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A SURVEY OF TRIAL FREE-BOUNDARY METHODS FOR THE NUMERICAL SOLUT--ETC(U)

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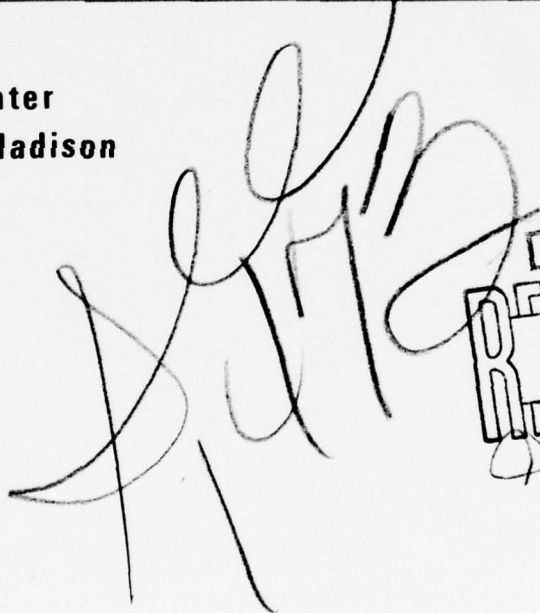
A SURVEY OF TRIAL FREE-BOUNDARY
METHODS FOR THE NUMERICAL
SOLUTION OF FREE BOUNDARY
PROBLEMS

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NUMERICAL SOLUTION OF FREE BOUNDARY PROBLEMS

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ABSTRACT

A free boundary problem is a (steady-state) boundary value problem involving differential equations on domains parts of whose boundaries, the free boundaries, are unknown and must be determined as part of the solution. A trial free-boundary method is a numerical method for solving free boundary problems in which the boundary is found by guessing the position of the boundary, solving a boundary-value problem in the resulting region, and using this solution to find a new guess for the position of the boundary, the procedure being repeated until the desired accuracy has been attained. This report gives a comprehensive survey of trial free-boundary methods.

AMS (MOS) Subject Classifications: 35J99; 35R99; 6502; 65N99

Key Words: Free boundary problems, Numerical methods, Trial free-boundary methods

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A SURVEY OF TRIAL FREE-BOUNDARY METHODS FOR THE NUMERICAL
SOLUTION OF FREE BOUNDARY PROBLEMS

Colin W. Cryer

0. Introduction

A free boundary problem (FBP; plural FBPS) is a (steady-state) boundary value problem involving differential equations on domains parts of whose boundaries, the free boundaries (FB; plural FBS), are unknown and must be determined as part of the solution. On such FBS the boundary conditions needed for a fixed boundary value problem (a problem for which the boundaries are known) are *supplemented by an additional boundary condition*.

The problem of seepage through a rectangular earth dam (see Figure 1) illustrates how FBPS arise.

In this problem water from an upstream reservoir (or head water) seeps through a rectangular earth dam to a lower reservoir (or tail water). The water only flows through the region Ω and the remainder of the dam remains dry. The air/water interface is denoted by Γ . It is assumed that the dam is so long that the flow is two-dimensional.

If Darcy's law holds then the flow is described by a velocity potential ϕ which is related to the hydraulic head h by

$$\phi = -Kh, \quad (1)$$

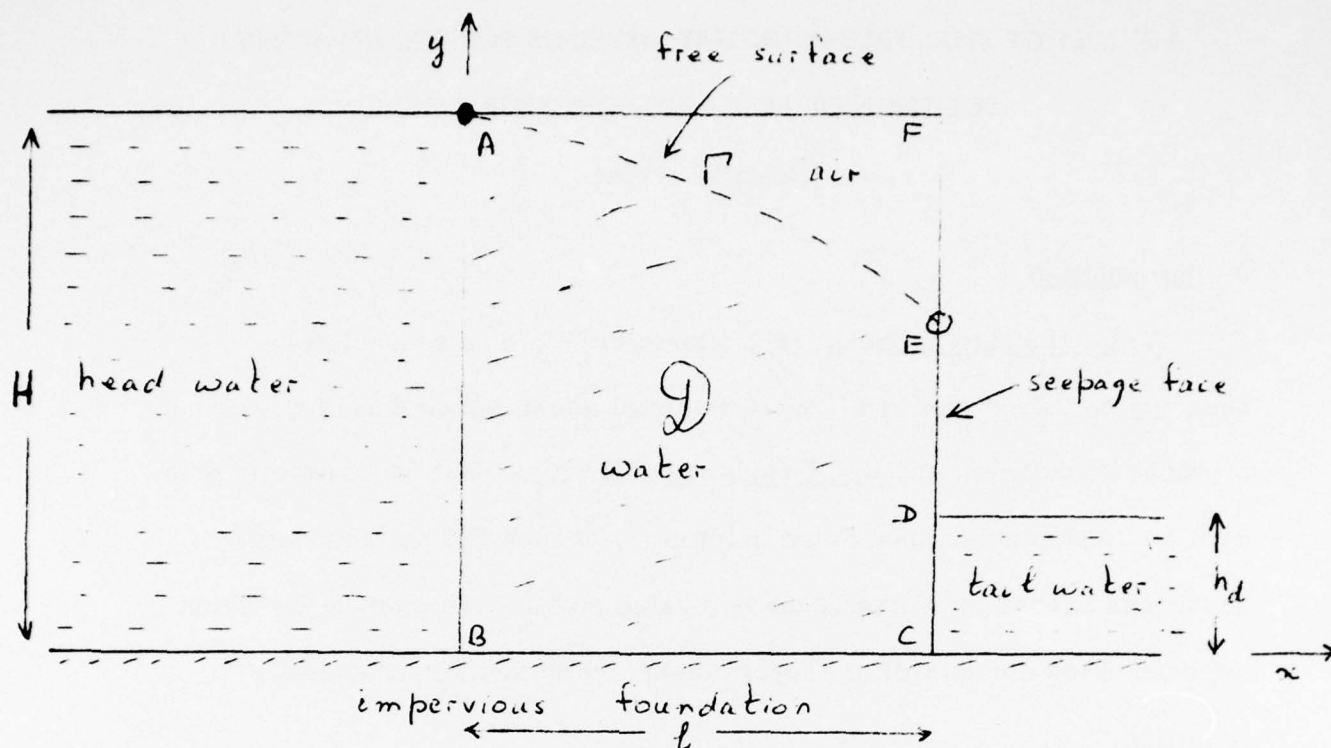


Figure 1: Seepage through a simple rectangular dam

where the constant K is the hydraulic conductivity. The hydraulic head satisfies Laplace's equation

$$h_{xx} + h_{yy} = 0, \text{ in } \Omega. \quad (2)$$

The boundary conditions are:

$$h = H, \text{ on } AB \text{ (interface with water at rest),}$$

$$\frac{\partial h}{\partial n} = 0, \text{ on } BC \text{ (impervious boundary),}$$

$$h = h_d, \text{ on } CD \text{ (interface with water at rest),} \quad (3)$$

$$h = y, \text{ on } DE \text{ (interface with air),}$$

$$\frac{\partial h}{\partial n} = 0, \text{ on } EA \text{ (streamline).}$$

If Γ were known then equation (2) with boundary conditions (3) would suffice to determine h . Since Γ is not known we have a FBP and an extra condition is required on Γ . This extra condition is:

$$h = \gamma, \text{ on EA (interface with air) .} \quad (4)$$

As illustrated by the above example, a FBP has the following structure:

$$\begin{aligned} \mathcal{A}u &= 0, \text{ in } \mathcal{D}, \\ \mathcal{B}u &= 0, \text{ on } \partial\mathcal{D}, \\ \mathcal{C}u &= 0, \text{ on } \Gamma, \end{aligned} \quad (5)$$

where u is the solution and where \mathcal{A} represents the linear or nonlinear elliptic differential equation(s) which must be satisfied by u on the open set \mathcal{D} . \mathcal{B} represents the linear or nonlinear boundary conditions which must be satisfied on the boundary $\partial\mathcal{D}$ of \mathcal{D} . If $\partial\mathcal{D}$ were completely known then the first two equations of (5) would define u completely. However, part of $\partial\mathcal{D}$, namely the FB Γ , is unknown and \mathcal{C} represents the additional linear or nonlinear boundary condition imposed on Γ .

In (5) it is to be understood that the open set \mathcal{D} may consist of the union of several domains, $\mathcal{D} = \bigcup_{i=1}^m \mathcal{D}_i$, in which case the first equation of (5) is to be interpreted as

$$\mathcal{A}_i u = 0 \text{ in } \mathcal{D}_i, \quad 1 \leq i \leq m.$$

The operators \mathcal{A}_i may or may not be the same.

The domains \mathcal{D}_i will be called free domains (FD; plural, FDS).

When it is necessary the number of domains will be indicated by saying that the problem is an m-domain FBP. When $m = 1$ we set $\mathcal{A} \equiv \mathcal{A}_1$, $\mathcal{D} \equiv \mathcal{D}_1$.

Using this terminology, we may describe the example of Figure 1 as a 1-domain FBP with FD $\partial\Omega$ and FB Γ .

Most of the FBPS which have been considered in the literature are 1-domain FBPS. A two-domain FBP would arise, for example, in a coastal region in which porous flow of fresh water occurred in a region Ω_1 and porous flow of salt water occurred in a region Ω_2 , the FB being the salt water/fresh water interface.

It is usually neither possible nor necessary to classify the boundary conditions on the FB into "fixed boundary conditions" and "supplementary boundary conditions", and the division of the boundary conditions on Γ into those represented by \mathcal{B} and that represented by \mathcal{C} is an arbitrary decision.

The FB Γ is defined to be that part of $\partial\Omega$ whose "shape" is unknown; the remainder of $\partial\Omega$ will be called the fixed boundary. Points at which the FB intersects the fixed boundary will be called points of detachment. If the location of a point of detachment is prescribed it is said to be fixed; otherwise it is said to be free. We illustrate these definitions by considering Figure 1. The point A is known so that A is a fixed point of detachment. The point E is not known so that E is a free point of detachment. The FB is EA; although the length of DE is not known its shape is known so that DE is part of the fixed boundary and not part of the FB.

FBPS have recently been considered in control theory, and this development is very important because it introduces new and exciting ideas. However, the vast majority of FBPS arise in continuum mechanics. In continuum mechanics a FBP arises in principle whenever there is a line or surface on which the solution is discontinuous, and the FB corresponds to this line of discontinuity or surface of discontinuity or singular surface. The discontinuities can arise for many reasons including:

- (i) Two different materials may be present: for example, air and water.
- (ii) The material may occur in different phases: for example, water and ice.
- (iii) The material properties may depend discontinuously upon certain variables: for example, the material may change from elastic to plastic when the stress reaches a critical value.
- (iv) There may be a jump in the pressure or velocity.

FBPS arise in the following branches of continuum mechanics: fluid mechanics, porous flow, mechanics of solids, heat conduction and diffusion, electromagnetism, and gravitation.

We are in the process of publishing a series of surveys on the various types of FBPS (the survey on porous flow FBPS has already appeared (Cryer [1976])). We are also in the process of publishing a series of surveys on the various mathematical and numerical aspects of FBPS.

The present survey considers the class of numerical methods known as trial free boundary methods, which are described in the following section.

1. General description of trial free boundary methods

We repeat the basic FBP:

$$\begin{aligned} \mathcal{A}u &= 0, \text{ in } \mathcal{D}, \\ \mathcal{B}u &= 0, \text{ on } \partial\mathcal{D}, \\ \mathcal{C}u &= 0, \text{ on } \Gamma. \end{aligned} \tag{1}$$

One approach to solving this FBP numerically is to generate a sequence of trial FBS $\Gamma^{(k)}$ and approximations $u_h^{(k)}$ to the corresponding trial solutions $u^{(k)}$ as follows:

Step 0. Guess an initial trial FB, $\Gamma^{(0)}$ say.

Step 1. Given $\Gamma^{(k)}$ let $\mathcal{D}^{(k)}$ be the corresponding domain. Compute an approximation, $u_h^{(k)}$ say, to the solution $u^{(k)}$ of the problem

$$\begin{aligned} \mathcal{A}u^{(k)} &= 0, \text{ in } \mathcal{D}^{(k)}, \\ \mathcal{B}u^{(k)} &= 0, \text{ on } \partial\mathcal{D}^{(k)}. \end{aligned} \tag{2}$$

Step 2. Given $\Gamma^{(k)}$ and $u_h^{(k)}$ compute a new trial FB $\Gamma^{(k+1)}$ by requiring that $\mathcal{C}u_h^{(k)}$ should be approximately equal to zero on $\Gamma^{(k+1)}$; i.e. "move the boundary" from $\Gamma^{(k)}$ to $\Gamma^{(k+1)}$.

Following Birkhoff [1961] we will call this method, and variations thereof, a trial free boundary method. (Previously, we have called methods of this type "move-the-boundary" methods (Cryer [1968, 1970a]) but Birkhoff's terminology is more descriptive.)

In Step 2 the computation of the approximate solution $u^{(k)}$ requires the approximate solution of a fixed boundary value problem, which can be done using any standard method such as finite elements or finite differences. We then speak of a trial free boundary method using finite elements, a trial free boundary method using finite differences, etc.

Steps 1 and 2 in a trial free boundary method, namely the computation of the approximate solution $u_h^{(k)}$ and the generation of the new trial FB $\Gamma^{(k+1)}$, are essentially independent and are considered separately in sections 2 and 3, respectively. In section 3.4 we summarize the numerical experiences of workers who have used trial free boundary methods and make some suggestions regarding their use. Finally, error estimates and related questions are considered in section 4.

The advantages of trial free boundary methods are:

- (i) Trial free boundary methods are, in principle, applicable to all FBPS and require no preliminary analysis.
- (ii) The main computational effort is in Step 1 which involves the solution of a linear problem if \mathcal{A} is linear.
- (iii) All computations are carried out in the physical domain so that "physical intuition" can be used and a "feeling" for the solution can develop as the computation proceeds.

Because of these advantages, trial free boundary methods are particularly well suited for hand computation, and before the advent of computers trial free boundary methods were the most widely used numerical methods for solving FBPS.

Unfortunately, trial free boundary methods also have several disadvantages:

- (a) To implement Step 1 one must be able to compute the approximate solutions $u_h^{(k)}$ for the general domains $\Omega^{(k)}$.
- (b) It is difficult to formulate Step 2 precisely. Each problem seems to require a different technique and while an individual research worker can accumulate experience, it is hard to transmit this knowledge to others. Many cases of non-convergence have been reported.
- (c) It is hard to achieve high accuracy and it is hard to estimate the error in $u_h^{(k)}$ and $\Gamma^{(k)}$. In some problems the shape of the FB is very sensitive to small errors in the condition $Cu = 0$, and it is hard to achieve high accuracy near points of separation. In general, it is difficult to obtain accuracy greater than about 1%.

Because of these disadvantages it proved relatively difficult to implement trial free boundary methods on computers, particularly on computers with small memories. There were a few successful implementations during the early 1960's, but it is only since about 1967 that there has been a substantial number of successful implementations.

For a number of years we felt that trial free boundary methods would and should be replaced by alternative methods such as numerical methods based upon variational inequalities. However, we have recently changed our opinion for the following reasons:

- (α) There are many FBPS, such as viscous flow fluid mechanics FBPS, for which trial free boundary methods are at present the only available method.
- (β) The difficulty of implementing Step 1 has been reduced with the use of finite elements and other techniques which are more flexible than finite differences.
- (γ) Step 2 has recently been successfully implemented in several applications using "integral methods" (see section 3.2.2).
- (δ) Although error estimates and convergence proofs are still lacking, there have been a large number of successful applications.

We, therefore, believe that trial free boundary methods will remain a useful and competitive approach to the numerical solution of certain classes of FBPS.

2. Step 1: Computation of the approximate trial solution

The approximate trial solution $u_h^{(k)}$ is the approximate solution of the fixed boundary value problem

$$\begin{aligned} \mathcal{A}u^{(k)} &= 0, \quad \text{in } \mathcal{D}^{(k)}, \\ \mathcal{B}u &= 0, \quad \text{on } \partial\mathcal{D}^{(k)}, \end{aligned} \tag{1}$$

and $u_h^{(k)}$ can be computed by any approximation method for fixed boundary value problems.

The term approximation method includes both numerical methods (such as the method of finite differences) and nonnumerical methods (such as analog methods). Each of the following subsections discusses the solution of $u_h^{(k)}$ by a particular approximation method.

Birkhoff [1971] gives a very readable introduction to numerical methods for fixed boundary value problems. Extensive bibliographies (not restricted to elliptic equations) are given by Voight [1967], Giese [1969], and Forsythe [1971]; the bibliography of Giese provides an exhaustive coverage of early work. Among the many symposia on the subject are Birkhoff and Varga [1969], and Hubbard [1976]. Specialized texts discussing particular approximation methods are referred to in the appropriate subsections below.

In the remainder of this section we make some general remarks about the relative efficiency of different methods of computing $u_h^{(k)}$. The relative efficiency of approximation methods for FBPS is, of course, closely related to their relative efficiency for fixed boundary problems, but there are some interesting differences which deserve further study:

- (i) There is growing interest in comparing the efficiency of numerical methods for solving the fixed boundary value problem (1).

Terry [1967] has compared integral equation methods and finite difference methods for Neumann boundary value problems and finds that the finite difference methods are faster. It is very likely that comparisons between finite difference methods and finite element methods for solving fixed boundary value problems will soon be available.

- (ii) All the numerical methods for solving (1) require the solution of a system of n algebraic equations of the form

$$A_h^{(k)} U_h^{(k)} = B_h^{(k)} \quad (2)$$

where $U_h^{(k)}$ and $B_h^{(k)}$ are in n -dimensional Euclidean space R^n and $A_h^{(k)} : R^n \rightarrow R^n$. The equations (2) are linear if \mathcal{A} and \mathcal{B} in (1) are linear; otherwise the equations (2) are nonlinear.

Basic references on the solution of systems of linear algebraic equations include: Varga [1962], Wilkinson [1965], Young [1971], and Stewart [1973]. Basic references on the solution of systems of nonlinear algebraic equations include: Ortega and Rheinholdt [1970], Rabinowitz [1970], Byrne and Hall [1973], and Rheinboldt [1974].

If $A_h^{(k)}$ is linear then the numerical methods for solving (2) fall into two categories: direct methods and iterative methods. The relative efficiency of direct methods and iterative methods for solving (2) depends upon the structure of $A_h^{(k)}$ and computing facilities which are available.

When solving (2) the previous solution $U_h^{(k-1)}$ provides a good initial approximation to $U_h^{(k)}$, so that the efficiency of iterative methods is greater for FBPS than for general fixed boundary value problems.

If $A_h^{(k)}$ is nonlinear the situation is more complex, but it seems likely that the efficiency of the nonlinear generalizations of iterative methods for linear equations will also be greater for FBPS than for general fixed boundary problems.

It is also possible that the efficiency of direct methods could be improved. For example, if the equations are suitably ordered, then many of the computations required to solve for $U_h^{(k)}$ using Gaussian elimination will already have been performed when solving for $U_h^{(k-1)}$, so that judicious preservation of previous results should lead to a substantial reduction in computing time.

(iii) There is a close relationship between the popularity (and efficiency ?) of a given approximation method for solving (1) and the computing facilities available. For example, integral equation methods and finite element methods only became popular when computers with large high speed memories became generally available. This relationship between approximation methods and computing facilities is clearly apparent from the references quoted in this survey which span a period of sixty years.

The relationship between numerical methods and computing facilities has not been very obvious in recent years because during the period 1960-1975 the computers which have been available have been rather similar from a numerical (as opposed to systems) viewpoint. A typical computer has had a microsecond arithmetic unit, a reasonably large microsecond core store, and a large secondary store. A new era is now beginning in which a much greater variety of computers will be available. Already, the parallel computer ILLIAC IV which has 64 separate arithmetic units has been used to solve FBPS (MacCormack and Stevens [1974]). A corresponding re-evaluation of numerical methods will occur. To mention but one possibility, when using a trial free boundary method it may be most efficient to make several guesses for the FB and solve the corresponding fixed boundary value problems simultaneously.

It is impossible to predict long-term future developments in computers, and it is quite possible that approximation methods which are at present regarded as out-of-date will again become popular. To mention one possibility, suppose that it became possible to build a picosecond (10^{-12} second) computer but that because of technical and financial restrictions the storage consisted of a 1000-word picosecond store with a million-word nanosecond (10^{-9} second) secondary store. This computer would be 10^9 times faster than the computers of 1950, but the storage allocation problems would be very similar. Iterative methods of solving (2), which require less storage than direct methods, would again become very popular.

(iv) Most approximation methods for fixed boundary value problems are based on the assumption that the solution is smooth. This assumption is usually true if the boundary is smooth, but most FBPS involve boundaries with corners in the neighbourhood of which the solution can have singularities. If the corner is a point of intersection of the FB with the fixed boundary, then the analysis of the singularity is even more difficult.

In the literature on trial free boundary methods there has been little discussion of the treatment of corner singularities. This is clearly a subject which requires further study, and we discuss it briefly in section 4.2.

(v) All numerical methods for the fixed boundary value problem (1) require that a decision be made about the location of certain special points which we will call "meshpoints". Examples of meshpoints are: netpoints (for finite difference methods); nodes (for finite element methods); quadrature points (for integral equation methods); and collocation points.

It is desirable that the choice of meshpoints should be made automatically by the computer rather than by hand for several reasons:

- (a) When using a trial-free-boundary method, the domain changes at every iteration.
- (b) If the solution is ill-behaved in a certain subregion of D , as is often the case for FBPS, it is desirable to be able to automatically refine the mesh in this subregion.

- (c) The most effective check of numerical results is to repeat the computation with a finer mesh. The labor involved in data preparation increases rapidly as the mesh is refined, and if the data is prepared by hand the necessary checks may not be carried out.

Although automatic mesh-generation programs are feasible, almost all programs for trial free boundary methods have used a substantial amount of manually prepared data. It is to be hoped that this will soon be remedied.

It is also desirable that the meshpoints be chosen so that the approximations $u_h^{(k)}$ change "continuously" as k increases. For example, Arms and Gates [1957] used a trial free boundary method with finite differences and the same grid for all k . Arms and Gates [1957, p. 6] observe that discontinuities occurred whenever "moving" $\Gamma^{(k-1)}$ to $\Gamma^{(k)}$ involved crossing a gridpoint, and these discontinuities invitiated their method.

- (vi) Finally, when $u_h^{(k)}$ has been computed it is necessary in Step 2 to compute $Cu_h^{(k)}$, and it is of course desirable that this computation should be relatively easy.

As is clear from the above remarks, it is not possible to indicate which approximation method is "best" for Step 1. For several years finite differences were widely used. However, it is relatively difficult to implement finite difference methods to handle a general domain $D^{(k)}$, and it is not surprising that finite element methods, which are much more

flexible, have recently become very popular. Integral equation methods and collocation methods are also very flexible, and we anticipate that their use will increase.

2.1 The trial free boundary method with finite differences

Finite difference methods for elliptic equations were introduced by Runge in 1908 and Richardson in 1910. One would have expected that finite difference methods would have been quickly applied to FBPS, particularly since Runge was familiar with the idea of trial free boundary methods for FBPS. (Blasius [1910, p. 97] acknowledges a suggestion made by Runge in the Göttingen seminar on hydrodynamics.) We have consulted many of the early papers of finite difference methods (such as those listed by Giese [1969]), but the first application of trial free boundary methods using finite differences is apparently due to Vandrey [1940] who computed the flow through a circular Borda mouthpiece.

Independently, Shaw and Southwell [1941] used finite differences to solve the problem of seepage through an earth dam for three geometries. Southwell and Vaisey [1946] applied this method to eleven fluid mechanics FBPS. The work of Southwell and his colleagues was followed by many similar computations by other workers. Not every author indicates how much effort was required, but some do: Yang [1949, p. 54], who solved two porous flow FBPS, states that his first problem took 224 hours and that subsequent problems took about 60 hours; Rouse and Abul-Fetouh [1950, p. 423]

who solved the axisymmetric wall jet FBP, state that their first problem took 150 hours. We should be duly grateful for computers.

The implementation of trial free boundary methods on early computers was not a straightforward matter for two reasons: (i) early computers had small memories which made the implementation of Step 1 difficult; (ii) it was necessary to automate Step 2, and it was far from clear how this should be done.

The first attempt to use a digital computer is apparently due to Young et al. [1955] and Arms and Gates [1957]. This work was not entirely successful. Jakobsson and Floberg [1957] solved several lubrication cavitation problems; they used a computer but moved the FB by hand (see Jakobsson and Floberg [1957, p. 33]). In the applications listed at the end of this section, all the results obtained after 1961 were obtained using a computer.

It is appropriate to mention here that we were once told that an IBM 650 program, Program 4.0.010 of February 1962, was capable of solving porous flow FBPS using trial free boundary methods with finite differences, but we have been unable to locate this program.

The basic texts on finite difference methods are the books of Forsythe and Wasow [1960] and Collatz [1960]. Greenspan [1968] discusses several applications. Remson, Hornberger, and Molz [1971] give applications to porous flow.

The finite difference method consists, in essence, of the following steps:

- (i) Choose a set η of n netpoints in $\Omega^{(k)} \cup \partial\Omega^{(k)}$.
- (ii) Set up a system of n equations connecting the values $U_h^{(k)}$ of the approximate solution $u_h^{(k)}$ at the netpoints,

$$A_h^{(k)} U_h^{(k)} = B_h^{(k)}. \quad (1)$$

- (iii) Solve this system of n equations.

The netpoints are usually taken to be the intersections of an array of gridlines with each other and with $\partial\Omega^{(k)}$. The implementation is simplified if the network is gridlike; that is, the gridlines intersect the boundary $\partial\Omega$ only at gridpoints (intersections of gridlines). The advantage of a gridlike network is that there are no awkward "special netpoints" near the boundary which require special treatment.

If the gridlines are equally spaced then it is obviously often not possible to choose the network so that it is gridlike. If $\partial\Omega^{(k)} = \bigcup_{j=1}^p \partial_j\Omega^{(k)}$ where $\partial_1\Omega^{(k)}$ is a curve and $\partial_2\Omega^{(k)}, \dots, \partial_p\Omega^{(k)}$ are line segments, one can choose a number of points on $\partial_1\Omega^{(k)}$ and define the gridlines to be the (unequally spaced) horizontal and vertical lines passing through these points; with luck, the resulting network will be gridlike. Figure 1 shows such a grid for the problem of a jet issuing from a container (Cryer [1968]). However, Cryer [1969, 1971] has given examples of quite simple domains $\Omega^{(k)}$ for which no gridlike network exists.

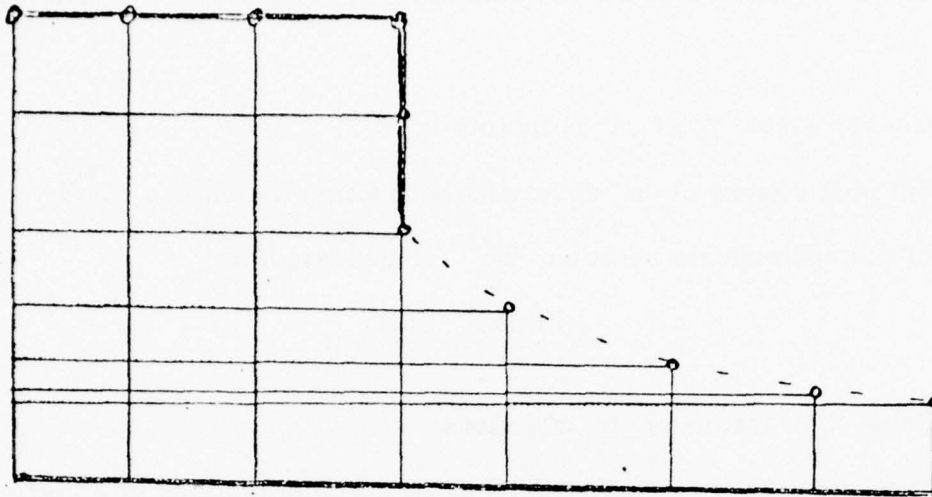


Figure 1: Gridlines defined by boundary points

Handworkers commonly used a graded grid, that is, a grid in which the basic grid consisted of equally spaced gridlines with gridlength h , but in which smaller gridlengths were used near corners and other singular points. Figure 2 (Southwell and Vaisey [1946, p. 140], Southwell [1946, Figure 116]) shows part of a graded grid for flow from an orifice.

By hand it is easier to use a graded grid (as in Figure 2) rather than an unequally spaced grid (as in Figure 1) because the finite difference formulas are simpler. On a computer, it is easier to use an unequally spaced grid rather than a graded grid because the computer can easily handle the more complicated finite difference formulas for an unequally spaced grid but encounters storage allocation problems with a graded grid.

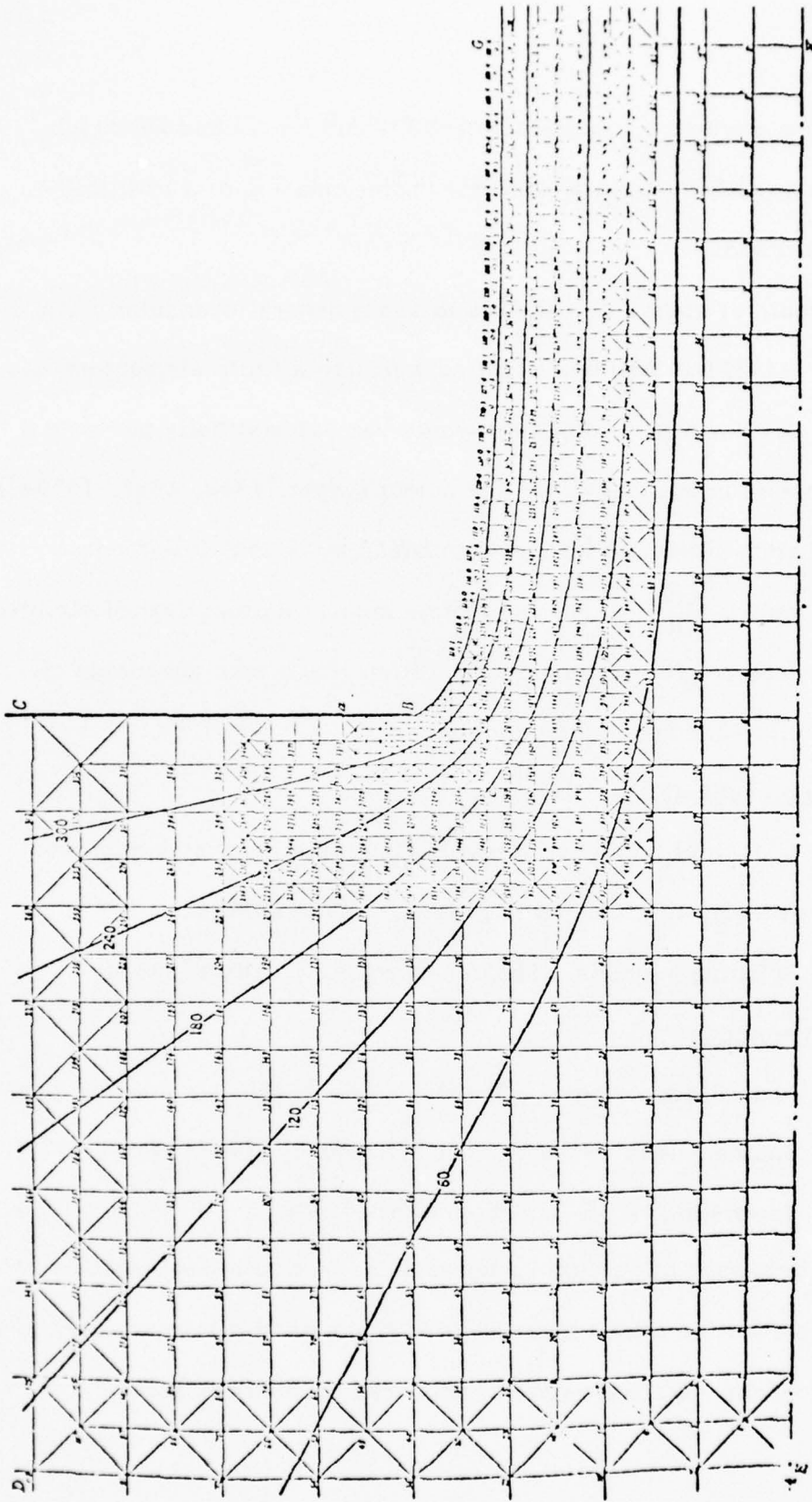


Figure 2: A graded grid

So far as we are aware, only Rippin [1959] and Rippin and Davidson [1967] have used a graded grid in a computer implementation of a trial free boundary method.

It would of course be possible to use a general triangular mesh (Kellogg [1964]) but then one might as well use a finite element method.

The development of programs which can automatically generate a mesh is laborious but feasible. The author (Cryer [1962, 1968, 1970a]) developed such a program for use with trial free boundary methods.

The matrix $A_h^{(k)}$ in (1) is sparse and has a great deal of structure. There are many possible methods of solution which take advantage of this structure. The main methods are:

(a) Iterative methods

(α) Classical pointwise methods such as SOR (systematic over relaxation). See Varga [1962], Young [1971].

(β) Splitting methods. See Yanenko [1971], Stone [1968].

(b) Direct methods

(α) Fast direct methods. See Dorr [1970], Buzbee and Dorr [1974].

(β) Sparse matrix methods. See Willoughby [1968], Reid [1971], Tewerson [1973], Bunch and Rose [1976].

As discussed in section 2, for each method, whether iterative or direct, the question arises as to whether the method can be adapted to take advantage of the iterative nature of trial free boundary methods.

The following FBPS have been solved using the trial free boundary method in conjunction with finite differences:

Fluid mechanics

Plane and axially symmetric cusped cavities: Southwell and Vaisey [1946].

Plane Riabouchinsky cavity: Mogal and Street [1972, 1974].

Axially symmetric Riabouchinsky cavity: Brunauer [1951], Young, Gates, Arms, and Eliezer [1955], Arms and Gates [1957], Sankar [1967], Fox and Sankar [1973].

Plane Borda mouthpiece: Southwell and Vaisey [1946].

Axially symmetric Borda mouthpiece: Vandrey [1940], Southwell and Vaisey [1946].

Plane jet from an orifice in a tube: Cryer [1968].

Axially symmetric jet from an orifice in a tube (with and without gravity): Southwell and Vaisey [1946], Abul-Fetouh [1949], Rouse and Abul-Fetouh [1950].

Flow over a plane weir: Southwell and Vaisey [1946].

Flow over a circular weir: Citrini [1951].

Flow over a submerged plate: Dumitrescu, Ionescu and Toth [1956].

Flow over a submerged arc: Dumitrescu, Ionescu, and Toth [1956].

Flow over a submerged vortex: von Kerczek [1965].

Flow under a planing surface: Southwell and Vaisey [1946], von Kerczek [1965] (unsuccessful).

Flow under a sluice gate: Southwell and Vaisey [1946].

Plane confined infinite symmetric bubbles: Stewart [1965], Stewart and Davidson [1967].

Axially symmetric confined infinite bubbles: Dumitrescu, Ionescu, and Toth [1956], Stewart [1965], Stewart and Davidson [1967].

Axially symmetric confined spherical cap bubbles with wake: Rippin [1959] and Rippin and Davidson [1967].

Periodic progressing gravity waves: Southwell and Vaisey [1946].

Solitary waves: Southwell and Vaisey [1946].

Two and three-dimensional aerofoils: Allen and Dennis [1953].

Swirling flow through an orifice: Binnie and Davidson [1949].

Flame in a channel: Ball [1951].

Lubrication cavitation: Jakobsson and Floberg [1957].

Porous flow

Simple rectangular dam: Shaw and Southwell [1941, p. 9], McNown, Hsu, and Yih [1955, p. 668], Finnemore and Perry [1968, p. 1066], J. M. Taylor [1971, p. 56] and Aitchison [1972], Comincioli, Guerri, and Volpi [1971].

Simple trapezoidal dam: Yang [1949, p. 48].

Polygonal dams: Shaw and Southwell [1941] and Murray [1960, p. 156] (trapezoidal dams with toe drains on pervious sublayers); Freeze [1971a] (inhomogeneous trapezoidal dams for saturated-unsaturated flow).

Drainage: van Deemter [1950].

Single penetrating well: Yang [1949], Boulton [1951], Kashef, Touloukian, and Fadum [1952], Kashef [1953], Murray [1960].

Single penetrating well with capillarity: Hall [1955].

Single partially penetrating well: Boreli [1955].

Elastic-plastic torsion

Hodge, Herakovich, and Stout [1968, Method III].

Magnetohydrodynamics

Stellar evolution: Cryer [1962, 1968].

Tokamak and toroidal compressor: Stevens [1974].

Electrohydrostatics

Fluid film in an electric field: Michael and O'Neill [1972a].

2.2. The trial free boundary method with finite elements

The finite element method is usually attributed to Courant [1943] although Le Roux [1914] derived the finite difference equations for Laplace's equation by minimizing the Dirichlet integral over piecewise linear functions. The first serious computations were performed by Turner, Clough, Martin, and Topp [1956] and Clough [1960]. The first application of finite elements to trial free boundary methods is probably due to R. L. Taylor [1966] who solved several porous flow FBPS.

Information about the method of finite elements is given by Birkhoff [1971], Varga [1971], Zienkiewicz [1971], Aubin [1972], Aziz [1972], Desai and Abel [1972], Oden [1972], Whiteman [1973, 1975], Strang and Fix [1973], de Boor [1974], Ciarlet [to appear].

Because of their greater flexibility, finite element methods encounter fewer topological difficulties than finite difference methods when used with trial free boundary methods. It is of course desirable to choose the finite elements so as to take account of any singularities. In addition to the discussions in textbooks (Zienkiewicz [1971], Desai and Abel [1972]), the paper of Iversen [1969] gives a clear discussion of the problems involved.

When using finite elements in conjunction with a trial free boundary method it is customary to choose the finite element array a-priori. The same finite array is used throughout the computation: as the trial free boundary changes only the coordinates of the finite elements adjacent to the FB change. Of course, if the trial free boundary changes a great deal, then an element may become very distorted or may even disappear, and this must be checked for. A typical finite element array is shown in Figure 1 for the problem of viscous flow from an axially symmetric pipe (Nickell, Tanner and Caswell [1974, Figure 5a]); the finite elements have been chosen so as to take account of the singularity at the point A where the fluid leaves the pipe.

There are 70 elements in Figure 1. Nickell, Tanner and Caswell [1974] do not indicate how many equations were involved, but in a related problem (the stick-slip problem) 150 elements involved 1000 equations, so that we may conclude that the array shown in Figure 1 involved about 500 equations.

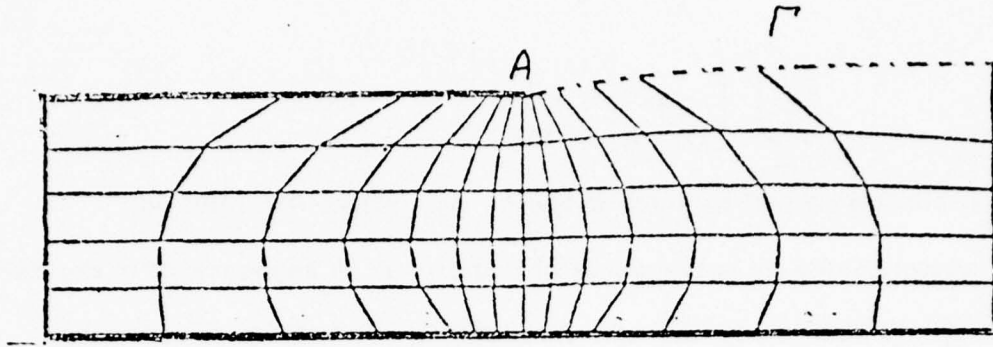


Figure 1: An array of quadrilateral elements

It is typical of finite element solutions of FBPS that relatively coarse arrays are used. For comparison, the finite difference grid part of which is shown in Figure 2 of section 2.1 involved 1028 equations which were solved by hand.

Winslow [1967] describes a method for the automatic triangulation of a region, and this method has been extended by Godunov [1971]; George [1971, p. 27] gives a critical discussion of Winslow's method.

George [1971, chapter 2] has an excellent discussion of the problems arising when automatically generating a triangulation of a domain. George reviews previous work in the literature, suggests two algorithms, and provides a program listing.

Sewell [1972] describes an algorithm for automatically refining a triangulation of a polygonal region in R^2 . The algorithm is designed so that: (i) no triangle Δ has a very small angle; (ii) the integral

$$\iint_{\Delta} \left[\max_{i+j=n} \left| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right| \right]^{2/(n+2)} dx dy \quad (1)$$

is approximately equal for each triangle Δ , where the solution u is being approximated by polynomials of degree n . A sequence of triangulations π_k is generated: the triangulation π_k is used to compute an approximation $u^{(k)}$ from which the integrand in (1) can be estimated prior to generation of the refinement π_{k+1} .

The finite element method leads to a system of linear equations

$$A_h^{(k)} U_h^{(k)} = B_h^{(k)}. \quad (2)$$

The matrix $A_h^{(k)}$ is sparse and has a band structure, but has in general much less structure than the corresponding matrix for the finite difference method.

In many programs the system (2) is solved using a band matrix version of Gaussian elimination. A number of different ideas have been used:

- (a) Instead of requiring a band of uniform width, one can allow bands of variable width. This is a straightforward generalization; among the first implementations was that of Cryer [1962, Chapter 5].
- (b) Instead of generating all the equations at once, one can use the frontal solution method (Irons [1970]) in which the equations are generated when needed in the Gaussian elimination.
- (c) One can try to re-order the equations so as to reduce the band width. George [1972, 1974] has suggested some interesting ideas here.

The solution by direct methods of sparse linear systems is at present a very active field of research. See the books and conferences: Willoughby [1968], Reid [1971], Tewerson [1973], Bunch and Rose [1976].

In a few cases the system (2) has been solved by an iterative methods. For trial free boundary methods this is a desirable approach because the previous solution $U_h^{(k-1)}$ is available. Unfortunately, the lack of detailed structure in the matrix $A_h^{(k)}$ makes it difficult to analyse. For example, Fix and Larsen [1971] have considered S.O.R. methods but in his review of the paper Varga (MR 45 (1973) #2935) observes that the main theorem of Fix and Larsen contains a nontrivial flaw, and adds that he doubts whether the convergence properties claimed are valid without additional hypotheses. Schultchen [1973] has tested an immense number of iterative methods for finite element matrices $A_h^{(k)}$ and concludes that a particular version of the conjugate gradient method is the most efficient procedure. See also Fried [1970].

It is also important to have information about the round-off errors which arise when solving the system (2). See Roy [1971], and S. K. Yang [1974].

The following FBPS have been solved using the trial free boundary method in conjunction with the finite element method:

Fluid mechanics FBPS

General references on the application of finite elements to fluid mechanics problems include: de Vries and Norrie [1973], Oden, Zienkiewicz, Gallagher, and Taylor (1974).

The following specific problems have been solved:

Cavitating hydrofoils: Dobray and Baker [1974]. (This reference was kindly drawn to our attention by B. E. Larock.)

Viscous flow from a tube (die-swell): Tanner [1973] and Nickell, Tanner, and Caswell [1974]. (These references were kindly drawn to our attention by J. Liu.)

Plane flow from symmetric polygonal nozzles: S. T. K. Chan [1971] and S. T. K. Chan, Larock and Herrman [1973].

Axisymmetric flows from orifices and valves: Chang and Larock [1973].

Three-dimensional jet in a transverse gravity field: Larock [to appear].

Circular crested weir (spillway): Tu [1971].

Gated spillway crest: Larock [to appear].

Plane jet impinging on a plane: Tu [1971].

Surface wave generated by a vortex (with linearized equations): Tu [1971].

Porous flow FBPS

R. L. Taylor [1966] gives a general program for solving plane and axisymmetric porous flow FBPS using finite elements. A later version of this program is given by Kealy and Busch [1971].

Variational principles for porous flow FBPS have been developed by Mauersberger [1965, 1965b]. Ponter [1972] has derived dual minimum principles for porous flow fixed boundary problems and used them in finite element computations; presumably these dual minimum principles could also be used for FBPS.

The following specific problems have been solved:

Simple rectangular dam: Kealy and Busch [1971, p. 9], Kealy and Williams [1971, p. 145].

Simple trapezoidal dams: W. D. L. Finn [1967]; S. P. Neuman and Witherspoon [1970]; Parkin [1971]; Pettibone and Kealy [1971] (inhomogeneous dam); Fenton [1972a].

Polygonal dams: W. D. L. Finn [1967] (trapezoidal dam with two sheetpiles); R. L. Taylor and Brown [1967] (with rock toe); Volker [1969] (with cut-off wall); Neumann and Witherspoon [1970] (with rock toe; with toe drain); Kealy and Busch [1971], Kealy and Williams [1971, p. 152].

Dams with general geometry: Nonlinear flow through a dam with curved downstream face, and nonlinear flow through a polygonal dam with cut-off wall (Volker [1969]).

Circular channel in a finite layer (pond): Neuman and Witherspoon [1970].

Single fully-penetrating well: Neumann and Witherspoon [1970].

Single partially-penetrating well: R. L. Taylor and Brown [1967].

2.3. Trial free boundary methods with Galerkin methods

By a Galerkin method we understand any method which involves approximating the solution $u^{(k)}(\underline{x})$ by a known smooth function $F(\underline{c}, \underline{x})$ depending upon an N-vector \underline{c} which is determined by minimizing the error in some sense.

References on Galerkin methods include: Kantorovich and Krylov [1958], Cavendish, Price, and Varga [1969], Mikhlin [1971], Birkhoff [1971], Varga [1971],

In all the applications of Galerkin methods to trial free boundary methods of which we are aware, the governing differential equation has been Laplace's equation and the approximate solution $u_h^{(k)}$ has been represented in the form

$$u_h^{(k)}(\underline{x}) = F(\underline{c}^{(k)}, \underline{x}) = p_0(\underline{x}) + \sum_{j=1}^N c_j^{(k)} p_j(\underline{x}), \quad (1)$$

or in an equivalent form, where:

- (i) Each function $p_j(\underline{x})$, $0 \leq j \leq N$, is a solution of Laplace's equation.
- (ii) $F(\underline{c}, \underline{x})$ satisfies some of the boundary conditions on the known part of $\partial\mathcal{D}$ for all \underline{c} .
- (iii) The function $p_0(\underline{x})$ incorporates the expected singularities of $u^{(k)}(\underline{x})$.

The values of the coefficients $c_j^{(k)}$ have been determined by satisfying the remaining boundary conditions in one of two ways:

- (a) Collocation: The boundary conditions $\mathcal{B}u^{(k)} = 0$ are imposed at N points thus leading to a system of N equations for the coefficients $c_j^{(k)}$.

(b) Least squares: The boundary conditions $\partial u^{(k)} = 0$ are imposed at $M \geq N$ points, and the resulting system of M equations is solved by least squares.

We now summarize the various applications of the Galerkin method.

Garabedian [1956; 1956a, p. 664] considers the axially symmetric Riabouchinsky cavity between circular disks. Garabedian takes great care in the choice of the functions p_j . He uses least squares to determine $\underline{c}_h^{(k)}$ with $N = 10$, $M = 24$ (Garabedian [1956a]) and $N = 20$, $M = 62$ (Garabedian [1956]).

Chappelear [1961] considers periodic progressing gravity waves in water of uniform depth. Chappelear takes

$$u_h^{(k)}(x, y) = c_0^{(k)} x + \sum_{j=1}^7 c_j^{(k)} \sin(2\pi j x / \lambda) \cosh(2\pi j y / \lambda) / (2\pi j / \lambda)$$

where λ is the wavelength. Chappelear uses least squares to determine $\underline{c}^{(k)}$ and it appears from the graphs in the paper that $M = 13$.

Kirkham [1964] considered the problem of axisymmetric porous flow towards a fully-penetrating well of radius r_w in a region of influence of depth h_e and radius r_e . Kirkham sets

$$u_h^{(k)}(r, y) = p_0(r, y) + \sum_{m=1}^{N-1} c_j^{(k)} \cosh\left(\frac{u_m y}{r_w}\right) C_0(u_m r / r_w) / \cosh\left(\frac{u_m h_e}{r_w}\right),$$

where

$$C_0 \left(\frac{u_m r}{r_w} \right) = J_0 \left(\frac{u_m r}{r_w} \right) Y_0 \left(\frac{u_m r_c}{r_w} \right) - J_0 \left(\frac{u_m r_c}{r_w} \right) Y_0 \left(\frac{u_m r}{r_w} \right),$$

the functions J_0 and Y_0 are Bessel functions, and the constants u_m are the roots of $C_0(u_m r_e / r_w) = 0$. The function $p_0(r, y)$ is a known function expressed as an infinite series. Kirkham set $N = 4$ and used collocation to determine $\underline{c}^{(k)}$. It should be observed that Kirkham explains his method in terms of a "fictitious region", but this in fact is merely a means of justifying the approximation for $u_h^{(k)}$.

Mason and Farkas [1971, 1972] consider the problem in which fresh water is supplied to a canal in the center of a long island with an impervious surface. Mason and Farkas use conformal mapping to "straighten out" a corner in the original domain and thereby remove a singularity in the solution. In the transformed $z = x + iy$ domain the solution is sought in the form

$$\phi^{(k)}(z) + i\psi^{(k)}(z) = i + \sum_{j=1}^N c_j^{(k)} z^{j-1/2}.$$

Collocation is used to determine $\underline{c}^{(k)}$. The collocation points are chosen in a special way which is explained and justified by Mason [1969]. Computations were carried out for $N = 8, 10, 12,$ and 14 .

We conclude with two remarks:

- (i) It is not necessary for the function F to have the form (1), nor is the Galerkin method necessarily restricted to Laplace's equation.

- (ii) The Galerkin method has several advantages:
- (a) It is easy to handle the trial FBS $\Gamma^{(k)}$.
 - (b) The approximate solution obtained is in a form which facilitates error estimates.
 - (c) The value of $Cu^{(k)}$ is readily obtained.

We, therefore, believe that the Galerkin method deserves greater attention particularly for problems with Laplace's equation and relatively simple geometry for which good choices of $F(\underline{c}, \underline{x})$ can be made.

2.4. The trial free boundary method with integral equations and surface singularities

One of the basic methods of solving a fixed boundary value problem is to reformulate the problem as an integral equation. It is, therefore, very natural to use this approach to compute $u_h^{(k)}$.

The first use of integral equations in connection with trial free boundary methods is due to Trefftz who in his thesis of 1914 (the greater part of which appears in Trefftz [1916]) considered the problem of an inviscid axially symmetric wall jet. Since then integral equations have been used by several authors. The approach has not been as widely used as might be expected for a number of reasons:

- (i) In general, the numerical solution of the integral equation requires the solution of a non-sparse system of linear algebraic equations, and this only became convenient after the introduction of computers. (Because of the special features of his problem Trefftz was able to avoid this.)

- (ii) The approach is most useful in axially symmetric problems, for which complex function techniques are not available. In axially symmetric problems, however, the kernels of the integral equations are non-trivial elliptic integrals.
- (iii) A particular FBP can be reformulated as an integral equation in several ways. Moreover, the integral equations are often derived in rather tortuous ways using physical ideas. In consequence, the literature is very fragmented.

Despite these difficulties, integral equations are being used increasingly often, and this trend will probably continue.

2.4.1. The method of Trefftz

The method described here was first used by Trefftz [1914, 1916] to solve the problem of an inviscid axially symmetric wall jet. Gilbarg [1960, p. 431] gives a useful summary of the work of Trefftz.

The geometry of an axially symmetric wall jet is shown in Figure 1.

Applying Green's identity we find that

$$2\pi\phi(0) = + \int_{\partial\mathcal{D}_3} \phi(q) \left(\frac{\partial}{\partial n_q} \frac{1}{r_{pq}} \right) dS_q - \int_{\partial\mathcal{D}_3} \frac{1}{r_{pq}} \frac{\partial\phi(q)}{\partial n_q} dS_q, \quad (1)$$

where \mathcal{D}_3 is the (three-dimensional) axisymmetric FD, p and q are points on $\partial\mathcal{D}_3$, r_{pq} is the distance between p and q , and ϕ is the velocity potential.

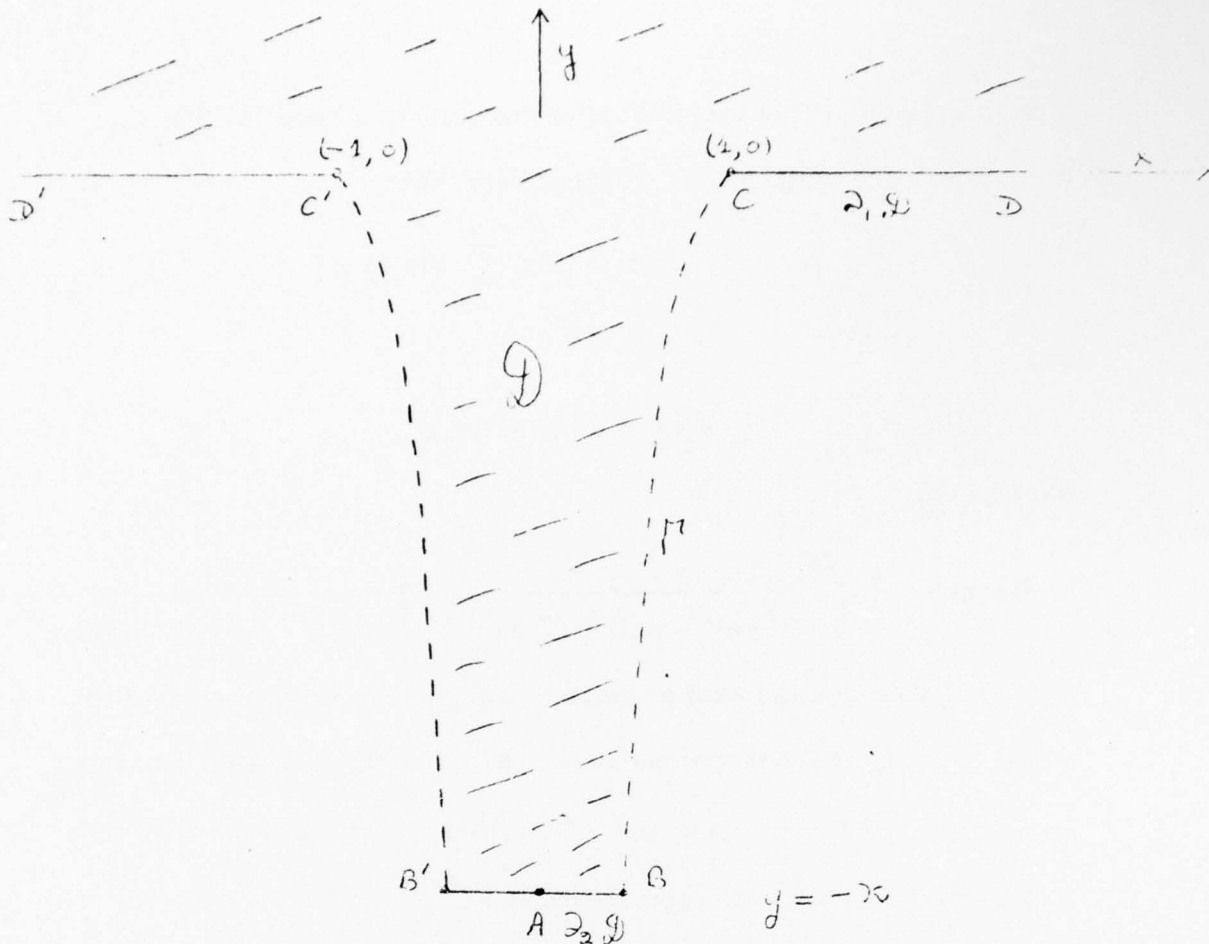


Figure 1: An axially symmetric wall jet

\mathcal{D}_3 is axially symmetric. Let \mathcal{D} be the cross-section of \mathcal{D}_3 in the xy -plane. \mathcal{D} is symmetric with respect to the y -axis, and we denote by $\partial\mathcal{D}$ the boundary of \mathcal{D} lying to the right of the y -axis. Then equation (1) can be reduced to an equation in the xy -plane of the form

$$2\pi\phi(s) = \int_{\partial\mathcal{D}} K_1(s,t)\phi(t)dt + \int_{\partial\mathcal{D}} K_2(s,t) \frac{\partial\phi(t)}{\partial n} dt, \quad (2)$$

where s and t denote arc-length along $\partial\mathcal{D}$, and K_1 and K_2 are kernels which depend upon s , t and the FB Γ (the boundary of the jet).

More precisely, if the coordinates of the points s and t are $(x(s), y(s))$ and $(x(t), y(t))$, respectively, then

$$K_1(s, t) = 2x(t) \frac{\partial}{\partial n_t} V(s, t),$$

and

$$K_2(s, t) = -2x(t)V(s, t),$$

where

$$V(s, t) = -\frac{1}{2} \int_0^{2\pi} \frac{d\omega}{\{[y(s) - y(t)]^2 + [x(s)]^2 + [x(t)]^2 - 2x(s)x(t) \cos \omega\}^{1/2}}.$$

To avoid dealing with an infinite integral the jet is truncated at a line $y = -\kappa$, so that the boundary $\partial\Omega$ consists of three components namely the GB Γ , the part $\partial_1\Omega$ of the line $y = 0$, and the segment $\partial_2\Omega$ of the line $y = -\kappa$. (see Figure 1).

Γ and $\partial_1\Omega$ are streamlines so that $\frac{\partial\phi}{\partial n} = 0$ on these curves. It is assumed that the velocity is constant on $\partial_2\Omega$, and the problem is so normalized that $\frac{\partial\phi}{\partial n} = 1$ on $\partial_2\Omega$. Thus, for given Γ , the second integral in equation (2) is a known function of s , $f(s; \Gamma)$ say, and the equation takes the form

$$2\pi\phi(s) = f(s; \Gamma) + \int_{\partial\Omega} K_1(s, t)\phi(t)dt. \quad (3)$$

A further simplification arises because $K_1(s, t) = 0$ if both the points s and t lie on the fixed boundary; this is best seen from equation (1) by noting that if p and q both lie on the plane $y = 0$ then $\frac{\partial r_{pq}}{\partial n_q} = 0$.

Thus,

$$2\pi\phi(s) = f(s; \Gamma) + \int_{\Gamma \cup \partial_2 \mathcal{D}} K_1(s, t)\phi(t)dt, \text{ for } s \in \partial_1 \mathcal{D},$$

$$2\pi\phi(s) = f(s; \Gamma) + \int_{\partial \mathcal{D}} K_1(s, t)\phi(t)dt, \text{ for } s \in \Gamma \cup \partial_2 \mathcal{D}.$$

The additional condition on the FB is that $\phi(s) = \phi_0 + s$ on Γ , where ϕ_0 is the value of ϕ at the point of separation. It follows that $\phi = \phi_0 + \ell$ on $\partial_2 \mathcal{D}$ where ℓ is the length of Γ .

The procedure follows by Trefftz is as follows:

- (i) Guess the FB $\Gamma^{(1)}$.
- (ii) Compute the value of ϕ on the known boundary $\partial_1 \mathcal{D}$ by means of

$$2\pi\phi^{(1)}(s) = f(s; \Gamma^{(1)}) + \int_{\Gamma^{(1)}} K_1(s, t)[\phi_0 + t]dt + \int_{\partial_2 \mathcal{D}^{(1)}} K_1(s, t)(\phi_0 + \ell)dt, \\ \text{for } s \in \partial_1 \mathcal{D}.$$

- (iii) Compute the value of ϕ on $\Gamma^{(1)}$ by means of

$$2\pi\phi^{(1)}(s) = f(s; \Gamma^{(1)}) + \int_{\partial_1 \mathcal{D}} K_1(s, t)\phi^{(1)}(t)dt + \\ + \int_{\partial_2 \mathcal{D}} K_1(s, t)(\phi_0 + \ell)dt + \int_{\Gamma^{(1)}} K_1(s, t)[\phi_0 + t]dt, \text{ for } s \in \Gamma^{(1)}.$$

- (iv) If $\phi^{(1)}(s) = \phi_0 + s$ on $\Gamma^{(1)}$ then the problem is solved. Otherwise we must adjust $\Gamma^{(1)}$. Trefftz [1916, p. 47] suggests the following strategy:

- (a) if $\phi^{(1)}(s) > \phi_0 + s$ on $\Gamma^{(1)}$, then $\Gamma^{(1)}$ should be moved away from the axis;
- (b) if $\phi^{(1)}(s) < \phi_0 + s$ on $\Gamma^{(1)}$, then $\Gamma^{(1)}$ should be moved towards the axis.

Trefftz used power series expansions and a planimeter to approximate the integrals. Because of the effort involved, Trefftz only carried out two iterations and he estimated that $.60 \leq C_c \leq .62$.

Steffen [1973] has implemented a modification of Trefftz's method on a computer. Steffen chooses n values y_i

$$-\kappa < y_n < y_{n-1} < \dots < y_1 < 0$$

and defines the k^{th} approximate FB $\Gamma^{(k)}$ to be the quadratic (sic) spline $x = x^k(y)$ passing through the points

$$(1, 0), (x_1^k, y_1), \dots, (x_n^k, y_n), (1.25x_n^k - .25x_{n-1}^k, -\kappa)$$

which has a horizontal tangent at $(1, 0)$. Given $\Gamma^{(k)}$ Steffen computes $\phi^{(k)}$ as done by Trefftz and then determines the error

$$F^k(y) = \phi^{(k)}(x^k(y), y) - (\phi_0 + l^k(y))$$

where $l^k(y)$ is the length along Γ^k expressed as a function of y .

The formulas of Steffen [1973, p. 14] differ from those of Trefftz in two minor respects: the formulas are expressed in terms of x and y rather than s and t ; and Steffen makes use of the identity (see equation (1))

$$2\pi = \int_{\partial \mathcal{D}_3} \left(\frac{\partial}{\partial n_q} \frac{1}{r_{pq}} \right) dS_q .$$

To find the n values $x_1^{k+1}, \dots, x_n^{k+1}$ Steffen uses the iteration scheme:

$$x_i^{k+1} = -c \sum_{\ell=1}^n F^k(y_\ell) |(i - \ell + 1)|^{-L} + x_i^k, \text{ for } i \leq j \leq n,$$

where c and L are constants chosen to improve convergence: the best values were found to be $c = .031$ and $L = 1$. Steffen [1973, p. 16] reports that the iterations did not converge completely, but oscillated. Steffen [1973, p. 16] also observes that he tried Newton's method without success.

At this point it is appropriate to emphasize the essential features of the method of Trefftz:

- (i) The method is based upon equation (1) which involves both ϕ and $\frac{\partial \phi}{\partial n}$, and use is made of the fact that both ϕ and $\frac{\partial \phi}{\partial n}$ are known on the FB.
- (ii) There is no need to solve an integral equation approximately because the geometry is such that the kernel $K_1(s, t)$ vanishes on the fixed boundary.

For many years the work of Trefftz was the basic work on axially symmetric problems. The method, or variations thereof, was used to solve the following problems:

- (a) Axially symmetric jet impinging normally on a plate: Schach [1935].
- (b) Wedge penetrating a fluid: Wagner [1932, p. 213], Pierson [1950]. (As Pierson [1950, p. 2] remarks, the work of Wagner is brief and cryptic. Pierson [1950, Appendix I] gives more details but is also not **readily** comprehensible.)

- (c) Cone penetrating a fluid: Shiffman and Spencer [1951] (the computations were performed by Hillman in 1946 in a report that we have been unable to obtain).
- (d) Axisymmetric magnetosphere under uniform external pressure: Slutz [1962].

The paper of Shiffman and Spencer [1951] is very clear. The method used differs from that of Trefftz in two respects: the kernel $K_1(s, t)$ does not vanish on the fixed boundary so that it is necessary to solve an integral equation to obtain the values of ϕ^k on the fixed boundary; the FB is moved by a global method.

The work of Slutz [1962] is an interesting contribution to a recent problem. There is no fixed boundary so that it is not necessary to solve an integral equation. The FB is moved by a global method.

2.4.2. The trial free boundary method with integral equations

A classical method for solving the Dirichlet problem for Laplace's equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad \text{in } \mathcal{D} \\ u &= f, \quad \text{on } \partial\mathcal{D} \end{aligned} \tag{1}$$

is to represent the solution u as the potential of a double layer of density μ on $\partial\mathcal{D}$; that is

$$u(\underline{x}) = \int_{\partial\mathcal{D}} F(\underline{x}, \underline{\xi}(t)) \mu(t) dt, \tag{2}$$

where t denotes distance along $\partial\Omega$, the density μ is an unknown function, $\underline{\xi} = \underline{\xi}(t)$ is the point on $\partial\Omega$ corresponding to t , and

$$F(\underline{x}, \underline{\xi}) = \frac{1}{\pi} \frac{\cos \phi}{r} \quad (3)$$

where ϕ and r are as in Figure 1.

For fixed $\underline{\xi}$,

$$F(\underline{x}, \underline{\xi}) = \frac{1}{\pi} \frac{\partial}{\partial n(\underline{\xi})} \ln |\underline{x} - \underline{\xi}| \quad (4)$$

so that $F(\underline{x}, \underline{\xi})$ is a solution of Laplace's equation. Thus the integral in (2), being the superposition of solutions, is also a solution of Laplace's equation. The term double layer refers to the fact that F can be interpreted as the potential due to a double layer.

In order to check that u satisfies the boundary conditions it is necessary to study the behaviour of the integral in (2) as \underline{x} approaches the boundary. If the boundary is smooth at the point t then it can be shown that

$$\lim_{\underline{x} \rightarrow \underline{\xi}(s)} \int_{\partial\Omega} F(\underline{x}, \underline{\xi}(t)) \mu(t) dt = \mu(s) + \int_{\partial\Omega} K(s, t) \mu(t) dt \quad (5)$$

where

$$K(s, t) = \frac{1}{\pi} \frac{d}{dt} \arctan \left[\frac{\eta(t) - \eta(s)}{\xi(t) - \xi(s)} \right]. \quad (6)$$

Thus, u given by (2) will solve the boundary value problem (1) if μ satisfies the Fredholm integral equation of the second kind,

$$\mu(s) + \int_{\partial\Omega} K(s, t) \mu(t) dt = f(s). \quad (7)$$

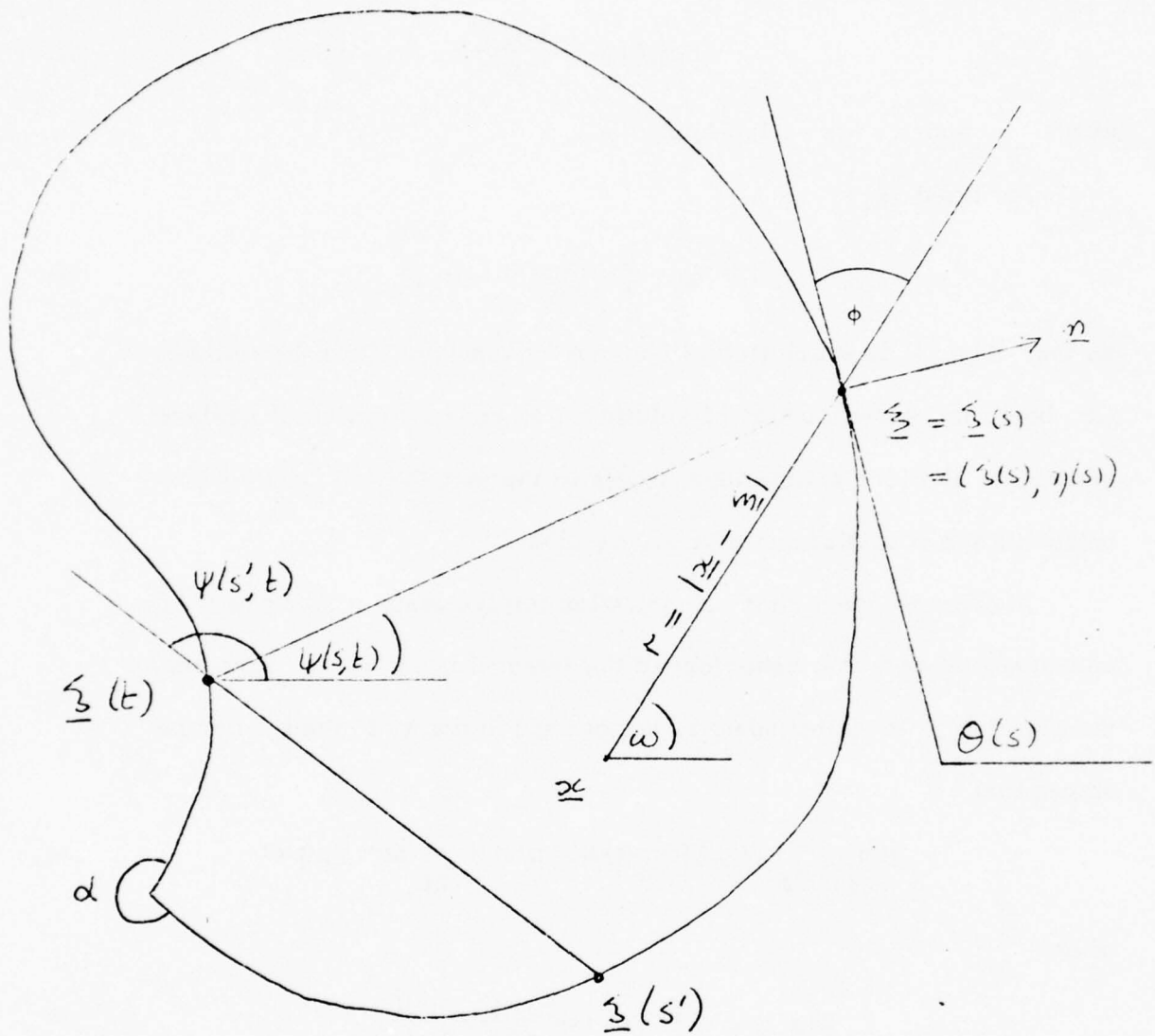


Figure 1: Notation for boundary potentials

It is possible to rewrite the above results in another form. The expression (2) is equivalent to the Stieltjes integral

$$u(\underline{x}) = \frac{1}{\pi} \int_{\partial\Omega} \mu(t) d\omega \quad (8)$$

where ω is shown in Figure 1. The limit (5) is readily seen to be a consequence of (8). Finally, equation (7) is equivalent to the Stieltjes integral equation

$$\mu(s) + \frac{1}{\pi} \int \mu(t) d_t \psi(s, t) = f(s) , \quad (9)$$

where ψ is as in Figure 1.

The derivation of the preceding results is described in many textbooks including: Kantorovich and Krylov [1958], Courant and Hilbert [1962, p. 298].

The integral equation (9) must be solved by some numerical method. General references on the numerical solution of integral equations include: Anselone [1971], Noble [1971], Delves and Walsh [1974], Atkinson [1976], Baker [to appear].

The numerical solution of equation (9) is considered by Kantorovich and Krylov [1958], Symm [1964], Jaswon and Symm [to appear]. Cryer [1970, p. 103] gives further references.

There is a close relationship between equation (9) and various integral equations which arise when constructing conformal maps: see Gaier [1964], Symm [1966, 1967, 1969].

Every numerical method for solving (9) leads to a system of n linear equations

$$A_h^{(k)} U_h^{(k)} = B_h^{(k)}, \quad (10)$$

where $U_h^{(k)}$ is an approximation to the density function μ . The matrix $A_h^{(k)}$ is a full matrix, and it seems that Gaussian elimination is the best method to solve (10). No comment seems called for except to observe that if the matrix is too large to be held in the main store, then the computations must be arranged so as to minimize transfers between the main store and the secondary store; an efficient way of doing this is described in a seldom-quoted paper of Barron and Swinnerton-Dyer [1960/61] which was intended for a magnetic-tape secondary store but is applicable to other types of secondary store. The time estimates given by Barron and Swinnerton-Dyer [1960/61] are too low because only the time required for data transfer is considered. When allowance is made for the computation time, the time estimates are very accurate (Cryer [1962, Chapter 5, p. 7]).

Since the system (10) is not sparse it might be thought that the solution of (10) would require much more computation time than the solution of the sparse linear equations arising when finite elements or finite differences are used to solve (1). However, equation (9) is a one-dimensional equation while equation (1) is a two-dimensional equation. Thus, for comparable accuracy, the order of the system (10) is far smaller than the order of the system arising from finite differences or finite elements.

We now indicate some related results:

(i) Three dimensions

The results can be extended to any number of dimensions. In three dimensions

$$F(\underline{x}, \underline{\xi}) = \frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{|\underline{x} - \underline{\xi}|}$$

and the kernel $K(s, t)$ is modified appropriately. See O. D. Kellogg [1953, p. 286], Garabedian [1964, p. 334].

General programs for solving the resulting integral equations have been developed by Hess and Smith [1967] and Hess [1973] and applied to the computation of the flow past obstacles such as aeroplanes and missiles.

For axisymmetric problems the axisymmetry can be used to reduce the integral equation to a one-dimensional equation. The surface singularities are often called ring sources or ring vortices. Unfortunately, the kernel of the one-dimensional equation involves elliptic integrals and this increases the computation time. See also Riegels [1952], Landweber [1951, 1959].

(ii) Non-smooth $\partial\Omega$

If $\partial\Omega$ has an exterior corner of angle $\alpha(s)$ at the point s on $\partial\Omega$ then (5) must be modified:

$$\lim_{\underline{x} \rightarrow \underline{\xi}(s)} \int_{\partial\Omega} F(\underline{x}, \underline{\xi}(t)) \mu(t) dt = \frac{\alpha(s)}{\pi} \mu(s) + \int_{\partial\Omega} K(s, t) \mu(t) dt .$$

The integral equation (7) becomes

$$\frac{\alpha(s)}{\pi} \mu(s) + \int_{\partial\Omega} K(s, t) \mu(t) dt = f(s) .$$

However, and this is its great advantage, equation (9) remains valid, that is

$$\mu(s) + \frac{1}{\pi} \int \mu(t) d_t \psi(s, t) = f(s) . \quad (11)$$

Equation (11) was first considered by Carleman, but Radon [1919] developed a much more satisfactory theory. Recently the work of Radon has been extended by Burago, Kral, and others; Radon defined $\partial\Omega$ to be of bounded rotation if the slope $\theta(s)$ (see Figure 1) is of bounded variation. Radon showed that if $\partial\Omega$ is of bounded rotation and has no cusps then (9) has a unique continuous solution μ for each continuous f . Cryer [1970] gives a detailed discussion of the theory and a comprehensive bibliography.

The general theory of Radon is not necessary in the present context, but most FBPS do involve boundaries $\partial\Omega$ with corners, so that the theory which is usually given in textbooks which assumes a smooth $\partial\Omega$ and is based on equation (7) is not adequate. Equation (11) has been used with trial free boundary methods by Cryer [1962, Chapter 1, p. 11], and Symm [1975]. Slutz [1962] used the same approach in a three-dimensional magnetohydrostatic problem which he solved by the method of Trefftz (see section 2.4.1); an interesting aspect of the problem treated by Slutz is that the FB has a corner.

(iii) Single-layer potentials

Instead of representing $u(\underline{x})$ as the potential of a double layer of density μ , one can represent μ as the potential of a single-layer

of density μ . For two-dimensional problems we have

$$u(\underline{x}) = \int_{\partial\Omega} \mu(t) \ell n |\underline{x} - \underline{\xi}(t)| dt, \quad (12)$$

instead of (4).

If the boundary condition is a Neuman condition, then equation (12) leads to a second order Fredholm integral equation for μ with the kernel $K(s,t)$ given by (6). However, if the boundary condition is a Dirichlet condition, then (12) leads to the first order equation

$$f(s) = \int_{\partial\Omega} \mu(t) \ell n |\underline{\xi}(s) - \underline{\xi}(t)| dt. \quad (13)$$

Conventional wisdom has it that equations such as (13) should be avoided. However, it has been found by Symm [1964], Jaswon and Symm [to appear], and others that equation (13) can be solved numerically without difficulty.

So far as we are aware, a rigorous treatment of the numerical solution of singular equations of the form (13) is not available. There is a considerable literature on the numerical solution of equations of the second kind with singular kernels of which we merely mention Hertling [1971] and McInnes [1972] (who generalizes the theory of collectively compact operators to the case of linear singular integral equations in a Hölder space.) There is also a considerable literature on the solution of first order integral equations; a recent reference is Strand [1974].

(iv) Combined potentials

It is possible to represent $u(\underline{x})$ in the form

$$u(\underline{x}) = \frac{1}{\pi} \int_{\partial_1 \mathcal{D}} \mu(t) \frac{\partial}{\partial \underline{n}} \ln |\underline{x} - \underline{\xi}(t)| dt + \int_{\partial_2 \mathcal{D}} \mu(t) \ln |\underline{x} - \underline{\xi}(t)| dt,$$

where $\partial_1 \mathcal{D} \cup \partial_2 \mathcal{D} = \partial \mathcal{D}$. We understand that this approach was used by Symm [1975] to solve the problem of porous flow through an earth dam.

(v) Biharmonic equations

Symm [1964] and Jaswon and Symm [to appear] show how the biharmonic equation can be solved using integral equations. This could be an effective method for solving FBPS for the biharmonic equation.

(vi) Connections with complex function theory

The solution u of (1) is the real part of an analytic function $w = w(z) = w(x + iy)$. There is a very close connection between the representation (2) and the representation of w as a Cauchy integral

$$w(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{w(\xi)}{\xi - z} d\xi. \quad (14)$$

The behavior of integrals of the form (14) is extensively studied in the literature on complex function theory. See Muskhelishvili [1953a] and Goluzin [1969]. For applications to the theory of elasticity see Muskhelishvili [1953].

The use of integral equations for trial free boundary methods has several advantages:

- (a) General domains $\Omega^{(k)}$ can be easily handled.
- (b) The approximate solution $u_h^{(k)}$ is expressed in terms of an integral involving an approximate density μ , so that the values of $u_h^{(k)}$ and its derivatives can be readily computed at any point of $\bar{\Omega}^{(k)}$. Thus the boundary condition $\nabla u^{(k)} = 0$ on $\partial\Omega^{(k)}$ is easily checked.
- (c) Corner singularities are easily handled by increasing the density of the meshpoints near the corners. For another approach see Symm [1973].
- (d) Finite difference methods and finite element methods immediately give values of the solution at mesh points, but if integral equations are used then the values of the solution at points in $\Omega^{(k)}$ must be computed, which is a straightforward but time-consuming computation which can require as much time as the solution of the integral equation. For fixed boundary value problems this additional computation is a definite disadvantage of the method of integral equations. However, for trial free boundary methods the values of the solution in $\Omega^{(k)}$ are only needed (if at all) after the final iteration, so that the additional computation is only a slight disadvantage.

The main disadvantage associated with the method of integral equations is that the method is limited to a few special differential equations such as Laplace's equation. This is a severe restriction, but it is not as severe as might be thought: we have compiled an extensive

bibliography of FBPS, and a very large number of these FBPS involve Laplace's equation.

The trial free boundary method with integral equations has been used to solve the FBPS listed below. Information about the type of surface singularity, the type of integral equation(s), and the numerical method used to solve the integral equation(s) is given in brackets.

Axisymmetric Riabouchinsky cavity flow: Armstrong and Dunham [1953] (ring vortices).

Axisymmetric cavity flow (Helmholtz model for cones and spheres; Riabouchinsky model for disks): Struck [1970] (ring sources; equations of first and second kind; Nyström method).

Axisymmetric wall jet: B. W. Hunt [1967, 1968] (ring vortices; equation of second kind; Krylov and Bogoliubov method).

Porous flow through rectangular dams: Symm [1975] (single-layer potential). Cryer [1962, chapter 1, p. 11] used a double-layer potential to check the solution of Shaw and Southwell [1941] but did not iterate.

Elastic-plastic torsion: Ponter [1966] and Jaswon and Ponter [1963] (double layer; equation of second kind).

2.4.3. The self-consistent method and the moment method

In this section we describe two methods which use surface singularities but not integral equations.

The self-consistent method was introduced by Mead and Beard [1964] to compute the earth's magnetopause for the symmetric case, and was later used by Olson [1968, 1969] for the general case.

The problem is as follows. A stream of charged particles, called the solar wind, flows from the sun towards the earth. The solar wind has density ρ and velocity $V\hat{v}$ where \hat{v} is a unit vector. The earth is represented as a magnetic dipole of strength M . The meridian plane is the plane containing the earth's dipole and \hat{v} . The equatorial plane contains \hat{v} and is perpendicular to the meridian plane. The earth's magnetic field deflects the solar wind and creates a three-dimensional region \mathcal{Q} near the earth which is not penetrated by the solar wind. The two-dimensional boundary Γ of \mathcal{Q} is a FB, the magnetopause.

The governing equations can be set up by noting that:

- (i) There is a magnetic field \underline{B}_t inside the magnetopause Γ which satisfies Maxwell's equations.
- (ii) There is no magnetic field outside Γ .
- (iii) The solar wind particles must behave in a prescribed fashion on Γ . The most common assumption is that there is specular reflection; that is, that the particles are reflected by the magnetopause Γ in the same way that light would be reflected by a mirror Γ .

In terms of appropriate polar coordinates (r, θ, ϕ) the FB is defined as the surface

$$r = R(\theta, \phi) .$$

The equatorial cross-section of the FB is smooth while the meridional cross-section contains two discontinuities - the neutral points where the magnetic field is zero.

The total magnetic field \underline{B}_t inside Γ may be thought of as consisting of two components namely the geomagnetic dipole field \underline{B}_g and a field \underline{B}_c caused by surface singularities, namely currents flowing in Γ . The field \underline{B}_g is known, and

$$\underline{B}_c = B_{cr} \hat{r} + B_{c\theta} \hat{\theta} + B_{c\phi} \hat{\phi}$$

where \hat{r} , $\hat{\theta}$, and $\hat{\phi}$ denote the unit vectors in the coordinate directions.

If specular reflection occurs then on the FB we have the condition

(Olson [1968, p. 31]),

$$\begin{aligned} \underline{c}u &= \left| \left(\hat{r} - \frac{1}{R} \left(\frac{\partial R}{\partial \theta} \right) \hat{\theta} - \frac{1}{R \sin \theta} \left(\frac{\partial R}{\partial \phi} \right) \hat{\phi} \right) \times \left((B_{cr} - 2 \sin \theta \sin \phi) \hat{r} + \right. \right. \\ &\quad \left. \left. + (B_{c\theta} + \cos \theta \sin \phi) \hat{\theta} + (B_{c\phi} + \cos \phi) \hat{\phi} \right) \right| - \\ &\quad - 8\pi\rho VM^{-1} R^3 \left(\cos \theta + \frac{\sin \theta}{R} \left(\frac{\partial R}{\partial \theta} \right) \right), \\ &= 0. \end{aligned} \tag{1}$$

The current field \underline{B}_c satisfies the equation (Olson [1968, p. 35]),

$$\underline{B}_c(p) = \frac{1}{2\pi} \iint_{\Gamma} \frac{(\hat{n}_q \times \underline{B}_t(q)) \times \underline{r}_{pq}}{(\underline{r}_{pq})^3} dS_q, \tag{2}$$

where \hat{n}_q is the unit normal to Γ at the point q , where dS_q is a surface element at q , and where \underline{r}_{pq} is the distance from p to q .

In the self-consistent method the field \underline{B}_c is guessed - usually $\underline{B}_c^{(0)} = \underline{B}_g$. Equation (1) is then integrated using finite differences to determine $\Gamma^{(0)}$. Next, equation (2) is used, with $\underline{B}_t = \underline{B}_g + \underline{B}_c^{(0)}$, to determine $\underline{B}_c^{(1)}$, and the process is repeated. The self-consistent method is thus a trial free boundary method, the trial FBS $\Gamma^{(k)}$ being found by an integral method (see section 3.2.2). The governing equation is Laplace's equation expressed in the form (2) using double-layer surface singularities.

In addition to the work mentioned above, Baker, Beard, and Young [1964] have applied the self-consistent method to some magnetopause problems with known solutions and Beard [1964] gives a good description of the method.

So far as we are aware, the self-consistent method has only been used to compute the magnetopause. The method would appear to have applications to many other FBPS, particularly problems of magnetic containment.

The moment method was introduced by Midgley and Davis [1962] to solve a magnetopause problem for the case of constant pressure, and was subsequently used by Midgley [1963] and Midgley and Davis [1963] for the case of specular reflection. The moment method involves the following steps:

- (i) The FB is represented in parametric form in terms of an unknown vector $\underline{c} = (c_i)$. Midgley and Davis [1962] assume that the FB is of the form

$$\rho = C \left[1 - \sum_{i=1}^N c_i (\pi/2 - \theta)^{2i} \right],$$

while Midgley and Davis [1963] assume that the FB is given in terms of the "flux function"

$$f(\rho, \phi) = a(v) \sin \phi e^{g(u, v)} (1 - [(1-u)^2 + h(u, v)]^{1/2})$$

where $v = \cos^2 \phi$, $a(v)$ is half the radius of the FB at $z = -\infty$, $u = \rho/a(v)$, and g and h are double power series in u and v with unknown coefficients c_i .

- (ii) For an assumed FB the surface current \underline{J} is determined and the resulting vector potential \underline{A} outside the magnetopause is expanded in a series of functions D_α which are related to the Legendre polynomials,

$$\underline{A}(p) = \int_{\Gamma} \int \frac{\underline{J}(q)}{|\underline{r}_{pq}|} dS_q = \sum_{\alpha} I_{\alpha}(\underline{c}) D_{\alpha}(p).$$

The coefficients I_{α} are called the moments.

- (iii) The condition that the magnetic field outside Γ should vanish is expressed in terms of the moments I_{α} . The resulting system of equation for the unknown coefficients \underline{c} is solved iteratively.

The moment method is thus a trial free boundary method, the trial free boundaries being found by a global method. (See section 3.2.3).

Although Midgley and Davis did obtain results they experienced considerable difficulties and the moment method, in this form at least, does not seem to be a competitive method.

2.5. Trial free boundary methods with analog methods

Analog methods have long been used to solve FBPS by the trial free boundary method. Bear [1972, chapter II] gives an excellent discussion of analog methods for porous flow problems, and Robertson [1965] describes analog methods for fluid mechanics problems. Basic texts on analog computers include Karplus [1958], Volynskii and Buhman [1965].

In the discussion below we do not consider analog methods which merely involve scaling, such as the modelling of porous flow problems using a "sand-box".

Two essential differences between analog and digital computers are:

- (i) Accuracy. In an analog computer it is difficult and expensive to achieve great accuracy, and one must usually be content with 1% accuracy. In a digital computer, far greater accuracy is possible. It is, however, important to note that the great accuracy

of a digital computer may not be fully utilized: very few FBPS have been solved with an accuracy greater than 1% , and in many practical problems the data is not known to 1% accuracy.

- (ii) An analog computer is special-purpose while a digital computer is general-purpose. An analog computer can thus sometimes be much faster than a digital computer on the class of problems for which the analog computer has been designed.

At present, analog computers have a dowdy and old-fashioned image compared to the glossy and modern image of digital computers. We suspect, however, that the value of analog computers as efficient special-purpose devices will be appreciated more in the future.

2.5.1. Trial free boundary methods with electrolytic tanks

An electrolytic tank is a tank filled with a conducting medium - the electrolyte - such as a copper sulfate solution. The electric potential in electrolyte satisfies Laplace's equation. When solving a FBP, the FB is represented by an adjustable surface, often made of wax. It is possible to solve two-dimensional problems involving equations of the form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho u) = 0;$$

for such problems the tank has a movable bottom which is adjusted so that the depth h of the electrolytic is proportional to ρ .

The following FBPS have been solved using the trial free boundary method with an electrolytic tank:

Fluid mechanics FBPS

Axially symmetric Riabouchinsky cavity behind a disk: Young, Gates

Arms, and Eliezer [1955, Figure 10] and Birkhoff and Zarantonello

[1957, p. 232] give some details of a solution obtained by

P. Marchet and M. Malavard in 1949.

Axially symmetric jet from a pipe: Abul-Fetouh [1949] and Rouse and

Abul-Fetouh [1950].

Axially symmetric jet striking a plate: P. G. Hubbard [1949] and
Leclerc [1950].

Weirs: Hay and Markland [1958] obtained results for the flow over a
weir for a large number of geometries.

Porous flow FBPS

Until recently, electrolytic tanks were widely used for studying
two dimensional and three dimensional porous flow problems. Bear [1972,
p. 706] and Polubarinova-Kochina [1962, p. 463] discuss the problem
of seepage through earth dams.

The adjustment of the FB in an electrolytic tank is quite
laborious: Rouse and Abul-Fetouh [1950, p. 423] remark that their
solution required two hundred hours of work. As regards the accuracy
attainable with an electrolytic tank, Hay and Markland [1958, p. 68 and
p. 69] state that their results are correct to 1%, and this figure seems
to be generally accepted.

2.5.2. Trial free boundary methods with the Hele-Shaw analog

It was observed by Hele-Shaw in 1898 that if a viscous fluid flows slowly between two parallel vertical plates which are a distance b apart, where b is small then the constitutive equation reduces to

$$\underline{v} = (u, v) = -\frac{1}{12} (gb^2/\mu)\text{grad}(y + p/\gamma),$$

where u and v denote the average velocity of the viscous fluid in the x and y directions. There is thus a direct analogy between viscous flow between parallel plates and porous flow. This is of course not surprising since all the theoretical derivations of Darcy's law in porous flow use models involving viscous flow through narrow passages. Bear [1972, p. 687] gives a detailed discussion of the Hele-Shaw analog which has been applied to many porous flow FBPS. The apparatus for observing viscous flow between parallel plates is often called a Hele-Shaw cell.

The Hele-Shaw analog is also of interest because it provides a link between porous flow FBPS and viscous fluid mechanics FBPS such as the two-dimensional flow of a viscous bubble (see Saffman and Taylor [1958]).

2.5.3. Trial free boundary methods with resistance networks

In sections 2.1 and 2.2 we considered the methods in which the fixed boundary value problem

$$\begin{aligned} \mathcal{A}u^{(k)} &= 0, \quad \text{in } \mathcal{D}^{(k)}, \\ \mathcal{B}u^{(k)} &= 0, \quad \text{on } \partial\mathcal{D}^{(k)}, \end{aligned}$$

was approximated by finite differences or finite elements, the resulting system of algebraic equations being solved by hand or on a computer.

An alternative approach is to replace the algebraic equations by an equivalent electrical network of resistances.

Resistance networks have been widely used for porous flow FBPS (Bouwer [1967], Bear [1972, p. 710]). An interesting aspect of such applications is that it has proved possible to automate the movement of the trial FBS:

- (i) Karplus [1956] automatically solves the problem of a water-cone by connecting a series of DC analog computing units to the resistance network along the network boundary corresponding to the FB.
- (ii) Herbert and Rushton [1966, p. 72] automatically solve the problem of flow through a porous dam by adding transistors which disconnect portions of the network where the condition $u^{(k)} \geq y$ is violated.

2.5.4. Trial free boundary methods with conducting paper and carbon

Wyckoff and Reed [1935] pioneered the use of conducting paper to solve porous flow FBPS. They used graphite-coated paper to solve the problem of seepage through an earth dam for four different geometries: rectangular dam with or without down-stream water, symmetric trapezoidal dam with faces of slope 30° and 45° . In the approach used by Wyckoff and Reed the velocity potential ϕ in the porous flow is represented by the electric potential in a sheet of resistance paper. The procedure used is shown schematically in Figure 1 for the case of flow through a rectangular dam.

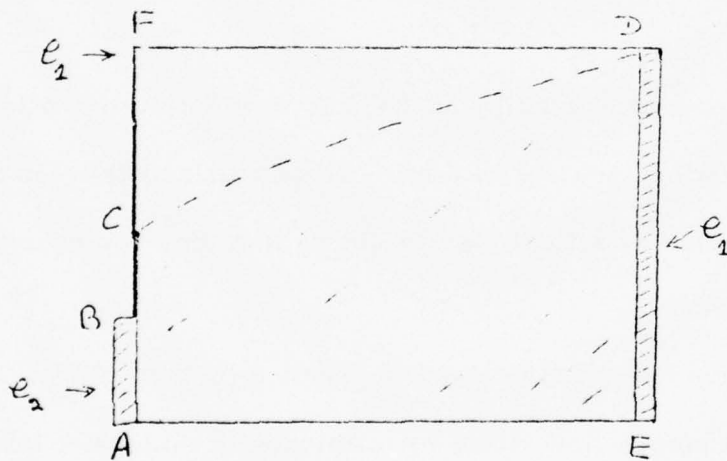


Figure 1: Resistance paper analog of porous flow through an earth dam (based on Wyckoff and Reed [1935, p. 396])

In Figure 1 the rectangular sheet of graphite-coated paper AEDF is provided with terminals which maintain parts of its boundary at prescribed electric potentials. The terminals DE and AB are good conductors and are kept at potentials e_1 and e_2 , respectively, so as to represent the upstream and downstream waters. The terminal BF is a resistance strip which represents the seepage face: the endpoints F and B are kept at potentials e_1 and e_2 and the potential varies linearly between F and B. The FB is the curve CD and is obtained by cutting the sheet of paper. All the boundary conditions are satisfied except the second boundary condition on the FB, namely that the potential should be linear along the FB, and the FB is gradually cut away until this boundary condition is satisfied.

The paper of Wyckoff and Reed is very carefully written. They discuss the various sources of error, and comparison between their analog results and exact analytical results shows that they can achieve an accuracy of about 1%.

Subsequently, Childs used the method of Wyckoff and Reed to solve several porous flow problems: drainage of rain water towards a row of drains (Childs [1943, 1945, 1945a]); flow down a slope interrupted by channel (Childs [1946]); the Ghyben-Herzberg lens (Childs [1950]).

The use of graphite paper is limited to plane FBPS. Babbitt and Caldwell [1948] studied the axisymmetric problem of a fully-penetrating well by cutting a thin carbon wedge, applying the appropriate electrical analogues of the boundary conditions, and then cutting the upper surface of the wedge (which corresponded to the FB) until both boundary conditions were satisfied.

We conclude with two remarks:

(i) In an era of high-speed computers the use of an analog method based on graphite-coated paper may seem prehistoric. However, it is not unknown for large amounts of computer time to be spent to obtain results which are little better than those of Wyckoff and Reed.

(ii) The work of Wyckoff and Reed [1935] and Babbitt and Caldwell [1948] provides valuable information about trial free boundary methods for porous flow problems: it is clear that such methods must be very stable; and the results of Wyckoff and Reed suggest that if the initial FB is too high the successive trial FBS form a monotonically decreasing sequence.

2.6. Trial free boundary methods with graphical methods

The earliest approximation methods for elliptic equations were graphical methods, and, according to Runge and Willers [1915, p. 165], Forchheimer, Blasius and von Mises all considered using trial free boundary methods in conjunction with graphical methods; we consulted these early references but found that none of the workers had implemented these ideas,

which was of course quite understandable because of the labor required.

L. Casagrande [1932, 1934] and A. Casagrande [1940], who solved several porous flow FBPS for dams, were apparently the first to solve FBPS using graphical methods. To our own knowledge, graphical methods were extensively used to solve porous flow problems up to 1960, but we are not aware of any applications to FBPS.

Shaw and Southwell [1941] were aware of the work of A. Casagrande, so that there has been a continuous development from graphical methods to finite difference methods and then to finite element methods.

3. Step 2: Movement of the boundary

We recall the definitions of Steps 1 and 2:

Step 1. Given $\Gamma^{(k)}$ let $\mathcal{D}^{(k)}$ be the corresponding domain. Compute an approximation, $u_h^{(k)}$ say, to the solution $u^{(k)}$ of the problem

$$\mathcal{A}u^{(k)} = 0, \quad \text{in } \mathcal{D}^{(k)},$$

$$\mathcal{B}u^{(k)} = 0, \quad \text{on } \partial\mathcal{D}^{(k)}.$$

Step 2. Given $\Gamma^{(k)}$ and $u_h^{(k)}$ compute a new trial FB $\Gamma^{(k+1)}$ by requiring that $\mathcal{C}u_h^{(k)}$ should be approximately equal to zero on $\Gamma^{(k+1)}$; i. e. "move the boundary" from $\Gamma^{(k)}$ to $\Gamma^{(k+1)}$.

Various aspects of Step 3 are considered in the following subsections: choice of boundary conditions; movement strategy; a simple convergence proof; and numerical experience.

3.1. Choice of boundary conditions

In most FBPS the boundary conditions \mathcal{B} and \mathcal{C} arise in a natural way and it is therefore natural to use them without modification. It is the purpose of this section to point out that it may be helpful to modify the boundary conditions on the FB before using a trial free boundary method.

Consider the case of a one-domain FBP when the governing equation is a general second order elliptic equation

$$\mathcal{A}u = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y = 0 \quad (1)$$

and when the boundary conditions on the FB Γ take the form:

$$\begin{aligned} \mathcal{B}u &= \beta_1 \frac{\partial u}{\partial n} + \alpha_1 u - \gamma_1 = 0, \\ \mathcal{C}u &= \beta_2 \frac{\partial u}{\partial n} + \alpha_2 u - \gamma_2 = 0, \end{aligned} \quad (2)$$

where β_i , α_i , and γ_i are smooth function of x and y . It is assumed that

$$\begin{vmatrix} \beta_1 & \alpha_1 \\ \beta_2 & \alpha_2 \end{vmatrix} \neq 0, \text{ on } \Gamma, \quad (3)$$

since otherwise the boundary conditions on Γ would be dependent or contradictory.

It is implied by (2) that the boundary condition

$$\beta_1 \frac{\partial u}{\partial n} + \alpha_1 u - \gamma_1 = 0, \text{ on } \Gamma, \quad (4)$$

is used to compute $u_h^{(k)}$, and that the boundary condition

$$\beta_1 \frac{\partial u}{\partial n} + \alpha_1 u - \gamma_1 = 0, \text{ on } \Gamma, \quad (5)$$

is used to move $\Gamma^{(k)}$.

The role of (4) and (5) could of course be interchanged. There are, however, many other possibilities. If $e_{ij}(x, y)$ are smooth functions satisfying

$$\begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \neq 0, \text{ on } \Gamma, \quad (6)$$

then instead of using \mathcal{B} and \mathcal{C} one could use the equivalent operators

$$\hat{\mathcal{B}} = \begin{cases} e_{11}\mathcal{B} + e_{12}\mathcal{C}, & \text{on } \Gamma, \\ \mathcal{B} & \text{on } \partial\mathcal{D}-\Gamma, \end{cases} \quad (7)$$

$$\hat{\mathcal{C}} = e_{21}\mathcal{B} + e_{22}\mathcal{C}, \quad \text{on } \Gamma.$$

The question which arises is whether $\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$ can be chosen so as to improve the convergence of a trial free boundary method.

Noting (3) we see that two possible choices for $\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$ take the form:

$$\hat{\mathcal{B}}_1|_{\Gamma} : \frac{\partial u}{\partial n} + \gamma_{11} = 0, \quad (8)$$

$$\hat{\mathcal{C}}_1 : u + \gamma_{12} = 0,$$

and

$$\hat{\mathcal{B}}_2|_{\Gamma} : u + \gamma_{21} = 0, \quad (9)$$

$$\hat{\mathcal{C}}_2 : \frac{\partial u}{\partial n} + \gamma_{22} = 0.$$

Of these two possibilities the first has the following advantages:

(i) The boundary value problem

$$\sigma u^{(k)} = 0, \quad \text{in } \mathcal{D}^{(k)},$$

$$\hat{\mathcal{B}}_1 u^{(k)} = 0, \quad \text{on } \partial\mathcal{D}^{(k)},$$

can often be reformulated as a variational problem in which the boundary condition

$$\frac{\partial u^{(k)}}{\partial n} + \gamma_{11} = 0, \text{ on } \Gamma^{(k)},$$

appears as the "natural" boundary condition.

- (ii) When $u_h^{(k)}$ has been computed, both $u_h^{(k)}$ and $\frac{\partial}{\partial n} u_h^{(k)} = -\gamma_{11}$ are known on $\Gamma^{(k)}$ without further computations. This makes it easy to use linear interpolation to find the next trial FB $\Gamma^{(k+1)}$ on which

$$\hat{C}_1 u^{(k)} = u_h^{(k)} + \gamma_{12} = 0. \quad (10)$$

In contrast, if the operators $\{\hat{B}_2, \hat{C}_2\}$ were used then only $u_h^{(k)}$ is known on $\Gamma^{(k)}$. To satisfy

$$\hat{C}_2 u^{(k)} = \frac{\partial u^{(k)}}{\partial n} + \gamma_{22} = 0, \text{ on } \Gamma^{(k+1)}, \quad (11)$$

using linear interpolation it would be necessary to compute both $\frac{\partial}{\partial n} u_h^{(k)}$ and $\frac{\partial^2}{\partial n^2} u_h^{(k)}$ on $\Gamma^{(k)}$ by numerical differentiation.

- (iii) In general, the solution of a boundary value problem involving Neumann boundary conditions is smoother than the solution of the corresponding problem involving Dirichlet boundary conditions. One might, therefore, hope to obtain somewhat greater accuracy using $\{\hat{B}_1, \hat{C}_1\}$ instead of $\{\hat{B}_2, \hat{C}_2\}$.

It would thus appear that it is always better to use $\{\hat{\beta}_1, \hat{c}_1\}$ rather than $\{\hat{\beta}_2, \hat{c}_2\}$. It should be remarked, however, that Southwell and Vaisey [1946, p. 127] in general preferred $\{\hat{\beta}_2, \hat{c}_2\}$ (which they called Method A) to $\{\hat{\beta}_1, \hat{c}_1\}$ (which they called Method B), because equation (10) provides less freedom than equation (11) in determining $I^{(k+1)}$.

It should perhaps also be remarked that the choice of β, c may sometimes be based on physical analogies, as for example when Shaw and Southwell [1941, p. 7] visualized the solution of porous flow problems for dams in terms of gas balloons held down by shot-bags. The mathematician may smile at this analogy (which was obviously influenced by the wartime conditions then prevailing), but the lack of more precise criteria makes the use of such analogies as good as any other method.

In the next two subsections two methods for choosing boundary conditions are discussed: a method due to Garabedian [1956] of constructing the boundary conditions $\hat{\beta}, \hat{c}$, so as to ensure quadratic convergence with respect to changes in the trial FBS; and a method due to Cryer [1968, 1970a] and McCorquodale and Li [1971] of using integral relations to derive alternative boundary conditions for problems where the boundary conditions on the FB involve an unknown constant (as is usually the case when Bernoulli's equation is used in fluid mechanics FBPS).

In conclusion it must be emphasized that although the above remarks have been concerned with one-domain FBPS for second order elliptic equations, the question of the choice of boundary conditions arises in every trial free boundary method. We have not discussed this question for other types of FBPS because we are not aware of any work on the subject.

3. 1. 1. Choice of boundary conditions: Garabedian's method

We now describe an ingenious way of choosing $\hat{\mathcal{B}}$ which is due to Garabedian [1956]. Recall that in a trial free boundary method the condition $\mathcal{R}u^{(k)} = 0$ is first satisfied on $\partial\mathcal{D}^{(k)}$. Then the FB $\Gamma^{(k)}$ is moved so that the condition $\mathcal{C}u^{(k)} = 0$ is satisfied, but of course the condition $\mathcal{R}u = 0$ is no longer satisfied on $\Gamma^{(k+1)}$.

The idea of Garabedian is to construct $\hat{\mathcal{B}}$ so that $\hat{\mathcal{B}}u$ is insensitive to movements of Γ . In the language of Garabedian, $\hat{\mathcal{B}}$ is constructed so that $\hat{\mathcal{B}}u$ is stationary with respect to normal displacements of the FB. More precisely, $\hat{\mathcal{B}}$ is constructed so that

$$(\hat{\mathcal{B}}u)_n = 0, \text{ on } \Gamma, \quad (1)$$

where u is the solution of the FBP. Garabedian's method may also be thought of as Newton's method applied to FBPS (Garabedian [1956a, p. 613]).

To construct \hat{R} we begin by observing that we may choose the coefficients e_{ij} of equation (7) of section 3.1 so that on Γ \hat{B} and \hat{C} are of the form

$$\hat{B}u = (u_n - \gamma_{11}) + \tau(u - \gamma_{12}) = 0,$$

$$\hat{C}u = u - \gamma_{12} = 0,$$

where τ is an arbitrary function. Thus,

$$\begin{aligned} (\hat{B}u)_n = 0 &= (u_n - \gamma_{11})_n + \tau(u - \gamma_{12})_n + \tau_n(u - \gamma_{12}), \\ &= u_{nn} - (\gamma_{11})_n + \tau[u_n - (\gamma_{12})_n], \end{aligned}$$

since $u - \gamma_{12} = 0$ on Γ . Condition (1) will thus be satisfied if

$$\tau = \frac{-u_{nn} + (\gamma_{11})_n}{u_n - (\gamma_{12})_n} = \frac{-u_{nn} + (\gamma_{11})_n}{\gamma_{11} - (\gamma_{12})_n}. \quad (2)$$

Let \underline{t} be the unit tangent to Γ , let s denote distance along Γ , and let κ denote the curvature of Γ . Then on Γ we have

$$u_n = \gamma_{11}, \quad (3)$$

$$u_t = u_s = (\gamma_{12})_s.$$

As special cases of the Frenet formulas,

$$u_{tt} - \kappa u_n = u_{ss} = (\gamma_{12})_{ss}, \quad (4)$$

$$u_{nt} + \kappa u_t = u_{ns} = (\gamma_{11})_s.$$

Finally, the equation $\mathcal{A}u = 0$ can be rewritten in the form

$$a'_{11}u_{nn} + 2a'_{12}u_{nt} + a'_{22}u_{tt} + b'_1u_n + b'_2u_t = 0, \quad (5)$$

where the coefficients a'_{ij} and b'_i are linear combinations of the a_{ij} and b_i . From (2), (3), (4), and (5) we find that

τ can be expressed in terms of γ_{11} , $(\gamma_{11})_n$, $(\gamma_{12})_n$, $(\gamma_{11})_s$, $(\gamma_{12})_s$, $(\gamma_{12})_{ss}$, a_{ij} , b_i , and κ . As an important special case, if γ_{11} and γ_{12} are constants and if \mathcal{A} is the Laplacian then equation (5) takes the form

$$u_{nn} + u_{tt} = 0,$$

and we find that $\tau = \kappa$.

In practice, the value of τ is not known because the values of κ , γ_{11} , etc. depend upon Γ which is unknown. The simplest strategy (used by Cryer [1968, 1970a]) is to replace τ by $\tau^{(k)}$ which is computed using $\Gamma^{(k)}$ instead of Γ . Then $u^{(k)}$ is defined to be the solution of the problem.

$$\mathcal{A}u^{(k)} = 0 \text{ in } \mathcal{D}^{(k)},$$

$$\hat{\mathcal{B}}^{(k)}u^{(k)} = (u_n^{(k)} - \gamma_{11}) + \tau^{(k)}(u^{(k)} - \gamma_{12}) = 0 \text{ on } \partial\mathcal{D}^{(k)}.$$

Cryer [1968, 1970a] found that the use of $\{\hat{\mathcal{B}}, \hat{\mathcal{C}}\}$ à la Garabedian instead of $\{\mathcal{B}, \mathcal{C}\}$ improved convergence but that it was not clear whether convergence was quadratic.

3.1.2. Choice of boundary conditions: integral relations

In a number of FBPS the boundary conditions on the FB involve an unknown constant λ which must be determined as part of the solution. For example, in fluid mechanics FBPS one often has the boundary condition:

$$C: u_n = \lambda \text{ on } \Gamma.$$

One approach which has often been used is to compute $\lambda^{(k)}$ by averaging:

$$\lambda^{(k)} = \text{average}_{\Gamma} u_n^{(k)}.$$

We believe that this is not a good approach, and in this section we indicate how the method of integral relations can be used to obtain useful expressions for λ .

We illustrate the idea by considering the problem of a plane jet of an ideal weightless fluid flowing from an orifice (Figure 1), following Cryer [1968, p. 64]. In terms of the stream function ψ the governing equation is

$$\nabla^2 \psi = 0,$$

and the boundary conditions are:

$$B: \begin{cases} \psi = 0, & \text{on } AB \\ \psi = AF, & \text{on } CDEF \\ \psi = y, & \text{on } AF, \\ \psi = \lambda y, & \text{on } BC, \end{cases}$$

$$C: |\text{grad } \psi| = \lambda, \text{ on } CD,$$

where λ is an unknown constant.

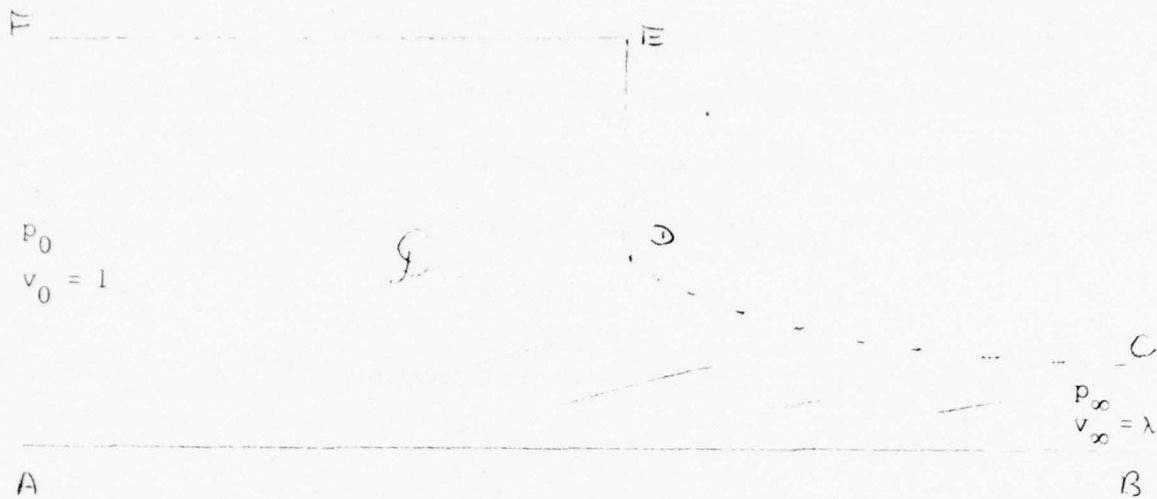


Figure 1. Plane jet flowing from an orifice

The pressure and velocity of the fluid will be denoted by p and v respectively. The limits of the fluid pressure and fluid velocity far upstream and far downstream are denoted by p_0 , p_∞ , v_0 , and v_∞ respectively.

Since (Milne-Thompson [1968, p. 114]),

$$v = |\text{grad } \psi|, \quad (1)$$

it follows that far upstream the fluid flows with velocity $v_0 = 1$ parallel to the x-axis, while far downstream the fluid flows with velocity $v_\infty = \lambda$ parallel to the x-axis.

The method of integral relations involves multiplying the governing differential equation by one or more functions and integrating over the domain D . The integral is then manipulated using integration by parts

together with the boundary conditions. In this way a number of identities satisfied by the solution can be obtained. These identities often have a physical meaning such as conservation of mass or momentum, and this is the case in the present problem.

From the principle of conservation of mass (Milne-Thompson [1968, p. 72]),

$$AF \cdot v_0 = BC \cdot v_\infty . \quad (2)$$

Then, since CDEF is a streamline, it follows from Bernoulli's equation (Milne-Thompson [1968, p. 10]) that

$$\frac{1}{2} \rho v^2 + p = \frac{1}{2} \rho v_\infty^2 + p_\infty \quad \text{on CDEF} , \quad (3)$$

where ρ is the density of the fluid.

Hence,

$$\frac{1}{2} \rho v_0^2 + p_0 = \frac{1}{2} \rho v_\infty^2 + p_\infty . \quad (4)$$

From (1) and the boundary condition C it follows that $v = v_\infty$ on CD. Thus, from (3),

$$p = p_\infty , \quad \text{on CD} . \quad (5)$$

Next, we apply Euler's momentum theorem (Milne-Thompson [1968, p. 79]) to the curve ABCDEFA:

Resultant thrust on ABCDEFA in the positive x-direction

$$\begin{aligned}
&= AF \cdot p_0 - \int_C^D p n_1 ds - \int_D^E p ds - BC \cdot p_\infty \\
&= \text{rate of flow of momentum in the x-direction} \\
&\quad \text{outwards across ABCDEFA} \\
&= \rho v_\infty^2 BC - \rho v_0^2 AF,
\end{aligned} \tag{6}$$

where n_1 is the component in the x-direction of the unit outward normal on CD.

Noting (3) and (4) and remembering that $n_1 ds = dy$, we see that, equation (6) is equivalent to

$$(AF - DE)(p_0 - p_\infty) - \frac{1}{2} \rho v_0^2 DE + \frac{1}{2} \rho \int_D^E v^2 ds = \rho v_\infty^2 BC - \rho v_0^2 AF. \tag{7}$$

Finally, from (7), (2), and (4), it follows that

$$(AF - DE)v_\infty^2 - 2v_0 v_\infty AF + \int_D^E v^2 ds + v_0^2 AF = 0. \tag{8}$$

Solving the quadratic equation (8) for v_∞ we find that

$$v_\infty = \frac{v_0 AF + [AF \cdot DE v_0^2 - (AF - DE) \int_D^E v^2 ds]^{1/2}}{AF - DE}. \tag{9}$$

Then, from (2),

$$AF = BC v_\infty / v_0. \tag{10}$$

In solving (8) the larger root was chosen because the smaller root leads to a physically unacceptable solution. For, let \bar{v}_∞ be the smaller root. Then, from (9),

$$\bar{v}_x < v_0 AF / (AF - DE)$$

and so, from (2),

$$BC > (AF - DE),$$

which is physically impossible (see Figure 1).

The arguments leading to equations (9) and (10) were originally used by Borda [1766]; a careful presentation of Borda's arguments is given by Gilbarg [1960, p. 340]. For the case considered by Borda v_0 could be determined exactly.

Rewriting (9) in terms of λ and ψ and remembering that $v_0 = 1$ in our case we find that

$$(AF - DE)\lambda = AF + [AF \cdot DE - (AF - DE) \int_D^E (\psi_x)^2 ds]^{1/2}. \quad (11)$$

Given an approximate solution $\psi^{(k)}$ we can use (11) to determine a corresponding approximation $\lambda^{(k)}$. We may expect $\lambda^{(k)}$ to be a good approximation since the right hand side of (11) only involves the values of ψ on the segment DE of the known boundary, and it is reasonable to expect $\psi_x - \psi_x^{(k)}$ to be small on DE even if $\psi - \psi^{(k)}$ is large on $\Gamma^{(k)}$.

McCorquodale and Li [1971] use the same idea for the problem of flow through a sluice gate. They are not too successful, but this may be because they adjust $\Gamma^{(k)}$ using a global method (see section 3.2.3).

3.2. Movement strategy

The most difficult aspect of trial free boundary methods involves the movement of $\Gamma^{(k)}$ to $\Gamma^{(k+1)}$. Unfortunately, many authors simply say that "the trial FB was adjusted until both boundary conditions were satisfied" and give no further details.

Handworkers such as Southwell and Vaisey [1946] did not use specific rules to move the boundary. It would of course be possible to emulate this by using a computer interactively: the computer would compute the approximation $u_h^{(k)}$ and the error $Cu_h^{(k)}$, while the human operator would adjust the boundary manually using a light pen at a console. It is also conceivable that the boundary could be moved by an "adaptive-learning" program which learnt from its mistakes. To our knowledge, however, in computer implementations the boundary has always been moved using a definite algorithm.

In computer implementations of trial free boundary methods it is usually convenient to regard the boundaries $\Gamma^{(k)}$ as being defined by a number of parameters, $a_1^{(k)}, \dots, a_p^{(k)}$, say. Possible parameterizations include:

- (i) $\Gamma^{(k)}$ is polygonal with $p/2$ vertices. The parameters $a_1^{(1)}, \dots, a_p^{(k)}$ then represent the x and y coordinates of the vertices of $\Gamma^{(k)}$.
- (ii) $\Gamma^{(k)}$ is the curve

$$y(x) = \sum_{j=1}^p a_j^{(k)} \phi_j(x)$$

where the functions ϕ_j are known.

(iii) $\Gamma^{(k)}$ is the curve

$$x(t) = \sum_{j=1}^{p/2} a_j^{(k)} \phi_j(t),$$

$$y(t) = \sum_{j=p/2+1}^p a_j^{(k)} \phi_j(t),$$

where the functions ϕ_j are known.

(iv) $\Gamma^{(k)}$ is defined implicitly by

$$f(x, y, a_1^{(k)}, \dots, a_p^{(k)}) = 0$$

where f is a given function.

We will denote the curve corresponding to $a_1^{(k)}, \dots, a_p^{(k)}$ by $\Gamma(a_1^{(k)}, \dots, a_p^{(k)})$ or $\Gamma(\underline{a}^{(k)})$ where $\underline{a}^{(k)} = (a_1^{(k)}, \dots, a_p^{(k)})$.

Methods for moving the boundary fall into three categories which we call local, integral, and global, respectively. To explain these methods we consider the problem of porous flow through a dam (see section 0).

In terms of the velocity potential $u \equiv \phi$ the FBP is:

$$\mathcal{A}u \equiv u_{xx} + u_{yy} = 0, \quad \text{in } \mathcal{D},$$

$$\mathcal{B}u \equiv u + Ky = 0, \quad \text{on } \Gamma,$$

$$\mathcal{C}u \equiv u_n = 0, \quad \text{on } \Gamma,$$

together with appropriate boundary conditions on the fixed boundary.

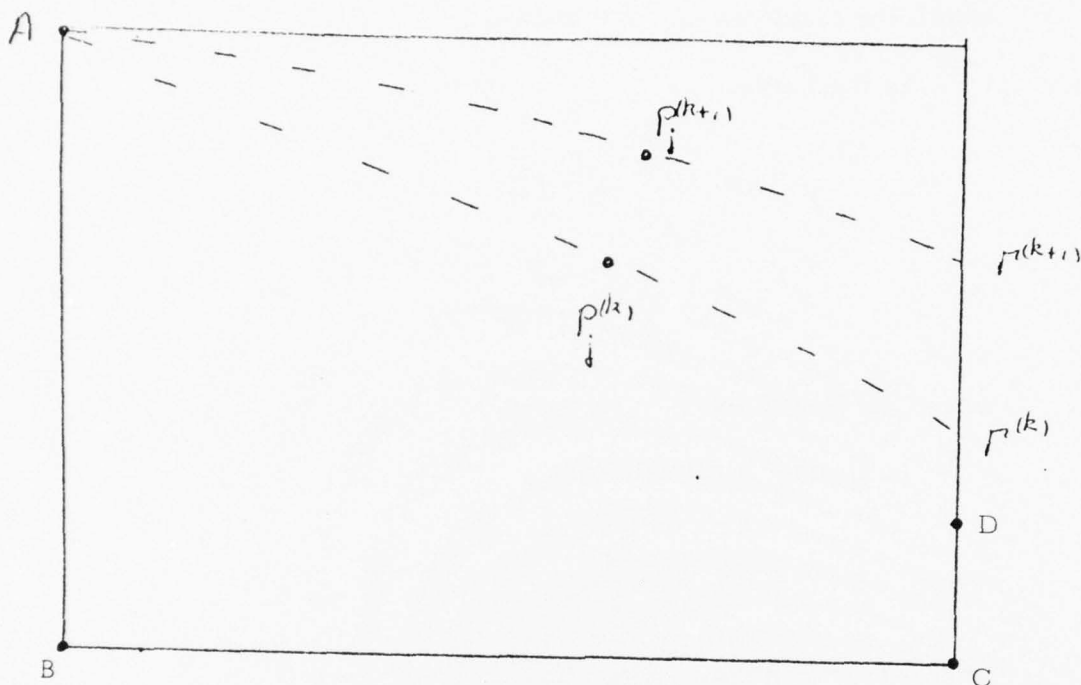


Figure 1. Moving the boundary

In a local method the adjustments to $\Gamma^{(k)}$ are made at individual points on the basis of the error $Cu^{(k)}$ at these points. Thus $Cu^{(k)}$, or rather $Cu_h^{(k)}$, is computed at m points $P_j^{(k)} \in \Gamma^{(k)}$, $1 \leq j \leq m$. If $Cu_h^{(k)}(P_j^{(k)}) \neq 0$ then a nearby point $P_j^{(k+1)}$ is determined so that $Cu_h^{(k)}(P_j^{(k+1)}) = 0$; that is, $P_j^{(k)}$ is "moved" to $P_j^{(k+1)}$ (see Figure 1). The curve $\Gamma^{(k+1)} = \Gamma(\underline{a}^{(k+1)})$ is obtained by fitting a curve of the form $\Gamma(\underline{a})$ through the points $P_j^{(k+1)}$. If $m = p$ then $\Gamma(\underline{a}^{(k+1)})$ will in general pass through all the points $P_j^{(k+1)}$. If $m > p$ then $\Gamma(\underline{a}^{(k+1)})$ is found by an appropriate curve-fitting procedure such as least squares.

In an integral method the curve $\Gamma^{(k+1)}$ is found by "integrating the boundary condition $\zeta u = 0$ ". In the present example, $u^{(k)}$ is a velocity potential from which the velocity field $\underline{v}^{(k)}$ can be found by differentiation. The condition $\zeta u \equiv u_n = 0$ corresponds to the physical condition that Γ is a streamline. Starting from the fixed point A, the streamline passing through A can be determined by integrating the velocity field $\underline{v}^{(k)}$, and this streamline is taken to be $\Gamma^{(k+1)}$.

In a global method a set of p perturbed boundaries $\Gamma^{(k,j)} = (a_1^{(k)}, \dots, a_{j-1}^{(k)}, a_j^{(k)} + \delta a_j^{(k)}, a_{j+1}^{(k)}, \dots, a_p^{(k)})$ and the corresponding solutions $u_h^{(k,j)}$ are generated. This information makes it possible to estimate the dependence of $\zeta u^{(k)}$ upon the parameters $\underline{a}^{(k)}$. The new approximate FB $\Gamma^{(k+1)}$ is chosen so as to minimize the error $\zeta u^{(k+1)}$. In particular, if it is assumed that $\zeta u^{(k)}$ depends linearly upon $\underline{a}^{(k)}$, then $\underline{a}^{(k+1)}$ can in general be chosen so that $\zeta u^{(k+1)}$ is zero at p points on $\Gamma^{(k+1)}$.

These three methods of moving Γ are discussed in greater detail in the following subsections. In the final subsection we discuss the case when the boundary conditions involve an unknown constant.

3.2.1. Movement strategy: local

In a local method of moving the boundary the error $\mathcal{C}u_h^{(k)}$ is computed at m points $P_j^{(k)} \in \Gamma^{(k)}$, $1 \leq j \leq m$. If $\mathcal{C}u_h^{(k)}(P_j^{(k)}) \neq 0$ then a nearby point $P_j^{(k+1)}$ is determined so that the boundary conditions are "satisfied better" at $P_j^{(k+1)}$; that is, $P_j^{(k)}$ is "moved" to $P_j^{(k+1)}$. The curve $\Gamma^{(k+1)}$ is found by fitting a curve through the points $P_j^{(k+1)}$.

Handworkers such as Southwell and Vaisey [1946] did not use specific rules to move the boundary and used trial and error. Indeed, when comparing the choice of a Dirichlet boundary condition for \mathcal{C} (which they call Method B)

$$\begin{aligned}\hat{\mathcal{B}}_1 u &= u_n + \gamma_{11} = 0, \\ \hat{\mathcal{C}}_1 u &= u + \gamma_{12} = 0,\end{aligned}\tag{1}$$

with the choice of a Neuman condition for \mathcal{C} (which they call method A)

$$\begin{aligned}\hat{\mathcal{B}}_2 u &= u + \gamma_{21} = 0, \\ \hat{\mathcal{C}}_2 u &= u_n + \gamma_{22} = 0,\end{aligned}\tag{2}$$

Southwell and Vaisey [1946, p. 127] comment that they prefer Method A to Method B because

"Method B ... yields a definite indication regarding the shape to be adopted in the next stage of the computations ... in Method B divergence is not under control".

Southwell [1946, p. 225] summarizes his viewpoint very well:

"The ability of Relaxation Methods to deal with problems such as these is due to the retention, throughout their development, of a tentative quality which orthodox methods do not possess. There is nothing new to the notion of continued approximation to a wanted solution, for this is the basis of all 'iterative' methods of attack. What is novel is the freedom that throughout is left to the computer, to decide the nature of his next step."

It is of interest to mention some of the details of trial-and-error methods:

- (i) Vaisey [1961] has informed us that in the computations the ratio (local correction)/(local error) sometimes changed sign during the process of solution. Furthermore, in some cases the error over a section of the boundary could only be reduced by adjusting a different section of the boundary. The problem of a waterfall (Southwell and Vaisey [1946, p. 144]) presented considerable difficulties, and we have been told that in some cases it was not possible to reduce the error by moving the boundary of the jet and that instead the edge of the waterfall was moved.
- (ii) Abul-Fetouh [1949] and B. W. Hunt [1967] considered the problem of an axially symmetric jet from an orifice. Following Southwell and Vaisey [1946, p. 134] they used the Neumann boundary condition for ψ which takes the form

$$\eta = \frac{1}{x} \frac{\partial \psi}{\partial n} - \lambda . \quad (3)$$

Abul-Fetouh used finite differences and Hunt used an integral equation to compute $u_h^{(k)}$, but both authors used trial-and-error to move the FB and give clear descriptions of their experience.

Abul-Fetouh [1949, p. 25] writes:

"The value of η indicates the difference between the asymptotic velocity and the local velocity at a certain point on the surface, and this should be reduced to zero. The tendency of the variation of η will determine the new assumed changes in the curve. A positive value of η shows that the local velocity is too large, and the jet section should increase. The curvature of the stream surface and the cross section of the flow affect the magnitude of the velocity at any point on that surface. An increase in cross-section decreases η and a decrease in curvature decreases η also. These two factors will work together to produce the final η . The method of choosing the right curve to make η approach zero is not easy; it requires considerable experience and personal judgment. However, some guidance is at hand if one knows that an increase in the jet radius at any point will decrease the value of η at that point and increase the values of η at the adjacent points on both sides of the considered point, since the curve becomes more concave. If the values of η alternate in sign, the shape of the curve should change. If η has the same sign, the asymptotic radius should change with the corresponding change in the curve. Trial and error will show the tendency and the amount of change in the ordinates of the curve. No definite value of the change of the ordinates can be expressed to produce a certain change in the surface velocity at a certain point, since the curvature of the profile affects the results to some extent. However, as a rough estimate, a change of ∓ 0.004 in the ordinate at a point will change the velocity at that boundary point by $\pm 1\%$. Near the orifice edge, the situation is more difficult, since the profile curves sharply in this zone and it is only by trial and error that one can arrive at the right shape. If two assumed curves happen to produce different signs of η for similar points, then the right curve lies between these two and a simple interpolation for the ordinate will be of great help."

Hunt [1967, p. 18] writes:

"Details of the method used for locating the free streamline consisted of initially choosing an asymptotic radius r_a . Next, the shape of the free streamline between the orifice and the region of uniform flow was adjusted by trial and error until velocities along the portion of the free surface farthest from the orifice were essentially constant. Finally, if free-surface velocities in the vicinity of the orifice were higher than the chosen asymptotic velocity, the asymptotic radius was decreased. Conversely, if velocities in this region were too low, the asymptotic radius was increased. The entire process had to be repeated a number of times until a satisfactory free-streamline geometry was found. This usually meant calculating velocities along the free surface for ten to twenty different geometries in order to obtain one solution."

In addition to the authors mentioned already, the following authors adjusted the FB by trial-and-error:

With finite differences: Vandrey [1940], Shaw and Southwell [1941],
Binnie and Davidson [1949], Yang [1949], Rouse and Abul-Fetouh
[1950], van Deemter [1950], Ball [1951], Boulton [1951],
Brunauer [1951], Citrini [1951], Kashef, Touloukian and Fadum
[1952], Allen and Dennis [1953], Boreli [1955], H. P. Hall [1955],
McNown, Hsu, and Yih [1955], Young, Gates, Arms, and Eliezer
[1955], Dumitrescu, Ionescu, and Toth [1956], Jakobsson and
Floberg [1957], J. A. Murray [1960].

With integral equations: Trefftz [1914, 1916], Wagner [1932],
Schach [1935].

With the advent of computers it became desirable to automate the movement of the FB. The approach usually followed has been to determine the points $P_j^{(k+1)}$ according to the condition that

$$C_{u_h}^{(k)}(P_j^{(k+1)}) = 0 \quad (4)$$

together with the condition that

$$P_j^{(k+1)} - P_j^{(k)} = \alpha_j^{(k)} \underline{d}_j^{(k)}, \quad (5)$$

where $\underline{d}_j^{(k)}$ is a specified unit vector.

Possible choices for $\underline{d}_j^{(k)}$ are:

- (a) The unit outward normal to $\Gamma^{(k)}$ at $P_j^{(k)}$.
- (b) The unit vector in one of the coordinate directions.
- (c) The unit outward conormal to $\Gamma^{(k)}$ at $P_j^{(k)}$. That is, if the governing equation is

$$\mathcal{A}u = (f_1 u_x)_x + (f_2 u_y)_y$$

and the unit outward normal is

$$\underline{n} = (n_1, n_2),$$

then the unit outward conormal is the vector

$$\underline{v} = (f_1 n_1, f_2 n_2) / [(f_1 n_1)^2 + (f_2 n_2)^2]^{1/2}.$$

(The advantage of using the conormal is that the boundary conditions for \mathcal{A} are often expressed in terms of the conormal.).

- (d) To specify $\underline{d}_j^{(k)}$ as part of the input data as is done by R. L. Taylor [1966] (see also Kealy and Busch [1971]).

The computation of $P_j^{(k+1)}$ so as to satisfy (4) depends, of course, upon the structure of \mathcal{C} . The most obvious approach (to us) is:

Method 1: Define

$$f_j^{(k)}(\alpha) = \mathcal{C} u^{(k)}(P_j^{(k)} + \alpha \underline{d}_j^{(k)});$$

compute or estimate

$$\left. \frac{d}{d\alpha} f_j^{(k)}(\alpha) \right|_{\alpha=0};$$

and then set

$$\alpha_j^{(k)} = -f_j^{(k)}(0) / \left(\left. \frac{d}{d\alpha} f_j^{(k)}(\alpha) \right|_{\alpha=0} \right). \quad (6)$$

However, various other approaches have been used:

Method 2: If $\mathcal{C}u(P)$ is of the form

$$\mathcal{C}u(P) \equiv F(u(P), u_n(P), P) = 0$$

then some authors define $\alpha_j^{(k)}$ by means of

$$F(u_j^{(k)}, u_n^{(k)}(P_j^{(k)}), P_j^{(k)} + \alpha_j^{(k)} \frac{d_j^{(k)}}{d_j}) = 0; \quad (7)$$

that is the dependence of $u^{(k)}(P)$ upon P is neglected.

Method 3: An even cruder approach is to introduce a function $G(x)$ (such as $G(x) \equiv x$) and set

$$\alpha_j^{(k)} = G(cu(P_j^{(k)})). \quad (8)$$

Although Method 1 is more elegant, Methods 2 and 3 have been used with great success in various problems, and it is conceivable that in some cases Methods 2 and 3 are more stable than Method 1.

We now illustrate the above remarks by some specific examples.

Most authors have chosen C to be the Dirichlet condition

$$Cu(\underline{P}) \equiv u(\underline{P}) + \gamma_{12}(\underline{P}) = 0. \quad (9)$$

With this choice of C the approaches which have been used include:

- (i) In the programs of R. L. Taylor for porous flow problems which have been used by Taylor and Brown [1967], Kealy and Busch [1971], Kealy and Williams [1971], and Pettibone and Kealy [1971], the boundary condition is of the form

$$C: u = 0$$

where u is the pressure. Taylor (see R. L. Taylor [1966, p. 6] and Kealy and Williams [1971, p. 62]) determines $P_j^{(k+1)}$ by

$$P_j^{(k+1)} - P_j^{(k)} = \beta u_h^{(k)}(P_j^{(k)}) \frac{d_j^{(k)}}{d_j}$$

where β is a correction factor. (This is Method 3 with $G(x) = \beta x$). Both β and the directions $\underline{d}_j^{(k)}$ must be provided by the user as data. R. L. Taylor [1966] suggests the choice $\beta = .25$ while Kealy and Williams [1971] suggest $\beta = .7$. No advice is given about the choice of the directions $\underline{d}_j^{(k)}$ but they should obviously make an angle of less than 90° with the positive y-axis.

This is a very crude approach - but it works.

- (ii) In porous flow problems the FB condition can also be written in the form (see section 0.)

$$C: u - y = 0$$

where u is the hydraulic head. This suggests that $P_j^{(k+1)} = (x_j^{(k+1)}, y_j^{(k+1)})$ be defined by moving $P_j^{(k)}$ in the direction of the y-axis,

$$\begin{aligned} x_j^{(k+1)} &= x_j^{(k)}, \\ y_j^{(k+1)} &= u_h^k(P_j^{(k)}), \end{aligned} \tag{10}$$

(Method 2) and this has been used by Neuman and Witherspoon [1970]. Neuman and Witherspoon also incorporate two other features: a preliminary calculation is made to determine the flow on the seepage surface; and certain restrictions (which they describe in some detail) are imposed on $P_j^{(k+1)}$.

Fenton [1972a, p. 44 and Appendix D] (see also Parkin [1971])

replaces (8) with

$$\begin{aligned}
 x_j^{(k+1)} &= x_j^{(k)} + d_j \omega [u_h^{(k)}(P_j^{(k)}) - y_j^{(k)}], \\
 y_j^{(k+1)} &= y_j^{(k)} + \omega [u_h^{(k)}(P_j^{(k)}) - y_j^{(k)}],
 \end{aligned}
 \tag{11}$$

where ω is taken to be 1.5 and where the constants d_j are chosen so that $P_j^{(k+1)}$ lies on a certain line. (This is Method 3 with $G(x) = \omega x$).

(iii) For the Dirichlet condition (9) Method 1 may be written in the form

$$\alpha_j^{(k)} = - \frac{u^{(k)}(P_j^{(k)}) + \gamma_{12}(P_j^{(k)})}{\frac{\partial}{\partial \underline{\alpha}} [u^{(k)}(\underline{P}) + \gamma_{12}(\underline{P})]_{P=P_j^{(k)}}}. \tag{12}$$

The quantity $\partial \gamma_{12} / \partial \underline{\alpha}$ is readily computed. If the first boundary condition is chosen to be of the form

$$\mathcal{B}u = \frac{\partial u}{\partial \underline{\alpha}} + \gamma_{22} = 0,$$

then

$$\frac{\partial u^{(k)}}{\partial \underline{\alpha}} = -\gamma_{22};$$

otherwise then $\frac{\partial u^{(k)}}{\partial \underline{\alpha}}$ can be estimated numerically or otherwise.

This approach was used by Garabedian [1956, 1956a], Cryer [1962, 1968, 1970a] and J. M. Taylor [1971, p. 35] (see also Aitchison [1972]).

The same approach was used in a special case by Tu [1971, p. 43] who introduced a relaxation parameter ω so that $\alpha_j^{(k)}$ was determined by

$$\alpha_j^{(k)} = -\omega \frac{u^{(k)}(P_j^{(k)}) + \gamma_{12}(P_j^{(k)})}{\frac{\partial}{\partial \underline{\alpha}} [u^{(k)}(\underline{P}) + \gamma_{12}(\underline{P})]_{P=P_j^{(k)}}}.$$

In a number of cases the boundary condition C has been taken to be in the Neumann form

$$Cu(P) = u_n(P) + \gamma_{22}(P) = 0. \quad (13)$$

With this choice of C the approaches which have been used include:

- (i) In fluid mechanics FBPS involving gravity we have

$$Cu = \left(\frac{\partial u}{\partial n}\right)^2 - 2gy = 0,$$

so that we may set (Method 2)

$$x_j^{(k+1)} = x_j^{(k)},$$

$$y_j^{(k+1)} = \left[\frac{\partial u}{\partial n}(P_j^{(k)}) \right]^2 / 2g.$$

Stewart [1965, p. 42] and Stewart and Davidson [1967] have used this method.

- (ii) Finnemore and Perry [1968] consider a porous flow problem and write the boundary condition in the form

$$Cu = \frac{\partial u}{\partial y} - \sin^2 \theta$$

where θ is the angle between the normal to Γ and the x-axis.

Finnemore and Perry move $P_j^{(k)}$ by setting $x_j^{(k+1)} = x_j^{(k)}$ and setting $y_j^{(k+1)}$ equal to a rather complicated and arbitrary function of $Cu^k(P_j^{(k)})$. (Method 3).

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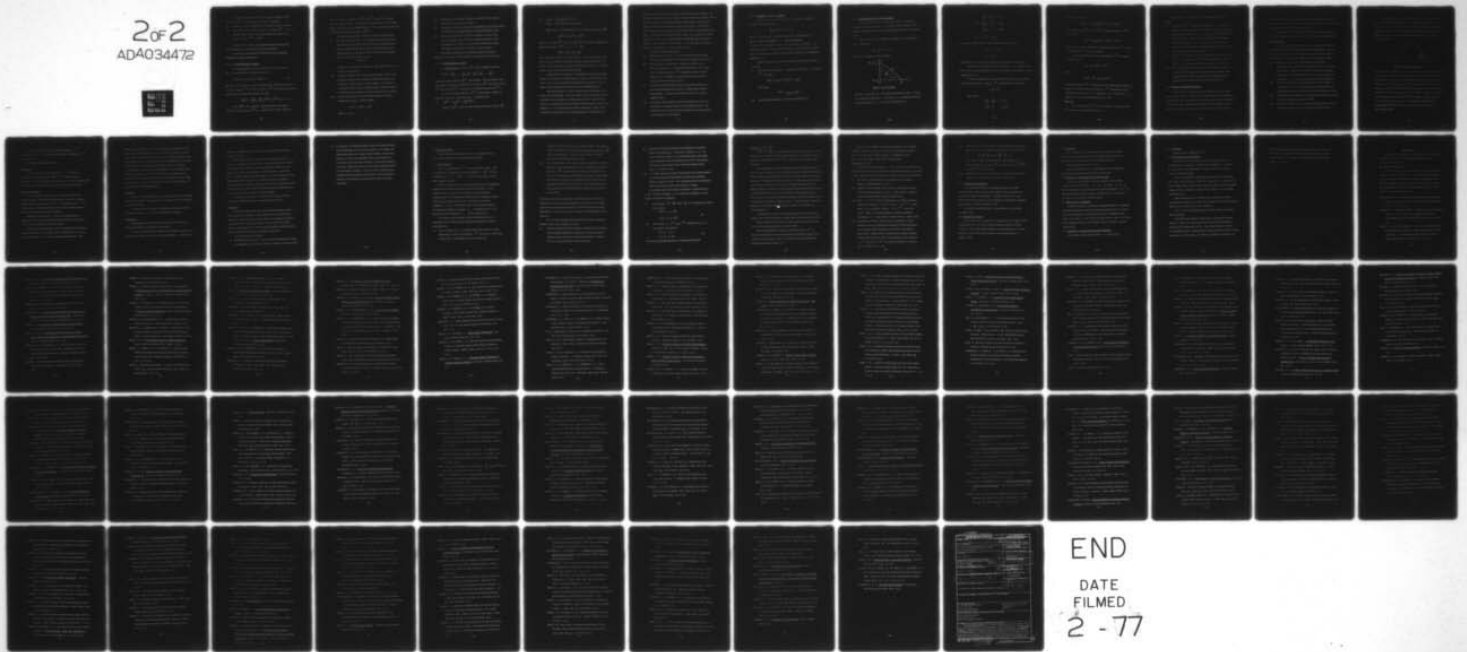
WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
A SURVEY OF TRIAL FREE-BOUNDARY METHODS FOR THE NUMERICAL SOLUT--ETC(U)
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In a number of instances it has been found desirable to smooth the points $P_j^{(k+1)}$ so as to prevent undesirable oscillations:

- (a) Finnemore and Perry [1968] used a standard smoothing subroutine.
- (b) Cryer [1962, 1968, 1970a] determined m points $P_j^{(k+1)}$, $1 \leq j \leq m$, and then fitted a curve $\Gamma(\underline{a})$ of prescribed form through these points to obtain $\Gamma^{(k+1)} = \Gamma(\underline{a}^{(k+1)})$.

Stevens [1974] solves an axisymmetric magnetohydrostatic FBP and apparently uses a local method (he gives few details).

In summary, local methods have been used very successfully despite their apparent arbitrariness.

3.2.2. Movement strategy: integral

In an integral method of moving the FB the boundary condition $Cu = 0$ is expressed in an implicit form such as

$$F(u(x), u_x(s), u_y(s), \underline{x}(s), \dot{\underline{x}}(s)) = 0, \quad (1)$$

where the FB is the curve $\underline{x} = \underline{x}(s)$ and where $\dot{\underline{x}}(s)$ denotes the derivatives of $\underline{x}(s)$. Given $\Gamma^{(k)}$ and $u^{(k)}$, the curve $\Gamma^{(k+1)}$ is obtained by integrating approximately the differential equation for $\underline{x}(s)$,

$$F(u_h^{(k)}(s), u_{x,h}^{(k)}(s), u_{y,h}^{(k)}, \underline{x}^{(k+1)}(s), \dot{\underline{x}}^{(k+1)}(s)) = 0. \quad (2)$$

In (2) $u_h^{(k)}(s)$ etc. represent approximations to the values of $u^{(k)}(s)$ etc. at the point $\underline{x}^{(k+1)}(s)$. If $\underline{x}^{(k+1)}(s) \in \mathcal{D}^{(k)}$ then $u_h^{(k)}(s)$ can

be obtained by interpolation. If $\underline{x}^{(k+1)}(s) \notin \mathcal{S}^{(k)}$ then $u_h^{(k)}(s)$ must be obtained by extrapolation. In the applications the method used to compute $u_h^{(k)}(s)$ does not seem to be critical; often, $u^{(k)}(s)$ is taken to be the value of $u^{(k)}$ at the gridpoint nearest to $\underline{x}^{(k+1)}(s)$.

The following applications of integral methods have been made:

- (i) The self-consistent method of computing the magnetopause which was introduced by Beard in 1960 and implemented numerically by Mead and Beard [1964], Baker, Beard, and Young [1964], is an integral method. In the general case considered by Olson [1968, 1969] the FB is expressed in the form

$$r = R(0, \phi)$$

where $(r, 0, \phi)$ are polar coordinates. The expression for $\mathcal{C}u \equiv F$ is given in section 2.4.3.

- (ii) von Kerczek [1965] used a version of the integral method for two problems, the flow over a submerged vortex (successfully) and flow under a planing surface (unsuccessfully). The method of von Kerczek differed from the integral method as described above in that during each iteration only one boundary point was moved.
- (iii) Mogal and Street [1972, 1974] solve the problem of a plane Riabouchinsky cavity. For this problem,

$$\mathcal{C}u \equiv F = \psi_x \dot{x}(s) + \psi_y \dot{y}(s)$$

where $\underline{x} = (x, y)$.

- (iv) Tanner [1973] and Nickell, Tanner, and Caswell [1974] compute the flow of a viscous jet from a tube.
- (v) Comincioli, Guerri, and Volpi [1971] and Baiocchi, Comincioli, Guerri, and Volpi [1973] solve the problem of porous flow through a dam.
- (vi) Michael and O'Neill [1972a] solve a problem involving a fluid film in an electric field. This is an interesting problem in that the equation (2) for $\underline{x}(s)$ is a second order differential equation which gives rise to a two-point boundary value problem.

As the above examples show, the integral method has proved of great value, and it seems clear that many more applications will be found.

3.2.3. Movement strategy: global

In a global method of moving $\Gamma^{(k)}$ a set of perturbed boundaries

$$\Gamma^{(k,j)} = \Gamma(a_1^{(k)}, \dots, a_{j-1}^{(k)}, a_j^{(k)} + \delta a_j^{(k)}, a_{j+1}^{(k)}, \dots, a_p^{(k)})$$

and corresponding solutions $u_h^{(k,j)}$ are generated. This information makes it possible to estimate the dependence of the error $\epsilon u^{(k)}$ upon the parameters $\underline{a}^{(k)}$. The new FB $\Gamma^{(k+1)}$ is chosen so as to minimize the error $\epsilon u^{(k+1)}$.

In order to minimize the error $\epsilon u^{(k+1)}$ we must have a measure for this error, $F(\underline{a})$ say. Two choices for F have been used:

$$(i) \quad F_1(\underline{a}^{(k)}) = (F_{1j}(\underline{a}^{(k)})) = (\epsilon u_h^{(k)}(P_j^{(k)}));$$

that is, $F_1(\underline{a}^{(k)})$ is the m -vector of the errors at the m points $P_j^{(k)}$.

(ii)
$$F_2(\underline{a}^{(k)}) = \left[\sum_{j=1}^m [cu_h^{(k)}(P_j^{(k)})]^2 \right]^{1/2};$$
 that is, F_2 is the least squares error.

When using F_1 it is assumed that F_1 depends linearly upon $\underline{a}^{(k)}$:

$$F_1(\underline{a}^{(k)} + \delta \underline{a}) = \underline{A}^{(k)} \delta \underline{a} + \underline{b}^{(k)}.$$

The $p \times p$ - matrix $\underline{A}^{(k)}$ and the p -vector $\underline{b}^{(k)}$ are found by computing p perturbed solutions $u^{(k,j)}$, $j = 1, \dots, p$. Then

$$\underline{a}^{(k+1)} = \underline{a}^{(k)} - (\underline{A}^{(k)})^{-1} \underline{b}^{(k)}.$$

This approach has been used by Sankar [1967, p. 153] and Fox and Sankar [1973] for the problem of an axially symmetric Riabouchinsky cavity. Tu [1971, p. 60] used the same method for the problem of flow over a circular-crested weir, but Tu also intervened manually.

Midgley [1963] and Midgley and Davis [1962, 1963] use a similar method for computing the magnetosphere in a plasma of constant pressure, and in a solar wind, the difference being that the components of F_1 are taken to be the moments of the surface current (see section 2.4.3).

Although the choice F_2 (least squares) seems reasonable, it has proved to be unsatisfactory. Arms and Gates [1957] were unable to obtain convergence for the problem of a Riabouchinsky cavity. We used least squares on a problem arising in the theory of stellar evolution (Cryer [1962, chapter 10, p. 7]) and also failed to obtain convergence. Rippin [1959] and Rippin and Davidson [1967] used least squares for the problem of an

axially symmetric spherical cap bubble, but had to intervene manually. The difficulties with the choice F_2 appear to be due in part to the fact that $Cu^{(k)}$ can only be computed to a few decimal places together with the fact that least square problems are ill-conditioned unless care is taken. McCorquodale and Li [1971] use least squares for the problem of flow under a bevelled sluice gate. McCorquodale and Li [1971, Figure 3] plot F_2 as a function of their parameters a_1 and a_2 and this illustrates a difficulty with least squares which we also experienced: the minimum of F_2 is not clearly defined and it is not clear that there is a point at which $F_2 = 0$.

We conclude with some remarks:

- (a) Whatever the choice for F , it is clear that it is very desirable to choose the perturbations so that they are orthogonal or nearly so.
- (b) It is necessary to ensure that F depends smoothly upon \underline{a} . Arms and Gates [1957, p. 6] used finite differences with a fixed grid. The perturbations of $\Gamma^{(k)}$ sometimes resulted in the FB moving across a gridpoint, and it was found that this caused unpleasantly large changes in $Cu^{(k)}$.
- (c) In comparing global methods with local and integral methods, it must be remembered that each step of a global method requires approximately p times as much work as one step of a local or integral method.
- (d) In summary, global methods of moving the FB have not been very successful in comparison with local and integral methods. We suspect that this is because global methods do not take advantage of any properties of the solution.

3.2.4. Estimation of unknown constant

In many inviscid fluid mechanics FBPS one boundary condition on the FB is the Bernoulli equation

$$\frac{1}{2} |\text{grad } u|^2 + \rho g y - \lambda = 0,$$

where λ is a constant. In a few problems, such as the problem of a Borda mouthpiece the value of λ can be determined a-priori. In most problems, however, it is necessary to determine λ as part of the solution. It is of interest that porous flow FBPS with sheet piles also have an unknown constant in the boundary conditions (see Cryer [1976, section 3.1.3]).

Methods which have been used to determine the unknown constant λ include:

- (a) Using a boundary condition B which does not involve λ to compute $u^{(k)}$, evaluating

$$\lambda(P_j^{(k)}) = \frac{1}{2} |\text{grad } u^{(k)}(P_j^{(k)})|^2 + \rho g y_j^{(k)},$$

and setting

$$\lambda^{(k+1)} = \text{average}_j \lambda(P_j^{(k)}).$$

- (b) Using integral identities as described in section 3.1.2.

3.3. A convergence proof for a model problem

So far as we are aware no proof of convergence of a trial free boundary method has been given. Cryer [1968, 1970a] analysed a very simple model problem and for lack of better results this work is described below.

The model FBP to be considered is as follows: it is required to find u satisfying

$$u_{xx} + u_{yy} = 0, \text{ in } \mathcal{D},$$

where \mathcal{D} is of the form shown in Figure 1.

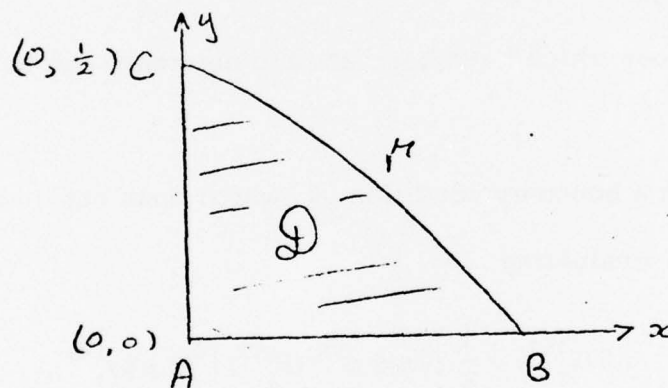


Figure 1: The model FBP

The FB Γ is the curve BC. The auxiliary restraints are that Γ should pass through the fixed point C and that on Γ y should be a monotone decreasing function of x . The boundary conditions are:

$$\mathcal{B}u = 0 = \begin{cases} u - (1-y), & \text{on AC,} \\ u_n + \sqrt{10}, & \text{on BC,} \\ u_n - 1 & \text{on AB,} \end{cases}$$

$$\mathcal{C}u = 0 = u - \frac{1}{2}, \quad \text{on BC.}$$

It is readily verified that a solution of this problem is given by

$$u : u = 1 - y - 3x,$$

$$\Gamma : y = \frac{1}{2} - 3x.$$

We believe that this solution is unique but have not tried to prove this.

Since the FB is a straight line it has curvature $\kappa = 0$. It follows that the boundary conditions are already in the form suggested by Garabedian [1956] (see section 3.1.1).

To solve the problem the approximate FBS are assumed to be straight lines passing through C. That is, $\Gamma^{(k)}$ is assumed to be of the form

$$y = \frac{1}{2} + c^{(k)} x.$$

The problem

$$u_{xx}^{(k)} + u_{yy}^{(k)} = 0, \quad \text{in } \mathcal{D}^{(k)}$$

$$\mathcal{B}u^{(k)} = 0, \quad \text{on } \partial\mathcal{D}^{(k)}$$

has the exact solution

$$u^{(k)} = 1 \cdot y + x \{ [10([c^{(k)}]^2 + 1)]^{1/2} - 1 \} / c^{(k)} .$$

The condition that $Cu^{(k)} = 0$ on $\Gamma^{(k+1)}$ is satisfied exactly if $c^{(k+1)}$ is defined by

$$c^{(k+1)} = \{ [10([c^{(k)}]^2 + 1)]^{1/2} - 1 \} / c^{(k)} .$$

Thus in this very simple problem the approximate solutions $u^{(k)}$ and the approximate FBS $\Gamma^{(k)}$ are known exactly.

To analyse the behaviour of the coefficients $c^{(k)}$ it is helpful to observe that if

$$f(c) = [1 + c^2]^{1/2} - 10^{1/2} ,$$

then

$$c^{(k+1)} = c^{(k)} - f(c^{(k)}) / f'(c^{(k)}) ,$$

so that the sequence $c^{(k)}$ is identical with the sequence which would be obtained by starting with the initial guess $c^{(0)}$ and applying Newton's method to the equation $f(c) = 0$. Noting that $f(c)$ is convex for $c \leq 0$ and that $f(0) < 0$ we have (Henrici [1964, p. 79]):

Theorem 1

For any initial guess $c^{(0)} < 0$ the sequence of approximate FBS $\Gamma^{(k)}$ converges quadratically to the FB Γ . \square

We conclude the discussion of the model problem with some remarks:

- (i) The fact that $\Gamma^{(k)}$ converges quadratically to Γ supports the assertion of Garabedian [1956, p. 223] that quadratic convergence is obtained by choosing \mathcal{B} and \mathcal{C} according to his method. The fact that sequence $c^{(k)}$ can be obtained by applying Newton's method to the equation $f(c) = 0$ reminds one that Garabedian's method can be thought of as an attempt to apply Newton's method to FBPS (Garabedian [1956a, p. 613]).
- (ii) Since $u^{(k)}$ is linear it can be computed by many methods. Cryer [1968, p. 46] computed $u_h^{(k)}$ using finite differences and found that round-off errors did not affect the quadratic convergence.
- (iii) The model problem was chosen because it resembles a much more complicated FBP arising in the study of stellar evolution. The numerical results for the more complicated problem behaved similarly to those for the model problem (Cryer [1968]).

3.4. Summary of numerical experience

In this section we summarize the available evidence about trial free boundary methods. Most authors are rather reticent about describing unsuccessful methods or methods which required a great deal of adjustment before they worked. This is understandable but it is unfortunate for two reasons: (i) later workers are not forewarned and repeat the mistakes;

(ii) the development of a satisfactory theory is hindered since such a theory must be formulated so as to exclude non-convergent examples. We, therefore, emphasize reported cases of non-convergence.

When the evidence is examined it becomes clear that the behavior of trial free boundary methods is heavily dependent upon the type of problem:

Porous flow FBPS

There is overwhelming evidence that trial free boundary methods are stable and converge satisfactorily for porous flow problems. The trial FBS can be moved using any local method and smoothing is not required. Most authors take the auxiliary condition to be the Dirichlet condition $h = y$ but this is not necessary. We summarize the evidence:

- (i) Finite difference methods were successfully used by a great many hand workers. Gillian Vaisey [1961] has said that she regarded the problem of porous flow through dams as an easy problem and assigned it as an exercise to her students for hand computation.
- (ii) Wyckoff and Reed [1935] and later workers such as Childs used conducting paper and adjusted the trial FB by cutting the paper. Clearly this approach requires that the method be extremely stable.
- (iii) Recently both finite difference and finite element methods have been used with automatic adjustment of the FB.
- (iv) Porous flow FBPS have been solved automatically using resistance networks by Karplus [1956] and Herbert and Rushton [1966].

To our knowledge only one case of divergence for a porous flow FBP has been reported in the literature. R. L. Taylor and Brown [1967, p. 30 and p. 32] mention an "ambiguity effect" at the point where the FB intersects the seepage surface. Taylor and Brown do not give details but Neuman and Witherspoon [1970, p. 895] used the program of Taylor for a dam with a toe drain (Figure 1).

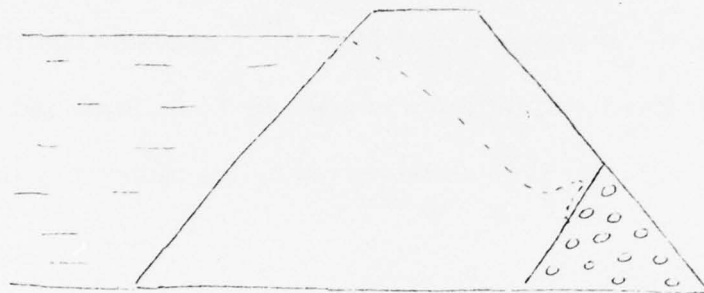


Figure 1: Dam with a toe drain

The successive approximations to the FB converged except in the neighbourhood of the toe dam where the FB oscillated. Neuman and Witherspoon [1970] implemented a modification of Taylor's method which did converge. The reason for the non-convergence of Taylor's results is not clear. Neuman and Witherspoon [1970, p. 891] state that the reason for the non-convergence is that the length of the seepage surface is not known. It seems to us that the non-convergence may be connected with the peculiar boundary conditions on an overhanging seepage surface (see Cryer [1976, section 2.2.]). Kealy and Busch [1971], who used Taylor's program, also mention the ambiguity effect.

Most authors use a reasonably good initial guess for $\Gamma^{(0)}$.
J. M. Taylor [1971, p. 37] mentions that difficulties occurred when $\Gamma^{(0)}$
was too far from Γ .

We make the following conjecture:

Conjecture I

In a porous flow FBP, if the initial guess $\Gamma^{(0)}$ for the FB is
horizontal then the successive trial FBS $\Gamma^{(k)}$ converge monotonely
downwards. (A similar assertion was made by F. S. Shaw and Southwell [1941,
p. 8] who asserted that it could be proved by the maximum principle.) \square

Fluid mechanics FBPS

Trial free boundary methods for fluid mechanics problems often
exhibit instability. Various smoothing techniques seem to be necessary:
obtaining the new approximate FB $\Gamma^{(k+1)}$ by moving the points on $\Gamma^{(k)}$
and then fitting a curve through these points; using a weighted average of
the new and old solutions; manual intervention.

Southwell and Vaisey [1946, p. 127] who solved eleven fluid
mechanics FBPS by hand observe that "usually the shape [of successive
 $\Gamma^{(k)}$] is found to diverge in some part at least".

Many authors have reported convergence difficulties when adjusting
the FB automatically: Young, Gates, Arms, and Eliezer [1955, p. 15] and
Arms and Gates [1957, p. 5] could not solve the Riabouchinsky cavity

problem; von Kerzcek [1965, p. 56] could not solve the problem of a flat plate hydroplane; Stewart [1965, p. 42] was forced to intervene manually when solving a bubble problem; Tu [1971, p. 87] obtained divergence when solving the problem of a curved weir with gravity; Steffen [1973] found that the iterates did not converge when solving the problem of an axially symmetric wall jet. We had personal experience of such instability when, around 1960, we tried to use a computer to solve the progressive-wave problem which had previously been solved by Southwell and Vaisey [1946]: many choices of \mathcal{B} and \mathcal{C} were tried, but in each case the curves $\Gamma^{(k)}$ rapidly developed undesirable wiggles.

Other FBPS

There is not sufficient evidence about other FBPS to draw any definite conclusions, with the possible exception of magnetohydrostatic FBPS which do seem to be stable.

It is natural to try and explain the stability characteristics of trial free boundary methods. L. Fox [1972] suggested the following:

Conjecture 2

The instabilities of the trial free boundary method are caused by physical instabilities of the problems being solved. \square

At the time we felt that this conjecture was unsatisfactory because it relied upon the rather vague concept "physical instability". However, the

evidence of instability described above does suggest quite strongly that Conjecture 2 is true.

A fairly decisive test for Conjecture 2 is provided by water-wave problems. Southwell and Vaisey [1946, p. 153] describe how in the solution of the problem of the flow past a planing surface the computations "took charge" and a solitary wave emerged. On the other hand, as described above, we were unable to solve the problem of a periodic progressing wave. This suggests that solitary waves are stable and periodic progressing waves are unstable. Recently, Benjamin and Feir [1965] and Benjamin [1972] have analysed wave waves using the Korteweg-de Vries approximation and have come to the same conclusions regarding stability.

An important consequence of Conjecture 2 is that since viscosity usually has a stabilizing effect in fluid dynamics we are led to:

Conjecture 3

Viscous fluid FBPS can be solved using trial free boundary methods. □
The importance of Conjecture 3 is that while there are many alternative methods for solving inviscid fluid FBPS, there are few methods for solving viscous fluid FBPS. The recent success of Tanner [1973] and Nickell, Tanner, and Caswell [1974] in solving viscous jet FBPS using the trial free boundary method offers hope that Conjecture 3 may be true.

We conclude with some remarks:

- (i) Southwell and Vaisey [1946, p. 122] use physical arguments to explain the instability of fluid mechanics FBPS as compared to porous flow FBPS.

(ii) In mechanics, the difference between stable and unstable problems often reveals itself in the governing equations: for example, the differential equation for a damped spring contains a damping term. However, for many FBPS instability seems to arise through in a more subtle way the boundary conditions: the governing equations for viscous, stable porous flow and for inviscid unstable fluid flow are both Laplace's equation. One can of course surmise that the instability arises for physical reasons such as the introduction of turbulence at the boundary, but this does not help one analyse the problem.

4. Concluding remarks

In this section we make some remarks on open questions about, and possible approaches to, trial free boundary methods.

4.1. Error estimates

Given an approximate FB $\Gamma^{(k)}$ and an approximate solution $u_h^{(k)}$ one would like to estimate the error in the solution, $u - u_h^{(k)}$, and also to estimate how far $\Gamma^{(k)}$ is from Γ . So far as we are aware, no such error estimates are known.

There is a definite need for error estimates because even experienced workers find it difficult to estimate the error in the solution. A good example of this is provided by the problem of an inviscid axially symmetric wall jet: several workers (Treffitz [1916], Southwell and Vaisey [1946], Rouse and Abul-Fetouh [1950]) estimated the contraction coefficient C_c to be .61 and Rouse [1962, p. 188] criticizes Birkhoff for supporting Garabedian's estimate $C_c = .58$ (Birkhoff [1961, p. 21]), but the best current estimate is $C_c = .591375$ (Snider [1971]).

Some authors have commented on the fact that the error can be quite large even when the boundary conditions are satisfied to within a small tolerance:

- (i) Arms and Gates [1957, p. 7] observe that for the problem of a plane Riabouchinsky cavity the initial guess $\Gamma^{(0)}$ was taken to be the exact analytical FB. Finite differences were used with 1145

gridpoints (see Young, Gates, et al. [1955, p.12]). The velocity on the FB should be constant, but the velocity computed from $u_h^{(0)}$ varied by more than 5% , and the average computed velocity was almost 2% higher than the true value.

- (ii) B. W. Hunt [1967, p. 18] , who used integral equations to solve the problem of an axially symmetric wall jet, compares the results for two curves $\Gamma^{(0)}$. The coordinates of the curves differed by less than 3% at every point, but the computed velocities behaved quite differently: for one curve the computed velocity differed from the asymptotic velocity by less than 1.5% , while for the other curve the computed velocity differed from the asymptotic velocity by as much as 108%.

The cases quoted above are perhaps exceptional because the flows are badly behaved near the point of separation, and because for the axially symmetric jet problem the fluid velocity on the FB is not known a-priori. Nevertheless, these cases serve as a warning that relatively high errors can occur.

In a few instances approximate solutions have been computed for problems whose exact solutions are known:

- (i) Southwell and Vaisey [1946, Figure 11] compare the exact solution for flow through a plane Borda mouthpiece with the approximate solution obtained using finite differences with about 700 points. The true and approximate FBS differ by about 5%.

- (ii) Analytical and numerical solutions are available for porous flow through a rectangular dam. (Cryer [1976, section 3.1.1.1]). For a dam of height 24, width 16, and downstream depth 4, the height of the downstream point of intersection is 12.7132 to four decimals, and the best value obtained using a trial free boundary method is 12.75 - an error of .03%.
- (iii) J. C. Baker, Beard, and Young [1964] use the self-consistent method to compute the earth's magnetopause for three simple problems where the solution is known. The computed results are indistinguishable from the exact results to the accuracy of a graph.

A substantial part of the machinery necessary to estimate the error $u - u_h^{(k)}$ is already available. For, to obtain error estimates, we must be able to estimate two quantities:

- (i) The difference $u^{(k)} - u_h^{(k)}$ where $u_h^{(k)}$ is the approximate solution of the problem

$$\begin{aligned} \mathcal{A}u^{(k)} &= 0, \text{ in } \mathcal{D}^{(k)}, \\ \mathcal{B}u^{(k)} &= 0, \text{ on } \partial\mathcal{D}^{(k)}. \end{aligned} \tag{1}$$

- (ii) The difference $u - u^{(k)}$ where $u^{(k)}$ satisfies (1) and u is the solution of the problem

$$\begin{aligned} \mathcal{A}u &= 0, \text{ in } \mathcal{D}, \\ \mathcal{B}u &= 0, \text{ on } \partial\mathcal{D}. \end{aligned} \tag{2}$$

We now consider these questions in somewhat greater detail.

Estimation of $u^{(k)} - u_h^{(k)}$

If $u_n^{(k)}$ is computed by any standard method such as the method of finite differences or the method of finite elements then there is an extensive literature devoted to estimating the error $u^{(k)} - u_h^{(k)}$. There are often several different approaches.

For finite difference methods the following classical approaches were used: Courant, Friedrichs, and Lewy [1928] (variational methods); Gerschgorin [1930] (maximum principle); Petrovskii [1941], Jamet and Parter [1967] (barrier functions). In addition there have been many other approaches to finite difference methods: Motzkin and Wasow [1953], Bramble and Hubbard [1964], Bramble, Hubbard, and Thomee [1969] (monotone-type approximations); Nitsche and Nitsche [1960]; Cea [1964], Aubin [1962] (external approximations); Lebaud [1969], Frehse [1969] (nonlinear equations); Lebaud and Raviart [1969] (discontinuous coefficients); Schaeffer [1972]. Diaz and Roberts [1952] draw attention to the similarity between iterative methods for continuous and discrete problems.

For finite element methods the standard approach is to use variational methods, but Ciarlet and Raviart [to appear] have used the maximum principle.

We have mentioned all the above methods because at the present time it is not clear which approach will be best for FBPS.

Much of the literature is based on the assumption that $\partial \mathcal{L}^{(k)}$ is smooth and is therefore not always applicable to FBPS which usually involve corners, but the case when $\partial \mathcal{L}^{(k)}$ has corners has been considered and references are given in section 4.2.

There is a considerable, but scattered, literature on the properties of elliptic difference equations, and some of these properties probably play a role in trial free boundary methods. See Heilbronn [1949], Kemeny and Snell [1961], Varga [1966], Ciarlet [1970a].

Estimation of $u - u^{(k)}$

The estimation of the difference $u - u^{(k)}$, that is, the estimation of the variation in the solution of a boundary value problem with respect to variations of the boundary has been considered from several viewpoints in the literature:

- (i) The classical variational formulas of Hadamard for the Green's function (Garabedian [1964, p. 558]).
- (ii) Estimates for the variation in the solution of the Dirichlet problem for linear elliptic equations of order m with respect to variations of the boundary have been obtained by Saak [1972].
- (iii) Since \mathcal{D} and $\mathcal{D}^{(k)}$ differ, it is desirable (and perhaps essential) to be able to continue u and $u^{(k)}$ across Γ and $\Gamma^{(k)}$, respectively, so as to be able to compare u and $u^{(k)}$ on the same domain. There is a growing literature on numerical analytical continuation: G. L. Lewis [1965], Henrici [1966], K. Miller [1970].
- (iv) The question of domain variations arises in the theory of finite elements, because in general the boundaries of the finite elements do not coincide with the boundary of the domain of the problem being solved. Berger, Scott, and Strang [1972], Strang and Berger [1974] and Thomee [1973, 1973a] give estimates for the differences $u - u^{(k)}$ and $\text{grad } (u - u^{(k)})$ for Poisson's equation in the plane. See also Necas [1967, p. 162].

- (v) Steffen [1972] considers FBPS for harmonic functions in the plane. Steffen considers the boundary conditions on Γ to be of the form

$$R u = \frac{\partial u}{\partial n} - \phi_1^{\mathcal{S}} = 0; \quad C u = \frac{\partial u}{\partial t} - \phi_2^{\mathcal{S}} = 0,$$

where $\phi_1^{\mathcal{S}}$ and $\phi_2^{\mathcal{S}}$ are given functions which depend upon \underline{x} and \mathcal{S} . Steffen shows that if $\mathcal{S}^{(k)}$ is close to \mathcal{S} then $R^{(k)} u^{(k)}$ and $C^{(k)} u^{(k)}$ are almost zero.

- (vi) Mühlig [1973] discusses the dependence of the solution upon the domain for axially symmetric jets from curved orifices.

4.2. Treatment of singularities

Most solutions of FBPS have singularities because most FBPS involve corners. Here, we make some remarks about the analytical and numerical treatment of singularities. We believe that this is an area which deserves study because of the possibility of substantially increasing the accuracy of numerical methods for FBPS.

As regards the analytical study of singularities, two approaches have been used:

(a) Asymptotic expansions

Lewy [1950, 1957], Wasow [1957a], Laasonen [1957], Lehman [1959], Wigley [1964, 1970, 1974] have derived asymptotic expansions for the solution of second order elliptic equations in the neighbourhood of corners. Kondrat'ev [1967, 1970] gives expansions for general equations (of arbitrary order).

(b) Integrability

It is of importance to know in which Sobolev space the solution lies. See Birman and Skvorcov [1962], Kondrat'ev [1967], Hanna and Smith [1967], and Grisvard [1971, 1972], Kudrjavcev [1974].

See also Kellogg [1966, 1970, 1971].

The approximate treatment of singularities is not well developed.

We list below several techniques which have been used:

(a) Elimination of a singularity by conformal mapping

If the governing equation is Laplace's equation in the complex z -plane and if it is known that $\psi \sim (z - z_0)^\alpha$ at a corner $z_0 = x_0 + iy_0$, then the singularity can be removed by the conformal mapping $w = (z - z_0)^\alpha$. This technique was applied by Mason and Farkas [1971, p. 19; 1972] in conjunction with a trial free boundary method.

(b) Subtracting off a singularity

If either the solution or the FB is being approximated by a combination of functions then the functions should obviously be chosen so as to match any known singularities. Examples will be found in Garabedian [1956; 1956a], Cryer [1968, p. 40], Mason and Farkas [1971, p. 22], J. M. Taylor [1971], Aitchison [1972], Fox and Sankar [1969], Fox [1971]. See also Wigley [1969].

(c) Construction of special finite difference equations

Motz [1946], Thom and Apelt [1961], L. C. Woods [1953].

(d) Dual series

Whiteman [1967], Whiteman [1969].

(e) Mesh refinement near singularity

Recently, Volkov [1966, 1966a, 1968] has analysed the error for composite meshes consisting of a square grid of gridlength h "patched" to "polar grids" in the neighbourhood of corners, but we are not aware of any computational results.

Proofs of convergence for finite difference methods have been given by: Walsh, and Young [1953, 1954, 1957], Wasow [1957], Laasonen [1957a, 1958], Wigley [1965], Kellogg [1966], Bramble, Hubbard, and Zlamal [1968], Veidinger [1968], Whiteman and Webb [1970], Sultanova [1971], Babuska and Kellogg [1972].

Finite element methods for problems with singularities have been considered by: Babuska [1970, 1970a, 1974], Babuska and Rosenzweig [1972], Birkhoff [1972], Strang and Fix [1973, chapter 8].

Symm [1973] has considered integral equation methods for problems with singularities.

Almost all of the references quoted above consider fixed boundary problems. There is of course a great need to consider the singularities at the points of detachment of the FB. This will be discussed in detail in a subsequent survey. Here we merely mention the work of Aitchison [1972] (see also J. M. Taylor [1971]) who obtains an expansion of the FB for the

porous flow through an earth dam in the neighbourhood of the point B at which the FB intersects the seepage surface, Aitchinson uses this result in two ways: (a) to help move the boundary (b) to compute the solution ϕ near B.

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DA: Dissertation Abstracts

MR: Mathematical Reviews.

If an author has more than one publication in year X, his publications are numbered X, Xa, Xb, This bibliography is part of a larger bibliography on FBPS, and to maintain compatibility we use the same numbering, so that if references X and Xa are not concerned with trial free boundary methods but reference Xb is, we only include reference Xb but retain the number Xb.

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