THE GENERALIZED VARIANCE OF A STATIONARY AUTOREGRESSIVE PROCESS

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The Generalized Variance of a Stationary Autoregressive Process

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An autoregressive process $\{y_t\}$ of order p with mean 0 is defined by

$$(1) \qquad y_t + \beta_1 y_{t-1} + \dots + \beta_p y_{t-p} = u_t , \quad t = \dots,-1,0,1,\dots ,$$

where the u_t are independent random variables with $\mathcal{E} u_t = 0, \mathcal{E} u_t^2 = \sigma^2$, $0 < \sigma^2 < \infty$. The stochastic process is stationary and y_t is independent of $u_{t+1}, u_{t+2}, \dots$ if and only if the β_j are such that the associated polynomial equation

$$(2) \qquad b(w) = \sum_{j=0}^{p} \beta_j \, w^{p-j} = 0 ,$$

where $\beta_0 = 1$, has roots $w_1, \dots, w_p$ less than 1 in absolute value. The purpose of this paper is to show that the generalized variance of the process is a power of the variance of u_t times $\Pi_{i,j=1}^{p}(1-w_i w_j)^{-1}$.

The covariance sequence of the process is composed of $\sigma_s = \mathcal{E} y_t y_{t+s} = \sigma_{-s}$, $s = 0,1,\dots$. Consider a sequence of observations $y_1, \dots, y_T$ for $T \geq p$ constituting a sample vector $\underset{\sim}{y}_T = (y_1,\dots,y_T)'$.

The covariance matrix of the sample vector is

$$\mathcal{E}\, \underset{\sim}{y}_T \underset{\sim}{y}_T' = (\sigma_{i-j}) \equiv \underset{\sim}{\Sigma}_T \equiv \sigma^2\, \underset{\sim}{Q}_T \;. \tag{3}$$

The determinant $|\underset{\sim}{\Sigma}_T| = (\sigma^2)^T |\underset{\sim}{Q}_T|$ is the <u>generalized variance</u> of $\underset{\sim}{y}_T$.

In the Gaussian case the joint density of $\underset{\sim}{y}_p$ and $u_{p+1}, \ldots, u_T$ $(T \geq p)$ is

$$\frac{|\underset{\sim}{Q}_p^{-1}|^{\frac{1}{2}}}{(2\pi)^{T/2}\, \sigma^T} \exp\left\{-\frac{1}{2\sigma^2}\left[\underset{\sim}{y}_p'\, \underset{\sim}{Q}_p^{-1}\, \underset{\sim}{y}_p + \sum_{t=p+1}^{T} u_t^2\right]\right\} \;. \tag{4}$$

Since the Jacobian of the transformation from $\underset{\sim}{y}_p, u_{p+1}, \ldots, u_T$ to $\underset{\sim}{y}_T$ is 1 , the constant of the density of $\underset{\sim}{y}_T$ is the same as of (4) and hence $|\underset{\sim}{Q}_p^{-1}| = |\underset{\sim}{Q}_T^{-1}|$, $T \geq p$. (See Walker (1961) and Siddiqui (1958).) Since $|\underset{\sim}{Q}_T| = |\underset{\sim}{Q}_p|$ for $T \geq p$, we call $|\underset{\sim}{Q}_p|$ the <u>normalized generalized variance</u> of the process.

For $T \geq p$ substitution for u_t from (1), $t = p+1, \ldots, T$, into (4) to obtain the density of $\underset{\sim}{y}_T$ yields the quadratic form $\underset{\sim}{y}_T'\, \underset{\sim}{Q}_T^{-1}\, \underset{\sim}{y}_T$, showing that every element of $\underset{\sim}{Q}_T^{-1}$ is a second degree polynomial in $\beta_1, \ldots, \beta_p$ except possibly elements of $\underset{\sim}{Q}_p^{-1}$. However, since the density of $y_1, \ldots, y_T$ is identical to the density of $y_T, \ldots, y_1$, the elements of $\underset{\sim}{Q}_p^{-1}$ must be second degree polynomials in $\beta_1, \ldots, \beta_p$. The components of $\underset{\sim}{Q}_p^{-1}$ are therefore polynomials in the roots $w_1, \ldots, w_p$ of degree at most $2p$. Hence, the determinant $|\underset{\sim}{Q}_p^{-1}|$ is a polynomial in $w_1, \ldots, w_p$ of degree at most $(2p)^p$.

Lemma 1. *If*

$$(5)\qquad \underset{\sim}{C}=(c_{ij}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ h_1 & h_2 & \cdots & h_p \\ h_1^2 & h_2^2 & \cdots & h_p^2 \\ \vdots & \vdots & & \vdots \\ h_1^{p-1} & h_2^{p-1} & \cdots & h_p^{p-1} \end{pmatrix},$$

then

$$(6)\qquad |\underset{\sim}{C}| = \prod_{i<j} (h_j - h_i)$$

and

$$(7)\qquad C_{1k} = (-1)^{k-1} \left(\prod_{i\neq k} h_i\right) \prod_{\substack{i<j \\ i\neq k\neq j}} (h_j - h_i) ,$$

where C_{1k} *denotes the cofactor of* c_{1k} *in* $\underset{\sim}{C}$.

Proof. $\underset{\sim}{C}$ is a Vandermonde matrix, and $|\underset{\sim}{C}|$ and $\underset{\sim}{C}^{-1}$ are given, for example, by Hamming (1962), Sections 8.2 and 10.3. A direct proof of (7) using (6) is as follows: to form C_{1k} delete row 1 and column k of $|\underset{\sim}{C}|$; in the cofactor, factor h_i out of the i-th column $(i\neq k)$ to obtain a Vandermonde determinant of order p-1 . Q.E.D.

Lemma 2. *The determinant of order* p ,

$$(8)\qquad D_p \equiv \left|\frac{1}{a_i + b_j}\right| = \frac{\prod_{i<j} (a_j - a_i)(b_j - b_i)}{\prod_{i,j=1}^{p} (a_i + b_j)} .$$

Proof. This is Cauchy's determinant; see, for example, Bellman (1960), Section 11.6, Exercise 1. A direct proof is as follows: To convert into 0 each element in the first column, except for that in the first row, we subtract from each row an appropriate multiple of its first row. The i,j-th element is thus converted into

$$(9) \qquad \frac{1}{a_i+b_j} - \frac{1}{a_1+b_j}\,\frac{a_1+b_1}{a_i+b_1} = \frac{a_i-a_1}{a_i+b_1}\,\frac{b_j-b_1}{a_1+b_j}\,\frac{1}{a_i+b_j}\ , \qquad i,j = 2, \ldots, p\ .$$

The first factor on the right-hand side is common to the i-th row and the second to the j-th column. Hence,

$$(10) \qquad D_p = \frac{1}{a_1+b_1}\left(\prod_{i=2}^{p} \frac{a_i-a_1}{a_i+b_1}\right)\left(\prod_{j=2}^{p} \frac{b_j-b_1}{a_1+b_j}\right) D_{p-1}\ ,$$

and the result follows. Q.E.D.

Theorem. *For* $\beta_1, \ldots, \beta_p$ *such that the roots of (2) are less than* 1 *in absolute value for* $T \geq p$

$$(11) \qquad |\underset{\sim}{\Sigma}_T| = (\sigma^2)^T \prod_{i,j=1}^{p} (1-w_i w_j)^{-1}\ .$$

Proof. We first consider the case where $w_1, \ldots, w_p$ are different and different from 0 . If $h_i = w_i^{-1}$, then in (5) $|\underset{\sim}{C}| \neq 0$. Anderson (1971) in (24) of Section 5.3 gives an expression for the elements of $\underset{\sim}{\Sigma}_p$ in terms of $w_1, \ldots, w_p$. (See also Problem 19 of Chapter 5.) Then

$$(12) \qquad |\underset{\sim}{Q}_p| = \frac{c_{11}^2 \cdots c_{1p}^2}{|\underset{\sim}{C}|^{2p-2}} |\underset{\sim}{V}| ,$$

where

$$(13) \qquad |\underset{\sim}{V}| = \left| \frac{1}{1 - w_i w_j} \right| = \left| \frac{w_i^{-1}}{w_i^{-1} + (-w_j)} \right| = \frac{1}{\prod_{i=1}^{p} w_i} \left| \frac{1}{w_i^{-1} + (-w_j)} \right|$$

$$= \frac{1}{\prod_{i=1}^{p} w_i} \frac{\prod_{i<j} (w_j^{-1} - w_i^{-1})(w_i - w_j)}{\prod_{i,j=1}^{p} (w_i^{-1} - w_j)}$$

$$= \left(\prod_{i=1}^{p} w_i \right)^{p-1} \frac{\prod_{i<j} (w_i - w_j)^2 \, w_i^{-1} w_j^{-1}}{\prod_{i,j=1}^{p} (1 - w_i w_j)} = \frac{\prod_{i<j} (w_i - w_j)^2}{\prod_{i,j=1}^{p} (1 - w_i w_j)} .$$

Further,

$$(14) \qquad \frac{c_{11}^2 \cdots c_{1p}^2}{|\underset{\sim}{C}|^{2p-2}} = \left[\frac{\left(\prod_{i=1}^{p} h_i^{p-1} \right) \prod_{i<j} (h_j - h_i)^{p-2}}{\prod_{i<j} (h_j - h_i)^{p-1}} \right]^2$$

$$= \frac{\prod_{i=1}^{p} h_i^{2p-2}}{\prod_{i<j} (h_j - h_i)^2} = \frac{1}{\prod_{i<j} (w_i - w_j)^2} ,$$

and (11) follows for the roots different and nonzero. The determinant $|\underset{\sim}{Q}_p^{-1}|$ is the polynomial $\Pi_{i,j=1}^{p}(1 - w_i w_j)$, which holds for all w_j such that $|w_j| < 1$, $j = 1, \ldots, p$. Q.E.D.

Discussion

1. If the process is Gaussian, the normalizing constant in the normal density of $\underset{\sim}{y}_T$ is $(2\pi)^{-T/2}$ times

$$|\underset{\sim}{\Sigma}_T|^{-\frac{1}{2}} = (\sigma^2)^{-T/2} \prod_{i,j=1}^{p} (1 - w_i w_j)^{\frac{1}{2}} . \tag{15}$$

2. If one or more of the roots approaches 1 in absolute value, $|\underset{\sim}{\Sigma}_T^{-1}| \to 0$ and $|\underset{\sim}{\Sigma}_T| \to \infty$. These facts agree with the nonexistence of a nontrivial stationary process satisfying (1) if one or more roots are equal to 1 in absolute value.

3. Grenander and Szegö (1958) in effect showed that $\lim_{T\to\infty} |\underset{\sim}{Q}_T| = \Pi_{i,j=1}^{p} (1 - w_i w_j)^{-1}$ by use of an integral of $|b(w)|^{-2}$, though they did not relate this result to the generalized variance of the autoregressive process. Walker (1961) noted that $|\underset{\sim}{Q}_T| = |\underset{\sim}{Q}_p| = 1/|\underset{\sim}{Q}_p^{-1}|$ for $T \geq p$. An alternate proof of the theorem can be assembled from the results of Grenander and Szegö and Walker. (See also Finch (1960).)

4. A moving average model of order q is defined by

$$x_t = v_t + \alpha_1 v_{t-1} + \ldots + \alpha_q v_{t-q} , \tag{16}$$

where the v_t are independent random variables with $\mathcal{E} v_t = 0$, $\mathcal{E} v_t^2 = \tau^2$, $0 < \tau^2 < \infty$. The associated polynomial equation

$$z^q + \alpha_1 z^{q-1} + \ldots + \alpha_q = 0 \tag{17}$$

has roots $z_1, \ldots, z_q$. Durbin (1959) conjectured that if $\tau^2 \underset{\sim}{N}_T$ is the covariance matrix of $x_1, \ldots, x_T$ generated by (16), and if all roots of (2) are less than 1 in absolute value, then

$$\lim_{T\to\infty} |\underset{\sim}{N}_T| = |\underset{\sim}{Q}_n| , \tag{18}$$

for some n sufficiently large compared with $p=q$ when $\alpha_j = \beta_j$, $j = 1, \ldots, p$. Finch (1960) showed that

$$\lim_{T\to\infty} |\underset{\sim}{N}_T| = \lim_{T\to\infty} |\underset{\sim}{Q}_T| \tag{19}$$

by use of some results of Grenander and Szegö (1958) and gave explicitly the limiting value of the generalized variance for an autoregressive moving average process. Walker (1961) used more algebraic methods to show (18) for $n = p = q$. As an example, these results for $p = q = 1$ are $Q_1 = 1/(1-\beta_1^2)$ and $|\underset{\sim}{N}_T| = (1-\alpha_1^{2T+2})/(1-\alpha_1^2) \to 1/(1-\alpha_1^2)$ as $T \to \infty$. Durbin (1959) considered the case $p = q = 2$ in detail.

For further discussion, see the recent paper by Shaman (1976).

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For a stationary autoregressive process of order p and disturbance variance σ^2 it is shown that the determinant of the covariance of $T(\geq p)$ consecutive random variables of the process is $(\sigma^2)^T \Pi_{i,j=1}^{p}(1 - w_i w_j)$, where $w_1,\dots,w_p$ are the roots of the associated polynomial equation.

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