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A Simple Model with Applications in Structural Reliability, Extinction of Species, Inventory Depletion, and Urn Sampling, I. Cut Set Failure.

by

E. El-Newehi<sup>a</sup>, F. Proschan<sup>1,2b</sup>, and J. Sethuraman<sup>2b</sup>  
University of Kentucky<sup>a</sup> and The Florida State University<sup>b</sup>

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Abstract

This is part I of a two-part paper devoted to the following model. A series-parallel system consists of  $(k+1)$  subsystems  $C_0, C_1, \dots, C_k$ , also called cut sets. Cut set  $C_i$  contains  $n_i$  components arranged in parallel,  $i = 0, 1, \dots, k$ . No two cut sets have a component in common. It is assumed that after  $t$  components have failed, each of the remaining components is equally likely to fail,  $t = 0, 1, \dots$ . We study the probability that the system fails because a specified cut set  $C_0$ , say, fails first. We obtain several alternative expressions and recurrence relations for this probability. Some of these formulae are useful in computations while others permit us to derive qualitative features like monotonicity, Schur-concavity, asymptotic limits, etc. These results are extended to the situation where some cut set size is first reduced to  $a$ , where  $a$  is a specified positive integer.

The above model has applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.

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1. Introduction. This is part I of a two-part paper devoted to the study of a simple model which has applications in reliability theory, extinction of species, inventory depletion, and urn sampling.

(a) Reliability. We state the model in a reliability context and also use the language of reliability theory in the solution of the problem. Consider a system consisting of  $(k+1)$  subsystems, called cut sets, arranged in series. The  $i^{\text{th}}$  cut set, called  $C_i$ , has  $n_i$  components arranged in parallel,  $i = 0, 1, \dots, k$ . No two cut sets have a component in common. Such a system is called a series parallel system. (A series system functions if and only if each component in it functions. A parallel system functions if and only if at least one of its components functions.) In the early stages of design of such a system, very little may be known about the reliability or life distribution of the components of the series-parallel system. Under these circumstances it is reasonable to assume that after  $t$  components have failed, each of the remaining components are equally likely to fail,  $t = 0, 1, \dots$ . We make this assumption throughout this paper. Let  $\underline{n} = (n_1, \dots, n_k)$  represent the vector of sizes of cut sets  $C_1, \dots, C_k$ . It is of interest to study  $P(n_0; \underline{n})$ , the probability that the system fails because cut set  $C_0$

fails first. In Section 2 we obtain several alternative expressions for  $P(n_0; \underline{n})$ .

We now briefly describe how this model finds applications in other areas.

(b) Extinction of species. Consider a lake which contains, in addition to other forms of life,  $(k+1)$  species of fish with species  $i$  containing  $n_i$  members,  $i = 0, 1, \dots, k$ . The fish are comparable in their vulnerability to capture; fish are caught one at a time in succession in a period when no births occur. It is then of interest to determine the probability that species 0 becomes depleted first. This probability is precisely the quantity  $P(n_0; \underline{n})$  defined above.

(c) Inventory depletion. A depot stocks  $(k+1)$  brands of a certain item. It is assumed all items are equally likely to be demanded. The probability that brand 0 becomes depleted first is  $P(n_0, \underline{n})$  and is again a quantity of interest.

(d) Sampling from urns. An urn contains  $n_i$  balls of color  $i$ ,  $i = 0, 1, \dots, k$ . Balls are removed at random one by one. The probability that balls of color 0 become exhausted first is again  $P(n_0; \underline{n})$ .

It is clear that the above problem occurs in other contexts as well. The organization of this paper is as follows. Section 2 contains several alternative expressions for  $P(n_0; \underline{n})$ . Section 3 studies qualitative features of this probability like monotonicity, Schur-concavity, asymptotic limits, etc. Section 4 deals with a generalization where instead of waiting for the series-parallel system to fail, we wait for an alarm to ring which happens as soon as some cut set is reduced to a components, where  $a$  is a positive integer.

In part II, we shall study the life length of the series-parallel system, that is the number of component failures leading to the failure of the system.

2. Expressions for  $P(n_0, \underline{n})$ .

In this section we obtain several alternative expressions for  $P(n_0; \underline{n})$ , the probability that cut set  $C_0$  fails before the rest. The subheadings in this section list the different techniques used. We will use the following notation throughout this paper.

$$n = \sum_0^k n_i = \text{the total number of components.}$$

$S$  denotes a subset of  $\{1, \dots, k\}$ .

$|S|$  = cardinality of  $S$ .

$S' = \{i: 1 \leq i \leq k, i \notin S\}$ .

$$n_S = \sum_{i \in S} n_i .$$

$$\underline{n}_S = (n_i: i \in S)$$

$$\underline{n}_{-i} = (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k) = \underline{n}_{S'} \text{ where } S = \{i\}.$$

(a) Direct method. Consider the case  $k = 1$ , that is, when there are only two cut sets  $C_0$  and  $C_1$ . We have the following theorem.

Theorem 2.1.

$$(2.1) \quad P(n_0; n_1) = \frac{n_1}{n_0 + n_1} .$$

Proof. Cut set  $C_0$  fails before cut set  $C_1$  if and only if the last component to fail is from cut set  $C_1$ . The probability that the  $i^{\text{th}}$  component to fail is from cut set  $C_1$  is  $n_1 (n_0 + n_1)^{-1}$ ,  $i = 1, \dots, n_0 + n_1$ . ||

Theorem 1 immediately gives an expression for the probability that the cut sets fail in a specified order and for  $P(n_0; \underline{n})$ , as can be seen from next theorem. For any series-parallel system with cut sets  $D_1, \dots, D_r$ , let  $\{D_1 < D_2 < \dots < D_r\}$  denote the event that cut set  $D_1$  fails first,  $D_2$  fails second,  $\dots$ , and  $D_r$  fails  $r$  th.

Theorem 2.2.

$$(2.2) \quad P(C_0 < C_1 < \dots < C_k) = \frac{n_1 n_2 \dots n_k}{(n_0 + n_1)(n_0 + n_1 + n_2) \dots (n_0 + n_1 + \dots + n_k)},$$

and

$$(2.3) \quad P(n_0; \underline{n}) = \sum_{\pi} \frac{n_1 n_2 \dots n_k}{(n_0 + n_{\pi_1})(n_0 + n_{\pi_1} + n_{\pi_2}) \dots (n_0 + n_{\pi_1} + \dots + n_{\pi_k})},$$

where the summation is over all permutations

$\pi = (\pi_1, \pi_2, \dots, \pi_k)$  of  $(1, 2, \dots, k)$ .

Proof. Using Theorem 1, notice that  $P(C_0 < C_1 < \dots < C_k)$

$$= P(C_0 \cup C_1 \cup \dots \cup C_{k-1} < C_k) P(C_0 < C_1 < \dots < C_{k-1} < C_k) |$$

$$C_0 \cup C_1 \cup \dots \cup C_{k-1} < C_k) = \frac{n_k}{n_0 + n_1 + \dots + n_k} P(C_0 < C_1 \dots < C_{k-1}).$$

This establishes (2.2). The expression for  $P(n_0; \underline{n})$  in (2.3) follows from (2.2) by summing the probabilities of the events  $\{C_0 < C_{\pi_1} < \dots < C_{\pi_k}\}$  over all permutations  $\pi$ . ||

(b) The union-intersection principle. Another expression for  $P(n_0; \underline{n})$

is obtainable from the union-intersection principle and Theorem 1.

Theorem 2.3.

$$(2.4) \quad P(n_0; \underline{n}) = n_0 \left[ \sum_S (-1)^{|S|} (n_0 + n_S)^{-1} \right]$$

where the summation is over all subsets  $S$  of  $\{1, \dots, k\}$ .

Proof. Let  $A_i$  denote the event that cut set  $C_0$  fails before cut set  $C_i$ ,  $i = 1, \dots, k$ . We may then write

$$(2.5) \quad P(n_0; \underline{n}) = P\left(\bigcap_{i=1}^k A_i\right) = \sum_i P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} \cup A_{i_2}) + \dots \\ + (-1)^{k-1} P(A_1 \cup \dots \cup A_k).$$

Now,

$$(2.6) \quad P(A_{i_1} \cup \dots \cup A_{i_r}) = 1 - P(A_{i_1}^c \cap \dots \cap A_{i_r}^c) \\ = 1 - P\left(\bigcup_{j=1}^r C_{i_j} < C_0\right) = 1 - \frac{n_0}{n_0 + n_S}, \text{ where } S = \{i_1, \dots, i_r\},$$

by using Theorem 1. Substituting (2.6) in (2.5) and using the identity

$$\sum_{r=1}^k (-1)^r - 1 \binom{k}{r} = 1,$$

one readily obtains (2.4). ||

(c) Conditioning. The next two theorems illustrate the use of a conditioning argument, which is a standard technique for a Markov process. The underlying Markov process here is that of the vector of the number of functioning components of the  $(k+1)$  cut sets at the  $t^{\text{th}}$  component failure,  $t = 0, 1, \dots, n$ .

Theorem 2.4.

$$(2.7) \quad P(n_0; \underline{n}) = \sum_{i=0}^k \frac{n_i}{n} P(n_0; (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k))$$

and

$$(2.8) \quad P(n_0; \underline{n}) = \sum_{i=1}^k \frac{n_i}{n} P(n_0; \underline{n}_i).$$

Proof. Relation (2.7) is obtained by conditioning on the outcome at the first component failure. Thus,

$$\begin{aligned} P(n_0; \underline{n}) &= P(C_0 \text{ fails first}) \\ &= \sum_{i=0}^k P(C_0 \text{ fails first} \mid 1^{\text{st}} \text{ component to fail comes from } C_i). \end{aligned}$$

$P(1^{\text{st}} \text{ component to fail comes from } C_i)$

$$= \sum_{i=0}^k \frac{n_i}{n} P(n_0; (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k)).$$

Relation (2.8) is likewise obtained by conditioning on the outcome at the last component failure. ||

Theorem 2.5. Let  $1 \leq r \leq k - 1$ . Then

$$(2.9) \quad P(n_0; \underline{n}) = \sum_{S: |S|=r} P(n_0; \underline{n}_S) \cdot P(n_0 + n_S; \underline{n}_S).$$

Proof. Consider the case  $r = 1$ . Relation (2.9) becomes

$$(2.10) \quad P(n_0; \underline{n}) = \sum_{i=1}^k \frac{n_i}{n_0 + n_i} P(n_0 + n_i; \underline{n}_i).$$

This is proved by conditioning on the second cut set to fail, as follows.

$$\begin{aligned}
 P(n_0; \underline{n}) &= \sum_{i=1}^k P(C_0 \text{ and } C_i \text{ fail before the rest and } C_0 \text{ fails before } C_i) \\
 &= \sum_{i=1}^k P(C_0 \text{ and } C_i \text{ fail before the rest). } P(C_0 \text{ fails before } C_i \mid \\
 &\quad C_0 \text{ and } C_i \text{ fail before the rest}) \\
 &= \sum_{i=1}^k P(n_0 + n_i; \underline{n}_i) P(n_0; n_i)
 \end{aligned}$$

which is (2.10). In the above we used the fact that

$$\begin{aligned}
 P(C_0 \text{ fails before } C_i \mid C_0 \text{ and } C_i \text{ fail before the rest}) \\
 &= P(C_0 \text{ fails before } C_i \text{ in the series-parallel system consisting of} \\
 &\quad C_0 \text{ and } C_i) \\
 &= P(n_0; n_i).
 \end{aligned}$$

Relation (2.9), for  $r = 2, \dots, k - 1$ , is likewise obtained by conditioning on the cut sets that fail second, third,  $\dots$ , and  $(r+1)$  st. ||

(d) Augmentation. We now illustrate the idea of augmentation to prove recurrence relations for  $P(n_0; \underline{n})$ .

Theorem 2.6. For any subset  $S$  of  $\{1, \dots, k\}$ ,

$$(2.11) \quad P(n_0; \underline{n}_S) = P(n_0; \underline{n}) + [1 - P(n_0; \underline{n}_{S^c})] P(n_0 + n_S, \underline{n}_S).$$

Proof. Relation (2.11) for  $S = \{i\}$  becomes

$$(2.12) \quad P(n_0; \underline{n}_i) = P(n_0; \underline{n}) + [1 - P(n_0; n_i)] P(n_0 + n_i; \underline{n}_i).$$

We establish (2.12) first. Consider a series-parallel system with cut sets  $\{C_j, j = 0, 1, \dots, i - 1, i + 1, \dots, k\}$  and augment it with the cut set

$C_i$  to form another series-parallel system B. Then  $C_0$  fails before the rest in system A if and only if  $C_0$  fails before  $C_1, \dots, C_k$  in system B or if  $C_i$  fails first followed by  $C_0$  next in system B. Thus,

$$\begin{aligned} P(n_0; \underline{n}_i) &= P(n_0; \underline{n}) + P(C_i \text{ fails first and } C_0 \text{ fails next}) \\ &= P(n_0; \underline{n}) + P(C_0 \text{ and } C_i \text{ fail before the rest}) \cdot P(C_i \text{ fails first} \\ &\quad \text{and } C_0 \text{ fails next} \mid C_0 \text{ and } C_i \text{ fail before the rest}) \\ &= P(n_0; \underline{n}) + P(n_0 + n_i; \underline{n}_i) P(n_i; n_0), \text{ which is equivalent to (2.12).} \end{aligned}$$

Relation (2.11) is proved similarly by augmenting the series-parallel system with cut sets  $\{C_0, C_i, i \in S\}$  with cut sets  $\{C_i, i \notin S\}$ . ||

(e) Order statistics. We now illustrate the use of order statistics in obtaining a compact and mathematically useful expression for  $P(n_0; \underline{n})$ .

Theorem 2.7.

$$(2.13) \quad P(n_0; \underline{n}) = \int_0^1 \prod_{i=1}^k (1 - x^{n_i}) n_0 x^{n_0-1} dx .$$

Proof. Associate a life time  $X_{ij}$  with the  $j^{\text{th}}$  component of the  $i^{\text{th}}$  cut set,  $j = 1, \dots, n_i, i = 0, 1, \dots, k$ . Assume that the  $X'_{ij}$ s are independent and identically distributed with a common continuous distribution. Then the  $n!$  patterns of component failures as determined by the life times  $X_{01}, \dots, X_{kn_k}$  of the  $n$  components are equally likely. Defining

$$X_i^* = \max_{1 \leq j \leq n_i} X_{ij}, \quad i = 0, 1, \dots, k, \text{ we have } P(n_0; \underline{n}) = P(X_0^* < \min(X_1^*, \dots, X_k^*)).$$

Assuming that the common distribution of the  $X'_{ij}$ s is uniform on  $(0, 1)$ , the right hand side of the above is easily evaluated as

$$\int_0^1 P(\min(X_1^*, \dots, X_k^*) > x) n_0 x^{n_0-1} dx = \int_0^1 \prod_{i=1}^k (1 - x^{n_i}) n_0 x^{n_0-1} dx. \quad \parallel$$

In the rest of this paper, we will find that (2.13) provides the most mathematically tractable formula. Relations (2.7), (2.8), (2.10), and (2.12) provide recurrence relations that will be useful in computations. Though relations (2.3) and (2.4) give direct expressions for  $P(n_0; \underline{n})$ , they may not be as useful in computations since they involve sums of a relatively large number of terms.

(f) Special cases. We now give exact formulae for  $P(n_0; \underline{n})$  in some special cases. Theorem 2.1 dealt with the case of two cut sets.

Lemma 2.8. When  $k = 2$ ,

$$(2.14) \quad P(n_0; n_1, n_2) = \frac{\binom{n_1}{n_0} \binom{n_2}{n_0} \binom{n+n_0}{n_0}}{n(n_0+n_1)(n_0+n_2)},$$

and when  $k = 3$ ,

$$(2.15) \quad P(n_0; n_1, n_2, n_3) = \frac{\binom{n_1}{n_0} \binom{n_2}{n_0} \binom{n_3}{n_0}}{n} \left[ \sum_{1 \leq i < j \leq 3} \frac{\binom{2n_0+n_i+n_j}{n_0} \binom{n_0+n_i+n_j}{n_0} \binom{n_0+n_i+n_j}{n_0}}{\binom{n_0+n_i}{n_0} \binom{n_0+n_j}{n_0} \binom{n_0+n_i+n_j}{n_0}} \right].$$

Proof. From relation (2.3),

$$P(n_0; n_1, n_2) = \frac{\binom{n_1}{n_0} \binom{n_2}{n_0}}{\binom{n_0+n_1}{n_0} n} + \frac{\binom{n_1}{n_0} \binom{n_2}{n_0}}{\binom{n_0+n_2}{n_0} n} = \frac{\binom{n_1}{n_0} \binom{n_2}{n_0} \binom{n+n_0}{n_0}}{n(n_0+n_1)(n_0+n_2)}.$$

Relation (2.15) follows from (2.14) and the recurrence relation (2.8).  $\parallel$

The next lemma treats the case where all the cut sets  $C_1, \dots, C_k$  or all but one of them have equal numbers of components.

Lemma 2.9.

$$(2.16) \quad P(n_0; \underbrace{m, \dots, m}_k) = \frac{n_0}{m} B\left(\frac{n_0}{m}, k+1\right),$$

and

$$(2.17) \quad P(n_0; \underbrace{p, m, \dots, m}_{k-1}) = \frac{n_0}{m} \left[ B\left(\frac{n_0}{m}, k\right) - B\left(\frac{n_0+p}{m}, k\right) \right], \text{ where } B(\dots) \text{ is}$$

the usual Beta function.

Proof. From Theorem 2.7,

$$P(n_0; m, \dots, m) = \int_0^1 (1 - x^m)^{n_0} x^{n_0-1} dx = \frac{n_0}{m} \int_0^1 (1 - y)^{k-1} y^{\frac{n_0}{m}-1} dy$$

$$= \frac{n_0}{m} B\left(\frac{n_0}{m}, k+1\right).$$

by substituting  $y = x^m$ . From the recurrence relation (2.12)

$$P(n_0; p, m, \dots, m) = P(n_0; m, \dots, m) - \frac{n_0}{n_0+p} P(n_0+p; m, \dots, m)$$

$$= \frac{n_0}{m} \left[ B\left(\frac{n_0}{m}, k\right) - B\left(\frac{n_0+p}{m}, k\right) \right], \text{ using (2.16). } \parallel$$

3. Qualitative Properties of  $P(n_0; \underline{n})$ . The expressions obtained in Section 2

are used in this section to express certain intuitively obvious properties of  $P(n_0; \underline{n})$  in a precise form. These can provide a theoretical basis for setting up suitable maintenance and inspection policies to guard against failure of "weaker" cut sets, that is those that have a higher probability of failing first.

We first notice that  $P(n_0; \underline{n})$  is permutation invariant in  $(n_1, \dots, n_k)$ . Next, since  $P(n_0; \underline{n})$  is the probability that cut set  $C_0$  fails before any of the cut sets  $C_1, \dots, C_k$ , it is fairly obvious that  $P(n_0; \underline{n})$  will decrease

if  $n_i$ 's,  $i = 1, \dots, k$  increase or if  $n_0$  decreases. This is the content of the next theorem.

**Theorem 3.1.** (a) For each  $n_0$ ,  $P(n_0; \underline{n})$  is strictly increasing in each of the arguments  $n_1, \dots, n_k$ . (b) For fixed  $n_1, \dots, n_k$ ,  $P(n_0; \underline{n})$  is strictly decreasing in  $n_0$ .

**Proof.** (a) Let  $n_1 \leq n'_1, \dots, n_k \leq n'_k$ , with at least one strict inequality.

Then

$$\prod_{i=1}^k (1 - x^{n_i}) < \prod_{i=1}^k (1 - x^{n'_i}) \quad \text{for } 0 < x < 1.$$

Relation (2.13) and the above inequality show that  $P(n_0; \underline{n}) < P(n_0; \underline{n}')$ .

(b) To prove this part, we use an alternative form of (2.13), namely

$$(3.1) \quad P(n_0; \underline{n}) = \int_0^1 x^{n_0} d\left(1 - \prod_{i=1}^k (1 - x^{n_i})\right),$$

and the inequality  $x^{n_0} > x^{n'_0}$  for  $0 < x < 1$ , whenever  $n_0 < n'_0$ . ||

The next theorem shows an ordering in the probabilities of cut set  $C_i$  causing the failure of the system,  $i = 0, 1, \dots, k$ , which agrees with the intuitive notion that smaller cut sets will fail first.

**Theorem 3.2.** Let  $n_0 \leq n_1 \leq \dots \leq n_k$ . Then

$$(3.2) \quad P(n_0; n_1, \dots, n_k) \geq P(n_1; n_0, n_2, \dots, n_k) \geq \dots \geq P(n_k; n_0, n_1, \dots, n_{k-1}).$$

**Proof.** It is clear from the permutation invariant property of  $P(n_0; \underline{n})$  that it is enough to prove that  $n_0 \leq n_1$  implies  $P(n_0; n_1, \dots, n_k) \geq P(n_1; n_0, n_2, \dots, n_k)$ . Rewriting relation (3.1), we obtain

$$(3.3) P(n_0; \underline{n}) = - \int_0^1 x^{n_0} d\left(\prod_{i=1}^k (1 - x^{n_i})\right) = \int_0^1 n_1 x^{n_0 + n_1 - 1} f(x) dx + \int_0^1 x^{n_0} (1 - x^{n_1}) (-f'(x)) dx,$$

where  $f(x) = \prod_{i=1}^k (1 - x^{n_i})$ . Notice that  $f(x) \geq 0$  and is decreasing in  $x$  and

thus  $-f'(x) \geq 0$ . For  $n_0 \leq n_1$ ,  $n_1 x^{n_0 + n_1 - 1} \geq n_0 x^{n_0 + n_1 - 1}$  and

$x^{n_0} (1 - x^{n_1}) \geq x^{n_1} (1 - x^{n_0})$ . Using these inequalities in (3.3), we obtain

$$P(n_0; n_1, \dots, n_k) \geq P(n_1; n_0, n_2, \dots, n_k). \quad ||$$

The next theorem establishes a homogeneity property of  $P(n_0; \underline{n})$ , which is not intuitively obvious.

Theorem 3.3. Let  $m$  be a positive integer and let  $m\underline{n} = (mn_1, \dots, mn_k)$ .

Then

$$(3.4) \quad P(n_0; \underline{n}) = P(mn_0; m\underline{n}).$$

Proof. From relation (2.13),

$$P(mn_0; m\underline{n}) = \int_0^1 \prod_{i=1}^k (1 - x^{mn_i}) mn_0 x^{mn_0 - 1} dx = \int_0^1 \prod_{i=1}^k (1 - y^{n_i}) n_0 y^{n_0 - 1} dy = P(n_0; \underline{n}).$$

wherein we have substituted  $y = x^m$ . ||

From theorem 3.3 it is natural to extend the domain of the function  $P(n_0; \underline{n})$  as follows.

$$P(\lambda_0; \underline{\lambda}) = \int_0^1 \prod_{i=1}^k (1 - x^{\lambda_i}) \lambda_0 x^{\lambda_0 - 1} dx, \quad \text{for } \lambda_0 \geq 0, \lambda_1 \geq 0, \dots, \lambda_k \geq 0.$$

$P(\lambda_0; \underline{\lambda})$  would represent the probability that a cut set  $C_0$  fails before cut sets  $C_1, \dots, C_k$  in some series-parallel system when all the  $\lambda$ 's are

rational numbers. When some of the  $\lambda$ 's are irrational,  $P(\lambda_0; \underline{\lambda})$  would represent a limit of such probabilities. This is the content of the next theorem which is stated without proof since it is obvious.

Theorem 3.4. Let  $n_0, n_1, \dots, n_k$  be such that  $n \rightarrow \infty$  and  $\frac{n_i}{n} \rightarrow \lambda_i$ ,

$\lambda_i \geq 0, i = 0, 1, \dots, k$ . Then

$$(3.5) \quad P(n_0; \underline{n}) \rightarrow P(\lambda_0; \underline{\lambda}) .$$

To state and prove the two remaining theorems, which show that  $P(n_0; \underline{n})$  has further special ordering properties we have to familiarize the reader with the concept of majorization, Schur functions, and functions decreasing in transposition. Majorization (see Definition 3.5) is a partial ordering in  $R_k$ ,  $k$ -dimensional Euclidean space. A Schur function is a function that is monotone with respect to this partial ordering. Many well known inequalities arising in probability and statistics are equivalent to saying that certain functions are Schur functions. Theorem 3.8 below shows that  $P(n_0; n_1, \dots, n_k)$  is a Schur-concave function in  $n_1, n_2, \dots, n_k$ . This property is then utilized to develop bounds on  $P(n_0; \underline{n})$ . Before stating and proving this result we now give definitions of majorization and Schur functions.

Definition 3.5. Given a vector  $\underline{x} = (x_1, \dots, x_k)$ , let  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[k]}$  denote a decreasing rearrangement of  $x_1, \dots, x_n$ . A vector  $\underline{x}$  is said to majorize a vector  $\underline{x}'$  (in symbols  $\underline{x} \underset{m}{\geq} \underline{x}'$ ) if

$$\sum_{i=1}^j x_{[i]} \geq \sum_{i=1}^j x'_{[i]}, \quad j=1, 2, \dots, k-1,$$

and

$$\sum_{i=1}^k x_{[i]} = \sum_{i=1}^k x'_{[i]} .$$

Notice that whenever  $(\pi_1, \dots, \pi_k)$  is a permutation of  $(1, \dots, k)$  and  $\underline{x}' = (x_{\pi_1}, \dots, x_{\pi_k})$ , we have  $\underline{x} \stackrel{m}{\geq} \underline{x}'$  and  $\underline{x}' \stackrel{m}{\geq} \underline{x}$ .

A useful characterization of majorization is given by Hardy, Littlewood, and Pólya (1952), p. 47.

Lemma 3.6.  $\underline{x} \stackrel{m}{\geq} \underline{x}'$  if and only if there exists a finite number, say  $r$ , of vectors  $\underline{x}^{(1)}, \dots, \underline{x}^{(r)}$  such that  $\underline{x} = \underline{x}^{(1)} \stackrel{m}{\geq} \dots \stackrel{m}{\geq} \underline{x}^{(r)} = \underline{x}'$  and such that  $\underline{x}^{(i)}$  and  $\underline{x}^{(i+1)}$  differ in two coordinates only,  $i = 1, 2, \dots, r-1$ .

Definition 3.7. A function  $f: R_k \rightarrow R$  is said to be a Schur-convex (Schur-concave) function if  $f(\underline{x}) \geq (\leq) f(\underline{x}')$  whenever  $\underline{x} \stackrel{m}{\geq} \underline{x}'$ .

Functions which are either Schur-convex or Schur-concave are called Schur functions. Note that a Schur function  $f$  is necessarily permutation-invariant.

The following are some examples of Schur-functions:

$$f_1(\underline{x}) = \prod_{i=1}^k \phi(x_i), \text{ where } \phi \text{ is log-convex,}$$

$$f_2(\underline{x}) = \sum_{i=1}^j x_{[i]}, \quad 1 \leq j \leq k.$$

The function  $f_3(\underline{x}) = \sum_{i=1}^k x_i$  is both Schur-convex and Schur-concave.

Theorem 3.8. For each  $\lambda_0$ ,  $P(\lambda_0; \underline{\lambda})$  is Schur-concave in  $\underline{\lambda}$  for  $\lambda_1 \geq 0, \dots, \lambda_k \geq 0$ .

Proof. Fix  $x \in [0, 1]$ . The function  $\phi_x(\lambda) = (1-x^\lambda)$  is easily checked to be log-concave in  $\lambda$ ,  $\lambda \geq 0$ . Thus, for each  $x$ ,  $h_x(\lambda_1, \dots, \lambda_k) =$

$$n_0 x^{n_0-1} \prod_{i=1}^k \phi_x(\lambda_i) \text{ is Schur-concave in } \underline{\lambda}.$$

Since positive linear combinations and limits of Schur-concave functions are Schur-concave functions,  $P(\lambda_0; \underline{\lambda}) = \int_0^1 h_x(\underline{\lambda}) dx$  is a Schur-concave function in  $\underline{\lambda}$ . ||

The implication of the above theorem is that cut set  $C_0$  is more likely to fail first if the remaining cut sets are homogeneous in size than if they are heterogeneous in size when the total number of components is fixed.

In the course of proving Theorem 3.2 we developed formula (3.4) for the probability  $P[C_0 < C_{\pi_1} < C_{\pi_2} < \dots < C_{\pi_k}]$ , where  $\pi = (\pi_1, \dots, \pi_k)$  is a permutation of  $\{1, 2, \dots, k\}$ . Next we show that the collection of these probabilities is partially ordered with respect to the  $n!$  permutations  $\pi$ . We need the following definition.

**Definition 3.9.** A function  $f: R^k \rightarrow R$  is said to be decreasing in transposition (DT) if  $f(\underline{x}) \geq f(\underline{x}')$  wherever  $\underline{x}$  and  $\underline{x}'$  are two vectors that differ in two coordinates only, say  $i$  and  $j$ ,  $i < j$ , with  $x_i = x'_j$ ,  $x_j = x'_i$ , and  $x_i \leq x_j$ . (See Hollander, Proschan, and Sethuraman (1975) for a study of DT functions and their applications to ranking problems.

**Theorem 3.10** Let  $n_0 \leq n_1 \leq \dots \leq n_k$ . Then  $P[C_0 < C_1 < \dots < C_k]$  is DT in  $(n_0, n_1, \dots, n_k)$ .

**Proof.** From Definition 3.9, it suffices to show that

$P[C_{i_0} < \dots < C_{i_r} < \dots < C_{i_s} < \dots < C_{i_k}] \geq P[C_{i_0} < \dots < C_{i_s} < \dots < C_{i_r} < \dots < C_{i_k}]$ , where the second sequence of C's differs from the first only in that  $i_r$  and  $i_s$  are interchanged,  $i_r < i_s$ , and  $n_{i_r} \leq n_{i_s}$ . From (2.2),

$$P[C_{i_0} < \dots < C_{i_r} < \dots < C_{i_s} < \dots < C_{i_k}] - P[C_{i_0} < \dots < C_{i_s} < \dots < C_{i_r} < \dots < C_{i_k}]$$

$$= \prod_{i=0}^k n_i \left[ \prod_{h=1}^k \left( \sum_{j=1}^h n_{i_j} \right)^{-1} - \prod_{h=1}^k \left( \sum_{j=1}^h n'_{i_j} \right)^{-1} \right],$$

where  $n'_{i_j} = n_{i_j}$  for  $j \neq r, s$  and  $n'_{i_r} = n_{i_s}$ ,  $n'_{i_s} = n_{i_r}$ . Since  $n_{i_r} \leq n_{i_s}$ , the difference above is nonnegative, and the desired result follows. ||

The next lemma gives an inequality for  $P(n_0; \underline{n})$  using the Schur-concave property established in Theorem 3.8.

Lemma 3.11. Let  $n_1, \dots, n_k$  be positive numbers. Let  $N = \sum_{i=1}^k n_i$ ,  $n^* = \min(n_1, \dots, n_k)$ ,  $r = N/k$ , and  $p = N - (k-1)n^*$ . Then

$$(3.6) \quad \frac{n_0}{n^*} \left[ E\left(\frac{n_0}{n^*}, k\right) - B\left(\frac{n_0+p}{n^*}, k\right) \right] \leq P(n_0; \underline{n}) \leq \frac{n_0}{r} B\left(\frac{n_0}{r}, k+1\right).$$

Proof. It is easy to see that  $\frac{p, n^*, \dots, n^*}{(k-1)} \geq (n_1, \dots, n_k) \geq (r, \dots, r)$ . From the Schur-concavity of  $P(n_0; \underline{n})$  in  $\underline{n}$ ,

$$P(n_0; \frac{p, n^*, \dots, n^*}{(k-1)}) \leq P(n_0; \underline{n}) \leq P(n_0; (r, \dots, r)).$$

Using the evaluation of the two bounds in the above inequality as given by relations (2.16) and (2.17) of Lemma 2.9 yields (3.6). ||

4. Extensions and Generalizations. In this section we present a straightforward generalization of the model studied in the previous sections.

Suppose that an alarm rings as soon as some cut set  $C_i$ , in our series-parallel system, reaches size  $a$ , where  $a$  is a non-negative integer,  $n_i \geq a + 1$ ,  $i = 0, 1, \dots, k$ . The level  $a$  may represent a critically small size for

cut sets in the system and the ringing of the alarm would then indicate that the system needs attention and repair. We wish to compute the probability  $P_a(n_0; \underline{n})$  that cut set  $C_0$  reaches size  $a$  when the alarm rings, that is, before any other cut set reaches size  $a$ . The model studied in the previous sections corresponds to the case  $a = 0$ . The next theorem provides a simple expression for  $P_a(n_0; \underline{n})$ .

Theorem 4.1. Let  $n_i$ , the number of components in  $C_i$ , satisfy  $n_i \geq a + 1$ ,  $i = 0, 1, \dots, k$ . Let  $P_a(n_0; \underline{n})$  be the probability that cut set  $C_0$  reaches size  $a$  before any of the other cut sets do. Then

$$(4.1) \quad P_a(n_0; \underline{n}) = \int_0^1 \prod_{i=1}^k [1 - \sum_{r=n_i-a}^{n_i} \binom{n_i}{r} x^r (1-x)^{n_i-r}] \cdot \binom{n_0}{n_0-a} x^{n_0-a-1} (1-x)^a dx$$

Proof. Associate independent and identically distributed life times,  $X_{01}, \dots, X_{0n_0}, \dots, X_{k1}, \dots, X_{kn_k}$  with the components of the system, as in the proof of Theorem 2.7. Assume that these random variables have the uniform distribution on  $[0,1]$ . Let  $X_i^{**}$  be the  $(n_i-a)^{th}$  order statistic from  $X_{i1}, \dots, X_{in_i}$ . Then

$$\begin{aligned} P_a(n_0; \underline{n}) &= P\{C_0 \text{ reaches size } a \text{ before any of the others do}\} \\ &= P(X_0^{**} < \min(X_1^{**}, \dots, X_k^{**})) \\ &= \int_0^1 \prod_{i=1}^k P(X_i^{**} > x) dP(X_0^{**} \leq x) \\ &= \int_0^1 \prod_{i=1}^k [1 - \sum_{r=n_i-a}^{n_i} \binom{n_i}{r} x^r (1-x)^{n_i-r}] \binom{n_0}{n_0-a} x^{n_0-a-1} (1-x)^a dx. \quad || \end{aligned}$$

The following theorem shows that  $P_a(n_0; \underline{n})$  is Schur-concave in  $\underline{n}$ .

Theorem 4.2. Fix  $a \geq 0$  and  $n_0$ . Then  $P_a(n_0; \underline{n})$  is a Schur-concave function of  $\underline{n}$ .

Proof. Let  $\phi_x(m) = 1 - \sum_{r=m-a}^m \binom{m}{r} x^r (1-x)^{n-r}$ . From Karlin (1968), Chapter 3, Theorem 5.4,  $\phi_x(m)$  is log-concave in  $m$  for each  $x$  in  $(0,1)$ . Thus

$\prod_{l=1}^k \phi_x(n_l)$  is log-concave in  $\underline{n}$ . From (4.1), we see that  $P_a(n_0; \underline{n})$  is an integral of this Schur-concave function with respect to a positive measure. Thus  $P_a(n_0; \underline{n})$  is Schur-concave in  $\underline{n}$ . ||

In the model considered so far we have assumed that our series-parallel system has cut sets which do not have common components. This is only a very special subclass of coherent systems. A general coherent system can be considered as a series-parallel system where cut sets are permitted to have overlapping sets of components. It would therefore be useful to study how far the results of this paper can be generalized in this direction.

We have assumed that all components are equally likely to fail or have the same reliability. It would be more realistic to assume that components of certain cut sets are more reliable than others. This is another area that needs to be studied.

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## 20 Abstract

→ parallel,  $i = 0, 1, \dots, k$ . No two cut sets have a component in common. It is assumed that after  $t$  components have failed, each of the remaining components is equally likely to fail,  $t = 0, 1, \dots$ . We <sup>consider</sup> study the probability that the system fails because a specified cut set  $C_0$ , say fails first. We <sup>are obtained</sup> obtain several alternative expressions and recurrence relations for this probability. Some of these formulae are useful in computations while others permit us to derive qualitative features like monotonicity, Schur-concavity, asymptotic limits, etc. These results are extended to the situation where some cut set size is first reduced to  $a$ , where  $a$  is a specified positive integer.

The above model has applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.

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(Abstract Continued)

It is assumed that after  $t$  components have failed, each of the remaining components is equally likely to fail,  $t = 0, 1, \dots$ . We study the probability that the system fails because a specified cut set  $C_0$ , say fails first. We obtain several alternative expressions and recurrence relations for this probability. Some of these formulae are useful in computations while others permit us to derive qualitative features like monotonicity, Schur-concavity, asymptotic limits, etc. These results are extended to the situation where some cut set size is first reduced to  $a$ , where  $a$  is a specified positive integer.

The above model has applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.