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INTERLACING PROPERTIES OF THE ZEROS OF THE ERROR FUNCTION IN BE--ETC(U)  
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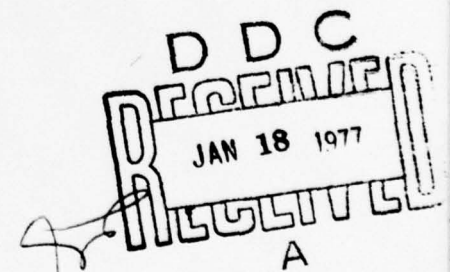
INTERLACING PROPERTIES OF THE ZEROS  
OF THE ERROR FUNCTION IN BEST  $L^p$ -  
APPROXIMATION,  $1 \leq p \leq \infty$

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ABSTRACT

Let  $\{u_i\}_{i=1}^n$ ,  $\varphi, \psi$  be functions in  $C(\bar{I})$ , where  $I$  is a finite interval. Let  $d\sigma$  be a finite, non-atomic, positive measure on  $I$ . For  $p \in [1, \infty]$ , let  $E_p(\varphi)$  and  $E_p(\psi)$  denote the error functions in the best  $L^p$ -approximation to  $\varphi$  and  $\psi$ , respectively, from  $[u_1, \dots, u_n]$ . For  $p < \infty$ , the  $L^p$ -approximation is taken with respect to  $d\sigma$ .

Theorem. Assume  $\{u_1, \dots, u_n\}$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  are Tchebycheff systems on  $I$ . Then for  $1 < p < \infty$ , the zeros of  $E_p(\varphi)$  and  $E_p(\psi)$  in  $I$  strictly interlace. For  $p = 1$ , either the zeros strictly interlace or  $E_1(\varphi)$  has exactly  $n$  sign changes and  $\text{sgn } E_1(\varphi)(t) = \text{sgn } E_1(\psi)(t)$  for all  $t \in \text{int}(I)$ . For  $p = \infty$ , the strict interlacing is present when  $I$  is a closed interval.

Applications to questions of interlacing of zeros in polynomial distortion problems are given.

AMS(MOS) Subject Classification - 41A50

Key Words - Tchebycheff systems, best  $L^p$ -approximation

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INTERLACING PROPERTIES OF THE ZEROS OF THE ERROR FUNCTION  
IN BEST  $L^p$ -APPROXIMATION,  $1 \leq p \leq \infty$

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1. Introduction

Let  $\{u_i\}_{i=1}^n, \varphi, \psi$  be functions in  $C(\bar{I})$ , where  $I$  is a finite interval. We shall henceforth assume that  $\bar{I} = [0, 1]$ . Let  $d\sigma$  be a finite, non-atomic, positive measure on  $I$ . For  $p \in [1, \infty]$ , let  $E_p(\varphi)$  and  $E_p(\psi)$  denote the error functions in the best  $L^p$ -approximation to  $\varphi$  and  $\psi$ , respectively, from  $[u_1, \dots, u_n]$ . For  $p < \infty$ , the  $L^p$ -approximation is taken with respect to  $d\sigma$ .

The main result of this paper, whose proof will be carried out separately for  $1 < p < \infty$ ,  $p = 1$ , and  $p = \infty$  in Sections 3-5, is:

Theorem 1.1. Assume  $\{u_1, \dots, u_n\}$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  are Tchebycheff (T) systems on  $I$ ,  $n \geq 1$ . Then, for  $1 < p < \infty$ , the zeros of  $E_p(\varphi)$  and  $E_p(\psi)$  in  $I$  strictly interlace. For  $p = 1$ , either the zeros strictly interlace, or  $E_1(\varphi)$  has exactly  $n$  sign changes and  $\text{sgn } E_1(\varphi)(t) = \text{sgn } E_1(\psi)(t)$  for all  $t \in \text{int}(I)$ . For  $p = \infty$ , strict interlacing is present when  $I$  is a closed interval.

For the case  $p = 1$ , the interlacing holds under somewhat less restrictive conditions on  $\varphi$  and  $\psi$  (see Theorem 4.2). On the other hand, for  $p = \infty$ , interlacing properties for the points of alternation are also available (see Theorem 5.2).

Assume  $\{u_1, \dots, u_k\}$  is a T-system on  $I$ , for  $k = n, n+1, n+2$ .

Let  $q_{k,p}$ ,  $k = n, n+1$  be the error function in the best  $L^p$ -approximation of  $u_{k+1}$  from  $[u_1, \dots, u_k]$ . If  $q_{n+1,p} = u_{n+2} - \sum_{i=1}^{n+1} \alpha_{i,p}^* u_i$ , then  $q_{n+1,p} = E_p(u_{n+2} - \alpha_{n+1,p}^* u_{n+1})$  where  $E_p$  is as previously defined. We observe next that  $\varphi = u_{n+1}$  and  $\psi = u_{n+2} - \alpha_{n+1,p}^* u_{n+1}$  satisfy the conditions of Theorem 1.1 and that  $q_{k,p}$  has exactly  $k$  sign changes in  $\text{int } I$ ,  $k = n, n+1$ . Hence, we now have

Theorem 1.2. The  $n$  zeros of  $q_{n,p}$  and the  $n+1$  zeros of  $q_{n+1,p}$  strictly interlace in  $I$  for  $1 \leq p < \infty$ . If  $I = \bar{I}$ , the interlacing holds for  $p = \infty$  as well.

When  $\{u_i\}_{i=1}^{n+2}$  are the monomials  $\{x^{i-1}\}_{i=1}^{n+2}$ , the interlacing properties for  $p = 2$  are a well known consequence of the fact the  $q_{k,2}$  are the orthogonal polynomials on  $I$  with respect to the measure  $d\sigma$ . For  $p = 1$  and  $d\sigma(t) = dt$ , the Lebesgue measure,  $q_{k,1}$  is the  $k^{\text{th}}$  Tchebycheff polynomial of the second kind, and the interlacing follows from the explicit expression for  $q_{k,1}$ . For  $p = \infty$ ,  $q_{k,\infty}$  is the  $k^{\text{th}}$  Tchebycheff polynomial of the first kind and the interlacing is known. Somewhat more general results for  $p = \infty$ , where  $u_i(t) = t^{i-1}$ ,  $i = 1, \dots, n$ , have been discussed in the literature (see Paszkowski [5], Rowland [6], and Shohat [7]). For  $p \neq 1, 2, \infty$ , the results are new even for the monomial case. For  $p = 2$ , a known result on the interlacing of roots of quasi-orthogonal polynomials (see [9, p. 5-10]) is subsumed by Theorem 1.1.

Some recent applications of the present results serve to establish the interlacing of the zeros in  $(0, 1)$  of  $P_{n,p}$  and  $P_{n+1,p}$ , where  $P_{n,p}$

is the unique solution of

$$\min\{\|P_{n,p}\| : P_n \in T_n, \|P_n\|_\infty = 1\}$$

normalized so that  $P_{n,p}(0) = 1$ , and  $T_n$  is the set of all trigonometric polynomials of degree  $\leq n$  (see [8]). An essentially similar property is available for the case where  $T_n$  is replaced by  $\pi_n$ , the set of algebraic polynomials of degree  $\leq n$  (see [1]).

## 2. Preliminaries

Let  $I$  and  $d\sigma$  be as previously indicated. In this section we recall some basic facts concerning continuous  $T$ -systems on  $I$ . These facts, with perhaps minor modifications, may all be found in Karlin and Studden [3], and Gantmacher and Krein [2].

Definition 2.1. The system  $\{u_i\}_{i=1}^n$  of continuous functions on an interval  $I$  is called a  $T$ -system if every non-trivial linear combination of  $\{u_i\}_{i=1}^n$  has at most  $n-1$  zeros in  $I$ .

The following concepts will prove relevant.

Definition 2.2. For any  $f \in C(I)$ , we call  $t_0 \in \text{int}(I)$  a non-nodal zero of  $f$  provided that  $f$  vanishes, but does not change sign at  $t_0$ . All other zeros are called nodal.

Definition 2.3. For  $f \in C(I)$ , let  $Z(f)$  denote the number of zeros of  $f$  in  $I$ , with the convention that non-nodal zeros are counted twice.

Definition 2.4. We say that a function  $f$  defined on  $I = [a, b]$  has  $k$  sign changes

on  $I$  if  $I$  is decomposable into  $k+1$  intervals  $I_i = [x_{i-1}, x_i]$ ,  $i=1, \dots, k+1$ ,  $x_0 = a < x_1 < \dots < x_{k+1} = b$ , such that  $\epsilon(-1)^i f(t) \geq 0$  for  $t \in I_i$ ,  $i=1, \dots, k+1$ ,  $\epsilon = +1$  or  $-1$  fixed, and  $f$  does not vanish identically on  $I_i$ . We denote the number of sign changes of  $f$  on  $I$  by  $S^-(f)$ .

Lemma 2.1.  $\{u_i\}_{i=1}^n$  is a T-system on  $I$  iff  $Z(u) \leq n-1$  for any  $u(t) = \sum_{i=1}^n a_i u_i(t)$ ,  $\sum_{i=1}^n a_i^2 > 0$ .

Lemma 2.2. Let  $\{u_i\}_{i=1}^n$  be a T-system on  $I$ . For any  $k$  prescribed distinct points in  $\text{int}(I)$ ,  $k \leq n-1$ , there exists a  $u(t) = \sum_{i=1}^n a_i u_i(t)$  with nodal zeros at these points, and which vanishes nowhere else in  $\text{int}(I)$ .

With the aid of the above lemma, it is a simple matter to prove the following result.

Lemma 2.3. Let  $\{u_i\}_{i=1}^n$  be a T-system on  $I$ ,  $u_i \in C(\bar{I})$ ,  $i=1, \dots, n$ . If  $v(t)$  is a bounded function on  $\bar{I}$ , with at most a finite number of discontinuities and

$$\int_I v(t) u_i(t) d\sigma(t) = 0, \quad i=1, \dots, n,$$

then  $S^-(v) \geq n$ .

### 3. $1 < p < \infty$

Let  $\{u_1, \dots, u_n\}$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  be T-systems on  $I$ , and assume  $\{u_i\}_{i=1}^n, \varphi, \psi \in C(\bar{I})$ . For fixed  $p \in (1, \infty)$ , let  $g_1(t) = E_p(\varphi)(t)$  and  $g_2(t) = E_p(\psi)(t)$ , where  $E_p(\varphi)$ ,  $E_p(\psi)$  are as defined in the introduction.

Theorem 3.1. The zeros of  $g_1(t)$  and  $g_2(t)$  in  $I$  strictly interlace.

The proof of Theorem 3.1 is divided into a series of lemmas and propositions. In the first part of this section, we prove the following result.

Proposition 3.1. For  $g_1(t)$  and  $g_2(t)$  as above,

$$n \leq S^-(\alpha g_1 + \beta g_2) \leq Z(\alpha g_1 + \beta g_2) \leq n+1$$

for all  $\alpha, \beta$  real,  $\alpha^2 + \beta^2 > 0$ .

Since  $\{u_1, \dots, u_n, \varphi, \psi\}$  is a T-system on  $I$ , we immediately obtain from Lemma 2.1,

Lemma 3.1. For  $g_1(t)$  and  $g_2(t)$  as above,

$$Z(\alpha g_1 + \beta g_2) \leq n+1,$$

for all  $\alpha, \beta$  real,  $\alpha^2 + \beta^2 > 0$ .

Set

$$(3.1) \quad \begin{aligned} h_1(t) &= [\operatorname{sgn} g_1(t)] |g_1(t)|^{p-1} \\ h_2(t) &= [\operatorname{sgn} g_2(t)] |g_2(t)|^{p-1}. \end{aligned}$$

From the orthogonality relations characterizing the unique best  $L^p$ -approximation on  $I$  from  $\{u_i\}_{i=1}^n$ , it follows that

$$(3.2) \quad \int_I h_j(t) u_i(t) d\sigma(t) = 0, \quad i = 1, \dots, n; j = 1, 2.$$

A direct application of Lemma 2.3 and (3.2) yields:

Lemma 3.2. For all  $\alpha, \beta$  real,  $\alpha^2 + \beta^2 > 0$ ,

$$S^-(\alpha h_1 + \beta h_2) \geq n .$$

Remark 3.1. For  $p = 2$ , Proposition 3.1 is an immediate result of Lemmas 3.1 and 3.2 since in this case  $h_i(t) = g_i(t)$ ,  $i = 1, 2$ .

From (3.1),  $h_i(t)$  and  $g_i(t)$  obviously have the same zero properties, with respect to both  $S^-$  and  $Z$ . Thus, from Lemma 3.2,

Lemma 3.3. For  $g_1(t)$  and  $g_2(t)$  as above,

$$n \leq S^-(g_i), \quad i = 1, 2 .$$

Therefore, in the proof of Proposition 3.1 we shall assume, without loss of generality, that  $\alpha = 1$ ,  $\beta \neq 0$ .

Lemma 3.4. For  $h_i(t)$  and  $g_i(t)$  as above,  $i = 1, 2$ , and  $t_0 \in I$ ,  $g_1(t_0) + \beta g_2(t_0) = 0$  iff  $h_1(t_0) + [\text{sgn } \beta] |\beta|^{p-1} h_2(t_0) = 0$ .

Proof. We disregard the case where  $h_i(t_0)$  or  $g_i(t_0)$  is zero for some  $i$ , since any one of the four terms being zero implies, since  $\beta \neq 0$ , that all the remaining terms are zero, and the lemma is proven.

Assume  $g_1(t_0) + \beta g_2(t_0) = 0$ . Thus,

$$|g_1(t_0)| = |\beta| |g_2(t_0)|, \text{ and } [\text{sgn } g_1(t_0)] = -[\text{sgn } \beta] [\text{sgn } g_2(t_0)] .$$

Therefore,

$$\begin{aligned} h_1(t_0) &= [\text{sgn } g_1(t_0)] |g_1(t_0)|^{p-1} \\ &= -[\text{sgn } \beta] [\text{sgn } g_2(t_0)] |\beta|^{p-1} |g_2(t_0)|^{p-1} \\ &= -[\text{sgn } \beta] |\beta|^{p-1} h_2(t_0) . \end{aligned}$$

The above analysis is totally reversible and the lemma is proven.

Lemma 3.5.  $g_1(t) + \beta g_2(t)$  has a non-nodal zero at  $t_0 \in \text{int}(I)$  iff  
 $h_1(t) + [\text{sgn } \beta] |\beta|^{p-1} h_2(t)$  has a non-nodal zero at  $t_0$ .

Proof. Assume, without loss of generality, that  $g_1(t) + \beta g_2(t) \geq 0$  for  $t$  in some small neighborhood of  $t_0$ , with equality holding for  $t = t_0$ .

Case 1.  $g_1(t_0) \neq 0$ .

Since  $g_1(t_0) \neq 0$ , we have  $g_2(t_0) \neq 0$ . Let  $N$  denote a neighborhood of  $t_0$  in which  $g_1(t)$ ,  $g_2(t)$  are never zero, and  $g_1(t) + \beta g_2(t) \geq 0$ .

Since equality holds for  $t = t_0$ , we have

$$(3.3) \quad [\text{sgn } g_1(t)] = -[\text{sgn } \beta] [\text{sgn } g_2(t)], \quad \text{for } t \in N.$$

If  $g_1(t_0) > 0$ , then, for  $t \in N$

$$(3.4) \quad g_1(t) = |g_1(t)| \geq |\beta| |g_2(t)|.$$

Raising both sides to the  $p-1$  st power, and multiplying by the expressions in (3.3), which are positive in this case, we obtain the desired result.

If  $g_1(t_0) < 0$ , then (3.4) is reversed, but the expressions in (3.3) are negative. The result once again follows.

Case 2.  $g_1(t_0) = 0$ .

Thus  $g_2(t_0) = h_2(t_0) = h_1(t_0) = 0$ . Since  $g_1(t)$  and  $h_1(t)$  exhibit the same zero structure, it follows from Lemmas 3.1 and 3.3 that  $g_i(t)$  and  $h_i(t)$  necessarily change sign at  $t_0$  for  $i = 1, 2$ .

Let  $N$  be a sufficiently small neighborhood of  $t_0$  such that the only zero of  $g_i(t)$ ,  $h_i(t)$ ,  $i = 1, 2$ , and  $g_1(t) + \beta g_2(t)$  in  $N$  is at  $t_0$ .

Assume  $g_1(t) > 0$  for  $t > t_0$ ,  $t \in N$ . Since  $|g_1(t)| \leq \beta g_2(t)$  for  $t \leq t_0$ ,  $t \in N$ , we have  $[\text{sgn } \beta] = [\text{sgn } g_2(t)]$  for  $t < t_0$ ,  $t \in N$ .

Since both  $g_1(t)$  and  $g_2(t)$  change sign at  $t_0$ ,  $|g_1(t)|^{p-1} \geq |\beta|^{p-1} |g_2(t)|^{p-1}$  for  $t \geq t_0$ ,  $t \in N$ , and  $|g_1(t)|^{p-1} \leq |\beta|^{p-1} |g_2(t)|^{p-1}$  for  $t \leq t_0$ ,  $t \in N$ .

The result now follows from the definitions of  $h_1(t)$  and  $h_2(t)$ . The case  $g_1(t) < 0$  for  $t > t_0$ ,  $t \in N$ , is totally analogous.

Since the analysis of Cases 1 and 2 is reversible, the lemma is proven.

Proof of Proposition 3.1. Apply Lemmas 3.1-3.5.

The next proposition is a modification of a result of Gantmacher and Krein [2], (see also Lee and Pinkus [4]).

Proposition 3.2. If  $\Phi, \Psi \in C(0,1)$ , and  $n \leq S^-(\alpha\Phi + \beta\Psi) \leq Z(\alpha\Phi + \beta\Psi) \leq n+1$  for all real  $\alpha, \beta$ ,  $\alpha^2 + \beta^2 > 0$ , then the zeros of  $\Phi$  and  $\Psi$  in  $(0,1)$  strictly interlace.

The proof of Proposition 3.2 is also divided into a series of lemmas. Note the important fact that the above inequalities imply that  $\alpha\Phi(t) + \beta\Psi(t)$  has no non-nodal zeros in  $(0,1)$ .

Let  $\{\xi_i\}_{i=1}^k$ ,  $\xi_0 = 0 < \xi_1 < \dots < \xi_k < \xi_{k+1} = 1$ , ( $k = n$  or  $n+1$ ) denote the zeros (sign changes) of  $\Phi(t)$  in  $(0,1)$ . Let  $I_i = (\xi_{i-1}, \xi_i)$ ,  $i = 1, \dots, k+1$ , and  $f(t) = \frac{\Psi(t)}{\Phi(t)}$ .

Lemma 3.6.  $f(t)$  is strictly monotone in each  $I_i$ ,  $i = 1, \dots, k+1$ .

Proof. If  $f(t)$  is a constant  $c$  on any interval of  $I_i$  of positive length, then  $Z(\Psi - c\Phi) = \infty$ , contradicting the hypothesis of the proposition. If  $f$  is not strictly monotone on  $I_i$ , then  $f$  has a relative extremum at some point  $x_i \in I_i$ . The function  $\Psi(t) - f(x_i)\Phi(t)$  has a non-nodal zero at  $x_i$ , contradicting the hypothesis. The lemma is proven.

Lemma 3.7.  $f(t)$  has exactly one zero in each  $I_i$ ,  $i = 2, \dots, k$ .

Proof. Since  $f(t)$  is monotone in each  $I_i$ ,  $i = 1, \dots, k+1$ , the limits

$$\lim_{t \rightarrow \xi_i^-} f(t) = \ell_i^- \quad \text{and} \quad \lim_{t \rightarrow \xi_i^+} f(t) = \ell_i^+$$

both exist as extended real numbers for  $i = 1, \dots, k$ . We shall show that none of the  $\{\ell_i^+\}_{i=1}^k$  and  $\{\ell_i^-\}_{i=1}^k$  is finite. Taken together with Lemma 3.6, this implies the statement of the lemma.

Let us assume that either  $\ell_i^-$  or  $\ell_i^+$  is finite. Since  $\Phi(\xi_i) = 0$ , it follows that  $\Psi(\xi_i) = 0$ . We are concerned with one of the following four cases.

- (i) exactly one of  $\ell_i^+$  and  $\ell_i^-$  is finite,
- (ii)  $\ell_i^+$  and  $\ell_i^-$  are finite and unequal
- (iii)  $\ell_i^+ = \ell_i^-$  (finite) and  $f$  is monotone in a neighborhood of  $\xi_i$ .
- (iv)  $\ell_i^+ = \ell_i^-$  (finite) and  $f$  is monotone in opposite senses for  $t \in I_i$  and  $t \in I_{i+1}$ .

If either cases (i) or (ii) occur, let  $c$  be any real number between

$l_i^+$  and  $l_i^-$ , while if case (iii) holds, let  $c = l_i^+ = l_i^-$ . Then  $\Psi(t) - c\Phi(t)$  has a non-nodal zero at  $\xi_i$  since  $\Phi(\xi_i) = \Psi(\xi_i) = 0$ , and  $\Phi(t)$  changes sign at  $\xi_i$ .

Assume case (iv) obtains. Let  $c = l_i^+ = l_i^-$  and assume, without loss of generality, that  $f(t) \leq c$  for  $t$  in a neighborhood of  $\xi_i$ . Now,  $\Psi(t) - c\Phi(t)$  has at least  $n$  sign changes in  $(0,1)$ , one of which is at  $\xi_i$ . Thus  $\Psi(t) - c\Phi(t) + \epsilon\Phi(t)$  has for  $\epsilon > 0$ ,  $\epsilon$  sufficiently small, at least  $n-1$  sign changes bounded away from  $\xi_i$ . Since  $f(t)$  is strictly monotone in  $I_i$  and  $I_{i+1}$ ,  $\Psi(t) - (c-\epsilon)\Phi(t)$  has a zero slightly to the left of  $\xi_i$ , a zero slightly to the right of  $\xi_i$ , and vanishes at  $\xi_i$ . Thus  $\Psi(t) - (c-\epsilon)\Phi(t)$  has at least  $n+2$  zeros in  $(0,1)$ . A contradiction. The lemma is proven.

Since  $\Phi(t)$  and  $\Psi(t)$  are interchangeable in the above analysis, Proposition 3.2, for  $n \geq 2$ , follows from Lemmas 3.6 and 3.7. For the cases  $n = 0$  and  $n = 1$ , the following lemma is also used.

Lemma 3.8.  $\Phi(t)$  and  $\Psi(t)$  have no common zeros in  $(0,1)$ .

Proof. Assume  $\Phi(\xi) = \Psi(\xi) = 0$ . Let  $f(t) = \frac{\Psi(t)}{\Phi(t)}$  and  $g(t) = \frac{\Phi(t)}{\Psi(t)}$ . Both  $f(t)$  and  $g(t)$  are, by Lemma 3.6, strictly monotone in some neighborhood to the left and in some neighborhood to the right of  $\xi$ . Furthermore, the limits as  $t \rightarrow \xi$ , from above and below, exist and are infinite by Lemma 3.7. However, both  $\Phi(t)$  and  $\Psi(t)$  change sign at  $\xi$  and a contradiction immediately ensues. The lemma is proven.

The proof of Proposition 3.2 is complete.

Proof of Theorem 3.1. If  $I$  is an open interval, then Theorem 3.1 is a consequence of Propositions 3.1 and 3.2.

Assume  $I = [0, 1)$  and  $g_1(0) = 0$ . Since  $n \geq 1$ , let  $\xi \in (0, 1)$  be such that  $g_1(\xi) = 0$  and  $g_1(t) \neq 0$  for all  $t \in (0, \xi)$ . From Proposition 3.2,  $g_2(\xi) \neq 0$ . We must prove that  $g_2(0) \neq 0$  and  $g_2(t)$  has a zero in  $(0, \xi)$ . Assume  $g_2(t)$  has no zero in  $[0, \xi]$ . This immediately contradicts the monotonicity of  $\frac{g_2(t)}{g_1(t)}$  in  $(0, \xi)$  (see Lemma 3.6). Now, assume  $g_2(0) = 0$ , and by interchanging  $g_1(t)$  and  $g_2(t)$ , if necessary, assume  $g_2(t) \neq 0$  in  $(0, \xi]$ . Assume also that  $g_1(t)g_2(t) > 0$  for  $t \in (0, \xi)$ . Therefore  $\lim_{t \rightarrow \xi^-} \frac{g_2(t)}{g_1(t)} = \infty$  and  $\lim_{t \rightarrow 0^+} \frac{g_2(t)}{g_1(t)} \downarrow c \geq 0$ ,  $c$  finite.  $g_2(t) - cg_1(t)$  has  $n$  sign changes in  $(0, 1)$  and thus for  $\epsilon > 0$ ,  $\epsilon$  sufficiently small,  $g_2(t) - (c+\epsilon)g_1(t)$  has  $n$  sign changes in  $(0, 1)$  bounded away from  $t = 0$ , a zero near  $t = 0$ , and a zero at  $t = 0$ . Therefore  $g_2(t) - (c+\epsilon)g_1(t)$  has at least  $n+2$  zeros in  $I = [0, 1)$ , a contradiction.

This same analysis applies where  $I = (0, 1]$  and  $I = [0, 1]$ . The theorem is proven.

#### 4. $p = 1$

As previously, let  $\{u_1, \dots, u_n\}$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  be T-systems on  $I$ , and assume  $\{u_i\}_{i=1}^n, \varphi, \psi \in C(\bar{I})$ . Let  $g_1(t) = E_1(\varphi)(t)$  and  $g_2(t) = E_1(\psi)(t)$ , where  $E_1(\varphi)$  and  $E_1(\psi)$  are as defined in the introduction.

In this section we prove the following result.

Theorem 4.1. The zeros of  $g_1(t)$  and  $g_2(t)$  on  $I$  strictly interlace unless  $S^-(g_1) = S^-(g_2) = n$ , in which case  $[\text{sgn } g_1(t)] = [\text{sgn } g_2(t)]$  for all  $t \in \text{int}(I)$ .

We define  $h_j(t)$ ,  $j = 1, 2$  on  $\bar{I}$  as follows. Set  $h_j(t) = \text{sgn } g_j(t)$  for  $t \in \text{int}(I)$ , and let  $h_j(t)$  be continuous at the endpoints,  $j = 1, 2$ . Since  $\{u_1, \dots, u_n\}$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  are T-systems on  $I$ ,  $g_1(t)$  and  $g_2(t)$  are uniquely defined, and since  $Z(g_j) \leq n+1$ ,  $j = 1, 2$ ,  $|h_j(t)| = 1$  a. e. on  $I$ ,  $j = 1, 2$ , and satisfy the orthogonality relations

$$(4.1) \quad \int_I h_j(t) u_i(t) d\sigma(t) = 0, \quad i = 1, \dots, n; j = 1, 2.$$

Lemma 4.1. For  $h_1(t)$  and  $h_2(t)$  as above,  $n \leq S^-(h_j) \leq n+1$ ,  $j = 1, 2$ , and  $n \leq S^-(h_1 \pm h_2)$ , unless  $h_1(t) \pm h_2(t) \equiv 0$  on  $I$ .

Proof. This is an immediate consequence of Definition 2.1 and Lemma 2.3.

Replacing  $h_j(t)$  by  $-h_j(t)$ , if necessary and letting  $\bar{I} = [0, 1]$ , we may assume the existence of  $\{\xi_i\}_{i=1}^k$  and  $\{\eta_i\}_{i=1}^m$ ,  $n \leq k, m \leq n+1$ , where

$$\xi_0 = 0 < \xi_1 < \dots < \xi_k < \xi_{k+1} = 1$$

$$\eta_0 = 0 < \eta_1 < \dots < \eta_m < \eta_{m+1} = 1$$

such that

$$(4.2) \quad \begin{aligned} h_1(t) &= (-1)^i, & \xi_i < t < \xi_{i+1}, & \quad i = 0, 1, \dots, k \\ h_2(t) &= (-1)^i, & \eta_i < t < \eta_{i+1}, & \quad i = 0, 1, \dots, m. \end{aligned}$$

Lemma 4.2. For  $h_1(t)$  and  $h_2(t)$  as above,  $S^-(h_1 \pm h_2) \leq \min\{k, m\}$ , and if  $k = m$ , then  $S^-(h_1 - h_2) \leq k - 1 = m - 1$ .

Proof. The above lemma is known. For completeness, we include a proof.

With no loss of generality, assume  $k \leq m$ . From the definition of

$h_1(t)$ ,

$$(h_1(t) \pm h_2(t)) (-1)^i \geq 0, \quad \xi_i < t < \xi_{i+1}, \quad i = 0, 1, \dots, k.$$

Thus  $S^-(h_1 \pm h_2) \leq k = \min\{k, m\}$ .

Assume  $k = m$  and  $\xi_1 \leq \eta_1$ . Since  $h_1(t) - h_2(t) \equiv 0$  on  $[0, \xi_1)$ ,  $S_{(0,1)}^-(h_1 - h_2) = S_{(\xi_1, 1)}^-(h_1 - h_2)$ . However  $h_1(t)$  has  $k-1$  sign changes on  $(\xi_1, 1)$ . Applying the previous result, the lemma is proven.

Lemma 4.3. If  $S^-(g_1) = S^-(g_2) = n$ , then  $\xi_i = \eta_i$ ,  $i = 1, \dots, n$ .

Proof. Since  $S^-(n_j) = S^-(g_j)$ ,  $j = 1, 2$ , then  $S^-(h_1 - h_2) \leq n-1$  by Lemma 4.2. From Lemma 4.1, it follows that  $h_1(t) \equiv h_2(t)$  for almost all  $t \in [0, 1]$ . Thus  $\xi_i = \eta_i$ ,  $i = 1, \dots, n$ , and the lemma is proven.

The above lemma is a restatement of the well-known fact that if  $\{u_1, \dots, u_n\}$  is a T-system on  $(0, 1)$ , then there exist  $n$  unique points  $\{\zeta_j\}_{j=1}^n$ ,  $\zeta_0 = 0 < \zeta_1 < \dots < \zeta_n < \zeta_{n+1} = 1$ , such that

$$\sum_{j=0}^n (-1)^j \int_{\zeta_j}^{\zeta_{j+1}} u_i(t) d\sigma(t) = 0, \quad i = 1, \dots, n.$$

To prove Theorem 4.1, it remains to consider the case where at least one of  $S^-(h_1)$ ,  $S^-(h_2)$  is  $n+1$ . Note that if  $S^-(h_j) = S^-(g_j) = n+1$ ,  $j = 1, 2$ , then we cannot have  $h_1(t) \equiv h_2(t)$  for almost all  $t \in I$ . This is a consequence of the fact that there exists a unique (up to a multiplicative constant) non-trivial linear combination of  $\{u_1, \dots, u_n, \varphi, \psi\}$  which changes sign at  $n+1$  given points in  $I$ , and it cannot be both of the form

$$g_1(t) = \varphi(t) - \sum_{i=1}^n a_i u_i(t), \quad \text{and}$$

$$g_2(t) = \psi(t) - \sum_{i=1}^n b_i u_i(t).$$

Lemma 4.4. Let  $h_1(t)$  and  $h_2(t)$  be as in (4.2). Then for each  
 $i = 1, \dots, k-1$ , there exists an  $\eta_j \in (\xi_i, \xi_{i+1})$  .

Proof. Assume that this is not the case. Replace  $h_2(t)$  by  $-h_2(t)$ , if necessary, in order that  $h_1(t) - h_2(t) \equiv 0$  for  $t \in (\xi_i, \xi_{i+1})$  . If  $i = 1$ , then  $h_1(t) - h_2(t)$  has no sign change in  $(0, \xi_3)$ , while  $S_{(\xi_3, 1)}^-(h_1 - h_2) \leq k-3$  by Lemma 4.2. Thus  $S_{(0, 1)}^-(h_1 - h_2) \leq k-2 \leq n-1$ , contradicting Lemma 4.1.

The analogous result holds for  $i = k-1$  . Assume  $1 < i < k-1$  . Then  $h_1(t) - h_2(t)$  has no sign change on  $(\xi_{i-1}, \xi_{i+2})$ , while  $S_{(0, \xi_{i-1})}^-(h_1 - h_2) \leq i-2$ , and  $S_{(\xi_{i+2}, 1)}^-(h_1 - h_2) \leq k-i-2$  . Therefore,  $S_{(0, 1)}^-(h_1 - h_2) \leq (i-2) + (k-i-2) + 2 = k-2 \leq n-1$ , a contradiction. The lemma is proven.

Proof of Theorem 4.1. If  $S^-(g_1) = S^-(g_2) = n$ , the result follows from Lemma 4.3. Assume this is not the case. Then Lemma 4.4 immediately implies that the zeros of  $g_1(t)$  and  $g_2(t)$  in  $(0, 1)$  strictly interlace. If  $I = [0, 1)$ , and  $g_1(0) = 0$ , then  $S^-(g_1) = n$  since  $g_1(t)$  has at most  $n+1$  zeros on  $I$ , and thus  $S^-(g_2) = n+1$  . The strict interlacing on  $I$  now follows. The same reasoning applies if  $I = (0, 1]$  or  $I = [0, 1]$ , and the theorem is proven.

A scrutiny of the proof of Theorem 4.1 reveals that the Tchebycheffian property of  $\{u_1, \dots, u_n, \varphi, \psi\}$  has not been used except to establish a bound on the number of sign changes of  $E_1(\varphi)$  and  $E_1(\psi)$  . Hence the same proof establishes the following.

Theorem 4.2. Let  $\{u_i\}_{i=1}^n$  be a continuous T-system on  $I$ , continuous on  $\bar{I}$ , and let  $\varphi$  and  $\psi$  be linearly independent continuous functions on  $I$

such that  $E_1(\varphi)$  and  $E_1(\psi)$  vanish on sets of measure 0 and change sign at no more than  $n+1$  points in  $I$ . Then either the two sequences of points of sign change strictly interlace, or  $\text{sgn } E_1(\varphi)(t) = \text{sgn } E_1(\psi)(t)$  for all  $t \in \text{int}(I)$ .

5.  $p = \infty$

The results for  $p = \infty$  parallel those obtained for  $p \in [1, \infty)$ . Moreover, the proofs, in this case, involve no more than a careful zero counting procedure (cf. [3, Chap. 2]). Accordingly, we state the results without proof. Note that in this case we assume, in order that the best approximation be unique, that  $I$  is closed. Let  $g_1(t)$  and  $g_2(t)$  denote the error function in best  $L^\infty$  (Tchebycheff) approximation to  $\varphi(t)$  and  $\psi(t)$ , respectively, from  $\{u_i(t)\}_{i=1}^n$  on  $I$ . We assume that  $\{u_i\}_{i=1}^n$  and  $\{u_1, \dots, u_n, \varphi, \psi\}$  are both T-systems on  $I$ . Then,

Theorem 5.1. The zeros of  $g_1(t)$  and  $g_2(t)$  in  $I$  strictly interlace.

From the characterization of best  $L^\infty$ -approximation, it is known that  $g_1(t)$  alternates at  $k$  points,  $k = n+1$  or  $n+2$ , between  $\|g_1\|_\infty$  and  $-\|g_1\|_\infty$ , and this same property obtains for  $g_2(t)$ . The fact that  $\{u_1, \dots, u_n, \varphi, \psi\}$  is a T-system on  $I$  immediately implies that  $g_1(t)$  and  $g_2(t)$  cannot both exhibit  $n+2$  points of alternation on  $I$ . Furthermore, the following result is also obtainable by zero counting.

Theorem 5.2. The points of alternation of  $g_1(t)$  and  $g_2(t)$  on  $I$  weakly interlace.

Remark 5.1. If  $\{u_1, \dots, u_n, \varphi, \psi\}$  is an Extended Tchebycheff system of

order 2, in which case the zero counting is modified (see Karlin and Studden [3, Chap. 1]), we have strict interlacing in Theorem 5.2 in the interior of  $I$ .

Properties of the error function of the type exhibited in this section were established under more restrictive conditions by Rowland [6], Shohat [7], and Paszkowski [5].

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