IDENTIFICATION OF PARAMETERS BY THE DISTRIBUTION OF A MAXIMUM RANDOM VARIABLE

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Identification of Parameters by the Distribution of a Maximum Random Variable

by

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1. Introduction and Summary.

Suppose $X_1, \ldots, X_n$ are independently distributed according to $N(\mu_1, \sigma_1^2), \ldots, N(\mu_n, \sigma_n^2)$, respectively. Does the distribution of the maximum of $X_1, \ldots, X_n$ uniquely determine $(\mu_1, \sigma_1^2), \ldots, (\mu_n, \sigma_n^2)$? In other terms, can more than one set of independent normal variables give rise to the same distribution of the random variable which is the maximum of the constituent random variables? Our paper answers this question; in fact, the relevant theorem answers this question about identification of parameters on the basis of the maximum of a set of random variables for a more general family of parent distributions.

Our attention was called to this problem by an inquiry from an econometrician (Professor Takeshi Amemiya, Department of Economics, Stanford University). In an econometric model X_1 is defined as the quantity of a commodity which consumers would be willing to purchase under specified circumstances including the price of a unit, and X_2 is the quantity that would be sold by the producers under certain conditions. A simple stochastic model specifies the demand and supply schedules as

$$X_1 = \alpha_1 + \beta_1 p + u_1 \ , \tag{1.1}$$

$$X_2 = \alpha_2 + \beta_2 p + u_2 \ , \tag{1.2}$$

where p is the price of the commodity and u_1 and u_2 are independent (unobservable) normal random variables with means 0 and variances σ_1^2 and σ_2^2, respectively. If the price is set independently of the market (for example, by an outside agency or by custom), then the quantity actually sold by the producers to the consumers at a given price p is the minimum of X_1 and X_2 given by (1.1) and (1.2). Since the econometrician wishes to determine the demand and supply schedules, he raises the question of whether observations on the quantity passing from producers to consumers at various prices determine the parameters of the model: the intercepts α_1 and α_2, the slopes β_1 and β_2, and the variances σ_1^2 and σ_2^2. Since the normal distribution is symmetric, the question is mathematically equivalent to the question posed in terms of the maximum of X_1 and X_2. (In the more customary equilibrium model there is a third equation equating demand to supply, $X_1 = X_2$. This model, in which $X_1 = X_2$ and p are observable random variables, represents a market in which the price adjusts so that at that price the quantity X_1 that buyers demand is equal to the quantity X_2 that sellers produce.)

In this paper we have studied the question in greater generality by considering the maximum of an arbitrary number of random variables.

Intuitively, the approach is that if the variance of one of $X_1, \ldots, X_n$ is larger than the variance of the others the distribution (or density) of the maximum in the upper tail is similar to the distribution (or density) in the upper tail of the component with maximum variance. It is therefore convenient for us to prove identifiability on the basis of a general condition on the upper tails of the densities in a certain family (Section 2). As applications of this general theorem we answer the question posed in the first paragraph in the affirmative and also assure our econometrician that the parameters in the model are identified.

It is natural to ask in what way this property of identification carries over to multivariate cases. In the econometric example a set of consumers may consider purchasing a number of commodities and a set of producers may propose to sell these commodities; for each commodity the quantity actually sold from producer to consumer at a given price is the minimum of the amount desired at that price and the amount offered by the producers at that price.

We study the multivariate case within the framework of the normal distribution, the parameters of which are the means of the variables, the variances, and the correlations between the variables. The maximum of the vector variable is defined as the vector of the maxima of the respective elements of the constituent vector variables. Consideration of the marginal distribution of the maximum of a particular coordinate of the constituent vector variables determines the set of means and variances of that coordinate (by Theorem 2.1). However, the one-to-one correspondence between the means and variances of two different coordinates of the constituent vector variables is not available from study of the several marginal distributions. Moreover, identification of the correlation coefficients in the distributions of the vector variable cannot be made from one-dimensional distributions.

In Section 3 we treat the multivariate normal distribution where all means are zero and all the correlations are nonnegative; in this case there is identification wherever the correlation is not zero. In Section 5 we treat the case of two vector random variables; then there is complete identification. In order to treat this case we develop a generalization of Mills' ratio, which is of interest in its own right. This is given in Section 4.

2. The Univariate Case.

Theorem 2.1. Let $\mathfrak{F}$ *be a family of probability density functions* f *on* R_1 *which are continuous and positive to the right of some point* A *and such that if* f *and* g *are any two distinct members of* $\mathfrak{F}$ *then* $\lim_{x\to\infty}\{f(x)/g(x)\}$ *exists and equals either* 0 *or* ∞ . *Let* $X_1, \ldots, X_n$ *be independent random variables with respective pdf's* $f_1, \ldots, f_n$ *in* $\mathfrak{F}$, *and* $Y_1, \ldots, Y_n$ *be independent random variables with pdf's* $\varepsilon\ \mathfrak{F}$. *If* $\max\{X_1, \ldots, Y_m\}$ *and* $\max\{Y_1, \ldots, Y_n\}$ *have identical distributions, then* $m = n$ *and there exists a permutation* $\{k_1, \ldots, k_m\}$ *of* $\{1, \ldots, m\}$ *such that the pdf of* Y_i *is* f_{k_i} , $i = 1, \ldots, m$.

Proof. By definition

$$(2.1)\qquad \Pr[\max\{X_1, \ldots, X_m\} \le x] = \prod_{i=1}^{m} \Pr\{X_i \le x\} = \prod_{i=1}^{m} F_i(x) \ ,$$

where $F_i(x) = \int_{-\infty}^{x} f_i(u)du$, and similarly,

$$(2.2)\qquad \Pr[\max\{Y_1, \ldots, Y_n\} \le x] = \prod_{i=1}^{n} G_i(x) \ ,$$

where $G_i(x) = \int_{-\infty}^{y} g_i(u)du$, and $g_1, \ldots, g_n\ \varepsilon\ \mathfrak{F}$. We are given

$$(2.3)\qquad \prod_{i=1}^{m} F_i(x) = \prod_{i=1}^{n} G_i(x) \ , \qquad -\infty < x < \infty \ .$$

Hence,

$$(2.4) \qquad \sum_{i=1}^{m} \ln F_i(x) = \sum_{i=1}^{n} \ln G_i(x) .$$

Differentiating with respect to x, we have for all $x > A$

$$(2.5) \qquad \sum_{i=1}^{m} f_i(x)/F_i(x) = \sum_{i=1}^{n} g_i(x)/G_i(x) .$$

By changing notation we can rewrite (2.5) as

$$(2.6) \qquad \sum_{i=1}^{m+n} a_i f_i(x)/F_i(x) = 0 ,$$

where $a_i = 1$, $i = 1,\ldots,m$, and $a_i = -1$, $i = m+1,\ldots,m+n$, and $f_{m+i} = g_i$, $i = 1,2,\ldots,n$.

There exists an f_i, say f_1, such that $\lim_{x\to\infty}[f_i(x)/f_1(x)]$ is either 0 or 1 for $i = 2,\ldots,m+n$. Let $I = \{i : f_i(x)/f_1(x) \to 1 \text{ as } x \to \infty\}$. Then dividing (2.6) by $f_1(x)$ and letting $x \to \infty$, we have $\Sigma_{i\varepsilon I}\, a_i = 0$, so that I contains an even number of elements, half of which are from $\{1, \ldots, m\}$. Thus a certain number of f_i in (2.5) are identical and are identical to the same number of g_i. Subtracting these identical terms from both sides of (2.5), we have a new equation of the same form but with fewer terms. The process can be iterated until each term on one side of (2.5) is matched with one term on the other. If $m = n$, the proposition is proved. On the other hand if $m \neq n$, say $m < n$, we have $n - m$ of the g_i such that $g_i(x) = 0$ for all $x > A$, contrary to the assumption about $\mathfrak{F}$. Hence, $m = n$. Q.E.D.

Example 2.1. The family of normal densities with arbitrary means and variances satisfies the assumptions of the theorem. In fact, if

$$\phi(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] , \tag{2.7}$$

then as $x \to \infty$

$$\frac{\phi(x|\mu_2,\sigma_2)}{\phi(x|\mu_1,\sigma_1)} \to \begin{cases} 0 \text{ if } \sigma_1^2 > \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 > \mu_2 , \\ \infty \text{ if } \sigma_1^2 < \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 < \mu_2 , \\ 1 \text{ if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 = \mu_2 . \end{cases} \tag{2.8}$$

Example 2.2. Let

$$\phi(x|\alpha,\beta,\sigma,p) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\alpha-\beta p)^2}{2\sigma^2}\right] . \tag{2.9}$$

Then as $x \to \infty$

(2.10)

$$\frac{\phi(x|\alpha_2,\beta_2,\sigma_2,p_2)}{\phi(x|\alpha_1,\beta_1,\sigma_1,p_1)} \to \begin{cases} 0 \text{ if } \sigma_1^2 > \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 > \alpha_2 + \beta_2 p_2 , \\ \infty \text{ if } \sigma_1^2 < \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 < \alpha_2 + \beta_2 p_2 , \\ 1 \text{ if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 = \alpha_2 + \beta_2 p_2 . \end{cases}$$

The parameters are identified if at least two of $X_1, \ldots, X_n$ have densities with the same pair α, β and different values of p. (In the econometric "equilibrium" model $X_1 = X_2$ and p are random variables determined as the solution of (1.1) and (1.2); any joint normal density of these two variables is consistent with various sets of β_1, β_2, σ_1^2 and σ_2^2 and hence the parameters cannot be identified.)

Example 2.3. Let $\phi_\lambda(x) = 0$, $x < 0$, $\phi_\lambda(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Then

$$\text{(2.11)} \qquad \frac{\phi_{\lambda_2}(x)}{\phi_{\lambda_1}(x)} \to \begin{cases} 0 & \text{if } \lambda_1 < \lambda_2 , \\ \infty & \text{if } \lambda_1 > \lambda_2 , \\ 1 & \text{if } \lambda_1 = \lambda_2 . \end{cases}$$

There exist many families of pdf's which do not have the property postulated in the theorem but the members of which are identified by the distribution of the maximum random variable.

Example 2.4. Let

$$\text{(2.12)} \qquad f_a(x) = \begin{cases} e^{-(x-a)}, & x \geq a , \\ 0 , & x < a . \end{cases}$$

If $f_i = f_{a_i}$, $i = 1,\dots,m$, $g_i = f_{b_i}$, $i = 1,\dots,n$, then

$$\text{(2.13)} \qquad \Pr[\max\{X_i, i = 1,\dots,m\} \leq \min\{a_i, i = 1,\dots,m\}] = 0$$

implies

(2.14) $$\min\{a_1, \ldots, a_m\} = \min\{b_1, \ldots, b_n\} .$$

Further, the form of the cdf of the maximum for values of x between the smallest and second smallest distinct a_i tells us that the number of $a_i = \min\{a_i\}$ is the same as the number of $b_i = \min\{a_i\}$; and also that the second smallest a_i = the second smallest b_i . Proceeding this way, we reach the conclusion that $m = n$ and $\{a_1, \ldots, a_m\}$ is a permutation of $\{b_1, \ldots, b_m\}$.

Example 2.5. Let

(2.15) $$f_a(x) = \frac{1}{2}e^{-|x-a|} .$$

In this case, if $f_i = f_{a_i}$ and $g_i = f_{b_i}$, then for all sufficiently large x , equation (2.3) becomes

(2.16) $$\prod_{i=1}^{m}\left(1 - \frac{1}{2}e^{-x+a_i}\right) = \prod_{i=1}^{n}\left(1 - \frac{1}{2}e^{-x+b_i}\right) .$$

Without loss of generality, we may assume $m \leq n$, write $z = e^x$, and multiply both sides of (2.16) by z^n , giving

(2.17) $$z^{n-m}\prod_{i=1}^{m}\left(z - \frac{1}{2}e^{a_i}\right) = \prod_{i=1}^{n}\left(z - \frac{1}{2}e^{b_i}\right) .$$

Equating the zeros of the two polynomials yields the conclusion that $m = n$ and $\{a_1, \ldots, a_m\}$ is a permutation of $\{b_1, \ldots, b_m\}$.

There are also examples of a family of densities whose elements are not identified by the distribution of the maximum. For example, let

$$f_a(x) = \begin{cases} ae^{ax}, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{2.18}$$

and $\mathfrak{F} = \{f_a : a > 0\}$. If $f_i = f_{a_i}$, $i = 1,\ldots,m$, the cdf of $\max\{X_i\}$ is

$$\begin{cases} e^{x\Sigma_{i=1}^m a_i}, & x \leq 0, \\ 1, & x > 0, \end{cases} \tag{2.19}$$

and the only inference from (1) is $\Sigma_{i=1}^m a_i = \Sigma_{i=1}^n b_i$.

3. A Special Multivariate Normal Case.

Suppose the vector $\underset{\sim}{X}_i = (X_{1i}, \ldots, X_{pi})'$ has the distribution $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i)$, $i = 1,\ldots,n$. Let $Y_j = \max(X_{j1}, \ldots, X_{jp})$ and let $\underset{\sim}{Y} = (Y_1, \ldots, Y_p)'$. Does the distribution of $\underset{\sim}{Y}$ determine the parameters $\underset{\sim}{\mu}_1, \underset{\sim}{\Sigma}_1, \ldots, \underset{\sim}{\mu}_n, \underset{\sim}{\Sigma}_n$? Consideration of the marginal distribution shows that the distribution of Y_j determines the values of the pairs $(\mu_{j1}, \sigma_{j1}^2), \ldots, (\mu_{jn}, \sigma_{jn}^2)$ for each j. (The j,k^{th} element of $\underset{\sim}{\Sigma}_i$ is $\sigma_{ji}\sigma_{ki}\rho_{jki}$.) But, the marginal distributions do not lead to correspondences between the different values of j. However, if bivariate distributions are identified, then the correspondences for $j = 1,\ldots,p$ can be made. Accordingly we turn our attention to bivariate distributions.

We consider an arbitrary number n of bivariate normal distributions but restrict them so as to minimize the kind of tedious computations which we are not able to avoid in Section 5 in the detailed discussion of the case $n = 2$.

Theorem 3.1. *If $\Phi_1, \ldots, \Phi_n, F_1, \ldots, F_m$ are nonsingular bivariate normal cdf's with zero means and the correlations in $\Phi_1, \ldots, \Phi_n$ are all nonnegative, then*

$$\prod_{i=1}^{n} \Phi_i(x,y) = \prod_{i=1}^{m} F_i(x,y) \tag{3.1}$$

implies that $m = n$, there are no F_i with negative correlations, the number of zero correlations is the same on both sides of (3.1), and the F_i with positive correlations can be identified one-for-one with the Φ_i having positive correlations.

Proof. By letting one variable go to infinity in the identity (3.1), we see from the one-dimensional result of Example 2.1 that the set of x-variances on the l.h.s. of (3.1) is a permutation of that on the r.h.s.; and the same is true of the y-variances. Also, $m = n$. Therefore, if the x-variance, correlation and the y-variance in Φ_i are respectively σ_i^2, ρ_i, τ_i^2, then we may assume, without loss of generality, that the corresponding parameters for F_i are σ_i^2, r_i, t_i^2, where $(t_1, \ldots, t_n)$ is a permutation of $(\tau_1, \ldots, \tau_n)$.

Taking logarithms in (3.1) and differentiating with respect to x, we get

$$(3.2) \qquad \sum_{i=1}^{n} \frac{1}{\sigma_i} n\left(\frac{x}{\sigma_i}\right) N\left[\left(\frac{y}{\tau_i} - \frac{\rho_i x}{\sigma_i}\right)\Big/\sqrt{1-\rho_i^2}\right] \Big/ \Phi_i(x,y)$$

$$= \sum_{i=1}^{n} \frac{1}{\sigma_i} n\left(\frac{x}{\sigma_i}\right) N\left[\left(\frac{y}{t_i} - \frac{r_i x}{\sigma_i}\right)\Big/\sqrt{1-r_i^2}\right] \Big/ F_i(x,y) ,$$

where $n(\cdot)$ and $N(\cdot)$ are, respectively, the one-dimensional standard normal pdf and cdf.

Now, let $\sigma = \max\{\sigma_i, i = 1,\ldots,n\}$, $I = \{i : \sigma_i = \sigma\}$, $I_0(\rho) = \{i : \rho_i = 0, i \in I\}$, $I_0(r) = \{i : r_i = 0, i \in I\}$. On dividing (3.2) by $(1/\sigma)n(x/\sigma)$ and letting $x \to \infty$, we get

$$(3.3) \qquad \sum_{i\in I_0(\rho)} 1 = \sum_{i\in I_0(r)} 1 + \sum_{\{i:i\in I, r_i<0\}} 1/N(y/t_i) .$$

Since the last term is strictly monotone unless $\{i : i \in I, r_i < 0\}$ is empty, we see that there is no $r_i < 0$ for $i \in I$, and that

$I_0(\rho)$ and $I_0(r)$ contain the same number of points. Hence, the terms corresponding to $(\sigma = \sigma_i, \rho_i = 0)$ and $(\sigma = \sigma_i, r_i = 0)$ cancel one another out in (3.2).

Next, let $\beta = \min\{\rho_i\tau_i : i \varepsilon I - I_0(\rho)\}$, and $I_1(\rho) = \{i : \rho_i\tau_i = \beta, i \varepsilon I\}$, $I_1(r) = \{i : r_i t_i = \beta, i \varepsilon I\}$. In (3.2), let $y = \beta x/\sigma + u$, divide by $1/\sigma\ n(x/\sigma)$ and let $x \to \infty$, holding u fixed. Then we obtain

$$(3.4) \qquad \sum_{i\varepsilon I_1(\rho)} N\left(\frac{u}{\tau_i\sqrt{1-\rho_i^2}}\right) = \sum_{i\varepsilon I_1(r)} N\left(\frac{u}{t_i\sqrt{1-r_i^2}}\right) + \sum_{\{i:r_it_i<\beta,i\varepsilon I\}} 1 \ .$$

Letting $u \to \infty$, we see that $\{i : r_it_i < \beta, i \varepsilon I\}$ is empty. Differentiating with respect to u in (3.4) we get

$$(3.5) \qquad \sum_{i\varepsilon I_1(\rho_1)} \frac{1}{\sqrt{\tau_i^2-\beta^2}}\, n\left(\frac{u}{\sqrt{\tau_i^2-\beta^2}}\right) = \sum_{i\varepsilon I_1(r)} \frac{1}{\sqrt{t_i^2-\beta^2}}\, n\left(\frac{u}{\sqrt{t_i^2-\beta^2}}\right) \ .$$

Let $\tau = \max\{\tau_i, t_j : i \varepsilon I_1(\rho), j \varepsilon \varepsilon I_1(r)\}$, divide by $n\left(u/\sqrt{\tau^2-\beta^2}\right)$ and let $u \to \infty$. We see that the number of $\tau_i = \tau$ is the same as that of $t_i = \tau$, and that for the corresponding Φ_i and F_i , we have $\rho_i = \beta/\tau = r_i$.

Eliminating the terms with $\tau_i = t_i = \tau$ from (3.5), we can repeat the process with the remaining largest value of the y-variance; and continuing this way, we see that we have identified $\{(\sigma_i, \rho_i, \tau_i), i \varepsilon I_1(\rho)\}$ with $\{(\sigma_i, r_i, t_i), i \varepsilon I_1(r)\}$ one-for-one. Eliminating these terms from (3.2), we can now repeat the process with the next larger value of $\rho_i \tau_i$, $i \in I - I_0(\rho)$, and continuing this way, we are able to identify $\{(\sigma_i, \rho_i, \tau_i), i \in I - I_0(\rho)\}$ with $\{(\sigma_i, r_i, t_i), i \in I - I_0(r)\}$ one-for-one. We are then left with an equation similar to (3.2) but with a smaller number of terms and with $\sigma_i < \sigma$. Iterating the procedure successively, we are thus able to identify $\{(\sigma_i, \rho_i, \tau_i) : \rho_i > 0, i = 1,\ldots,n\}$ with $\{(\sigma_i, r_i, t_i) : r_i > 0, i = 1,\ldots,n\}$ one-for-one.

Finally, removing these identical terms from both sides of (3.1), we are left with products of one-dimensional normal cdfs in x and y with the same set of x-variances appearing on both sides and also the same set of y-variances, but there is no unique matching of (x,y) pairs among these.

4. A Bivariate Mills' Ratio.

For the investigation in Section 5 we need a generalization of the univariate Mills' ratio

$$(4.1) \qquad n(x)\left(\frac{1}{x} - \frac{1}{x^3}\right) < 1 - N(x) < n(x)\frac{1}{x} \;, \qquad x > 0 \;,$$

where $n(\cdot)$ and $N(\cdot)$ are the standard normal pdf and cdf. (See, e.g., Feller [1], p. 175.)

Theorem 4.1. Let

$$(4.2) \qquad \bar{\Phi}(x,y) = \int\limits_{\substack{u > x \\ v > y}} \phi(u,v)\, du dv \;,$$

where

$$(4.3) \qquad \phi(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{u^2 - 2\rho\, uv + v^2}{2(1-\rho^2)}\right] .$$

If $\rho > 0$, $A = (x-\rho y)/\sqrt{1-\rho^2} > 0$, $B = (y-\rho x)/\sqrt{1-\rho^2} > 0$, then

$$(4.4) \qquad \phi(x,y)\frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)}\left(1 - \frac{1}{A^2}\right)\left(1 - \frac{1}{B^2}\right) \le \bar{\Phi}(x,y) \le \phi(x,y)\frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} .$$

If $\rho \le 0$, $A > 0$, $B > 0$, then

$$(4.5) \qquad \phi(x,y)\frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)}\left(1 - \frac{1-\rho}{A^2}\right)\left(1 - \frac{1-\rho}{B^2}\right) \le \Phi(x,y) \le \phi(x,y)\frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} .$$

Proof:

$$(4.6) \qquad \Phi(x,y) = \int_{s,t>0} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{(x+s)^2-2\rho(x+s)(y+t)+(y+t)^2}{2(1-\rho^2)}\right] dsdt$$

$$= \phi(x,y) \int_{s,t>0} \exp\left[-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right] \exp\left[-\frac{s(x-\rho y)+t(y-\rho x)}{(1-\rho^2)}\right] dsdt$$

$$= \phi(x,y)(1-\rho^2)\, J\,(A,B)\ ,$$

where

$$(4.7) \qquad J(A,B) = \int_{u,v>0} \exp\left[-\frac{u^2-2\rho uv+v^2}{2}\right] \exp[\,-\,Au\,-\,Bv]dudv\ .$$

We have (for $u \geq 0$, $v \geq 0$)

$$(4.8) \qquad (1-\rho)(u^2+v^2) \leq u^2 - 2\rho uv + v^2 \leq u^2 + v^2\ , \qquad \rho > 0\ ,$$

$$(4.9) \qquad u^2 - 2\rho uv + v^2 = u^2 + v^2\ , \qquad \rho = 0\ ,$$

$$(4.10) \qquad u^2 + v^2 \leq u^2 - 2\rho uv + v^2 \leq (1-\rho)(u^2+v^2)\ , \qquad \rho < 0\ .$$

Let

$$(4.11) \qquad J_a = \int_{u,v>0} \exp\left[-\frac{a^2(u^2+v^2)}{2}\right] \exp(-\,Au\,-\,Bv)dudv\ .$$

Then for $a^2 = 1 - \rho$

$$(4.12) \qquad J_1 \leq J\,(A,B) \leq J_a\,, \qquad \rho > 0\,,$$

$$(4.13) \qquad J\,(A,B) = J_1\,, \qquad \rho = 0\,,$$

$$(4.14) \qquad J_a \leq J\,(A,B) \leq J_1\,, \qquad \rho < 0\,.$$

Note that J_a is the product of two terms of the type

$$(4.15) \qquad \int_{u>0} \exp\left[-\frac{a^2u^2 + 2cu}{2}\,du\right],$$

where $c = A$ or B. The expression (4.15) equals

$$(4.16) \quad \exp\left[\frac{c^2}{2a^2}\right] \int_{u>0} \exp\left[-\frac{(au+\frac{c}{a})^2}{2}\right]du = \exp\left[\frac{c^2}{2a^2}\right] \frac{\sqrt{2\pi}}{a} \int_{v>\frac{c}{a}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)dv\,.$$

If $c > 0$, this is bounded above and below, respectively, by $1/c$ and $(1/c)(1-a^2/c^2)$ according to (4.1). For $\rho = 0$ the theorem follows directly from (4.1) because $\bar{\Phi}(x,y) = \bar{\Phi}(x)\bar{\Phi}(y)$. For $\rho > 0$ and $\rho < 0$ the theorem follows from (4.6), (4.12), and (4.14). Q.E.D.

Remarks. Notice that the conditions $A > 0$, $B > 0$ for $x > 0$, $y > 0$ correspond to a point (x,y) in the first quadrant that is above the regression line (Y on X) and to the right of the other (X on Y). If $\rho \leq 0$, then $A > 0$ and $B > 0$ are automatically satisfied if $x,y > 0$.

It is of interest to compare Theorem 2 with the bound of Harkness and Godambe ([2]). The principal term in our estimate (which is also our upper bound) is

$$(4.17) \qquad \phi(x,y)\frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} ,$$

subject to the assumption that $x - \rho y$, $y - \rho x > 0$ (which is a restriction only if $\rho > 0$). Since (4.17) is symmetric in (x,y), let $y = cx$, with $1 \leq c < 1/\rho$ if $\rho > 0$. Then (4.17) is

$$(4.18) \qquad \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2(1-2\rho c+c^2)}{2(1-\rho^2)}\right] \frac{(1-\rho^2)^2}{x^2(1-\rho c)(c-\rho)} .$$

In the Harkness and Godambe estimates, the principal terms in the upper and lower bounds are, respectively,

$$(4.19) \quad \frac{(1+\rho)^2}{x^2} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2(1-2\rho+1)}{2(1-\rho^2)}\right] = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{(1+\rho^2)}{x^2} \exp\left[-\frac{x^2 2(1-\rho)}{2(1-\rho^2)}\right] ,$$

and

(4.20)

$$\frac{(1+\rho)^2}{c^2x^2} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2c^2(1-2\rho+1)}{2(1-\rho^2)}\right] = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{(1+\rho)^2}{c^2x^2} \exp\left[-\frac{x^2c^2 2(1-\rho)}{2(1-\rho^2)}\right] .$$

Now, (4.18) = (4.19) = (4.20) if $c = 1$, but for $1 < c < 1/\rho$, (4.18)/(4.19) $\to 0$ and (4.20)/(4.18) $\to 0$ as $x \to \infty$.

Finally, if $\rho > 0$ and $y > x/\rho$, we can still get quite good bounds from (4.4) by using $\bar{\Phi}(x',y) \leq \bar{\Phi}(x,y) \leq \bar{\Phi}(x,y')$ where $x < x' < y$ and $x < y' < y$, which is better than using $\bar{\Phi}(y,y) \leq \bar{\Phi}(x,y) \leq \bar{\Phi}(x,x)$.

Since the denominator in the last factor in (4.18) is maximized by choosing $c = (\rho + 1/\rho)/2$, $\bar{\Phi}(x,y')$, with $y' = [(\rho^2+1)/(2\rho)]x$, might be a reasonably good upper bound for $\bar{\Phi}(x,y)$ when $y > x/\rho$, $\rho > 0$; similarly, $\bar{\Phi}(x',y)$, with $x' = [2\rho/(\rho^2+1)]y$ might be a reasonably good upper bound. In the case of a general bivariate normal distribution, we have

$$(4.21) \qquad \int_a^\infty \int_b^\infty \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}} \exp\left[\frac{\frac{(s-\mu)^2}{\sigma^2} - 2\rho\frac{(s-\mu)(t-\nu)}{\sigma\tau} + \frac{(t-\nu)^2}{\tau^2}}{2(1-\rho^2)}\right] dt\ ds$$

$$= \int_{\frac{a-\mu}{\sigma}}^\infty \int_{\frac{b-\nu}{\tau}}^\infty \phi(u,v)dv\ du$$

$$= \bar{\Phi}\left[\frac{a-\mu}{\sigma}, \frac{b-\nu}{\tau}\right]$$

Corollary 4.1. Let (4.21) be denoted by $\bar{\Phi}(a,b;\ \mu,\nu,\sigma,\tau,\rho)$, _let the integrand be denoted by_ $\phi(s,t;\ \mu,\nu,\sigma,\tau,\rho)$, _and let_

$$(4.22) \qquad A = \frac{\frac{a-\mu}{\sigma} - \rho\frac{b-\nu}{\tau}}{\sqrt{1-\rho^2}} , \quad B = \frac{\frac{b-\nu}{\tau} - \rho\frac{a-\mu}{\sigma}}{\sqrt{1-\rho^2}} .$$

_If_ $\rho > 0,\ A > 0,\ B > 0,$ _then_

$$(4.23) \qquad \left(1 - \frac{1}{A^2}\right)\left(1 - \frac{1}{B^2}\right) < \frac{\bar{\Phi}(a,b;\ \mu,\nu,\sigma,\tau,\rho)AB}{\sigma\tau\phi(a,b;\ \mu,\nu,\sigma,\tau,\rho)(1-\rho^2)} < 1 .$$

_If_ $\rho \leq 0,\ A > 0,\ B > 0,$ _then_

$$(4.24)\qquad \left(1-\frac{1-\rho}{A^2}\right)\left(1-\frac{1-\rho}{B^2}\right) < \frac{\bar{\Phi}(a,b;\ \mu,\nu,\sigma,\tau,\rho)AB}{\sigma\tau\phi(a,b;\ \mu,\nu,\sigma,\tau,\rho)(1-\rho^2)} < 1\ .$$

<u>Corollary 4.2</u>. <u>Let</u> $y-\nu = c(x-\mu)$, <u>where</u> c <u>is a positive constant</u>. <u>Then, as</u> $x \to \infty$, <u>we have</u>

$$(4.25)\qquad \frac{\bar{\Phi}(x,cx;\ \mu,\nu,\sigma,\tau,\rho)(x-\mu)^2(\tau-c\rho\sigma)(c\sigma-\rho\tau)}{\sigma^3\tau^3\ \phi(x,cx;\ \mu,\nu,\sigma,\tau,\rho)(1-\rho^2)^2} \to 1$$

<u>for all</u> $c > 0$ <u>if</u> $\rho \le 0$ <u>and for</u> $c \in (\rho\tau/\sigma,\ \tau/[\rho\sigma])$ <u>if</u> $\rho > 0$.

5. The General Case of Two Multivariate Normal Distributions.

As was explained in Section 3, the solution of identifiability for the multivariate normal case depends on the solution for the bivariate normal case. We treat the case of $n = 2$ in terms of the notation of Section 3.

Theorem 5.1. *If Φ_1, Φ_2, F_1, F_2 are bivariate normal cdf's such that*

$$\Phi_1(x,y)\ \Phi_2(x,y) = F_1(x,y)\ F_2(x,y) \ , \tag{5.1}$$

then one of the following relations holds:

(i) $\Phi_1 = F_1$ *and* $\Phi_2 = F_2$, *or*

(ii) $\Phi_1 = F_2$ *and* $\Phi_2 = F_1$, *or*

(iii) $\Phi_i(x,y) = A_i(x)\ B_i(y)$, $i = 1,2$, *and*

$$F_1(x,y) = A_1(x)\ B_2(y),\ F_2(x,y) = A_2(x)\ B_1(y) \ .$$

Proof. We shall give a detailed proof under the assumption that the means are all zero. The proof for the general case of arbitrary means is not different in spirit, but only involves additional tedious computations of the same kind. As in Section 3, we shall assume the parameters to be $(\sigma_i^2, \rho_i, \tau_i^2)$, $i = 1,2$, on the lhs of (5.1) and (σ_i^2, r_i, t_i^2), $i = 1,2$, on the rhs. From (5.1) we obtain

$$(5.2) \qquad g(x,y) = \frac{\partial^2}{\partial x \partial y} [\Phi_1(x,y)\ \Phi_2(x,y)]$$

$$= \phi_1\Phi_2 + \frac{\partial}{\partial x}\Phi_1 \frac{\partial}{\partial y}\Phi_2 + \frac{\partial}{\partial y}\Phi_1 \frac{\partial}{\partial x}\Phi_2 + \Phi_1\phi_2$$

$$= \phi_1(x,y)\ \phi_2(x,y)$$

$$+ \frac{1}{\tau_1\sigma_2}\, n\left(\frac{y}{\tau_1}\right) n\left(\frac{x}{\sigma_2}\right) N\left[\left(\frac{x}{\sigma_1} - \frac{\rho_1 y}{\tau_1}\right)\Big/\sqrt{1-\rho_1^2}\right] N\left[\left(\frac{y}{\tau_2} - \frac{\rho_2 x}{\sigma_2}\right)\Big/\sqrt{1-\rho_2^2}\right]$$

$$+ \frac{1}{\sigma_1\tau_2}\, n\left(\frac{x}{\sigma_1}\right) n\left(\frac{y}{\tau_2}\right) N\left[\left(\frac{y}{\tau_1} - \frac{\rho_1 x}{\sigma_1}\right)\Big/\sqrt{1-\rho_1^2}\right] N\left[\left(\frac{x}{\sigma_2} - \frac{\rho_2 y}{\tau_2}\right)\Big/\sqrt{1-\rho_2^2}\right]$$

$$+ \Phi_1(x,y)\ \phi_2(x,y)\ ,$$

where $n(\cdot)$ and $N(\cdot)$ are univariate standard normal pdf and cdf, respectively. Also,

$$(5.3) \qquad f(x,y) = \frac{\partial^2}{\partial x \partial y} [F_1(x,y)\ F_2(x,y)]\ .$$

I. First consider the simple case $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$. By consideration of the marginal distributions we have $\sigma_1 = \sigma_2 = s_1 = s_2$, $\tau_1 = \tau_2 = t_1 = t_2$, so that the problem can be reduced to standard form by scale transformations on x and y. In this case (5.2) and (5.3) give us

$$(5.4) \qquad g_{\underline{\rho}}(x,y) = \frac{1}{2\pi\sqrt{1-\rho_1^2}} \exp\left(-\frac{x^2-2\rho_1 xy+y^2}{2(1-\rho_1^2)}\right) \Phi_2(x,y)$$

$$+ n(x)n(y)\, N\left[\frac{(x-\rho_1 y)}{\sqrt{1-\rho_1^2}}\right] N\left[\frac{(y-\rho_2 x)}{\sqrt{1-\rho_2^2}}\right]$$

$$+ n(x)n(y)\, N\left[\frac{(y-\rho_1 x)}{\sqrt{1-\rho_1^2}}\right] N\left[\frac{(x-\rho_2 y)}{\sqrt{1-\rho_2^2}}\right]$$

$$+ \Phi_1(x,y) \frac{1}{2\pi\sqrt{1-\rho_2^2}} \exp\left(-\frac{x^2-\rho_2 xy+y^2}{2(1-\rho_2^2)}\right)$$

$$= g_{\underline{r}}(x,y) \; .$$

On putting $y = x$, we have

$$(5.5) \qquad \frac{g_{\underline{\rho}}(x,x)}{n^2(x)} = \frac{1}{\sqrt{1-\rho_1^2}} \exp\left(\frac{\rho_1}{1+\rho_1} x^2\right) \Phi_2(x,x) + 2N\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}\, x\right) N\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}}\, x\right)$$

$$+ \frac{1}{\sqrt{1-\rho_2^2}} \exp\left(\frac{\rho_2}{1+\rho_2} x^2\right) \Phi_1(x,x) \; .$$

Suppose $\max(\rho_1, \rho_2, r_1, r_2) = \rho_1 > 0$ and let $r_1 \geq r_2$. Then as $x \to \infty$

$$(5.6)\qquad \frac{g_{\underline{\rho}}(x,x)}{n^2(x)}\Big/\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right) \to \begin{cases} \dfrac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 > \rho_2 , \\ \dfrac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = \rho_2 . \end{cases}$$

On the other hand

$$(5.7)\qquad \frac{g_{\underline{r}}(x,x)}{n^2(x)}\Big/\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right) \to \begin{cases} 0 & \text{if } \rho_1 > r_1 \geq r_2 , \\ \dfrac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = r_1 > r_2 , \\ \dfrac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = r_1 = r_2 , \end{cases}$$

Hence, if $\rho_1 = \rho_2$, then $r_1 = r_2 = \rho_1 = \rho_2$ and $\Phi_1 = \Phi_2 = F_1 = F_2$. If $\rho_1 > \rho_2$, then $r_1 = \rho_1$ and $r_2 = \rho_2$; thus $\Phi_1 = F_1$ and $\Phi_2 = F_2$.

Now suppose $\max(\rho_1, \rho_2, r_1, r_2) = \rho_1 = 0$. If $\rho_2 = 0$, (5.5) $\to 4$; then $g_{\underline{r}}(x,x)/n^2(x) \to 4$, which implies $r_1 = r_2 = 0$. If $\rho_2 < 0$, (5.5) $\to 3$; then $g_{\underline{r}}(x,x)/n^2(x) \to 3$, which implies $r_1 = 0 > r_2 = \rho_2$.

Now suppose $\max(\rho_1, \rho_2, r_1, r_2) = \rho_1 < 0$. Then (5.5) $\to 2$. Consider

$$(5.8)\qquad \frac{g_{\underline{\rho}}(x,x)}{n^2(x)} - 2 = \frac{1}{\sqrt{1-\rho_1^2}}\exp\left(\frac{\rho_1}{1+\rho_1}x^2\right)\Phi_2(x,x)$$

$$+ \frac{1}{\sqrt{1-\rho_2^2}} \exp\left(\frac{\rho_2}{1+\rho_2} x^2\right) \Phi_1(x,x)$$

$$- 2\bar{N}\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}\, x\right) - 2\bar{N}\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}}\, x\right)$$

$$+ 2\bar{N}\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}}\, x\right) N\left(\sqrt{\frac{1-\rho_2}{1-\rho_2}}\, x\right) .$$

By Mills' ratio

$$\bar{N}\left(\sqrt{\frac{1-\rho_i}{1+\rho_i}}\, x\right) \sim \sqrt{\frac{1+\rho_i}{1-\rho_i}}\, \frac{1}{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1-\rho_i}{2(1+\rho_i)} x^2\right) . \tag{5.9}$$

Then as $x \to \infty$

$$\left[\frac{g_{\underline{\rho}}(x,x)}{n^2(x)} - 2\right] \sqrt{1-\rho_1^2} \exp\left(\frac{-\rho_1}{1+\rho_1} x^2\right) \to \begin{cases} 1 & \text{if } \rho_1 > \rho_2 , \\ 2 & \text{if } \rho_1 = \rho_2 . \end{cases} \tag{5.10}$$

Similarly as $x \to \infty$

$$\left[\frac{g_{\underline{r}}(x,x)}{n^2(x)} - 2\right] \sqrt{1-\rho_1^2} \exp\left(\frac{-\rho_1}{1+\rho_1} x^2\right) \to \begin{cases} 0 & \text{if } \rho_1 > r_1 , \\ 1 & \text{if } r_1 = \rho_1 > \rho_2 , \\ 2 & \text{if } r_1 = r_2 = \rho_1 . \end{cases} \tag{5.11}$$

We obtain identification.

II. Having disposed of the case $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$, we now consider the situation where there is at least one inequality. Without loss of generality, we may suppose $\sigma_1 > \sigma_2$. Then in Equation (5.2), if we keep y fixed and let $x \to \infty$, we have

$$(5.12) \qquad \frac{g(x,y)\sigma_1}{n\left(\frac{x}{\sigma_1}\right)} \to \begin{cases} 0 & \text{if } \rho_1 > 0 , \\ \frac{1}{\tau_1} n\left(\frac{y}{\tau_1}\right) N\left(\frac{y}{\tau_2}\right) + \frac{1}{\tau_2} n\left(\frac{y}{\tau_2}\right) N\left(\frac{y}{\tau_1}\right) & \text{if } \rho_1 = 0 , \\ \frac{1}{\tau_2} n\left(\frac{y}{\tau_2}\right) & \text{if } \rho_1 < 0 . \end{cases}$$

In the same way, from (5.3) we get

$$(5.13) \qquad \frac{f(x,y)\sigma_1}{n\left(\frac{x}{\sigma_1}\right)} \to \begin{cases} 0 & \text{if } r_1 > 0 , \\ \frac{1}{t_1} n\left(\frac{y}{t_1}\right) N\left(\frac{y}{t_2}\right) + \frac{1}{t_2} n\left(\frac{y}{t_2}\right) N\left(\frac{y}{t_1}\right) & \text{if } r_1 = 0 , \\ \frac{1}{t_2} n\left(\frac{y}{t_2}\right) & \text{if } r_1 < 0 . \end{cases}$$

Consequently, from (5.1) we see that

(α) $\quad \rho_1 > 0 \Longrightarrow r_1 > 0$,

(β) $\quad \rho_1 = 0 \Longrightarrow r_1 = 0$,

(γ) $\quad \rho_1 < 0 \Longrightarrow r_1 < 0$ and $t_2 = \tau_2$ (so that $t_1 = \tau_1$) .

In Case (α), let $y = \gamma x$, where $\gamma = \rho_1\tau_1/\sigma_1$, and $x \to \infty$; then, noting that $1/\sigma^2 - 2\rho\gamma/(\sigma\tau) + \gamma^2/\tau^2 = (1-\rho^2)/\sigma^2 + (\rho/\sigma - \gamma/\tau)^2$, from (5.2) we obtain

$$(5.14) \qquad \frac{g(x,\gamma x)}{n\left(\frac{x}{\sigma_1}\right)}\sigma_1 \to \frac{1}{\tau_1\sqrt{2\pi(1-\rho_1^2)}} .$$

On the other hand, looking at the expression for (5.3) similar to that for (5.2), we observe that

$$(5.15)\qquad f_i(x,\gamma x) = \frac{1}{\sigma_i t_i\sqrt{2\pi(1-r_i^2)}}\, n\left(\frac{x}{\sigma_i}\right) \exp\left[-\frac{\left(\frac{\gamma}{t_i}-\frac{r_i}{\sigma_i}\right)^2 x^2}{2(1-r_i^2)}\right] ,$$

so that

$$(5.16)\qquad \frac{g(x,\gamma x)}{n\left(\frac{x}{\sigma_1}\right)}\,\sigma_1 \to \begin{cases} 0 & \text{if } \frac{\gamma}{t_1} \neq \frac{r_1}{\sigma_1} , \\ \frac{1}{t_1\sqrt{2\pi(1-r_1^2)}} & \text{if } \frac{\gamma}{t_1} = \frac{r_1}{\sigma_1} . \end{cases}$$

Hence, on account of (5.1), we conclude that

$$(5.17)\qquad \frac{\gamma}{t_1} = \frac{r_1}{\sigma_1} \quad\text{and}\quad t_1^2(1-r_1^2) = \tau_1^2(1-\rho_1^2) .$$

This implies $t_1 = \tau_1$ and $r_1 = \rho_1$. So, in Case (α), $\Phi_1 = F_1$ and hence $\Phi_2 = F_2$.

Next, in Case (β), the original equation (5.1) becomes

$$(5.18)\qquad N\left(\frac{x}{\sigma_1}\right) N\left(\frac{y}{\tau_1}\right) \Phi_2(x,y) = N\left(\frac{x}{\sigma_1}\right) N\left(\frac{y}{t_1}\right) F_2(x,y) .$$

If we now remove the common factor $N(x/\sigma_1)$ from both sides, differentiate with respect to x and remove the common factor $(1/\sigma_2)n(x/\sigma_2)$ from both sides, we are left with

$$(5.19) \qquad N\left(\frac{y}{\tau_1}\right) N\left[\frac{\frac{y}{\tau_2} - \rho_2 \frac{x}{\sigma_2}}{\sqrt{1-\rho_2^2}}\right] = N\left(\frac{y}{t_1}\right) N\left[\frac{\frac{y}{t_2} - r_2 \frac{x}{\sigma_2}}{\sqrt{1-r_2^2}}\right] .$$

If $\rho_2 = 0$, the lhs of (5.19) is independent of x, and hence $r_2 = 0$; in this case, both sides of (5.1) are products of univariate cdfs, and there is no unique matching of (x,y) pairs. On the other hand, if $\rho_2 \neq 0$ then $r_2 \neq 0$; and differentiating (5.19) with respect to x; we have

$$(5.20) \qquad N\left(\frac{y}{\tau_1}\right) \frac{\rho_2}{\sigma_2\sqrt{1-\rho_2^2}}\, n\left[\frac{\frac{y}{\tau_2} - \rho_2 \frac{x}{\sigma_2}}{\sqrt{1-\rho_2^2}}\right] = N\left(\frac{y}{t_1}\right) \frac{r_2}{\sigma_2\sqrt{1-r_2^2}}\, n\left[\frac{\frac{y}{t_2} - r_2 \frac{x}{\sigma_2}}{\sqrt{1-r_2^2}}\right] .$$

Setting $y = 0$ yields $\rho_2 = r_2$, and with arbitrary y_1 putting $x = 0$ gives $t_2 = \tau_2$. Hence, $\Phi_1 = F_1$ and $\Phi_2 = F_2$.

Finally, in Case(γ), $\rho_1 < 0$, $r_1 < 0$, $t_i = \tau_i$, $i = 1,2$. If, in (5.1), we take $x,y < 0$ and use the fact that $\Phi(x,y) = \bar{\Phi}(-x,-y)$ for a normal cdf, we obtain

$$(5.21) \qquad \prod_{i=1}^{2} \bar{\Phi}_i(x,y) = \prod_{i=1}^{2} \bar{F}_i(x,y) , \qquad x,y > 0 .$$

Now, if we set $y = cx$ and let $x \to \infty$, we see from Corollary 4.2 that the asymptotic relation (4.24) holds for $\bar{\Phi}_1$ and $\bar{F}_1$ for all $c > 0$

and also holds for Φ_2 and $\bar{\bar{F}}_2$ at least for all c in an interval of positive length containing the point τ_2/σ_2. Hence, we have

$$(5.22)\qquad \prod_{i=1}^{2} \frac{\phi_i(x,cx)(1-\rho_1^2)^2}{x^2\left(\frac{1}{\sigma_i}-\frac{c\rho_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{\rho_i}{\sigma_i}\right)} \Bigg/ \prod_{i=1}^{2} \frac{f_i(x,cx)(1-r_i^2)^2}{x^2\left(\frac{1}{\sigma_i}-\frac{cr_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{r_i}{\sigma_i}\right)} \to 1$$

as $x \to \infty$, for all c in an interval of positive length containing the point τ_2/σ_2. But

$$(5.23)\qquad \prod_{i=1}^{2}\left[\frac{\phi_i(x,cx)}{f_i(x,cx)}\right] = \prod_{i=1}^{2} \frac{(1-r_i^2)^{\frac{1}{2}}}{(1-\rho_i^2)^{\frac{1}{2}}} \exp\left[-\frac{1}{2}Q(c)\,x^2\right],$$

where $Q(c)$ is a quadratric polynomial, and (5.22) implies that the rhs of (5.23) has a finite positive limit for all c in an interval of positive length. This can happen only if $Q(c) \equiv 0$; the limit is then $\Pi_{i=1}^{2}\sqrt{(1-r_i^2)/(1-\rho_i^2)}$. Thus we have

$$(5.24)\qquad \sum_{i=1}^{2}\left[\frac{1}{\sigma_i^2}-\frac{2\rho_i c}{\sigma_i\tau_i}+\frac{c^2}{\tau_i^2}\right](1-\rho_i^2)^{-1} = \sum_{i=1}^{2}\left[\frac{1}{\sigma_i^2}-\frac{2r_i c}{\sigma_i\tau_i}+\frac{c^2}{\tau_i^2}\right](1-r_i^2)^{-1},$$

and

$$(5.25)\qquad \prod_{i=1}^{2}(1-\rho_i^2)^{-3/2}\left(\frac{1}{\sigma_i}-\frac{c\rho_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{\rho_i}{\sigma_i}\right) = \prod_{i=1}^{2}(1-r_i^2)^{-3/2}\left(\frac{1}{\sigma_i}-\frac{cr_i}{\tau_i}\right)\left(\frac{c}{\tau_i}-\frac{r_i}{\sigma_i}\right),$$

both relations holding for all c in an interval of positive length. Consequently, from (5.24) we obtain

$$(5.26)\qquad \frac{1}{\sigma_1^2}\left(\frac{1}{1-\rho_1^2}-\frac{1}{1-r_1^2}\right)+\frac{1}{\sigma_2^2}\left(\frac{1}{1-\rho_2^2}-\frac{1}{1-r_2^2}\right)=0,$$

$$(5.27) \qquad \frac{1}{\tau_1^2}\left(\frac{1}{1-\rho_1^2}-\frac{1}{1-r_1^2}\right)+\frac{1}{\tau_2^2}\left(\frac{1}{1-\rho_2^2}-\frac{1}{1-r_2^2}\right)=0\ ,$$

$$(5.28) \qquad \frac{1}{\sigma_1\tau_1}\left(\frac{\rho_1}{1-\rho_1^2}-\frac{r_1}{1-r_1^2}\right)+\frac{1}{\sigma_2\tau_2}\left(\frac{\rho_2}{1-\rho_2^2}-\frac{r_2}{1-r_2^2}\right)=0\ .$$

If $r_1 = \rho_1$, then $\Phi_1 = F_1$, and hence $\Phi_2 = F_2$. So, it remains only to investigate the possibility $r_1 \neq \rho_1$; in this case, $(5.26) \Longrightarrow r_2 \neq \rho_2$, and from (5.26) and (5.27) we have

$$(5.29) \qquad \frac{\tau_1}{\sigma_1} = \frac{\tau_2}{\sigma_2} = \tau\ , \text{ say}$$

But from (5.25) we know that the polynomials in c on the two sides of the equation have the same zeros. The zeros of the lhs are $\{\tau/\rho_1, \tau\rho_1, 0\}$ if $\rho_2 = 0$ and $\{\tau\rho_1, \tau/\rho_1, \tau\rho_2, \tau/\rho_2\}$ if $\rho_2 \neq 0$ and of the rhs $\{\tau/r_1, \tau r_1, 0\}$ if $r_2 = 0$ and $\{\tau r_1, \tau/r_1, \tau r_2, \tau/r_2\}$ if $r_2 \neq 0$. Hence, the assumption that $r_1 \neq \rho_1 < 0$ leads to the conclusion $r_1 = \rho_2$ and $r_2 = \rho_1$. This, together with (5.26), contradicts the assumption that $\sigma_1 > \sigma_2$. Thus in Case (γ) also, we must have $r_1 = \rho_1$, $r_2 = \rho_2$, so that $\Phi_1 = F_1$, $\Phi_2 = F_2$. Q.E.D.

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20. ABSTRACT

Under what conditions does the distribution of the maximum of a set of independent random variables uniquely determine the distributions of the component random variables? In the univariate case a sufficient condition is roughly that for every two distinct densities $f(x)$ and $g(x)$ in the family of possible densities $f(x)/g(x) \to 0$ or ∞ as $x \to \infty$. Hence, the distribution of $\max X_i$, $i = 1, \ldots, n$, when X_i has the distribution $N(\mu_i, \sigma_i^2)$ uniquely determines μ_i, σ_i^2, $i = 1, \ldots, n$ (except for indexing). The identifiability property has been proved for multivariate normal distributions for $n = 2$ and for every n when all correlations are positive; each component of the vector of maxima consists of the maximum of that component of the n constituent vectors. Inequalities for the probability in the upper right-hand quadrant of the bivariate normal distribution have been developed; these are generalizations of Mills' ratio.