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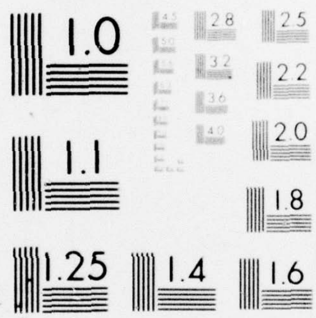


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# DISCONNECTED SOLUTIONS

by

W. F. Lucas



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# DISCONNECTED SOLUTIONS

BY W. F. LUCAS[1]

1. Introduction. In the book, Theory of Games and Economic Behavior (1944), J. von Neumann and O. Morgenstern introduced a theory of solutions (or stable sets) for multi-person cooperative games in characteristic function form. A longstanding conjecture has been that the union of all solutions of any particular game is a connected set. (E.g., see [3].) This announcement describes a twelve-person game for which this conjecture fails. The essential definitions for an n-person game will be reviewed briefly before the counterexample is presented. A sketch of the proof is presented here, and the details will appear elsewhere.

2. The Model. An n-person game is a pair $(N,v)$ where $N = \{1,2,\dots,n\}$ is the set of players and $v$ is a characteristic function on $2^N$, i.e., $v$ assigns the real number $v(S)$ to each subset $S$ of $N$ and $v(\emptyset) = 0$. The set of imputations is

$$A = \{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

where $x = (x_1, x_2, \dots, x_n)$ is a vector with real components. For any $S \subset N$, let $x(S) = \sum_{i \in S} x_i$. For any $X \subset A$ and nonempty $S \subset N$, define $\mathrm{Dom}_S X$ to be the set of all $x \in A$ such that there

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exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $y(S) \leq v(S)$. Let $\text{Dom } X = \bigcup_{\emptyset \neq S \subset N} \text{Dom}_S X$. A subset $V$ of $A$ is a <u>solution</u> if $V \cap \text{Dom } V = \emptyset$ and $V \cup \text{Dom } V = A$. The <u>core</u> of a game is

$$C = \{x \in A\colon x(S) \geq v(S) \text{ for all nonempty } S \subset N\}.$$

For any solution $V$, $C \subset V$ and $V \cap \text{Dom } C = \emptyset$.

A characteristic function $v$ is <u>superadditive</u> if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. The game below does not have a superadditive $v$ as is assumed in the classical theory, but it is equivalent solutionwise to a game with a superadditive $v$. (See [1, p. 68].)

3. <u>Example</u>. The 13 vital coalitions for our example consist of $N = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and elements from three classes:

$$\mathcal{B} = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,12\}\},$$
$$\mathcal{S} = \{\{1,3,6,7,9,11\}, \{1,4,5,7,9,11\}, \{2,3,5,7,9,11\}\},$$
$$\mathcal{T} = \{\{1,3,8\}, \{1,5,10\}, \{3,5,12\}\}.$$

And $v$ is given by: $v(N) = 6$, $v(S) = 1$ for all $S \in \mathcal{B}$, $v(S) = 4$ for all $S \in \mathcal{S}$, $v(S) = 1$ for all $S \in \mathcal{T}$, and $v(S) = 0$ for all other $S \subset N$. For this game $A = \{x\colon x(N) = 6 \text{ and } x_i \geq 0 \text{ for all } i \in N\}$. Consider also the six-dimensional hypercube

$$B = \{x \in A\colon x(S) = 1 \text{ for all } S \in \mathcal{B}\}.$$

The core $C$ is the intersection of $C(\mathcal{S})$ and $C(\mathcal{T})$ where

$$C(\mathcal{S}) = \{x \in B: x(S) \geq 4 \text{ for all } S \in \mathcal{S}\},$$

$$C(\mathcal{T}) = \{x \in B: x(S) \geq 1 \text{ for all } S \in \mathcal{T}\}.$$

C is a proper superset of the convex hull of the six vertices of B which have $x_i = 1$ for i equal to five of the six odd indices 1, 3, 5, 7, 9 and 11, and $x_{i+1} = 1$ when i is the remaining odd numbered player. Let $\text{Dom}_{\mathcal{B}}X = \cup_{S \in \mathcal{B}} \text{Dom}_S X$. Note that $\text{Dom}_{\mathcal{B}}C \supset A - B$, and hence any solution V for our game is a subset of B.

4. Outline of Proof. First, note that any component of an $x \in B$ has a maximum value of $x_i = 1$. Consequently, the following three sets are contained in any solution V, i.e., they are subsets of $\cap V$:

$$E = \{x \in B: x_i = x_j = 1 \text{ for } i \neq j \text{ and } \{i,j\} \subset \{1,3,5\}\},$$

$$F = \{x \in C(\mathcal{T}): x_p = 1 \text{ for } p = 7, 9 \text{ or } 11\},$$

$$P = \{(0,1,0,1,0,1,0,1,0,1,0,1)\}.$$

Next, we can show that $\cup V$ must be a disconnected set. Let $G = \{x \in B: x(\{7,9,11\}) \leq 1\}$, $G^0 = \{x \in B: x(\{7,9,11\}) < 1\}$, and $P' = \{x \in G: x_2 = x_4 = x_6 = 1\}$. Throughout this section the indices i, j and k represent some ordering of the distinct indices 1, 3 and 5. The subset H of E consisting of the three triangular regions

$$H_i = \{x \in G: x_{i+1} = x_j = x_k = 1; x_7 + x_9 + x_{11} = 1\}$$

is in $\cap V$ and $\text{Dom}_{\mathcal{S}}H \supset G^0 - (E \cup P')$. The subset J of F consisting of the three triangular regions

$$J_1 = \{x \in F: x_1 = x_7 = x_9 = 1, x_3 + x_5 + x_{12} = 1\},$$
$$J_3 = \{x \in F: x_3 = x_7 = x_{11} = 1, x_1 + x_5 + x_{10} = 1\},$$
$$J_5 = \{x \in F: x_5 = x_9 = x_{11} = 1, x_1 + x_3 + x_8 = 1\}$$

is also in $\cap V$ and $\text{Dom}_T J \supset B - C(T) \supset P' - P$. So any $x \in \cup V - P$ either has $x \in E$ or $x \in B - G^0$, i.e., $x_i = x_j = 1$ or $x(\{7,9,11\}) \geq 1$. Such $x$ are clearly disconnected from the singleton $P \subset \cap V$.

Finally, it is necessary to demonstrate that this game does possess at least one solution. $V' = C \cup E \cup F \cup P$ is in any solution $V$, and $V'$ can be enlarged to a solution in two steps. First, include the set of imputations $L$ in $C(T) - (V' \cup \text{Dom } V')$ which is simultaneously maximal with respect to all three of the relations "$\text{Dom}_S$" for $S \in \mathcal{S}$. Clearly $L \subset \cap V$. Next, pick a particular $S^i = \{i+1,j,k,7,9,11\} \in \mathcal{S}$ and then add in those elements $L^i$ in $C(T) - (V' \cup L \cup \text{Dom}(V' \cup L))$ which are maximal with respect to the relation "$\text{Dom}_{S^i}$" and are at the same time symmetrical in the sense that $x_j = x_k$. It requires some detail to describe the sets $L$ and $L^i$ explicitly, and to verify that the resulting sets $V^i = V' \cup L \cup L^i$ are solutions for our example. These will appear elsewhere.

5. Remarks. At one time it was apparently believed that proving the union of all solutions connected could be a major step in showing that every game has a solution. It is now known [2] that a solution need not exist for every game. On the other hand, it is possible that results on disconnecting $\cup V$ might be useful in the resolution of important open questions about whether solutions

always exist for games with full-dimensional cores, with empty cores, or which are constant-sum.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes a twelve-person cooperative game in characteristic function form (with side payments) for which the union of all solutions (stable sets) is a disconnected set. This disproves a longstanding conjecture in the classical theory of coalitional games.

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