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ASYMPTOTIC PROBABILITIES IN A CRITICAL AGE-DEPENDENT BRANCHING --ETC(U)

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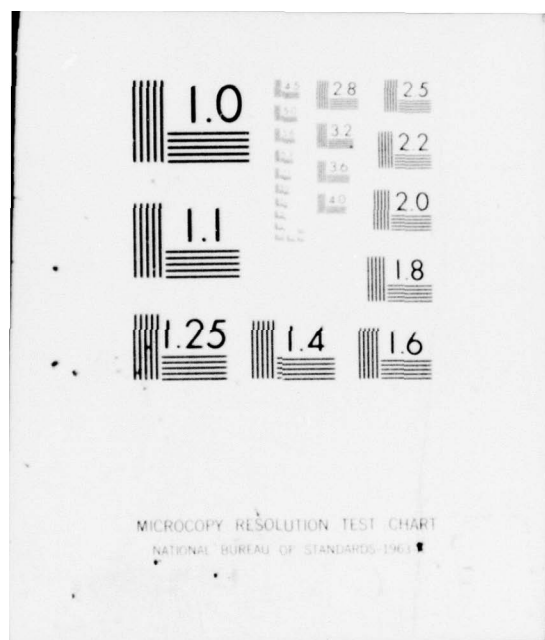
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6 ASYMPTOTIC PROBABILITIES IN A
CRITICAL AGE-DEPENDENT BRANCHING PROCESS

by

10 Howard J. Weiner

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by Howard J. Weiner

Let $Z(t)$ denote the number of cells alive at time t in a critical Bellman-Harris age-dependent branching process, that is, where the mean number of offspring per parent is one. A comparison method is used to show for $k \geq 1$, and a high-order moment condition on $G(t)$, where $G(t)$ is the cell lifetime distribution, that $\lim_{t \rightarrow \infty} t^2 P[Z(t)=k] = a_k > 0$, where $\{a_k\}$ are constants.

The method is also applied to the total progeny in the critical process.

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Asymptotic Probabilities in a
Critical Age-Dependent Branching Process

by Howard J. Weiner

1. Introduction.

Let $Z(t)$ denote the number of cells alive at time t in a critical Bellman-Harris [2] age-dependent branching process. That is, at time $t = 0$, the process starts with one new cell with lifetime distribution $G(t)$, $G(0+) = 0$, non-lattice, and

$$(1.1) \quad G'(t) = g(t),$$

the density exists, and for some $\delta > 0$,

$$(1.2) \quad \int_0^{\infty} t^{4+\delta} g(t) dt < \infty.$$

At the end of its life, the cell disappears and is immediately replaced by k new cells with probability $p_k \geq 0$, with

$$(1.3) \quad \sum_{k=0}^{\infty} p_k = 1$$

and criticality,

$$(1.4) \quad \sum_{k=1}^{\infty} k p_k = 1.$$

Each new cell proceeds identically as the parent cell and independently of all other cells and the state of the system. Assume that for all integers k ,

$$(1.5) \quad \sum_{n=1}^{\infty} n^k p_n < \infty.$$

It is the purpose of this note to show that for $k \geq 1$,

$$(1.6) \quad \lim_{t \rightarrow \infty} t^2 P[Z(t)=k] = a_k > 0,$$

where $\{a_k\}$ are constants.

The method is also applied to total progeny in the critical process.

2. Integral Equation and Result.

Define the offspring generating function, for $0 \leq s \leq 1$,

$$(2.1) \quad h(s) \equiv \sum_{k=0}^{\infty} p_k s^k.$$

Define the generating function for $Z(t)$ as, for $0 \leq s \leq 1$, all $t \geq 0$,

$$(2.2) \quad F(s, t) \equiv E(s^{Z(t)}).$$

Then [2] $F(s, t)$ satisfies

$$(2.3) \quad F(s, t) = 1 - G(t) + \int_0^t h(F(s, t-u)) dG(u),$$

with $F(s, 0) = s$.

Theorem. Let $Z(t)$ denote the number of cells at time t in a critical Bellman-Harris branching process satisfying (1.1) - (1.5). Assume also that the derivatives of h satisfy

$$(2.4) \quad 0 < h''(1) < \infty, \quad 0 < h'''(1) < \infty.$$

Then for $k \geq 1$

$$(2.5) \quad \lim_{t \rightarrow \infty} t^2 P[Z(t)=k] = a_k > 0$$

where $\{a_k\}$ are constants.

Proof. Consider first the case $k = 1$, and write for simplicity

$$(2.6) \quad P(t) \equiv P[Z(t)=1].$$

From the representation (2.3), derivatives of $F(s,t)$ with respect to s exist, and note that

$$(2.7) \quad \left. \frac{\partial F(s,t)}{\partial s} \right|_{s=0} = P(t).$$

From (2.7), (2.3) one obtains the relationship

$$(2.8) \quad P(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))P(t-u)dG(u)$$

$$\text{and } P(0) = 1,$$

where

$$(2.9) \quad Q(t) = P[Z(t) > 0].$$

From [2],

$$(2.10) \quad \lim_{t \rightarrow \infty} tQ(t) = 2\mu(h''(1))^{-1},$$

where

$$(2.11) \quad \mu \equiv \int_0^{\infty} tdG(t).$$

Assume for simplicity in all the following that

$$(2.12) \quad \mu \equiv 1.$$

The proof will proceed by a number of claims.

Claim I. Let $R(t)$ be continuous, $R(0) = 0$, and satisfy, for $t > 0$, that

$$(2.13) \quad R(t) \begin{matrix} > \\ (<) \end{matrix} \int_0^t h'(1-Q(t-u))R(t-u)dG(u).$$

Then for all $t > 0$,

$$(2.14) \quad R(t) \begin{matrix} > \\ (<) \end{matrix} 0.$$

Proof. Assume the upper inequality of (2.13).

Note first that for all $t > 0$,

$$(2.15) \quad R(t) \neq 0.$$

This follows from (2.13) by assuming there is a t_0 such that $R(t_0) = 0$, and a contradiction is clear.

If the upper inequality in (2.13) is false, (2.15) requires that $R(t) < 0$ for $t > 0$. Then on an arbitrary interval $[0, t]$, assume that $R(t)$ assumes its minimum at $t_1 \leq t$. Since $0 \leq h'(x) \leq 1$, setting $t = t_1$ in the upper inequality (2.13) yields a contradiction. The lower inequality of (2.13) implies the lower inequality of (2.14) similarly, completing the claim.

Claim II. For some $0 < \epsilon < 1$, all $t \geq 0$, define

$$(2.16) \quad K(t) = \begin{cases} 1 & , \quad t < 1 \\ \frac{1}{t^{2-\epsilon}} & , \quad t \geq 1 \end{cases}$$

$$(2.17) \quad L(t) = \begin{cases} 1 & , \quad t < 1 \\ \frac{1}{t^{2+\epsilon}} & , \quad t \geq 1. \end{cases}$$

Then for all t sufficiently large,

$$(2.18) \quad K(t) > 1 - G(t) + \int_0^t h'(1-Q(t-u))K(t-u)dG(u)$$

and

$$(2.19) \quad L(t) < 1 - G(t) + \int_0^t h'(1-Q(t-u))L(t-u)dG(u).$$

Proof. To show (2.18), note that the right side yields the inequality

$$(2.20) \quad \int_0^t h'(1-Q(t-u))K(t-u)dG(u) \leq G(t) - G(t/2) \\ + \int_0^{t/2} h'(1-Q(t-u))K(t-u)dG(u).$$

A Taylor expansion of the rhs of (2.20) yields,

$$(2.21) \quad \int_0^{t/2} h'(1-Q(t-u))K(t-u)dG(u) \\ = \int_0^{t/2} (1-Q(t-u)h''(1) + o(Q(t-u)))K(t-u)dG(u).$$

A Newton binomial expansion applied to the rhs integrand of (2.21) using (2.10), Theorem 4, p. 406 of [3], (2.16), and that $\mu \equiv 1$ yields that the rhs of (2.21) equals

$$\begin{aligned}
(2.22) \quad & \int_0^{t/2} \left(1 - \frac{2}{t-u} + o\left(\frac{1}{t-u}\right)\right) \frac{dG(u)}{(t-u)^{2-\epsilon}} \\
&= \frac{1}{t^{2-\epsilon}} \int_0^{t/2} \frac{dG(u)}{\left(1 - \frac{u}{t}\right)^{2-\epsilon}} - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right) \\
&= \frac{1}{t^{2-\epsilon}} \int_0^{t/2} \left(1 + \frac{(2-\epsilon)u}{t} + o\left(\frac{1}{t}\right)\right) dG(u) - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right) \\
&= \frac{1}{t^{2-\epsilon}} + \frac{(2-\epsilon)}{t^{3-\epsilon}} - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right) \\
&< \frac{1}{t^{2-\epsilon}}
\end{aligned}$$

for all t sufficiently large. In view of (2.20), (1.1), this suffices for (2.18), and (2.19) follows similarly.

Claim III. For $0 < \epsilon < 1$, and all t sufficiently large,

$$(2.23) \quad \frac{1}{t^{2+\epsilon}} < P(t) < \frac{1}{t^{2-\epsilon}}.$$

Proof. This follows from Claims I and II, where $R = K - P$, for example.

Claim IV. Let $R(t)$ satisfy

$$(2.24) \quad R(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))R(t-u)dG_0(u),$$

with $R(0) = 1$, where $Q(t)$ is the probability of non-extinction for the process with lifetime $G(t)$, and

$$(2.25) \quad 1 - G_0(t) = e^{-t}.$$

Then as $t \rightarrow \infty$,

$$(2.26) \quad R(t) \sim \frac{c}{t^2}$$

for some constant $c > 0$.

Proof. Write (2.24) as

$$(2.27) \quad R(t) = 1 - G(t) + e^{-t} \int_0^t h'(1-Q(u))R(u)e^u du.$$

Differentiating, with respect to t ,

$$(2.28) \quad R'(t) = -g(t) + (1-G(t)-R(t)) + h'(1-Q(t))R(t).$$

Then, expanding h in a Taylor series, one obtains

$$(2.29) \quad R'(t) + \frac{2}{t} R(t) = f(t)$$

where

$$(2.30) \quad f(t) = 1 - G(t) - g(t) + \frac{h'''(*)}{2} Q^2(t)R(t) + (Q(t) - \frac{2}{h''(1)t})R(t),$$

with $1 - Q(t) \leq * \leq 1$. In view of (1.1), (1.2), (1.5), it follows from Theorem 4 of [3], p. 406, that

$$(2.31) \quad (Q(t) - \frac{2}{h''(1)t}) = O(\frac{\log t}{t^2}).$$

An analysis similar to that in Claims I - III yields that for $0 < \epsilon < 1$, for t sufficiently large,

$$(2.32) \quad \frac{1}{t^{2+\epsilon}} < R(t) < \frac{1}{t^{2-\epsilon}}.$$

Then (1.1), (2.31), (2.32) used in (2.30) imply that

$$(2.33) \quad t^2 f(t)$$

is integrable with respect to Lebesgue measure on the positive line.

To solve (2.29) for t large, let

$$(2.34) \quad Q(t) \equiv t^2 R(t).$$

Then (2.29) becomes

$$(2.35) \quad Q'(t) = t^2 f(t).$$

For fixed $t_0 > 0$, one then obtains

$$(2.36) \quad Q(t) - Q(t_0) = \int_{t_0}^t \xi^2 f(\xi) d\xi$$

or

$$(2.37) \quad R(t) \sim \frac{c}{t^2}$$

for t sufficiently large, and $c > 0$, completing this claim.

Claim V. Let

$$(2.38) \quad T(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))R(t-u)dG(u).$$

Then, as $t \rightarrow \infty$,

$$(2.39) \quad t^2 |T(t) - R(t)| \rightarrow 0.$$

Proof.

$$(2.40) \quad |T(t) - R(t)| \leq \left| \int_0^{t/2} h'(1-Q(t-u))R(t-u)(dG(u) - dG_0(u)) \right| \\ + G(t) - G(t/2) + G_0(t) - G_0(t/2).$$

A Taylor series and Newton binomial expansion on the right side integrand of (2.40) yields

$$(2.41) \quad \int_0^{t/2} \left[1 - Q(t-u)h''(1) + Q^2(t-u)h'''(k(t-u)) \right] \left[\frac{c}{(t-u)^2} + o\left(\frac{1}{(t-u)^2}\right) \right] (dG(u) - dG_0(u))$$

where

$$(2.42) \quad 1 - Q(t-u) \leq k(t-u) \leq 1,$$

and expression (2.41) equals

$$\begin{aligned}
(2.43) \quad & \int_0^{t/2} \left[1 - \frac{2}{t-u} + o\left(\frac{1}{t-u}\right) \right] \left[\frac{c}{(t-u)^2} + o\left(\frac{1}{(t-u)^2}\right) \right] (dG(u) - dG_0(u)) \\
&= \int_0^{t/2} \left[1 - \frac{2}{t}\left(1 + \frac{u}{t}\right) + o\left(\frac{1}{t}\right) \right] \left[\frac{c}{t^2} + o\left(\frac{1}{t^2}\right) \right] (dG(u) - dG_0(u)) \\
&= o(t^{-4}),
\end{aligned}$$

for t sufficiently large, using that

$$\mu = \int_0^\infty u dG(u) = \int_0^\infty u dG_0(u) = 1.$$

This completes Claim V.

The proof of the theorem for $P(t)$ may now be completed.

Define the iterative scheme, for $n \geq 0$

$$(2.44) \quad P_{(n+1)}(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u)) P_{(n)}(t-u) dG(u),$$

and let

$$(2.45) \quad P_{(0)}(t) \equiv R(t).$$

Then from (2.38), it follows that

$$(2.46) \quad P_{(1)}(t) = T(t).$$

Claims IV and V imply that, for t large,

$$(2.47) \quad P_{(0)}(t) \sim \frac{c}{t^2} \quad \text{and} \quad P_{(1)}(t) \sim \frac{c}{t^2}.$$

An induction argument along the lines of Claim V establishes that,
for large t ,

$$(2.48) \quad P_{(n)}(t) \sim \frac{c}{t^2}.$$

Note that for $n \geq 0$,

$$(2.49) \quad P(t) - P_{(n+1)}(t) = \int_0^t h'(1-Q(t-u))(P(t-u) - P_{(n)}(t-u))dG(u).$$

Denote

$$(2.50) \quad \Delta_n(t) \equiv P(t) - P_{(n)}(t),$$

then from (2.49),

$$(2.51) \quad |\Delta_{n+1}(t)| < \int_0^t |\Delta_n(t-u)|dG(u) < |\Delta_0|^* G_{n+1}(t),$$

where $*$ denotes convolution, and $G_n(t)$ is the n^{th} convolution of G with itself.

Let $\{X_i\}$ denote I.I.D. random variables, with distribution function $G(t)$.

Set

$$(2.52) \quad S_n \equiv \sum_{i=1}^n X_i.$$

Then

$$(2.53) \quad G_n(t) \equiv P[S_n \leq t] = P[S_n - n \leq t - n],$$

and if one takes $n > t$, by (1.2) and Chebyshev's inequality,

$$(2.54) \quad G_n(t) \leq \frac{\text{Var } S_n}{(n-t)^2} = \frac{n}{(n-t)^2}.$$

Also,

$$(2.55) \quad |\Delta_o| * G_n(t) \leq G(t) - G(t/2) + \int_0^{t/2} |P(t-u) - R(t-u)| dG_n(u)$$

and (1.2), Claims III, IV, and (2.54) yields that

$$(2.56) \quad |\Delta_o| * G_n(t) \leq \frac{Kn}{t^{2-\epsilon}(n-t)^2},$$

where K is a constant.

For fixed large t , let

$$(2.57) \quad n > [t^{1+\epsilon}].$$

This suffices for the proof of the theorem for $P(t) \equiv P[Z(t) = 1]$.

For $k \geq 2$, note that

$$(2.58) \quad k!P[Z(t)=k] \equiv k!P_k(t) = \left. \frac{\partial^k F(s,t)}{\partial s^k} \right|_{s=0}.$$

Then (2.58) applied to (2.3) for $k = 2$ yields

$$(2.59) \quad P_2(t) = \frac{1}{2} \int_0^t h''(1-Q(t-u)) P^2(t-u) dG(u) + \int_0^t h'(1-Q(t-u)) P_2(t-u) dG(u).$$

Since the theorem is true for $P(t)$, then as (2.59) is of the same form as (2.8), a similar analysis as that used to establish the result for $P(t)$ also establishes it for $P_2(t)$. Assume the result true for $P_k(t)$, $k \leq n$. Applying (2.58) to (2.3) for $k = n+1$ yields an equation of the form

$$(2.60) \quad P_{n+1}(t) = f_{n+1}(t) + \int_0^t h'(1-Q(t-u))P_{n+1}(t-u)dG(u)$$

where the induction hypothesis yields that

$$(2.61) \quad f(t) = o(t^{-4}).$$

In view of (2.61), equation (2.60) is of the same form as (2.8), and the analysis for $P(t)$ applies to $P_{n+1}(t)$, yielding the theorem.

Remark. This result is known for the critical discrete-time or Galton-Watson process. See [2] Ch. 1, for example. It may be possible that a series of comparison sequences of the type given in (2.44), (2.45) relating the critical Bellman-Harris process with a corresponding critical Galton-Watson process could yield this result.

3. Total Progeny.

For another application of the method, let

(3.1) $N(t)$ = total number of progeny born by t in a critical age-dependent branching process with $\infty > h'(1) \equiv \sum_{k=1}^{\infty} kp_k > 0$.

Denote, for $0 \leq s \leq 1$, $t \geq 0$

$$(3.2) \quad H(s, t) \equiv E(s^{N(t)}) = \sum_{k=1}^{\infty} P[N(t)=k]s^k.$$

Then (e.g. [4], p. 394)

$$(3.3) \quad H(s, t) = s \left[1 - G(t) + \int_0^t h(H(s, t-u)) dG(u) \right].$$

Since, for $k \geq 1$,

$$(3.4) \quad \left. \frac{\partial^k H(s, t)}{\partial s^k} \right|_{s=0} = k! P[N(t)=k] \equiv k! Q_k(t)$$

one may apply (3.4) to (3.3) to obtain the limiting values of $P[N(t)=k] \equiv Q_k(t)$.

For $k = 1, 2$ one obtains, since $H(0, t) \equiv 0$, that

$$(3.5) \quad Q_1(t) = 1 - G(t) + \int_0^t h(0) dG(u) \rightarrow p_0 \quad \text{as } t \rightarrow \infty.$$

$$(3.6) \quad Q_2(t) = \int_0^{t/2} + \int_{t/2}^t h'(0) Q_1(t-u) dG(u) \rightarrow p_0 p_1 \quad \text{as } t \rightarrow \infty.$$

Similarly, by induction and an application of Leibniz' lemma for successive differentiation, the limiting values of the $\{Q_k(t)\}$ may be obtained. This result is implied by a representation essentially as that in [1], pp. 275-276. No assumptions on higher moments of G are required, but all moments of $h(s)$ are needed in this approach.

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* in the limit as t approaches infinity t squared $P[...$

