <resp. does not start> with 1, then  $\epsilon_2 = \epsilon'_2 = 0$ , and for all  $n \geq 4$ ,

$$\epsilon_n = \epsilon'_n = \begin{cases} 0 & \text{when } n \equiv 0 \pmod{4} \\ 2^{\lfloor n/4 \rfloor - 1} & \text{when } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. By 3.6,  $\epsilon_n = \epsilon'_n$  for all  $n \geq 4$ . To complete the proof, note that  $\epsilon_2 = \epsilon'_2 = 0$ ,  $\epsilon_4 = 0$ ,  $\epsilon_6 = 1$ , and use induction in conjunction with the recursion  $\epsilon_{n+4} = \epsilon_n + \epsilon'_n$ . That  $\epsilon_{n+4} = \epsilon_n + \epsilon'_n$  for all  $n \geq 8$  is a consequence of the following property of the operators  $\rho$ ,  $\alpha$  and  $\beta$ :

If the sequence  $H = (h_1, \dots, h_b)$  of positive integers satisfies the condition that  $H^\rho = H^{\alpha\beta}$  then the same condition is satisfied by the sequence  $(1, h_1, \dots, h_{b-1}, h_b + 1)$ ; when  $b \geq 2$  and  $h_1 = 1$  the condition is satisfied also by the sequence  $(h_2, \dots, h_{b-1}, h_b - 1)$ .  $\square$

3.11 THEOREM Let  $\phi_n$ ,  $\psi_n$  and  $\omega_n$  denote the respective numbers of ambiguous n-seines, of symmetric ambiguous n-seines, and of skew symmetric ambiguous n-seines. Then  $\phi_2 = \psi_2 = \phi_4 = \psi_4 = 2$ ,

$\omega_2 = \omega_4 = 0$ , and

for all even  $n \geq 4$ ,  $\psi_n = 2^{\lfloor n/4 \rfloor - 1}$

for all even  $n \geq 4$ ,  $\omega_n = 0$  or  $2^{\lfloor n/4 \rfloor - 1}$  according as  $n \equiv 0$  or  $2 \pmod{4}$ ,

for all even  $n \geq 6$ ,  $\phi_n = 2^{(n-6)/2} + 2^{\lfloor (n-2)/4 \rfloor}$ .

Proof. Use 3.9 and 3.10 for  $\psi_n$  and  $\omega_n$ . With  $\lambda_n = \phi_n - \psi_n - \omega_n$ , it follows from 3.5 and 3.7 that

$$4\lambda_n + 2\psi_n + 2\omega_n = 2^{(n-2)/2}$$

whence

$$\phi_n = 2^{(n-6)/2} + \frac{1}{2}\psi_n + \frac{1}{2}\omega_n$$

and the stated equality follows.  $\square$

3.12 THEOREM. Let  $\sigma_n$  denote the number of isomorphism types of n-seines and  $\tau_n$  the number of isomorphism types of symmetric n-seines. Then for all even  $n \geq 6$ ,

$$\sigma_n = \begin{cases} \frac{1}{2}(1 + 21\zeta_j) - 44\zeta_{j-1} + 30\zeta_{j-2} - 2^{j-1} & \text{when } n = 4j \\ 1 + \frac{1}{2}(3\zeta_{j+1} - 13\zeta_j) + 5\zeta_{j-1} - 2^{j-1} & \text{when } n = 4j + 2 \end{cases}$$

and

$$\tau_n = \begin{cases} \frac{3}{4}\zeta_{2j} - \frac{5}{2}\zeta_{2j-2} + \frac{21}{4}\zeta_j - 22\zeta_{j-1} + 15\zeta_{j-2} - 4^{j-2} & \text{when } n = 4j \\ \frac{3}{4}\zeta_{2j+1} - \frac{5}{2}\zeta_{2j} + \frac{3}{4}\zeta_{j+1} - \frac{13}{4}\zeta_j + \frac{5}{2}\zeta_{j-1} - 4^{j-1} - 2^{j-1} + \frac{1}{4} & \text{when } n = 4j+2 \end{cases}$$

Proof. Let  $\phi_n$ ,  $\psi_n$ ,  $\omega_n$  and  $\lambda_n$  be as in 3.11, and let  $\rho_n$  denote the number of isomorphism types of reversible rooted n-seines. Each unambiguous symmetric n-seine yields a single type of reversible rooted n-seine, and by 3.7 each ambiguous symmetric n-seine yields two such types (assuming  $n \geq 4$ ). Hence  $(\sigma_n - \psi_n) + 2\psi_n = \rho_n$  and the above equation for  $\sigma_n$  follows from 2.6 and 3.11. Similarly, there is a single type of rooted n-seine associated with each unambiguous symmetric n-seine, there are two such types associated with each n-seine that is unambiguous and asymmetric or ambiguous and symmetric or ambiguous and skew-symmetric, and there are four such types associated with each n-seine that is ambiguous but neither symmetric nor skew-symmetric. It follows that for even  $n \geq 4$ ,

$$(\sigma_n - \psi_n) + 2(\tau_n - \phi_n - (\sigma_n - \psi_n)) + 2\psi_n + 2\omega_n + 4\lambda_n$$

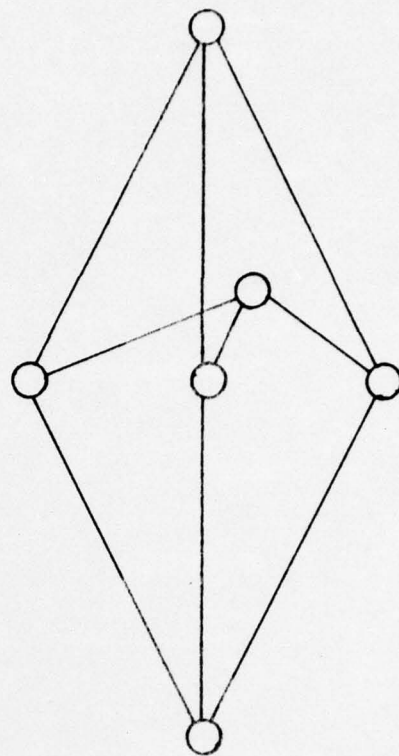
is the number  $r_n$  of isomorphism types of rooted n-seines, whence

$$2\tau_n = r_n - 2\phi_n + \psi_n + 2\omega_n + \sigma_n.$$

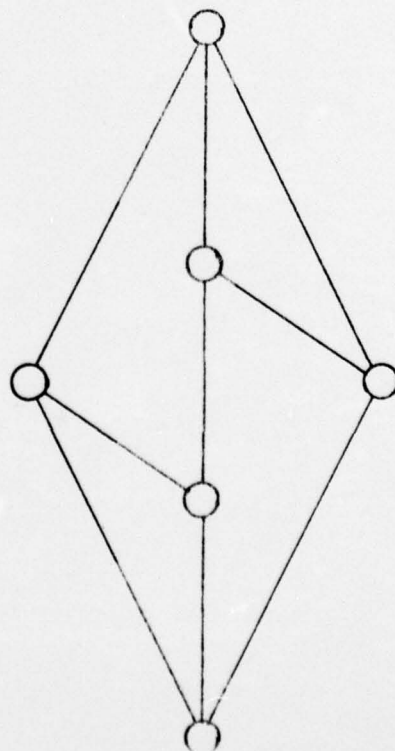
The stated conclusion then follows with the aid of 2.5, 3.11 and the value of  $\sigma_n$  already established.  $\square$

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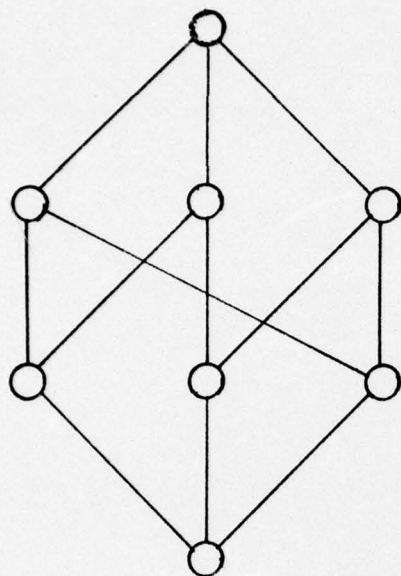
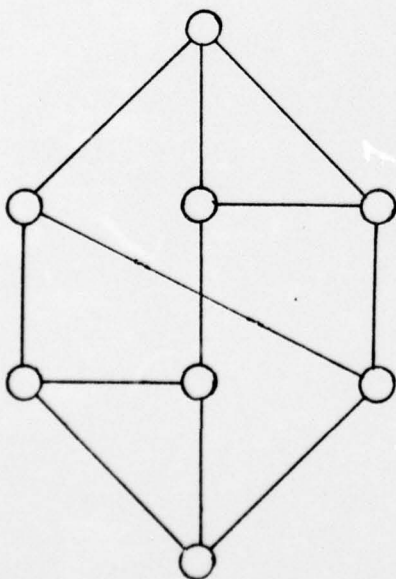
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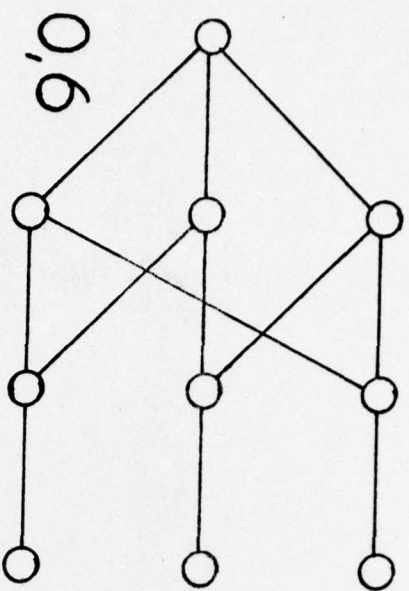
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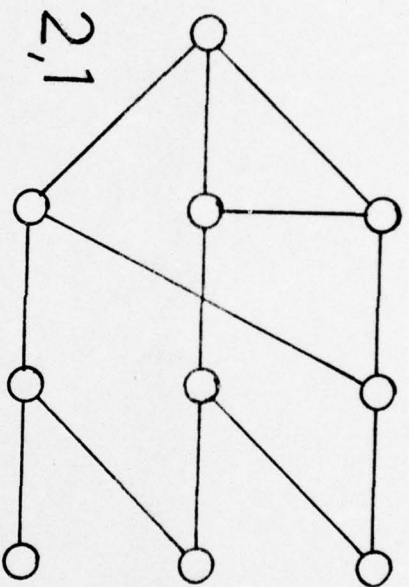
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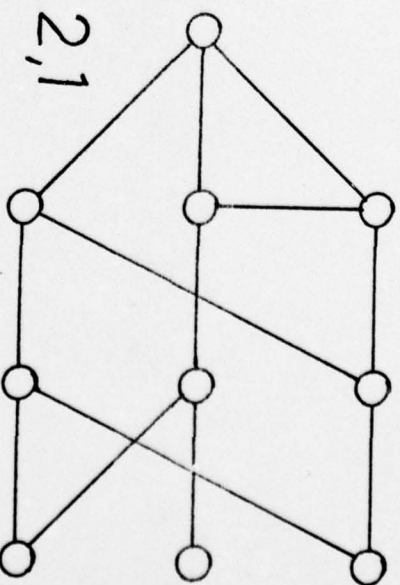
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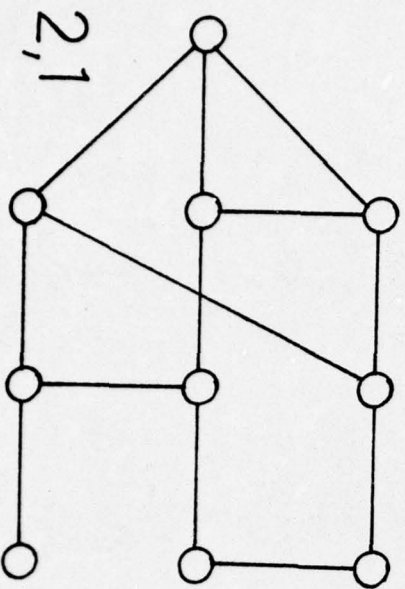
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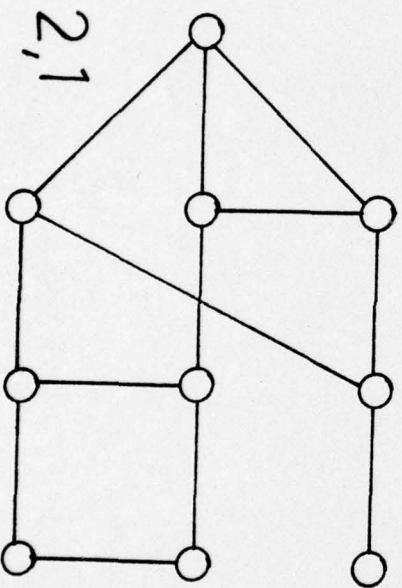
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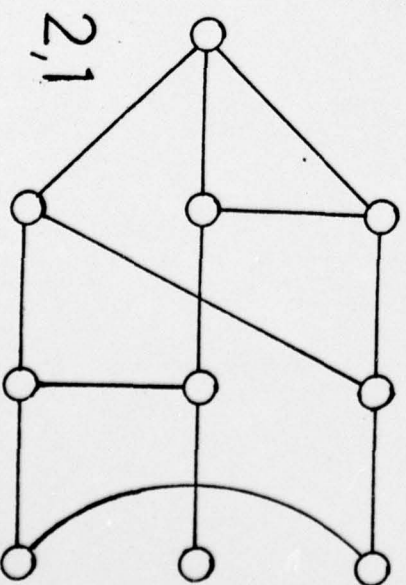
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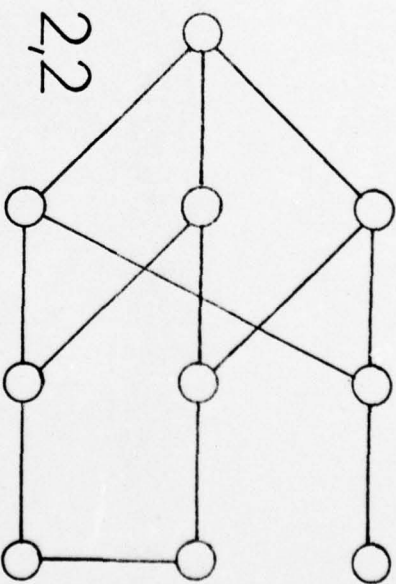
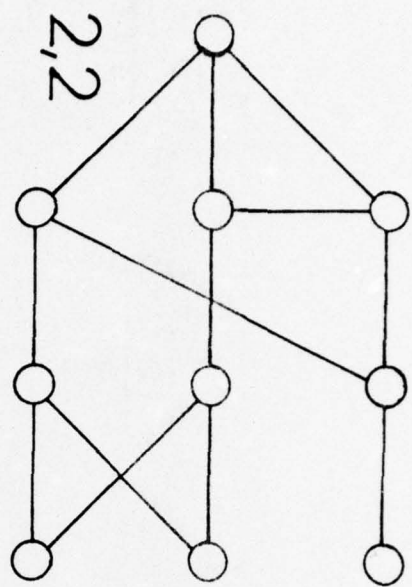


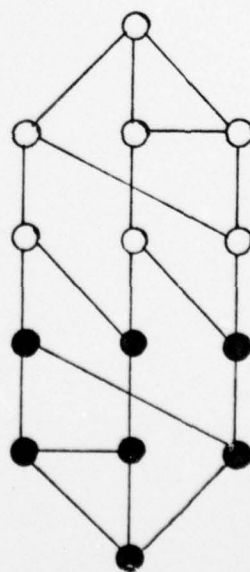
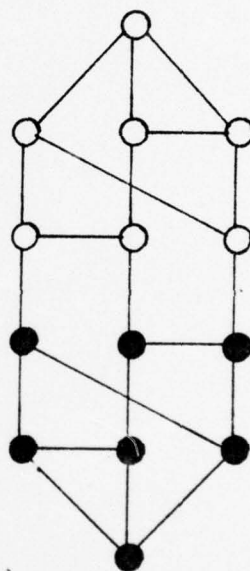
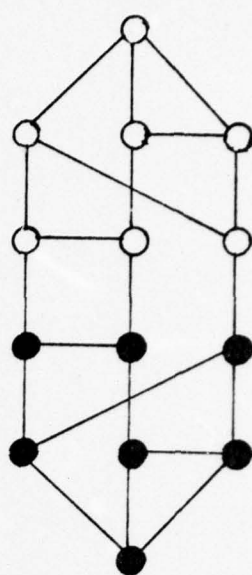
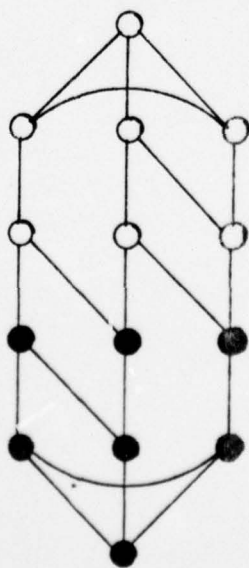
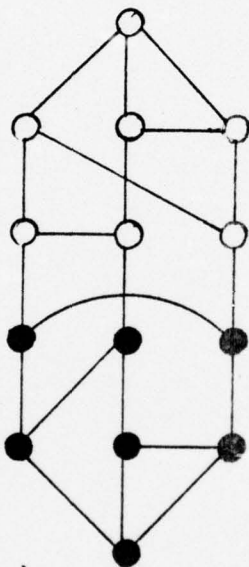
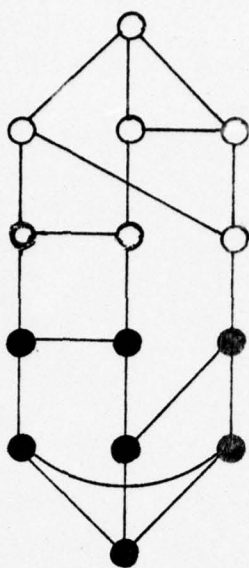
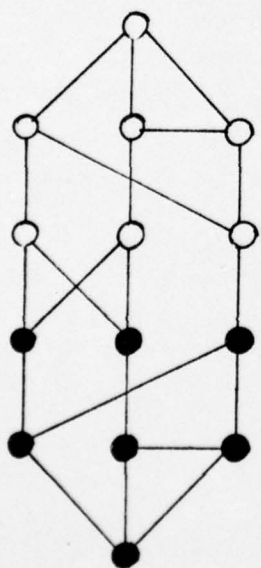
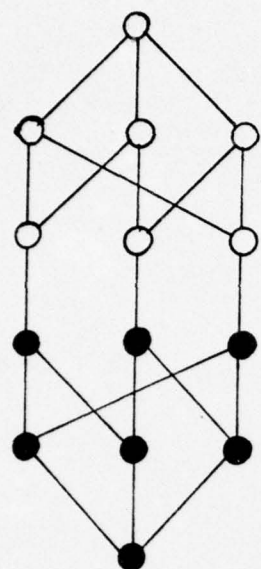
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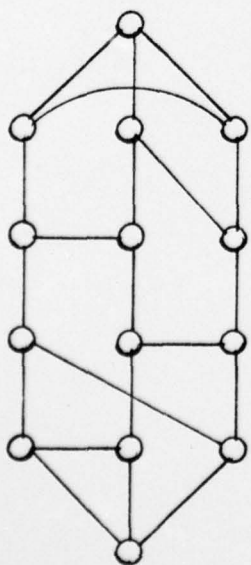
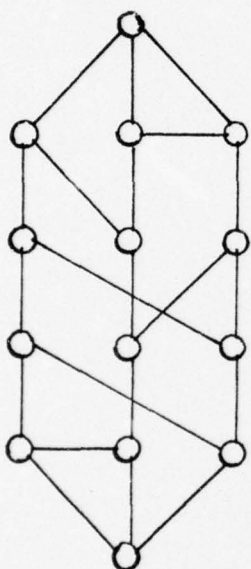
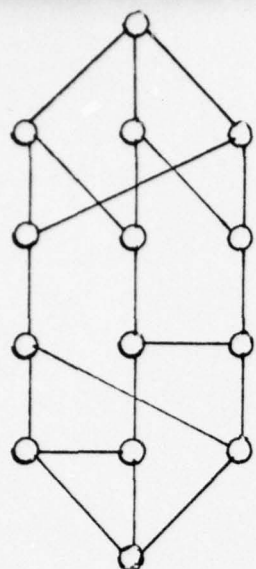
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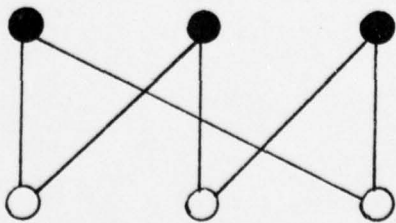
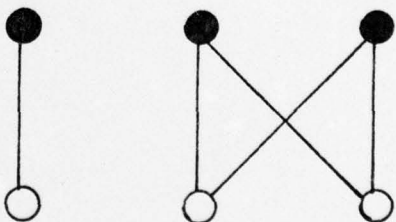
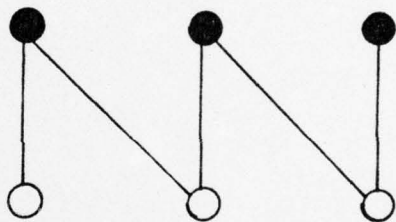
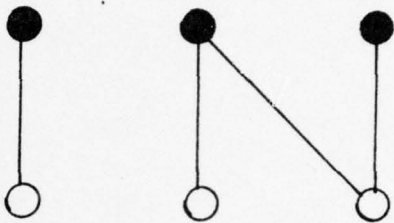
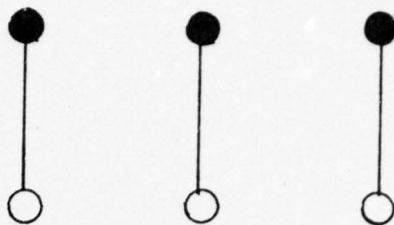




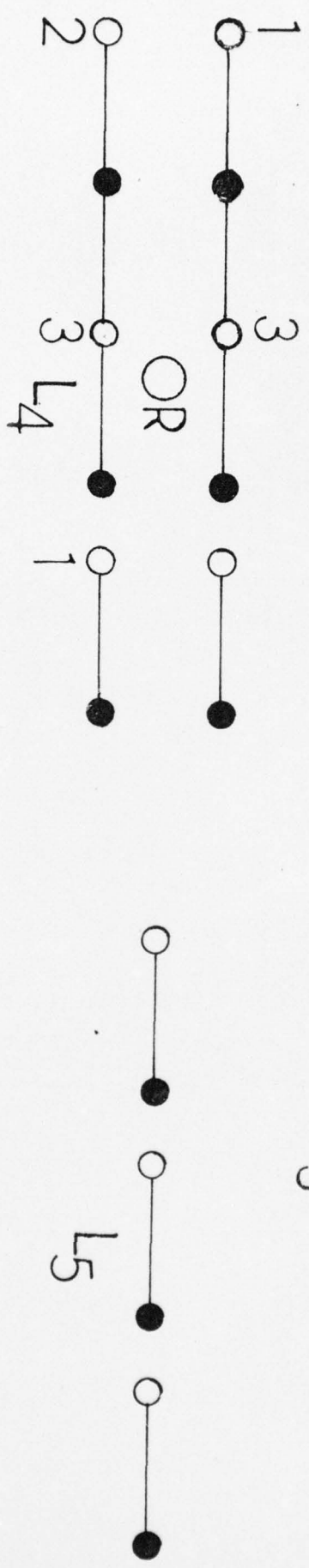
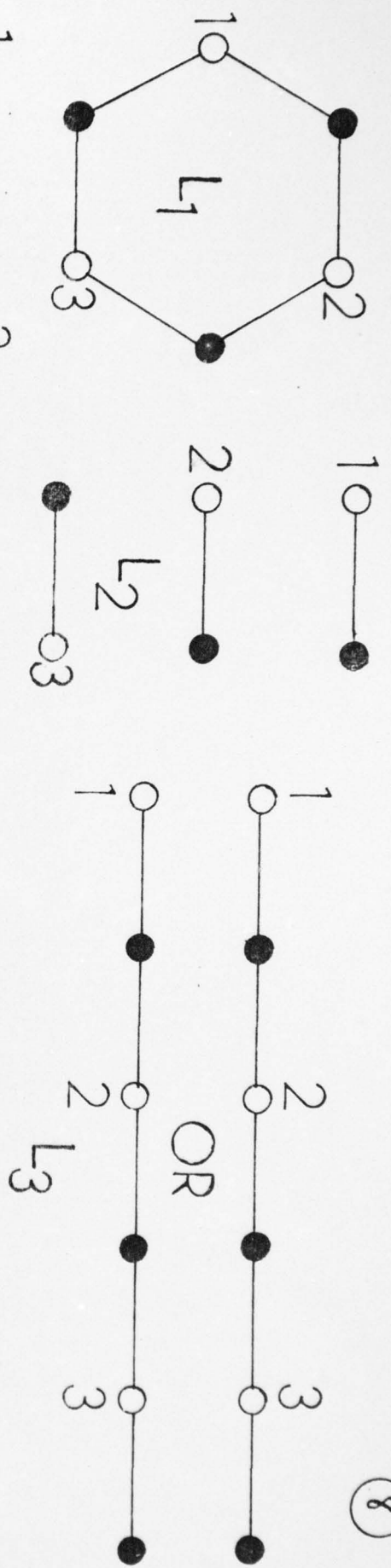
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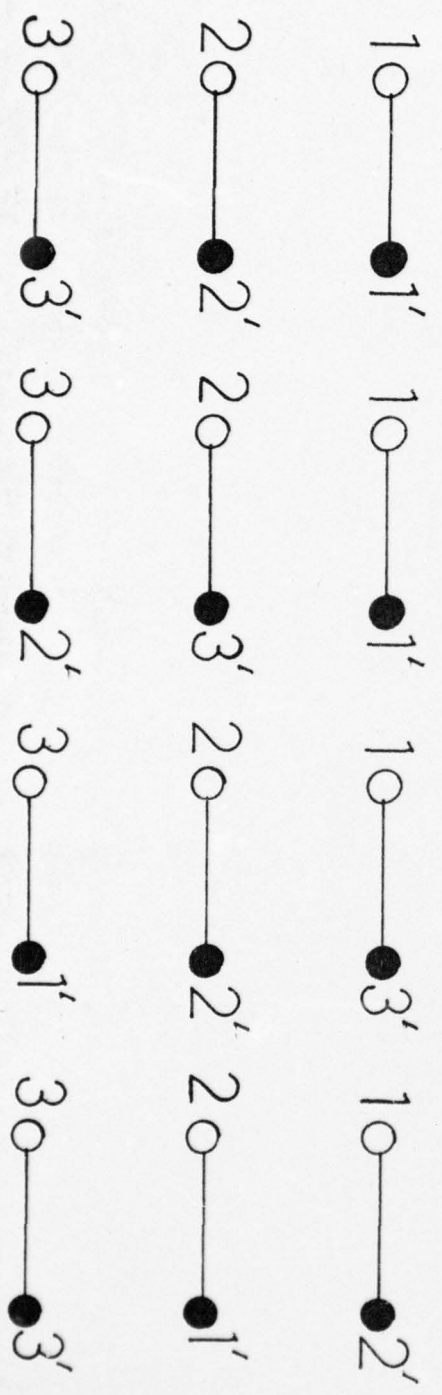
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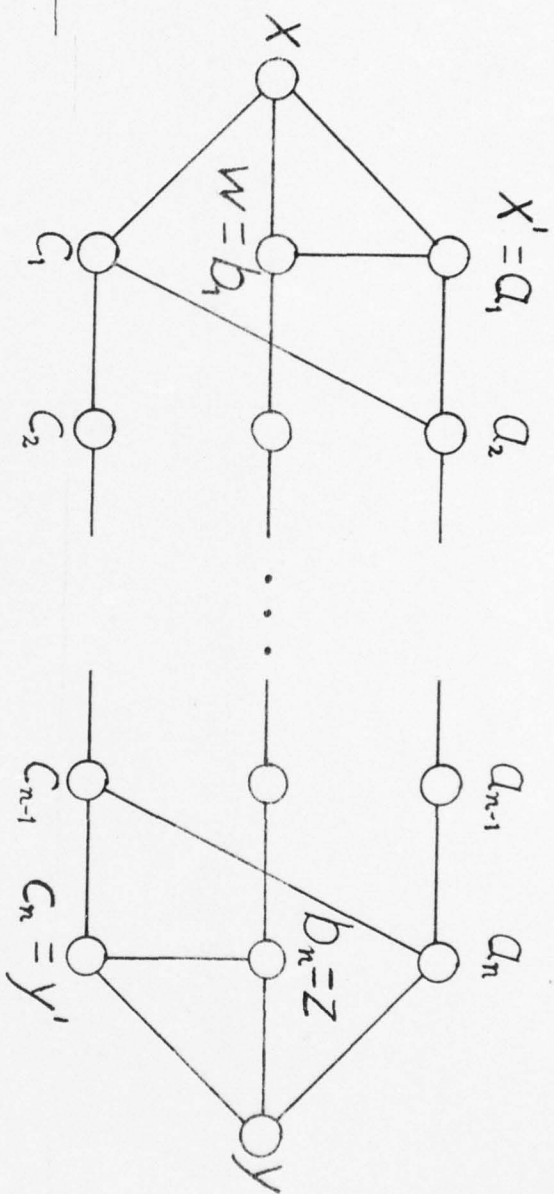


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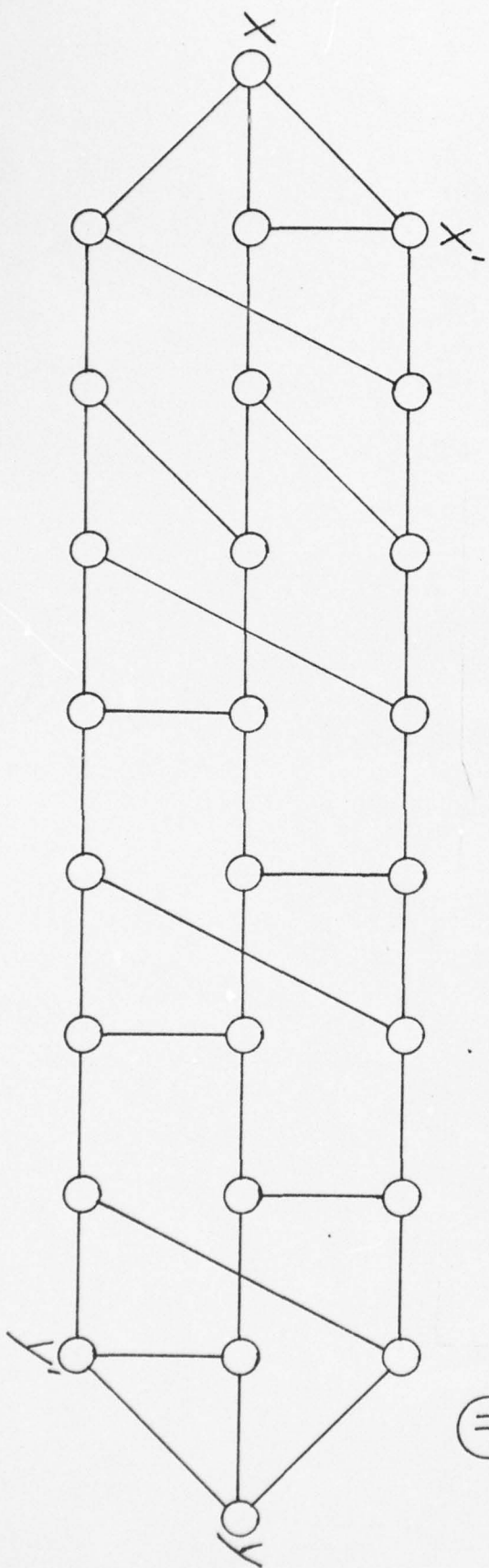


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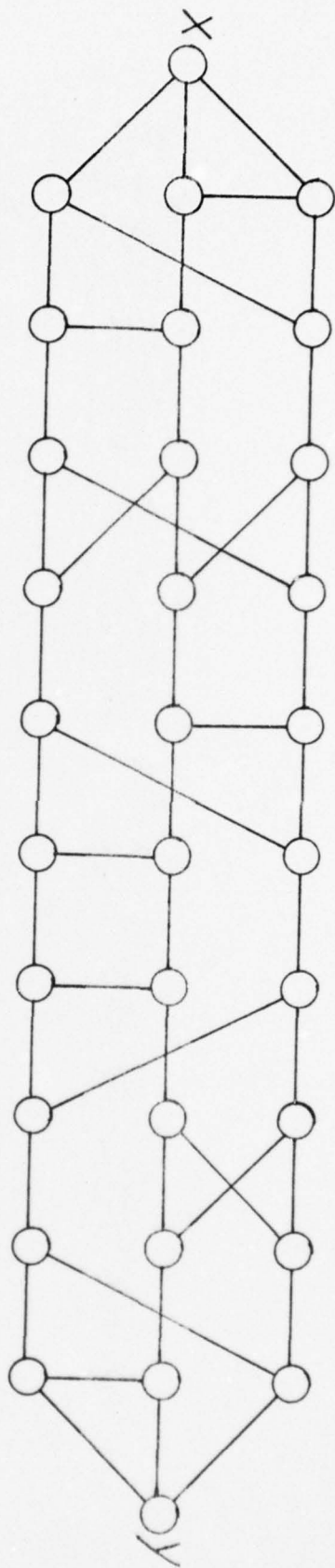




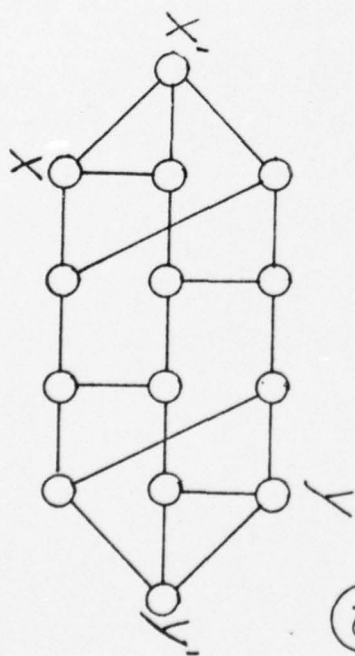
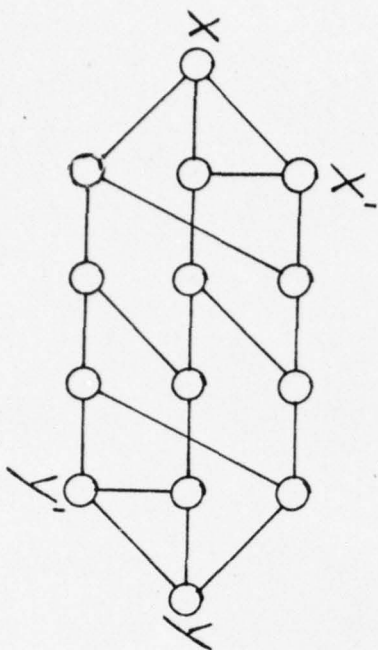
10



11



(12)



(13)

