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OPTIMAL CONTROL OF A BROWNIAN MOTION.(U)
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by

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TECHNICAL REPORT NO. 5

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Optimal Control of a Brownian Motion

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Herman Chernoff
and
Albert John Petkau

1. Introduction

In a recent paper, Rath [9] characterizes the solution of an optimal stochastic control problem where the controller can switch from either one of two modes to the other and in each mode, a diffusion process $Z(t)$ evolves according to a reflected Brownian motion with drift and diffusion parameters determined by the mode. In this problem, one possible application of which concerns the queue length $Z(t)$ of operations waiting to be performed in a computer, there are different costs per unit time for each mode of operation, there are switching costs for changing modes, and there is a linear holding cost per unit time, $c_0 Z(t)$. The object is to determine a policy which minimizes the long run average cost or more precisely the infinite horizon expected average cost. Rath demonstrates that the optimal policy among all stationary policies consists of switching at two key levels of the process. The proof involves approximating the problem by a sequence of discrete time discrete space random walk problems, solving the latter and going to the limit.

In this paper we consider a generalization of this problem. Our objective is to demonstrate that the general potential function or value difference approach developed by Bather [1,2,3] and subsequently used by various authors [5,6,7,8,11] enables one to work with the diffusion process directly. This approach is more analytic and thus has potential advantages in adding general insights on the behavior of solution to the original problem and its generalizations. It also lends itself easily to numerical techniques.

The major drawbacks in this approach are that in complicated versions of the problem, e.g. those involving more than two modes, the analytic approach becomes cumbersome. Finally while it is easy to show that the candidate solutions which satisfy the optimality conditions are optimal in the class of all stationary procedures, there is difficulty, due to lack of compactness in demonstrating that such a candidate is optimal among all possibly non-stationary procedures.

2. The Model

Informally, we assume that there are k modes. At any given time we may switch instantaneously from mode i to mode j at a cost of $K_{ij} \geq 0$. While we are in mode i , the cost is c_i per unit time. Also the queue length $Z(t)$

changes according to reflected Brownian motion with mean drift μ_i and variance σ_i^2 per unit time. The holding cost is $h[Z(t)]$ per unit time.

More formally, for $i = 1,2,\cdots,k$ let $W_i(t)$ be a Brownian motion with $W_i(0) = 0$, $E\,W_i(t) = \mu_i t$, and $E\{[W_i(t+s)-W_i(s)]^2 | W_i(s)\} = \sigma_i^2 t$ for all $s,t \geq 0$. If mode i_j is selected for the j^{th} time period $(t_{j-1},t_j]$ where $0 = t_0 < t_1 < \cdots$, the basic diffusion process originating at y_0 is

$$Y(t) = y_0 + \sum_{j=1}^{j'-1} [W_{i_j}(t_j)-W_{i_j}(t_{j-1})] + W_{i_{j'}}(t)-W_{i_{j'}}(t_{j'-1})$$

$$t_{j'-1} \leq t \leq t_{j'}$$

Since we wish to represent a queue length which cannot go below 0, the description of the current state $X(t) = (i(t),Z(t))$ should contain the level $Z(t)$ of the <u>reflected</u> controlled process where

$$Z(t) = Y(t) - \min(0,Y(s);0 \leq s \leq t)$$

as well as the current mode $i(t) = i_j$ if $t_{j-1} < t \leq t_j$.

If mode i is in continuous use during the time interval $(s,t]$, $t > s$, a cost given by

$$\int_s^t \left\{c_i + h[Z(t')]\right\} dt'$$

is incurred. We shall assume that $h(z) = o(e^{az})$ for all $a > 0$ as $z \to \infty$. With no loss of generality we may assume $h(0) = 0$ and $K_{i_1,i_2} + K_{i_2,i_3} \geq K_{i_1,i_3}$. The model with $k = 2$ modes and $h(z) = c_0 z$, $c_0 > 0$ corresponds exactly to the case considered by Rath. We shall restrict ourselves to the case of 2 modes after first discussing the relatively simple case of $k = 1$ mode where there is no control problem.

It is desired to select a policy which will minimize

$$\lim_{t\to\infty} t^{-1} E\left\{C(x_0, 0, t)\right\}$$

where $C(x,s,t)$ represents the total cost incurred over the time interval $[s,t)$ when the state at time s is $X(s) = x$.

This problem has a stationary or time-homogeneous character which suggests that an optimal strategy should consist of decomposing the set of possible current states $x = (i,z)$ into subsets $C^*_{i,i} = \left\{(i,z): z \in C_{i,i}\right\}$ of continuation states where one remains in mode i and $C^*_{i,j} = \left\{(i,z): z \in C_{i,j}\right\}$ of switching states where one switches from mode i to mode j , if

$i \neq j$ for $i,j = 1,2,\cdots,k$. There is some literature [4,10] which describes conditions under which there are optimal policies which are stationary. We shall confine our attention to such policies and later comment briefly on the more general question when the "good" policies we select are optimal among all policies according to our long run expected average cost criterion.

The Rath solution for the linear cost function consists of (a,b) switching policies with $0 \leq a \leq b \leq \infty$ where one switches from $i=1$ to $i=2$ if $z \geq b$ and one switches from $i=2$ to $i=1$ if $z \leq a$. Thus $C_{11} = [0,b)$, $C_{12} = [b,\infty)$, $C_{21} = [0,a]$ and $C_{22} = (a,\infty)$. For simplicity we shall confine our attention to those stationary policies where (1) each C_{ij} consists of a finite number of non-degenerate intervals; (2) C_{ii} is an open subset of $[0,\infty)$ (0 is regarded as an inner point); and (3) $C_{ij} \subset C_{jj}$. We shall call such stationary policies <u>regular</u>.

3. The Potential Function

The basic advantage of dealing with stationary policies is that the state $X(t)$ becomes a Markov Process when such a policy is applied. The stationary distribution which derives from such Markov Processes provides an alternative basis for the proof of the existence of the analytic tool, the potential function, which we introduce under the heuristic assumption that for any stationary policy with finite long run expected average cost γ

$$(3.1) \qquad E\{C(x,t,T)\} = \gamma(T-t) + v(x) + o(1) \quad \text{as} \quad T \to \infty$$

Then the function $v(x) = v(i,z) = v_i(z)$, the <u>potential function</u> provides the relative disadvantage of the initial state $x = (i,z)$ compared with any other state $x' = (i',z')$. (In this paper we shall leave v determined up to an unknown constant which will not be required and whose calculation is more difficult than the analysis we require.)

Suppose now that $z \neq 0$ is a point of C_{ii}. Then the following backward induction argument demonstrates that

$$(3.2) \qquad \mu_i v_i'(z) + \frac{1}{2}\sigma_i^2 v_i''(z) + c_i + h(z) = \gamma \quad \text{for} \quad z \in C_{ii}, z \neq 0$$

The argument is that

$$E\{C((i,z),t-dt,T))\} = [c_i+h(z)]dt + E\{C((i,z+dZ),t,T))\} + o(dt)$$

where $dZ(t) = dW_i(t)$ has mean $\mu_i dt$ and variance $\sigma_i^2 dt$. Substituting in (3.1), expanding $v_i(z+dZ)$ about z, and neglecting terms of order $o(dt)$, Equation (3.2) follows. Equation (3.2) may be interpreted as expressing the overall cost rate γ as the sum of the current cost rate $c_i + h(z)$ plus one due to the expected movement of the diffusion process.

If $z = 0 \varepsilon C_{ii}$, the reflected nature of the process leads to

(3.3) $$v_i'(0) = 0 \qquad \text{for } 0 \varepsilon C_{ii}$$

since $dZ = O_p(dt)^{1/2}$. For switching states we have

(3.4) $$v_i(z) = K_{ij} + v_j(z) \qquad \text{for } z \varepsilon C_{ij}, \quad j \neq i .$$

Finally the condition $h(z) = o(e^{az})$ as $z \to \infty$ for $a > 0$ implies

(3.5) $$v_i(z) = o(e^{az}) \quad \text{as } z \to \infty \quad \text{for } a > 0, \quad i = 1,2,\cdots k$$

so long as the policy does not permit z to drift off to ∞.

The heuristic introduction (3.1) to the potential function can be replaced by a more precise and rigorous result which can be expressed in terms of the cost D of a "game" starting in state x at time t and terminating at time $T > t$ with a terminal cost of $v[X(T)]$. This is presented below as follows:

If v satisfies (3.2) - (3.5) for a regular policy, then

$$(3.6)\quad D(x,t,T) \equiv E\{C(x,t,T)\} + E\{v[X(T)]\,|\,X(t)=x\} = \gamma(T-t) + v(x)$$

can be established by a "backward induction" argument on t .

4. Solutions of the differential equation and applications.

It is instructive to see what (3.2), (3.3), and (3.5) imply in the one mode case where there is no optimal control problem. Here we may as well drop the subscript i . A solution $f(z)$ of the differential equation (3.2) has the form

$$(4.1)\quad f'(z) = \alpha e^{-2(\mu/\sigma^2)z} + \frac{\gamma-c}{\mu}\,[1-e^{-2(\mu/\sigma^2)z}]$$

$$- \frac{2}{\sigma^2}\int_0^z h(w)e^{-2(\mu/\sigma^2)(z-w)}dw \quad \text{for } \mu \neq 0 \ .$$

If $\mu > 0$, our process drifts off to ∞ and for $h(z) \to \infty$ as $z \to \infty$, we would have $\gamma = \infty$, and this solution of the differential equation would be irrelevant for the one mode problem. On the other hand it will be useful later for the more general control problem, as will be the solution of (3.2) for the case $\mu = 0$. If $\mu < 0$ an alternate representation of the solution of (3.2) is

$$(4.2) \quad f'(z) = \alpha^{*} e^{-2(\mu/\sigma^2)z} + \frac{\gamma - c}{\mu} + \frac{2}{\sigma^2} \int_z^{\infty} h(w) e^{-2(\mu/\sigma^2)(z-w)} dw$$

for $\mu < 0$.

Finally if $\mu = 0$, the solution of (3.2) is

$$(4.3) \quad f'(z) = \alpha^{**} + \frac{2(\gamma - c)z}{\sigma^2} - \frac{2}{\sigma^2} \int_0^{z} h(w)\,dw \quad \text{for} \quad \mu = 0 .$$

The condition (3.3) leads to $\alpha = 0$ in (4.1). On the other hand if $\mu < 0$ and (3.5) applies, we have $\alpha^{*} = 0$ in (4.2). These two "boundary" conditions imply

$$(4.4) \quad \gamma = c + I$$

where

$$(4.5) \quad I = \frac{-2\mu}{\sigma^2} \int_0^{\infty} h(w) e^{2(\mu/\sigma^2)w} dw \qquad \mu < 0$$

In the special case where $h(z) = c_0 z$, the expressions in equation (4.1) - (4.5) are easily evaluated. Thus, in the one mode case $\gamma = \infty$ for $\mu \geq 0$ and $\gamma = c + I$ for $\mu < 0$ where

$$(4.5)' \qquad I = I^* \equiv -c_0\sigma^2/2\mu \qquad \text{for} \quad \mu < 0 .$$

The solution of (3.2) may be expressed by

$$(4.1)' \qquad f'(z) = \frac{1}{\mu}[\gamma-(c+I^*)] - \frac{c_0 z}{\mu} + \tilde{\alpha}e^{-2(\mu/\sigma^2)z} \qquad \text{if} \quad \mu \neq 0$$

and

$$(4.3)' \qquad f'(z) = \alpha^{**} + \frac{2(\gamma-c)}{\sigma^2} z - \frac{c_0}{\sigma^2} z^2 \qquad \text{if} \quad \mu = 0 .$$

Note that the condition $f'(0) = 0$ in (4.1)' implies $\tilde{\alpha} = -[\gamma-(c+I^*)]/\mu$. Thus in the one mode case with $\mu < 0$,

$$(4.4)' \qquad \gamma = c - c_0\sigma^2/2\mu ,$$

$f'(z) = -c_0 z/\mu$ and except for an unknown constant $v(z) = -c_0 z^2/2\mu$.

To illustrate the evaluation of α and Γ_i' for a two mode problem, suppose that $C_{11} = (a_1,a_2)$ and $C_{22} = [0,b_1) \cup (b_2,\infty)$

where $0 < a_1 < b_1 < b_2 < a_2$ and $\mu_2 < 0$. Then v_1' and v_2' involve 4 unknown constants. These are γ and α_{21} (α for mode 2 with $z \leq b_1$), α_{22} (α for mode 2 with $z \geq b_2$) and α_1 (α for mode 1 with $a_1 < z < a_2$). On the other hand $v_2'(0) = 0$ implies $\alpha_{21} = 0$, and $v_2(z) = o(e^{az})$ implies $\alpha_{22} - \frac{\gamma - c_2}{\mu_2} + \frac{I_2}{\mu_2} = 0$ (i.e., $\alpha_{22}^* = 0$). Furthermore, the conditions $v_1(a_j) = v_2(a_j) + K_{12}$ and $v_2(b_j) = v_1(b_j) + K_{21}$, imply that

$$\int_{a_j}^{b_j} [v_2'(z) - v_1'(z)]dz = K_{21} + K_{12} \quad \text{for } j = 1,2$$

which imposes two more conditions on the four constants. These lead to the determination of γ and $v_i'(z)$ on the continuation sets, C_{ii}. On the switching states C_{ij}, $v_i(z) = v_j(z) + K_{ij}$. From all of this calculus, v is determined up to an unknown additive constant, the determination of which, even in the one mode problem, requires analysis of the stationary distribution of the Markov Process governing the state (i,z) as a function of time.

5. Optimality Conditions

Two very natural optimality conditions on the policy are

$$(5.1) \qquad v_i(z) \leq K_{ij} + v_j(z) \qquad \text{for } z \varepsilon C_{ii}$$

and

$$(5.2) \qquad c_j + \mu_j v_j'(z) + \frac{\sigma_j^2}{2} v_j''(z) \le c_i + \mu_i v_j'(z) + \frac{\sigma_i^2}{2} v_j''(z)$$

$$\text{for } z \in \mathcal{I}(C_{ij})$$

where $\mathcal{I}$ represents the interior and the latter condition derives from considering the consequence of staying in mode i for a short period of time while $z \in \mathcal{I}(C_{ij})$. During this period $v_i(z) = K_{ij} + v_j(z)$ and hence $v_i'(z) = v_j'(z)$ and $v_i''(z) = v_j''(z)$. Related to these conditions is the smoothness condition that the right hand and left hand derivatives of the potential function are equal on $\mathcal{B}(C_{ij})$ the boundary of C_{ij} . If we label this common derivative $v_i'(z)$ and again use the fact that $v_i(z) = K_{ij} + v_j(z)$ on C_{ij} we may write

$$(5.3) \qquad v_i'(z) = v_j'(z) \qquad \text{for} \quad z \in \mathcal{B}(C_{ij})$$

Applying these optimality conditions with a backward induction argument yields part of a sufficiency result. Details of such an argument appear in [7]. To be more explicit, suppose that $\mathcal{P}_0$ is a regular stationary policy and v_0 and γ_0 satisfy the equations (3.2) to (3.5) and (5.1) to (5.3). Then for any alternative measurable policy $\mathcal{P}$ (not necessarily stationary)

$$(5.4) \quad D(x,t,T) \equiv E\{C(x,t,T)\} + E\{v_0[X(T)] \,|\, X(t) = x\} \ge \gamma_0(T-t) + v_0(x)$$

(where E represents expectation with respect to policy $\mathcal{P}$) with equality for $\mathcal{P} = \mathcal{P}_0$. It is clear that if conditions are such that the second term on the left of (5.4) must be $o(T-t)$ then $\mathcal{P}_0$ is optimal and γ_0 is the optimal long run average cost. It is somewhat peculiar that the main obstacle in establishing the optimality of $\mathcal{P}_0$ lies in the possibility of a better non-stationary $\mathcal{P}$ for which $v_0(X(T))$ may occasionally be very large, a possibility that is intuitively associated only with poor policies.

If μ_1 and μ_2 are both negative, it is clear that $E\{v_0[X(T)] | X(T)=x\} = 0(1)$ and a candidate stationary policy $\mathcal{P}_0$ satisfying the optimality conditions 5.1 to 5.3 will be optimal. If $\mu_1 > 0$ it is easy to see that $\mathcal{P}_0$ is optimal among the class of regular stationary alternatives. For a regular alternative which uses only mode 2 when $Z(t)$ is large, $E\{v_0[X(T) | X(t)=x\} = 0(1)$. On the other hand, for a regular alternative which allows one to stay in mode 1 when $Z(t)$ is large, $\gamma = \infty$.

To establish optimality among all measurable alternatives when $\mu_1 > 0$ is more difficult. In Section 8 a proof is outlined for the linear holding cost case. That proof can be generalized somewhat but a clear understanding seems to require a different approach.

6. The case of one transient mode

Our general approach will be to consider some simple policies and to catalog those parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, c_1, c_2$, K_{12} and K_{21} for which such policies satisfy the optimality conditions. We shall pay special attention to the conditions (5.1) and (5.3).

To avoid undue attention to fussy details we shall assume that at least one mean, say μ_2, is negative, that $h(z) \geq 0$ for $z \geq 0$ and that for large z, $h(z)$ is large enough to make the continued use of a mode with positive mean prohibitive. We shall be more explicit about this last condition shortly.

Let

$$\beta_i = c_i + I_i \tag{6.1}$$

where I_i is the weighted average of h,

$$I_i = \frac{-2\mu_i}{\sigma_i^2} \int_0^\infty e^{2(\mu_i/\sigma_i^2)w} h(w)\,dw \quad , \text{ if } \quad \mu_i < 0 \; . \tag{6.2}$$

Then β_i is the long run average cost corresponding to the exclusive use of mode i if $\mu_i < 0$. If $\mu_i \geq 0$, the long run average cost would be at least $c_i + \lim\inf_{z\to\infty} h(z)$.

Thus combining all of our conditions, we shall hereafter assume

H_0: $\mu_2 < 0$

H_1: $h(0) = 0$ and $h(z) \geq 0$ for $z \geq 0$

H_2: $h(z) = o(e^{az})$ as $z \to \infty$ for $a > 0$

H_3: $h(z)$ is continuous

H_4: $\liminf_{z \to \infty} h(z) > \max(I_2, I_2 + c_2 - c_1)$

Assumption H_4 states that the average holding cost in mode 2 is less than that for large z. Further if $\mu_1 > 0$, using mode 1 would lead to an average cost which would exceed $c_2 + I_2$, that of using mode 2 exclusively. If both μ_1 and μ_2 are negative we shall find it convenient to relabel the subscripts so that

H_5: $I_1\sigma_1^{-2} > I_2\sigma_2^{-2}$ if $\mu_1, \mu_2 < 0$,

assuming for minor convenience that equality does not obtain. Note that under these assumptions, $I_i \geq 0$ for $\mu_i < 0$ and that in principle we allow $c_i < 0$.

The simplest policy is that where one mode is transient. By this we mean that the policy is such that after shifting from that mode, it is never revisited. In particular we shall assume that mode 1 is transient and hence $\gamma = \beta_2 = c_2 + I_2$.

Theorem 6.1. The optimality conditions (5.1) and (5.3) are satisfied for a transient policy with $C_{11} = [0,b)$, and

$\gamma = \beta_2$ *if*

$$(6.3) \qquad c_1 - c_2 < I_2(1-\sigma_1^2/\sigma_2^2)$$

and

$$(6.4) \qquad c_1 - c_2 > I_2 - I_1 \quad (\text{i.e., } \beta_1 > \beta_2) \text{ if } \mu_1 < 0$$

and

$$(6.5) \qquad K = K_{12} + K_{21} \geq L$$

for some positive L *depending on* $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, c_1, c_2$ and h. *The optimality condition* (5.2) *is satisfied for the linear holding cost* $h(z) = c_0 z$.

Proof: We shall defer the proof of the last sentence till the next section, and we shall first treat

Case A: $\mu_1 \geq 0$

Let $g(z) = f_1(z) - f_2(z)$ where f_1 and f_2 are the special solutions of (3.2) represented by $f_1(0) = f_2(0) = 0$,

$$(6.6) \qquad f_1'(z) = \frac{2}{\sigma_1^2} \int_0^z e^{2(\mu_1/\sigma_1^2)(w-z)} [\gamma - c_1 - h(w)] dw$$

$$(6.7) \qquad f_2'(z) = \frac{-2}{\sigma_2^2}\int_z^{\infty} e^{2(\mu_2/\sigma_2^2)(w-z)}[\gamma-c_2-h(w)]dw$$

and if $\mu_1 \neq 0$

$$(6.8) \qquad g'(z) = f_1'(z) - f_2'(z) = \frac{\gamma-c_1}{\mu_1}(1-e^{-2(\mu_1/\sigma_1^2)z})$$

$$- \frac{2}{\sigma_1^2}\int_0^z e^{2(\mu_1/\sigma_1^2)(w-z)}h(w)dw$$

$$- \frac{\gamma-c_2}{\mu_2} - \frac{2}{\sigma_2^2}\int_z^{\infty} e^{2(\mu_2/\sigma_2^2)(w-z)}h(w)dw$$

If $\mu_1 = 0$, the first term on the right must be replaced by $2z(\gamma-c_1)/\sigma_1^2$. Since $g'(0) = 0$ and (6.3) implies $g''(0) = 2[\sigma_1^{-2}(\gamma-c_1)-\sigma_2^{-2}I_2] > 0$, $g'(z)$ is positive for small positive z . On the other hand H_4 implies that as $z \to \infty$, $f_1'(z)$ becomes negative and $f_2'(z)$ becomes positive. Hence $g'(z)$ becomes negative and there is a positive $b > 0$ which is the minimum positive root of $g'(z) = 0$. Let

$$(6.9) \qquad L = g(b) = \int_0^b g'(z)dz > 0$$

We shall now show that the potential function for $C_{11} = [0,b)$ and $C_{12} = [b,\infty)$ will satisfy conditions (5.1) and (5.3). Let $v_2(z) = f_2(z) + \xi_2$, and let $v_1(z) = f_1(z) + \xi_1$ for $0 \le z \le b$ and $v_1(z) = v_2(z) + K_{12}$ for $z > b$. Then v is our potential function (except for an additive constant) provided $v_1(b) = v_2(b) + K_{12}$ or $\xi_2 - \xi_1 = L - K_{12}$. Since $g'(z) > 0$ for $0 < z < b$, $v_1 - v_2$ attains its minimum of $\xi_1 - \xi_2 = K_{12} - L$ at $z = 0$, (6.5) implies (5.1). Since $g'(b) = 0$, the smoothness condition (5.3) is satisfied.

Note that if we increase c_1 so that $c_1 - c_2 \to I_2(1-\sigma_1^2/\sigma_2^2)$ $g'(z)$ decreases and the corresponding values of b and L approach zero monotonically.

Case B: $\mu_2 < 0$, $\mu_1 < 0$.

While the sign of $g''(0)$ is determined to be positive with the same argument as in Case A, the sign of $g'(z)$ for large z is that of $(\beta_1-\gamma)/\mu_1 = (\beta_1-\beta_2)/\mu_1$ since

$$f_1'(z) = \frac{\gamma-c_1}{\mu_1}[1-e^{-2(\mu_1/\sigma_1^2)z}] + \frac{I_1}{\mu_1}e^{-2(\mu_1/\sigma_1^2)z}$$

$$+ \frac{2}{\sigma_1^2}\int_z^\infty e^{2(\mu_1/\sigma_1^2)(w-z)}h(w)dw .$$

Hence (6.4), which is a natural requirement for mode 1 to be transient, implies that the sign of $g'(z)$ is negative for large z . The remainder of the argument in Case A applies equally well to Case B.

Thus as we promised we have proved all but the last sentence of *Theorem 6.1.* □

It follows from this proof that as c_1 increases so that $c_1 - c_2$ approaches $I_2(1-\sigma_1^2/\sigma_2^2)$, $g'(z)$ decreases and the corresponding values of b and L approach zero monotonically. Thus one would anticipate that larger values of c_1 would yield optimal policies with C_{11} null. In Case B where $\mu_1 < 0$, as c_1 decreases so that $c_1 - c_2$ approaches $I_2 - I_1$, b and L increase monotonically. (The limiting value of b may be $+\infty$. This is the case for linear holding cost as is easily derived from the analysis that follows shortly.) As c_1 decreases below this level, a transient policy to be optimal clearly would have to use mode 2 as the transient mode. In the next section we shall show that this is not the case, i.e., that the optimal policy has no transient mode when $c_1 + I_1 < c_2 + I_2$.

The particular values of K_{12} and K_{21} which yield a given sum K have no influence on γ for a non-transient two mode policy. This is more or less obvious since any such policy which involves more than one switch pays K for every pair of switches. For policies with a transient mode, increasing K does not affect γ . On the other hand, decreasing K below L leads to the possible optimality of non-transient policies.

In summary, for given h satisfying the assumptions $H_1 - H_5$, and for essentially the entire class of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, the set of c_1, c_2, K_{12}, K_{21} have been classified according to values for which our special transient policy satisfies the optimality conditions.

If the holding cost is linear $h(z) = c_0 z$, $c_0 > 0$, and

$$f_1'(z) = \frac{1}{\mu_1}[\gamma - (c_1 + I_1^*)][1 - e^{-2(\mu_1/\sigma_1^2)z}] - \frac{c_0 z}{\mu_1} \quad \text{for } \mu_1 \neq 0$$

$$f_1'(z) = \frac{2(\gamma - c_1)}{\sigma_1^2} z - \frac{c_0}{\sigma_1^2} z^2 \quad \text{for } \mu_1 = 0$$

while*

$$f_2'(z) = [-c_0 z + (\gamma - \beta_2)]/\mu_2 \quad .$$

Thus g'' is strictly monotone decreasing in both cases A and B and hence the positive solution b of $g'(z) = 0$ is unique. Moreover the conditions (6.3), (6.4) and (H.5) translate to

(6.3') $$c_1 - c_2 + c_0(\sigma_2^2 - \sigma_1^2)/2\mu_2 < 0$$

(6.4') $$c_1 - c_2 + \frac{c_0}{2}\left(\frac{\sigma_2^2}{\mu_2} - \frac{\sigma_1^2}{\mu_1}\right) > 0 \quad \text{if} \quad \mu_1 < 0$$

and

(H.5') $$\mu_2 < \mu_1 < 0 \ .$$

*The term $\gamma - \beta_2$ is zero here and is inserted as a covenience for reference in the more general case of Section 7 where it appears as a negative quantity.

7. Two non-transient modes

The next simplest type of policy to consider is that where $C_{11} = [0,b)$ and $C_{22} = (a,\infty)$ where $0 < a < b$. For such a policy neither mode is transient and both modes will recur infinitely often in the long run. We shall treat three cases, including two in parallel with those of Section 6. We shall determine conditions under which such policies satisfy the optimality conditions (5.1) and (5.3) and show that (5.2) is also satisfied in the case of linear holding cost. These results provide a complete classification of optimal policies in the linear case.

Theorem 7.1 The optimality conditions (5.1) and (5.3) are satisfied for a policy with $C_{11} = [0,b)$ and $C_{22} = (a,\infty)$ with $0 < a < b$ for appropriate $\gamma < \beta_2$ in the following cases.

Case A: $\mu_1 \geq 0$ and $0 < K = K_{12} + K_{21} < L$

Case B: $\mu_1 < 0$, $c_1 - c_2 > I_2 - I_1$, and $0 < K < K_{12} + K_{21} < L$

Case C: $\mu_1 < 0$, $c_1 - c_2 < I_2 - I_1$, and $0 < K < K_{12} + K_{21} < L_0$

for some appropriate L_0 depending on c_1, c_2, μ_1, μ_2, σ_1^2 and σ_2^2. If the holding cost is linear, $h(z) = c_0 z$, $L_0 = \infty$ and the optimality condition (5.2) is satisfied in all three cases.

Proof: We shall treat the first two cases in parallel with those of Section 6. Our proof of the last sentence will be adequate to cover the last sentence of Theorem 6.1. In Section 6 we took the precaution of using γ in place of β_2 in our formulae even though they were equal. As a result equations (6.6) to (6.8) may still be applied.

Hence it is clear that

$$\frac{\partial g'(z)}{\partial \gamma} = \frac{1}{\mu_1}\left[1-e^{-2(\mu_1/\sigma_1^2)z}\right] - \frac{1}{\mu_2} \qquad \text{if} \quad \mu_1 \neq 0$$

(7.1)

$$\frac{\partial g'(z)}{\partial \gamma} = \frac{2z}{\sigma_1^2} - \frac{1}{\mu_2} \qquad \text{if} \quad \mu_1 = 0$$

and that $\partial g'(z)/\partial\gamma > 0$. If we regard $g'(z)$ as a function of z and γ, say $g_1(z,\gamma)$ then we are concerned with how the values of z for which $g_1(z,\gamma) = 0$ change as γ decreases from $\gamma = \beta_2$. At $\gamma = \beta_2$ these are the values 0 and b of the cases treated in Theorem 6.1. As γ decreases, one root $z_1(\gamma)$ is monotonically increasing from 0 and the next positive root $z_2(\gamma)$ is monotonically decreasing from b until they meet at a common value z_0 corresponding to a value γ_0 of γ and such that $g_1(z_0,\gamma_0) = \partial g_1(z_0,\gamma_0)/\partial z = 0$. These roots $z_1(\gamma)$ and $z_2(\gamma)$ represent values of a and b for which the optimality conditions (5.1) and (5.3) are satisfied with K_{12} and K_{21} values for which $K = K_{12} + K_{21} = \int_{z_1}^{z_2} g_1(z,\gamma)dz$ is monotonically decreasing from L to 0 as γ decreases from β_2 to γ_0. The case where $z_1 = z_2 = z_0$ and

$\gamma = \gamma_0$ corresponds to the limiting case of zero switching costs. This disposes of Cases A and B of Theorem 7.1

Now let us treat

Case C: $\mu_1 < 0$, $c_1 - c_2 < I_2 - I_1$

First let c_1 decrease in Case B so that $c_1 - c_2 \to I_2 - I_1$ or equivalently β_1 decreases toward β_2 . Then, in the transient case where $\gamma = \beta_2 = c_2 + I_2$, $g'(z)$ increases monotonically to a limiting function $g_0'(z)$ which vanishes at $z = 0$ and is positive for positive z close to 0 . The corresponding values of b and L increase monotonically to possibly infinite limits b^* and L^* .

As c_1 passes below $I_2 - I_1 + c_2$, (Case C), β_1 decreases below β_2 and policies where mode 1 is transient can no longer be optimal. Moreover $\gamma < \beta_1 < \beta_2$ and $\beta_2 - \gamma$ is small. Then $g'(0) = (\beta_2 - \gamma)/\mu_2 < 0$ and $\lim_{z\to\infty} g'(z) = -\infty$. But for moderately small positive values of z , $g'(z)$ is sufficiently close to the positive limit of the transient case mentioned above that $g'(z)$ will be positive for some positive z . Thus $g'(z)$ has at least two positive roots, one of which is close to zero and the first two of which can be labeled a and b and correspond to the optimality conditions (5.1) and (5.3) for some $K = \int_a^b g'(z)dz$. As c_1 and $\gamma - c_1$ decrease, $g'(z)$ decreases and K, a, and b will behave monotonically until K reaches 0 and a and b come together as in the discussion of cases A and B. For fixed c_1, as γ increases to β_1, K increases to some limit L_0 depending on c_1, c_2,

$\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$.

The value of L_0 depends upon $g_0'(z)$ which can be written

$$(7.2) \qquad g_0'(z) = \mu_1^{-1}\{I_1 - I_1(z)\} - \mu_2^{-1}\{I_2 - I_2(z)\}$$

where

$$I_i(z) = \frac{-2\mu_i}{\sigma_i^2} \int_z^\infty e^{2(\mu_i/\sigma_i^2)(w-z)} h(w)dw \ , \quad i=1,2,$$

are exponentially weighted averages of $h(w)$ for $w > z$. Let us now consider the case where c_1 is decreased by δ from $I_2 - I_1 + c_2$ and $\gamma - c_1$ stays fixed at $\beta_2 - c_1$. Then $g'(z) = g_0'(z) + \delta/\mu_2$ and for δ sufficiently small $g'(z) = 0$ has a root $a_o > 0$ close to 0 . Let b_o be the second positive root if there is one and ∞ otherwise. Then $L_0 = \int_{a_o}^{b_o} g'(z)dz$. If $g_0'(z)$ is strictly monotonically increasing, $L_0 = \infty$ and $b_o = \infty$ for all δ . Otherwise L_0 will vanish for c_1 sufficiently small. If the holding cost is linear, $h(z) = c_0 z$, then $g_0'(z) = c_0 z(\mu_1^{-1} - \mu_2^{-1})$ and $L_0 = \infty$ for all $c_1 < I_2 - I_1 + c_2$.

It remains only to prove that the optimality condition (5.2) is satisfied in all three cases if $h(z) = c_0 z$. Our proof will also apply to Theorem 6.1. We begin with some general considerations and specialize to the linear case. Let us apply condition (5.2) to the policy $C_{11} = [0,b)$ and $C_{22} = (a,\infty)$ in which case $v_2'(z) = f_2'(z)$ on C_{12} and $v_1'(z) = f_1'(z)$ on C_{21} . Both $f_1'(z)$ and $f_2'(z)$ satisfy (3.2) and hence

$$c_j + \mu_j f_j'(z) + \frac{\sigma_j^2}{2} f_j''(z) = c_i + \mu_i f_i'(z) + \frac{\sigma_i^2}{2} f_i''(z) .$$

Substituting in (5.2) we have

$$\mu_1 g'(z) + \frac{\sigma_1^2}{2} g''(z) \leq 0 \qquad \text{for} \quad z \geq b \tag{7.3}$$

and

$$\mu_2 g'(z) + \frac{\sigma_2^2}{2} g''(z) \geq 0 \qquad \text{for} \quad z \leq a \tag{7.3'}$$

where (7.3') disappears in the transient case of Section 6. If $g'(z) < 0$ and $g''(z) > 0$ for $z \leq a$ (7.3') applies. If $g'(z) < 0$ and $g''(z) < 0$ for $z \geq b$ and $\mu_1 > 0$ (7.3) applies.

We now proceed to the special case of $h(z) = c_0 z$ to show that both (7.3) and (7.3') apply. As we noted in Section 6, $g''(z)$ is monotone decreasing in both cases A and B. It is easily seen to be monotone in Case C also and hence must be monotone decreasing with a root between a and b. Hence (7.3') applies with strict inequality. On the other hand $\mu_1 g' + \sigma_1^2 g''/2$ is linear in z with slope $c_0(\mu_1-\mu_2)/\mu_2 < 0$ and hence it suffices to establish (7.3) for $z = b$. But $g'(b) = 0$ and $g''(b) < 0$ and thus (7.3) applies with strict inequality. □

Let us review the current status. For the linear holding cost we have shown that the simple policies of Sections 6 and 7 apply to a large set of possible parameter values. What have we omitted? To minimize discussion we have avoided "boundary" cases where a or b or K are zero or where $\beta_1 = \beta_2$ when $\mu_1 < 0$ or when $\mu_1 = \mu_2$, but these cases are not particularly deep. The case $\mu_1 < \mu_2 < 0$ was covered in the case $\mu_2 < \mu_1 < 0$ by interchanging subscripts. (In the non-linear case, the condition $\mu_2 < \mu_1 < 0$ is replaced by (H.5)). Considering the fact that as c_1 increases to $c_2 + I_2(1-\sigma_1^2/\sigma_2^2)$, b and $L \to 0$ in the transient case, if one could establish the optimality, i.e., the sufficiency of the optimality conditions, one would be led to the conclusion that when $c_1 \geq c_2 + I_2(1-\sigma_2^2/\sigma_2^2)$ an optimal policy requires a null C_{11}. Thus we have shown that the policies of Sections 6 and 7 are the class of optimal policies for all parameter values involving linear holding costs, provided we can establish the sufficiency of the optimality conditions (5.1) to (5.3). This is accomplished in Section 8.

What is the situation for the non-linear case? If $g_0'(z)$ is not monotone increasing, we must seek more complex policies for some parameter values. If $g_0'(z)$ is strictly monotone increasing, it is possible that the simple policies will suffice if we could establish (7.3) and (7.3'). In Section 9 we present an example involving a non-monotone holding cost where an optimal policy requires $C_{22} = [0,b) \cup (b_2,\infty)$

while $C_{11} = (a_1, a_2)$ where $0 < a_1 < b_1 < b_2 < a_2$. One may conjecture that the simple "two point" policies of Sections 6 and 7 contain all optimal policies if $h(z)$ is monotone and approaches infinity as $z \to \infty$. However that conjecture is not valid in general. Computations were carried out for the case where $\mu_1 = -.2$, $\sigma_1^2 = 0.1$, $c_1 = 0.23$, $\mu_2 = -1.0$, $\sigma_2^2 = 1.0$, $c_2 = 0.0$, and $\gamma = 0.46$ while $h(z)$ is constructed with 3 line segments which have slope 1 for $0 \leq z \leq 0.8$, 0.1 for $0.8 \leq z \leq 2.0$ and 20.0 for $z \geq 2.0$. Then $g'(z) \leq 0$ except in the interval $(0.04, 0.85)$ but $\mu_1 g'(z) + \sigma_1^2 g''(z)/2$ is positive in $(1.722, 2.025)$. Thus condition 7.2 fails for the only candidate for a "two-point" optimal policy. Note that in this example $\sigma_1^{-2} I_1 = 2.425$, $\sigma_2^{-2} I_2 = 0.591$, and $0.23 = c_1 - c_2 < I_2 - I_1 = 0.349$. Thus we are in Case C but $g_0'(z)$ is not monotonic. It attains a local maximum of 1.967 at $z = 0.767$ but starts to increase again after $z = 1.137$.

8. Sufficiency of the Optimality Conditions

In Section 5 we found it relatively easy to establish the optimality of a candidate stationary policy $\mathcal{P}_0$ satisfying the optimality conditions (5.1) to (5.3) with $\gamma = \gamma_0$ and $v = v_0$ when this policy is compared with other regular stationary policies and, in the case where $\mu_1 < 0$, when it is compared with all measurable policies. If $\mu_1 \geq 0$, another proof is required to establish optimality in the class of all measurable policies. We shall outline such a proof for the special case of linear holding cost. Since the proof is clumsy and seems to have limited prospects for generalization and does not appear to confront the main issues and conditions which should be illuminated by an insightful proof our's will be informal and sketchy.

The main points of our proof consist of showing first that given an arbitrary policy one can do almost as well over a long time period T with a policy that applies mode 2 for a substantial time period whenever $Z(t)$ exceeds T^r for some r between .5 and 1 . Second, such a restricted policy can do better than γ_0 over $(0,T)$ only if $T^{-1}Ev_0[X(T)]$ is not small, in which case the expected holding cost over the interval $(T-T^{r+\delta},T)$ with $r + \delta < 1$ is so substantial that the average expected cost over $(0,T-T^{r+\delta})$ is less than what is attainable and a contradication results.

Let $\delta > 0$ and $1 + 3\delta < 2r < 2r + 4\delta < 2$. We shall use the fact that for $0 \leq t \leq s \leq T$, and any $n > 0$

$$P\{\sup_{i,|t-s|<T^r} |W_i(s)-W_i(t)| > kT^{\delta+r/2}\} = o(T^{-n}) \quad \text{as} \quad T \to \infty$$

Given any policy $\mathcal{P}$ with state process $X(t)$, we define a modified version $\mathcal{P}_T$ over the interval $(0,T)$ which follows $\mathcal{P}$ until $Z(t) \geq T^r$ at which time mode 2 is applied for a time interval of length $-T^r/2\mu_2$ after which one matches the modes which would have been used had $\mathcal{P}$ been followed. As soon as the modified queue length $Z_T(t) \geq T^r$ once again, one repeats mode 2 for another time interval length $-T^r/2\mu_2$. Between the <u>era</u> consisting of the time from the first arrival to T^r and the next arrival of $Z_T(t)$ to 0, the time duration in mode 2 has been increased by an amount τ_1 because of our modification. At the end of the era $Z_T(t) \leq Z(t)$. Succeeding eras from the arrivals to T^r followed by the returns to 0 involve increased durations in mode 2 of $\tau_2, \tau_3, \cdots$.

The modified policy may lead to certain increases of cost. That due to additional switching is $0(T^{1-r})$ since there are at most $0(T^{1-r})$ additional switches of mode. The additional cost due to the difference in c_i is $0(T)$. The difference in holding cost is bounded as follows. Let t^* represent the current value at time t of the increased time duration in mode 2 in the current or i-th era. Then

$$Z_T(t) \leq Z(t) + (\mu_2-\mu_1)t^* + T^\delta(kt^{*1/2}+1) \quad \text{for} \quad 0 \leq t^* \leq \tau_i ,$$

with probability exceeding $1-o(T^{-n})$. But $-k_1t^* + k_2t^{*1/2}$ attains a maximum value of $k_2^2/4k_1$ and hence the additional cost due to the difference in holding cost is with large probability $0(T^{1+2\delta})$.

None of these extra costs are incurred if $Z(t) < T^r$ for $0 \leq t \leq T$. If $Z(t) \geq T^r$ for some $t \leq T$, then the holding cost using $\mathcal{P}$ is very likely to exceed $T^{2r-\delta}$ which is large compared with the possible additional expense incurred by using $\mathcal{P}_T$. It follows that the expected average cost of $\mathcal{P}_T$ over $(0,T)$ exceeds that of $\mathcal{P}$ by at most a relatively small amount.

If $\mathcal{P}_0$ is not optimal then there is an infimum $\tilde{\gamma}$ of the expected average costs over long time intervals where $\tilde{\gamma} < \gamma_0$. Then there is a sequence of times T_i and policies $\mathcal{P}_i$ so that the $T_i^{-1}EC_i(x,0,T_i) \to \tilde{\gamma} < \gamma_0$. In that case (5.4) implies that $\underline{\lim}\ T_i^{-1}Ev_0[X(T_i)] \geq \gamma_0 - \tilde{\gamma}$ and hence $\underline{\lim}\ T_i^{-1}E[Z_i^2(T_i)] \geq -2\mu_2(\gamma_0-\tilde{\gamma})/c_0$ since $v_0(x) \approx -c_0z^2/2\mu_2$ as $z \to \infty$. Moreover the same inequality applies for the restricted policy $\mathcal{P}_{i,T_i}$. Now let $\tilde{T}_i = T_i - T_i^{r+\delta}$. With very large probability

$$Z_{T_i}(T_i-t) \geq [Z_i(T_i)-\mu_1t-T_i^{\delta+r/2}]^+ \qquad 0 \leq t < T_i^{r+\delta}$$

and $Z_{T_i}(T_i) < T_i^{r+\delta}$. Then if $\mu_1 > 0$, the holding cost over $(\tilde{T}_i, T_i)$ exceeds $(2\mu_1)^{-1}\{Z_{T_i}(T_i)-T_i^{\delta+r/2}]^+\}^2$ and if $\mu_1 = 0$, it exceeds $[Z_{T_i}(T_i)-T_i^{\delta+r/2}]^2$. It follows that the ezpected holding cost over $(\tilde{T}_i, T_i)$ is a substantial multiple of T_i. But then

$$\overline{\lim}\ \tilde{T}_i^{-1} EC_{T_i}(x,0,\tilde{T}_i) < \tilde{\gamma}$$

which contradicts our definition of $\tilde{\gamma}$. □

9. Computation of Optimal Solutions

The number of essentially independent parameters for this problem is so great that it is unfeasible to tabulate the solutions even in the linear case. It is preferable to use a numerical method for computing solutions for specific values of the parameters. There are many numerical approaches that can be used including even backward induction. Consistent with the general analytic approach of this paper are several methods which apply the smoothness condition (5.3).

For example in the case of the linear holding cost, one approach that was used successfully is described below for $\mu_1 \neq 0$. If one anticipates the solution of the form $C_{11} = [0,b)$ and $C_{22} = (a,\infty)$, then

$$v_1'(z) = \frac{-c_0 z}{\mu_1} + \frac{\gamma-(c_1+I_1^*)}{\mu_1}\,[1-e^{-2(\mu_1/\sigma_1^2)z}]$$

$$v_2'(z) = \frac{1}{\mu_2}\,[\gamma-(c_2+I_2)] - \frac{c_0 z}{\mu_2} .$$

When the switching costs $K_{12} = K_{21} = 0$, this solution degenerates to the form where $a = b = z_0$, $v_1'(z_0) = v_2'(z_0)$, and $v_1''(z_0) = v_2''(z_0)$. These last two equations are easily solved for γ and z_0 . As $K_{12} + K_{21}$ increases, the optimal γ , a and b change monotonically and the following four equations

$$v_2(a) = v_1(a) + K_{21} , \quad v_1(b) = v_2(b) + K_{12} ,$$

$$v_1'(a) = v_2'(a) , \quad v_1'(b) = v_2'(b)$$

involve the unknowns γ , a , b , and two constants of integration, one of which can be arbitrarily set, say by the equation $v_1(0) = 0$. This leaves four equations in four unknowns which can be solved iteratively by Newton's method (using z_0 for an initial approximation to a and b) . One may check in advance to see whether one is in the transient case by checking the inequalities (6.3) and (6.4) and determining b and L for $\gamma = \beta_2 = c_2 + I_2$.

An alternative approach was used for the following example where the holding cost was non-linear, i.e.,

$$h(z) = z + 2.5[(z^2+2z+2)e^{-z}-2].$$

This example was constructed to illustrate a solution of the form $C_{11} = (a_1,a_2)$, $C_{22} = [0,b_1)\cup(b_2,\infty)$ where $a_1 < b_1 < b_2 < a_2$. The function $h(z)$ is always positive but after a brief rise near $z = 0$, dips close to zero at $z = 3.31$ and then rises again, behaving asymptotically like $z - 5$ as $z \to \infty$ and like z near $z = 0$. With the parameters $\mu_1 = -.5$, $\mu_2 = -1$, $\sigma_1^2 = .81$, $\sigma_2^2 = .49$, $c_1 = .9$ and $c_2 = 1.0$ and $K = K_{12} + K_{21} = .03$, some preliminary calculations suggested that for z close to zero and z close to ∞ , the approximate linearity and the choice of parameters would make mode 2 preferable. However for z close to 3 the low holding cost suggested that it would be desirable to let z decrease slowly (i.e., to use mode 1).

The approach used was to select an approximation to a_1, a_2,b_1,b_2 and to compute the potential function using the method described at the end of Section 4. Then $v_1'(z) - v_2'(z)$ was evaluated at a_1,b_1,b_2 and a_2 . Then a_1, b_1, b_2 and a_2 were changed to reduce $|v_1'(z) - v_2'(z)|$ at these four points. A positive value of $v_1'(z) - v_2'(z)$ at $z = a_1$ and $z = a_2$ and a negative value at $z = b_1$ and $z = b_2$ suggests increasing these values of a_i and b_i. The value of $v_1'(z) - v_2'(z)$ at each of these points depends mainly on that point and after starting gingerly with small changes of a_1, b_1, b_2, a_2 in the appropriate directions, subsequent appropriate changes leads to rapid convergence to $a_1 = .7786$, $b_1 = 1.4973$, $b_2 = 3.6673$ and $b_2 = 4.5185$. The resulting value of $\gamma = 1.206496$ is a rather slight improvement in effect over the value $\gamma = 1.206736$ for our choice of the initial approximation $a_1 = 1$, $b_1 = 2$, $b_2 = 3$, $a_2 = 4$.

Additional computation confirmed that the optimality conditions (5.1) and (5.2) are satisfied.

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The potential function methods of Bather are applied to give a more analytic approach to a problem of Rath on stochastic control of a Brownian motion. This approach is applied also to a generalization of the Rath problem, an application of which involves deciding between two costly modes for reducing a queue length when there is a holding cost and a cost of switching from one mode to another. Some computational methods are indicated.