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VIBRATIONS OF THIN ELASTIC SHELLS - A NEW APPROACH.(U)

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VIBRATIONS OF THIN ELASTIC SHELLS — A NEW APPROACH

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ABSTRACT

A previous investigation by one of the present authors extended the method of Edge-Functions to the determination of natural frequencies and associated mode shapes of free vibration of thin elastic plates with a variety of boundary conditions. The present investigation further extends this approach to the case of a shallow elastic spherical shell of n-sided polygonal plan-form. Natural frequencies and associated mode shapes together with boundary residuals (indicating how well the approximate solution has satisfied boundary conditions) are readily displayed by the computer program offered. Results obtained in the present work are shown to be in excellent agreement with existing values for the specific boundary conditions treated. The present approach involves only modest computer efforts but offers the significant feature of rapid determination of how precisely the boundary conditions have been satisfied along each edge of the shell.

INTRODUCTION

The objective of the present investigation is to present a new generalized approximate approach to the problem of free vibrations of thin, shallow, elastic shells. The technique is applicable to shell structures having an arbitrary polygonal contour in the base plane and arbitrary boundary conditions along this contour. The specific example of a shallow spherical shell is investigated in detail. The present investigation employs the Edge-Function method recently successfully applied to determination of natural frequencies of free vibration of initially flat thin elastic plates by I.H. Tai and W.A. Nash [1]. The study [1] and the present work permit rapid determination of plate and shell natural frequencies (respectively) and associated mode shapes with the expenditure of only very modest amounts of computer effort. Further, the present approach permits comprehensive evaluation of the errors involved in this approximate technique; a feature that is absent in most other approaches for determination of plate and shell natural frequencies.

FUNDAMENTAL EQUATIONS

The basic differential equations employed in this work to describe the behavior of a thin elastic shell in the absence of applied loads are those derived in Kraus [2]. The system of equations reduces to two simultaneous differential equations for the normal deflection w and the stress function ϕ of the form:

$$D\nabla^2\nabla^2 w + \nabla_k^2\phi = -\rho h \frac{\partial^2 w}{\partial t^2} \quad (1)$$

$$\frac{1}{Eh}\nabla^2\nabla^2\phi - \nabla_k^2 w = 0 \quad (2)$$

where ∇^2 is the Laplace operator, k_x and k_y are curvatures in the x and y directions respectively, $\nabla_k^2 = k_y \frac{\partial^2}{\partial x^2} + k_x \frac{\partial^2}{\partial y^2}$, E represents Young's modulus, h the shell thickness, ν is Poisson's ratio, ρ the shell density, t denotes time, and $D = Eh^3/12(1-\nu^2)$. Normal and shear stress resultants as well as bending and twisting moments may be expressed in terms of w and ϕ as indicated in [2]. The tangential displacements u, v along the x and y axes respectively may be expressed in terms of w and ϕ from relations presented in [2]. For example, the first of these relations is:

$$\frac{\partial u}{\partial x} + k_x w = \frac{1}{Eh}\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right) \quad (3)$$

Let us consider the case of a shallow spherical shell for which $k_x = k_y = k = 1/R$ where R is the radius of curvature of the shell middle surface. If both w and ϕ exhibit a harmonic dependence on time, i.e. if

$$w = W(x,y)\sin\omega t \quad (4)$$

$$\phi = \Phi(x,y)\sin\omega t \quad (5)$$

then Equations (1) and (2) reduce to

$$D\nabla^4 W + k\nabla^2\Phi = \rho h\omega^2 W \quad (6)$$

$$\frac{1}{Eh}\nabla^4\Phi - k\nabla^2 W = 0 \quad (7)$$

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where ω is the natural circular frequency of free vibration. It is possible to uncouple (6) and (7) into the form

$$\nabla^6 w - p^4 \nabla^2 w = 0 \quad (8)$$

$$\nabla^8 \phi - p^4 \nabla^4 \phi = 0 \quad (9)$$

where

$$p^4 = \frac{h}{D}(\rho\omega^2 - Ek^2) \quad (10)$$

Since there are four conditions to be specified on each edge of the polygonal boundary the solutions for w and ϕ must contain four arbitrary constants.

For convenience let us set

$$\psi = \nabla^4 w - p^4 w \quad (11)$$

Substitution of this relation into (8) leads to

$$\nabla^2 \psi = 0 \quad (12)$$

FORMULATION OF SOLUTION

The projection of the shell onto the x-y plane is considered to be a two-dimensional convex simply-connected region R, the closed boundary B of R being a polygon of J sides. As in Figure 1, a set of rectangular Cartesian coordinate axes Oxy is chosen and the

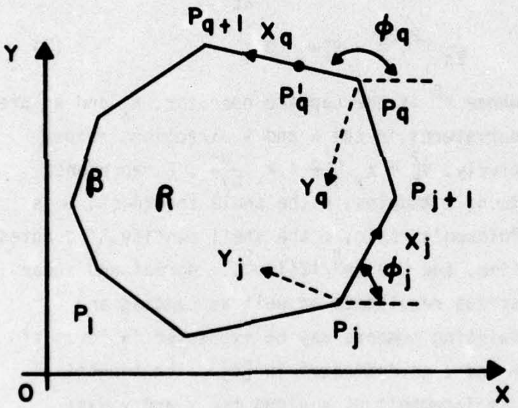

FIGURE 1

vertices and sides of the polygon B are numbered 1 to J. The typical vertex P_j has coordinates (X_j, Y_j) in the Oxy reference frame and the typical side, the j^{th}, is P_jP_{j+1}, having length a_j and making an angle ϕ_j with the positive direction of the Ox axis. Associated with each side j is a set of <u>edge-axes</u> x_jy_j, having origin at P_j and directed along the (inward) normal to that side.

The fundamental series solution of equation (12) in rectangular coordinates, derived using the separation of variables technique, is

$$\psi = \sum_{(m)} \{[A_m \sin(mx + \alpha_m) + B_m \cos(mx + \beta x)]e^{-my} + [C_m \sin(mx + \alpha_m) + D_m \cos(mx + \beta x)]e^{+my}\} \quad (13)$$

where m, A_m, B_m, C_m and D_m are as yet undetermined constants. The phase shifts α_m and β_m, although unnecessary here, are included for sake of generality later. Since the Laplacian operator ∇^2 is invariant under transformations which produce translation and rotation of axes, it follows that

$$\frac{\partial^2 \psi}{\partial x_j^2} + \frac{\partial^2 \psi}{\partial y_j^2} = 0 \quad (14)$$

The solution of this equation will then take the form

$$\psi = \sum_{(m_j)} \{[A_m^j \sin(m_jx_j + \alpha_m) + B_m^j \cos(m_jx_j + \beta_m)]e^{-m_jy_j} + [C_m^j \sin(m_jx_j + \alpha_m) + D_m^j \cos(m_jx_j + \beta_m)]e^{+m_jy_j}\} \quad (15)$$

in terms of the coordinates (x_jy_j) of a point referred to the j^{th} system of axes. Subscripts and superscripts j have been added to m and to the constants A_m, B_m, C_m, and D_m to indicate association with the j^{th} side of the boundary B. Clearly, similar expressions arise for solutions associated with each of the J sides of the polygon and all such solutions may be superposed, the governing differential equation being linear. Thus, a general solution may be put in the form

$$\psi = \sum_{j=1}^{J} \sum_{(m_j)} \{[A_m^j \sin(m_jx_j + \alpha_m) + B_m^j \cos(m_jx_j + \beta_m)]e^{-m_jy_j} + [C_m^j \sin(m_jx_j + \alpha_m) + D_m^j \cos(m_jx_j + \beta_m)]e^{m_jy_j}\} \quad (16)$$

Analogous to the solution for a simple rectangular region, the constants m_j in (16) are chosen so that

$$m_j = 2M\pi/a_j \quad (17)$$

for M a non-negative integer, in order to facilitate generation of the boundary condition equations later. Obviously the function ψ contributes to the determination of the displacement components and stress couples. Thus, in general, in formulating the boundary conditions for the typical, j^{th}, side, the coefficients in (16) associated with the j^{th} side can be made implicitly dependent on the relevant boundary conditions. In the general solution at any point within the region R the displacement components and stress resultants will be composed of contributions from each side of B, each contribution being dependent on the boundary conditions imposed on that particular side. Thus, invoking St. Venant's principle, it is physically desirable that such contributions decay with increasing distance from the corresponding side. So, in (16) the coefficients C_m^j and D_m^j are set equal to zero. Consequently ψ is chosen to be

$$\psi = \sum_{j=1}^{J} \sum_{M} [A_{1M}^j \sin(m_jx_j + \alpha_m) + B_{1M}^j \cos(m_jx_j + \beta_m)]e^{-m_jy_j} \quad (18)$$

in which m_j is defined in (17).

Each of the fundamental terms $e^{-m_jy_j}\sin(m_jx_j + \alpha_m)$ and $e^{-m_jy_j}\cos(m_jx_j + \beta_m)$ in ψ now possesses the advantageous properties of being directly associated with the j^{th} side of the boundary and of decaying in contribution to the overall solution with increasing distance

y_j into the region from that side. Such functions, similar both in form and notation to those employed by Tai and Nash [1], are termed edge-functions after Quinlan [3]. Comparable expressions for Φ may be written but are omitted for brevity. If one substitutes (18) into (11) and also carries out a comparable procedure for Φ, one finally obtains

$$W = \sum_{j=1}^{J} \sum_{M} \{[-\frac{1}{p^4} A_{1M}^j e^{-m_j y_j} + A_{3M}^j e^{-(m_j^2 + p^2)^{1/2} y_j} + A_{4M}^j e^{-(m_j^2 - p^2)^{1/2} y_j}] \sin(m_j x_j + \alpha_m) + [-\frac{1}{p^4} B_{1M}^j e^{-m_j y_j} + B_{3M}^j e^{-(m_j^2 + p^2)^{1/2} y_j} + B_{4M}^j e^{-(m_j^2 - p^2)^{1/2} y_j}] \cos(m_j x_j + \beta_m)\} \quad (19)$$

and

$$\Phi = \sum_{j=1}^{J} \sum_{M} \{[-\frac{1}{p^4}(A_{2M}^j - \frac{\rho h \omega^2}{2m_j^2 k} m_j y_j A_{1M}^j) e^{-m_j y_j} + \frac{Ehk}{p^4}(A_{3M}^j e^{-(m_j^2 + p^2)^{1/2} y_j} - A_{4M}^j e^{-(m_j^2 - p^2)^{1/2} y_j})] \cdot \sin(m_j x_j + \alpha_m) + [-\frac{1}{p^4}(B_{2M}^j - \frac{\rho h \omega^2}{2m_j^2 k} m_j y_j B_{1M}^j) \cdot e^{-m_j y_j} + \frac{Ehk}{p^2}(B_{3M}^j e^{-(m_j^2 + p^2)^{1/2} y_j} - B_{4M}^j e^{-(m_j^2 - p^2)^{1/2} y_j})] \cos(m_j x_j + \beta_m)\} \quad (20)$$

The expression for the transverse deflection (19) reduces in the case of zero curvature, $k = 0$, to the corresponding expression obtained by Tai and Nash [1] for the transverse deflection of a thin flat plate.

Shell boundary conditions are expressed in terms of various combinations of the parameters W, Φ, the in-plane displacements u_q and v_q along and perpendicular to the q-th side of the boundary, the rotation β about the tangent to the q-th side, the rotation β_q' about the normal to the q-th side, the in-plane stress resultant N_q, the bending moment M_q about the q-th side, the twisting moment M_{nt}^q about that same side, the Kirchhoff effective shearing stress resultants T_q and V_q, and the in-plane shearing stress resultant N_{nt}^q, all associated with the q-th edge. Expressions for each of these twelve parameters are derivable in terms of the coefficients A_{iM}^j and B_{iM}^j but are omitted for brevity. However, it is convenient to symbolize these boundary functions in the form

$$\Lambda_t$$

where t is an integer ranging from 1 to 12 for the above parameters W, Φ, u_q etc. respectively. Further, let us introduce the function $\Lambda_t E_M^j$ to denote the edge-function contribution to Λ_t stemming from the j-th coordinate system and associated with the particular value M. In this case each of the above boundary functions may be written in the form

$$\Lambda_t = \sum_{j=1}^{J} \sum_{M} \Lambda_t E_M^j \quad (21)$$

It is necessary to determine values for the frequency as well as the coefficients A_{iM}^j and B_{iM}^j so as to satisfy boundary conditions. Thus, boundary conditions require that four of the functions Λ_t be specified on each edge of the polygonal boundary. This is accomplished as follows.

If P_q' is any point $(x_q, 0)$ on the q^{th} side of the boundary, as in Figure 1, the function $\Lambda_t(P_q')$ is specified for four values of the parameter $t = t_q^1$, t_q^2, t_q^3 and t_q^4, say. Consequently, equation (21) implies that

$$\Lambda_t(P_q') = \sum_{j=1}^{J} \sum_{M} \Lambda_t E_M^j(P_q'), \quad t = t_q^1, t_q^2, t_q^3, t_q^4;\ q = 1,2,\ldots,J \quad (22)$$

In order to obtain identity equations for the coefficients A_{iM}^j and B_{iM}^j, it is noted that the series of terms

$$\sum_{(M)} \Lambda_t E_M^q(P_q')$$

in the summation (22) does not contain negative exponential factors since $y_q = 0$, so that each term involves only x_q. Since the negative exponentials tend to diminish boundary influences this series of terms is dominant and consequently (22) may be written

$$\Lambda_t(P_q') - \sum_{j=1}^{J}{}' \sum_{M} \Lambda_t E_M^j(P_q') = \sum_{M} \{ \sum_{i=1}^{4} (h_i A_{iM}^q + g_i B_{iM}^q) \sin(m_q x_q + \alpha_m) + \sum_{i=1}^{4} (h_{i+r} A_{iM}^q + g_{i+4} B_{iM}^q) \cos(m_q x_q + \beta_m)\} \quad (23)$$

The dash in the summation $\sum_{j=1}^{J}{}'$, indicates that the term $j = q$ is omitted. The factors h and g in (23) are independent of x_q and may readily be obtained from the expressions for $\Lambda_t E_M^q$ by setting $y_q = 0$. Thus the left hand side of (23) is some function $H(x_q)$ of x_q and the identity can be satisfied provided that $H(x_q)$ can be expanded in the trigonometric form of the series on the right-hand side. On choosing

$$m_q = 2M\pi/a_q, \quad M = 0,1,2,\ldots \quad (24)$$

and

$$\alpha_m = \beta_m = 0 \quad (25)$$

the right-hand side of (23) may be considered the Fourier series expansion of $H(x_q)$ in the range $[0, a_q]$.

Several methods of approximately satisfying the boundary identities (23) may be employed. Perhaps the most obvious is to multiply (23) by

$$\sin \frac{2N}{a_q} x_q dx_q$$

and integrate from 0 to a_q, then repeat using the cosine function. This leads to a set of equations of the form

$$G(x_q) = 0 \quad (26)$$

which is actually an infinite set of equations in an infinite number of unknowns. It is of course necessary to truncate these equations at some level, say L_j, for each side j of the polygon.

Because the truncation of the harmonic series expansion of $G(x_q)$ at $N = L_q$ implies that the approximation

$$G(x_q) = \frac{1}{2}E_0 + \sum_{N=1}^{L_q}[E_N \cos \frac{2N\pi}{a_q} x_q + F_N \sin \frac{2N\pi}{a_q} x_q] \quad (27)$$

is being used, another way to approximate $G(x_q)$ is to use trigonometric interpolation to fit a finite trigonometric series to $G(x_q)$ by a discrete least squares method. As is shown in reference [4], if a discrete Fourier series interpolation, based on discrete least squares fitting at 2k*-1 equidistant internal points in the interval $[0,a_q]$ is used in approximation (27), then

$$E_N = \frac{1}{k^*}\sum_{k=0}^{2k^*} w_k G(x_k)\cos \frac{2N\pi}{a_q} x_k \qquad N = 0,1,2,\ldots,L_q$$

$$F_N = \frac{1}{k^*}\sum_{k=1}^{2k^*-1} w_k G(x_k)\sin \frac{2N\pi}{a_q} x_k \qquad N = 1,2,\ldots,L_q \quad (28)$$

where

$$x_k = ka_q/2k^* \quad (29)$$

and w_k are weight factors defined by

$$w_k = 1/2: \quad k = 0, \quad k = 2k^* \qquad (30)$$
$$= 1: \quad 0 < k < 2k^*$$

The expressions (28) for E_N and F_N may be set to zero thus generating the requisite set of simultaneous boundary equations. This interpolation method of setting up the simultaneous equations has the advantage, over the first method above, of requiring less computational time. Furthermore, it is found in practice that there is very little difference in accuracy between the results obtained from either method.

The set of equations arising from expression (27) in which the coefficients of the expansion must be set to zero, ensures that the boundary conditions (22) are satisfied at all points of the q^{th} side, $q = 1,2,\ldots,J$, except possibly at the vertices. If the boundary conditions are not automatically satisfied at the end points of each side of the polygon, they must be imposed at these points. Thus it is required to set to zero both the boundary function and its derivative along the relevant side, so that

$$G(x_q) = 0 \quad (31)$$

and

$$\frac{\partial}{\partial x_q} G(x_q) = 0 \quad (32)$$

for both ends $x_q = 0$ and $x_q = a_q$ of the interval and for each of the boundary functions corresponding to $t = t_q^1, t_q^2, t_q^3$, and t_q^4. The effect of these point equations, which are referred to as vertex equations, is not alone to ensure that the boundary conditions are satisfied at the end points of the interval, but also to increase the rate of convergence of the Fourier series from order 1/M to at least order $1/M^3$. Such equations clearly give rise to the necessity of introducing additional unknowns in the basic solutions (19) and (20). Since, in general, there are four vertex equations arising for each boundary condition on each side, a total of sixteen additional unknowns associated with each side are required. These are provided by using fractional-edge-functions. The latter are edge functions of the types in expressions (19) and (20) within the summation signs. Whereas in the above harmonic edge-functions

$$m_q = 2M\pi/a_q, \quad M = 0,1,\ldots,L_q, \ \alpha_m = 0; \ \beta_m = 0 \quad (33)$$

for the fractional-edge-functions new values of these parameters are defined, so that

$$m_q = 2V_M\pi/a_q; \quad \alpha_m \neq 0; \quad \beta_m \neq 0 \quad (34)$$

where V_M is non-integral. The sixteen requisite unknowns can be provided by including in the solution two fractional-edge-functions generated simply by choosing two different values for V_M. They may be incorporated in the general solution by adopting the convention that they correspond to M = -1 and M = -2 in the summations in (19) and (20. It may be noted that, in theory, the solution is independent of the choice of the vertex numbers V_M and of α_m and β_m for the fractional-edge-functions. However, in practice it is found that a judicious choice of values will increase the convergence of the truncated series solution.

It was found in [1] that the problem of free vibrations of flat plates could be successfully treated by use of a combination of edge functions and fractional edge functions. That is, the prescribed boundary conditions could be satisfied to the desired degree of accuracy. In the case of shell vibrations this is not the case and it becomes necessary to employ not only the two types of functions used in plate analysis, but in addition a third type of function termed a shell polar function. This function contributes to the time-dependent deflections in the interior regions of the shell away from the boundaries, whereas the edge functions contribute largely in the immediate vicinity of the shell edges. The shell polar functions are of the form

$$\Phi = Re[EZ^\lambda + F\,\bar{Z}Z^{\lambda+1}] \quad (35)$$
$$W = Re[F'.Z^\lambda]$$

where z = x + iy (origin at any convenient interior point), $\bar{z}$ is the complex conjugate, E, F, and F' are complex constants, and λ is an integer. Details of the derivation and accompanying properties of these special functions are presented in the Appendix.

The shell polar functions are appended to the solutions to replace some (or possibly all) of the fractional edge functions. The selection of the ratio of fractional edge functions to shell polar functions is made on the basis of experience and judgment in use of this technique but with the requirement that boundary conditions be satisfied to the prescribed degree of accuracy. In the present investigation a combination of half edge functions and half polar functions was found best for satisfying the prescribed boundary conditions.

Application of the boundary conditions leads to a system of homogeneous equations in the unknowns A^j_{iM} and B^j_{iM} of the form

$$\Sigma C_{ab}(\omega)A_b = 0, \quad b = 1,2 \tag{36}$$

where A_b represents A^j_{iM} and B^j_{iM}. The C_{ab} are coefficients obtained from the interpolation method mentioned previously and are functions of the unknown frequency ω. The frequencies are found by setting the determinant of the system equal to zero, viz:

$$|C_{ab}(\omega)| = 0 \tag{37}$$

Values of ω are determined by an suitable iteration method, such as the bisection procedure.

NUMERICAL RESULTS

Let us consider the free vibrations of a thin, elastic shallow spherical steel shell of square plan-form. The radius of curvature is 30.0 inches, the shell thickness 0.05 inches, the length of each side 12.0 inches (in plan-form), $E = 30 \times 10^6$ lb/in^2, Poisson's ratio = 0.3 and the material density is 0.732×10^{-3} lb-sec^2/in^4. Boundary conditions to be considered are (a) all sides simply supported, and (b) all sides clamped. Natural frequencies and associated mode shapes are desired. The natural frequencies are indicated by Equation (37) and a computer program for determination of these frequencies, associated mode shapes, and boundary residuals is available upon request from the authors.

For the case of simply supported edges the first four natural frequencies are indicated below. Results are compared to those due to Vlasov [5] as well as Malkina [6]. The investigation [6] applies to the case of a spherical dome with circular plan-form but is employed here as an approximation by considering the greatest circular base to be inscribed in the square plan-form shell under consideration.

Frequency	1	2	3	4
Present method				
L=1	6748.86	6759.30	6763.15	6829.48
L=7	6748.85	6759.30	6863.63	6795.35
Vlasov [5]	6749.51	6759.29		6827.35
Malkina [6]	6749.42		6762.31	

The computer program developed indicates the small residual bending moments along each of the sides of the shell. For example, for the first natural frequency if the peak bending moment at the mid-point of the shell is taken to be unity, the root-mean-square-boundary residual bending moment (for L=7) is found to be 0.37×10^{-2}. This same function for the fourth natural frequency is found to be 0.88×10^{-5}. Clearly these constitute very adequate satisfaction of boundary conditions.

For the case of a clamped edge spherical shell the first four natural frequencies are indicated below, as well as those found by the Malkina method [6] again using the greatest inscribed circular base.

Frequency	1	2	3	4
Present Method				
L=1	6746.24	6794.72		
L=7	6746.24	6788.39	6992.20	7135.09
Malkina [6]	6746.24	6796.89	6998.91	

Again, the computer program displays the small residual deflections along the shell boundary. For example, for the first natural frequency the root-mean-square boundary deflection is 0.99×10^{-3} (for L=3) and for the fourth natural frequency it is 0.75×10^{-5} (for L=3) compared to peak mid-point deflection taken to be unity. Mode shapes are found by calculating the relative deflections of closely spaced points on the shell then connecting all zero-deflection points at a given frequency.

DISCUSSION AND CONCLUSIONS

The present investigation has indicated that, for the geometry considered, the Edge Function Method is well-suited to determination of natural frequencies and associated mode shapes of thin elastic shallow shells undergoing free vibration. For the cases discussed through specific examples, the present analysis yielded results in excellent agreement with existing analytical results.

The computer program developed during the course of the present investigation is applicable to any spherical shell of n-sided plan-form. Further, the boundary conditions along each edge are completely arbitrary and may be different on the various edges without giving rise to complications in frequency determination.

The use of the present technique of analysis for determination of natural frequencies and mode shapes of thin elastic shells yields an additional bonus not orginarily found in other approximate methods, namely rapid determination of how well the specified boundary conditions have been satisfied. This is accomplished through digital evaluation of significant structural parameters along the boundary, say deflection and slope for a clamped edge, which should, of course vanish identically but the small residuals corresponding to each of these parameters are routinely printed out by the computer and root-mean-square values along any of the polygonal edges displayed. Thus, one may readily ascertain exactly how well the specified boundary conditions have been satisfied along all boundaries of the plate.

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APPENDIX

SHELL POLAR FUNCTIONS

Equation (9) gives for

$$\nabla^4(\nabla^4 - p)\phi = 0 \qquad \text{(A-1)}$$

one solution of which is

$$\nabla^4\phi = 0$$

from which, following O'Callaghan [4]

$$\phi = \mathrm{Re}[Ez^{\lambda} + F\bar{z}z^{\lambda+1}]; \quad z = x + iy \qquad \text{(A-2)}$$

where E and F are complex constants and λ is arbitrary. Accordingly on substituting for $\nabla^4\phi$ in (7), we obtain

$$\nabla^2 W = 0 \qquad \text{(A-3)}$$

and equation (6) then gives

$$W = \frac{k}{\rho h\omega^2}\nabla^2\phi \qquad \text{(A-4)}$$

On introducing the operators

$$\frac{\partial}{\partial x_q} = e^{i\phi_q}\frac{\partial}{\partial z} + e^{-i\phi_q}\frac{\partial}{\partial \bar{z}}$$
$$\frac{\partial}{\partial y_q} = ie^{i\phi_q}\frac{\partial}{\partial z} - ie^{i\phi_q}\frac{\partial}{\partial \bar{z}} \qquad \text{(A-5)}$$

from which

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar{z}}$$

and on operating with ∇^2 on (A-2) and substituting in (A-4) we obtain

$$W = \frac{4(\lambda+1)}{\rho hr\omega^2}\mathrm{Re}[Fz^{\lambda}] = \mathrm{Re}(F'z^{\lambda}) \qquad \text{(A-6)}$$

where

$$F' = qF; \quad q = \frac{4(\lambda+1)}{\rho hr\omega^2} \qquad \text{(A-7)}$$

To fit into the general polar form for the t^{th} derived function

$$Q_t = \mathrm{Re}[EAz^{\lambda^*} + F(B\bar{z}z^{\lambda^*+1} + Cz^{\lambda^*+2})] \qquad \text{(A-8)}$$

where A, B and C are complex functions of ϕ and λ, and where necessary will have a subscript t - A_t, B_t, C_t - to distinguish the different functions. Equation (A-6) for W then gives:

$$A = 0; \; B = 0; \; C = 2; \; \lambda^* = \lambda - 2 \qquad \text{(A-9)}$$

Values of A, B, and C for each of the remaining eleven parameters ϕ, u_q, v_q, etc. are tabulated below.

Function	t	λ^*	A_t	B_t	C_t
W	1	$\lambda-2$	q	0	0
ϕ	2	λ	1	1	0
u_q	3	$\lambda-1$	$A_o e^{i\phi_q}$	$B_o e^{i\phi_q}$	$C_o e^{-i\phi_q}$
v_q	4	$\lambda-1$	iA_3	iB_3	$-iC_3$
β_q	5	$\lambda-1$	A_4/r	B_4/r	$C_4/r - i\,C_1 e^{i\phi_q}/z^2$
N_q	6	$\lambda-2$	$(\lambda-1)e^{2i\phi_q}$	$(\lambda+1)e^{2i\phi_q}$	$2(\lambda+1)$
M_q	7	$\lambda-4$	0	0	$D^*\lambda(\lambda-1)e^{2i\phi_q}$
M^q_{nt}	8	$\lambda-4$	0	0	iC_7
T_q	9	$\lambda-2$	A_{12}	B_{12}	$C_{12}+C_8/rz^2$
V_q	10	$\lambda-5$	0	0	$(\lambda-2)i\phi_q\,C_8/r$
β'_q	11	$\lambda-1$	A_4/r	B_4/r	$C_4/r - \lambda e^{i\phi_q}\,C_1/z^2$
N^q_{nt}	12	$\lambda-2$	$-i\lambda(\lambda-1)e^{2i\phi_q}$	$-i\lambda\lambda'$	0

$$D^* = D(1-\nu)$$
$$\lambda' = (\lambda+1)e^{2i\phi_q}$$