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# SOUND RANGING FOR ARTILLERY

## VOLUME I. A THEORY OF SOUND PROPAGATION THROUGH THE ATMOSPHERE AND AN APPLICATION TO SOUND RANGING

SIGNAL CORPS ENGINEERING LABORATORIES
FORT MONMOUTH, NEW JERSEY

20 DECEMBER 1940

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# SOUND RANGING FOR ARTILLERY

VOLUME I

## THEORY OF SOUND PROPAGATION THROUGH THE ATMOSPHERE AND AN APPLICATION TO SOUND RANGING



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FORT MONMOUTH, NEW JERSEY
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**SIGNAL CORPS LABORATORIES**
**FORT MONMOUTH, NEW JERSEY**

# S O U N D   R A N G I N G   F O R   A R T I L L E R Y

**Volume I**

**A Theory of Sound Propagation Through the Atmosphere**
**and**
**an Application to Sound Ranging**

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## GLOSSARY OF TERMS

*azimuth* - the direction of approach of the sound wave front, determined by the projection of the wave front normal on the horizontal plane tangent to the earth at the point of observation.

*diffraction* - deviation of sound waves from a normal course in an unobstructed medium when passing the angle or edge of a large body.

*drift-increment* - the total displacement of the sound wave, normal to its direction of propagation, within the layer of atmosphere considered.

*drift-wind* - the horizontal wind component normal to the direction of sound propagation.

*drift-wind gradient* - the rate of change with elevation of horizontal wind component normal to the direction of sound propagation.

*inclination* - the angle of the sound wave normal with the horizontal plane tangent to the earth at the point of observation.

*law of refraction* - the relation connecting the velocity of sound, wind, and inclination of the sound wave normal at one point with their values at another.

*locus* - a number of possible positions satisfying a given set of conditions.

*parameter* - a quantity fixed for a given set of meteorological conditions, but varying from one set of conditions to another.

*parametric solution* - a solution in terms of the variables and a parameter, the parameter being constant once the set of conditions is defined.

*range-increment* - the total displacement of the sound wave, in its direction of propagation, within the layer of atmosphere considered.

*refractive wind* - the horizontal wind component in the direction of sound propagation.

*refractive wind gradient* - the rate of change with elevation of the horizontal wind component in the direction of sound propagation.

*sound path* -
*sound ray* - the space curve traversed by a point on a sound wave surface.

*sound velocity gradient* - the rate of change of the velocity of sound with elevation.

*space array* - any arrangement of microphones such that all are not in the same plane.

*specific humidity* - the mass of water vapor in unit mass of moist air.

*tetrahedron* - a solid bounded by four plane triangular faces.

*time-increment* - the time of travel of the sound wave along a particular path through the atmospheric layer considered.

*virtual temperature* - the temperature of absolutely dry air having the same density as moist air at a slightly lower temperature.

## LIST OF SYMBOLS

**Additional symbols are defined in the text.**

| | |
|---|---|
| $\alpha$ | stratification parameter, defined by ratio of sound velocity gradient to refractive wind gradient |
| $\alpha, \beta, \gamma$ | direction angles of normal to sound wave front |
| $\gamma$ | ratio of specific heat at constant pressure to specific heat at constant volume |
| $\gamma_d, \gamma_h, \gamma_w$ | $\gamma$ as defined above for dry air, humid (or moist) air, and water vapor respectively |
| $\Delta$ | third order determinant |
| $\varepsilon$ | a small positive quantity; angular correction |
| $\zeta$ | true azimuth angle from north (positive clockwise) |
| $\eta$ | cosine of angle of inclination of wave front normal to horizontal plane |
| $\theta$ | azimuth of sound approach |
| $\lambda, \mu, \nu$ | direction cosines of normal to sound wave front |
| $\pi$ | 180 degrees |
| $\rho$ | density |
| $\rho_d$ | density of dry air |
| $\rho_w$ | density of water vapor |
| $\sigma$ | drift parameter |
| $\psi$ | angle between two lines |

| | |
|---|---|
| $A, B$ | lower and upper boundaries, respectively, of a layer of atmosphere |
| $a$ | velocity of sound in a still atmosphere |
| $C$ | constant |
| $c_p$ | specific heat at constant pressure |
| $c_v$ | specific heat at constant volume |
| $D$ | third order determinant |
| $d$ | index of reliability (shortest distance between two lines) |
| $\frac{d}{dt}, \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ | total differentials with respect to time and space coordinates, respectively |
| $\frac{da}{dz}$ | sound velocity gradient |
| $\frac{dT'}{dz}$ | virtual temperature gradient (or lapse rate) |
| $\frac{du_\ell}{dz}$ | refractive wind gradient |
| $\frac{dv_e}{dz}$ | drift-wind gradient |
| $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ | partial differentiation |
| $\delta x, \delta y, \delta z$ | infinitesimal increments in space coordinates |
| $\Delta t$ | time-increment |
| $\Delta X_e$ | range-increment |
| $\Delta Y_e$ | drift-increment |
| $\Delta z$ | depth of atmospheric layer |

| | |
|---|---|
| $e$ | vapor pressure |
| $i$ | $\sqrt{-1}$ |
| $k$ | constant of integration; range parameter |
| $m_d$ | molecular weight of dry air |
| $m_w$ | molecular weight of water vapor |
| $P$ | point in space; distance in $x$-direction between two coordinate systems |
| $P, N, E$ | coordinates of origin of one coordinate system in relation to another |
| $p$ | distance of plane from origin of coordinate system; atmospheric pressure |
| $p_d$ | atmospheric pressure of dry air |
| $q$ | specific humidity |
| $q_{max}$ | saturation specific humidity (at given $T$ and $p$) |
| $R$ | gas constant, equal to $2.8703 \times 10^6$ c.g.s units for dry air; distance between sound source and observation point |
| $R'$ | universal gas constant |
| $r$ | relative humidity |
| $r_1$ | distance between two points |
| $T$ | time of arrival of sound wave; temperature of air in degrees absolute |
| $T'$ | virtual temperature in degrees absolute |

| | |
|---|---|
| $t$ | time |
| $u, v, w$ | wind components in $x, y, z$ directions respectively |
| $u_*$ | refractive wind |
| $V$ | instantaneous velocity of propagation of sound, motion of medium included |
| $\frac{V_0}{n_0}$ | apparent velocity of sound along the horizontal, constant for a particular sound ray |
| $v_*$ | drift-wind |
| $X_0, Y_0$ | grid-coordinates of point of observation |
| $X_s, Y_s$ | grid-coordinates of sound source |
| $X_*, Y_*$ | integral of $dx_*$ and $dy_*$, respectively |
| $x, y, z$ | space coordinates |
| $x_*, y_*$ | coordinates after clockwise rotation about $Z$-axis, of amount $\frac{\pi}{2} + \theta$, of previous coordinate system |
| $Z_{max}$ | level of total refraction |

**A Theory of Sound Propagation Through the Atmosphere and an Application to Sound Ranging**

## I. INTRODUCTION

This constitutes Volume I of a series of reports dealing with (a) the effect of meteorological conditions on the propagation of sound through the atmosphere, and (b) the development of a technique of sound ranging whereby existent meteorological conditions can be quantitatively evaluated in terms of their net effect on sound propagation from source to microphone position. Specifically, it is the intent of the present report to formulate a theory of sound propagation from basic analytical considerations, adequate for *any* meteorological situation whatsoever, and so presented that it may readily be applied to military sound ranging on artillery.

An adequate, though not necessarily feasible, method of approach to the problem of the determination of meteorological corrections for sound ranging would be one concerned with an analysis of sound propagation along the *particular path* traversed by the sound wave. The concept of the sound path, or ray, is not new. For example, Milne[1] had occasion to investigate an analogous problem, that of the location of aircraft by means of sound propagation through the atmosphere. Gutenberg[2] and others have been concerned with the abnormal propagation of sound to great distances, and have utilized the concept of the sound path in computing the approximate temperature in the upper portion of the stratosphere. Esclangon[3], in a publication for the French Army, has studied the problem of sound propagation along theoretical lines. The present investigation, though carried on independently of Esclangon's, parallels his work in certain fundamental derivations. However, none of these investigations carries to a complete solution the net effect of the meteorological conditions along the entire sound path in a manner suitable for application in sound ranging. This is due in part to the different nature of the problem with which they are concerned, and in part to assumptions which are not justifiable from the standpoint of the accuracies desired in the present sound ranging program in the U. S. Army. *The paramount concern of the present investigation is the evolution of theory of sound propagation, sparing no generality wherever this endeavor is compatible with an explicit solution, yet within full sight of the fact that its eventual utility can only be that of a practical field sound ranging technique.* The solutions obtained permit *any* manner of variation with elevation of temperature, humidity, wind direction, and wind velocity, these four elements considered in *any* combination whatsoever.

It is shown in subsequent sections of this report that, in long range propagation of sound, the sound path is somewhat similar to the trajectory of a projectile. That is, sound propagation through the atmosphere is a three-dimensional problem. Hence a study is required of the microphone array which affords means of determining the direction of sound approach. To this end Section II is devoted to a mathematical analysis of the geometrical arrangement of microphones sufficient to determine uniquely the space orientation of the direction of arrival of sound.

[1] E.A. Milne, *Sound Waves in the Atmosphere*, Phil. Mag. 42, pp. 96-114, July 1921.

[2] B. Gutenberg, *Die Schallausbreitung in der Atmosphäre*, Handbuch der Geophysik, Band 9, pp. 89-145.

[3] E. Esclangon, *L'Acoustique des Canons et des Projectiles*, *Memorial de L'Artillerie Francaise*, IV, 3e Fasc.

The solution of the problem requires also a knowledge of the velocity of sound and its variation with elevation. Section III includes a discussion of equations relating the velocity of sound as a function of temperature and humidity. These meteorological elements can be determined by meteorograph ascents in the atmosphere. The final equations are put in such form as to utilize quantities used in current meteorological practice.

The nonhomogeneity of the medium of sound transmission, viz., the atmosphere, is considered in Section IV. The general equations of motion are derived for the propagation of sound in such a medium, after the manner of development by Milne.

These equations of motion are rendered to an explicit solution in Section V for the case of a stratified atmosphere. A technique is introduced in the integration of the equations of motion which permits of parametric solutions for the time of travel of the sound, and for the coordinates, $x$, $y$, $z$, of a point on the sound wave front at any position along its path of propagation. As a consequence the method of analysis, with but slight modification, lends itself to the employment of graphical or mechanical devices;[*] obviously advantageous in that laborious computations are terminated once the parametric solutions are adequately represented.

Certain simple cases of atmospheric stratification are considered in Section VI.

In Section VII a mathematical solution is derived for the position of the sound source. The solution leads to the determination of a parameter which may be utilized as an index of reliability for the ranging operations performed.

No attempt has been made in this preliminary report to incorporate correction terms for the time of travel and the range and azimuth of the source of sound due to abnormally high sound velocities in the vicinity of the sound source, when that source is an explosion of considerable magnitude. Treatment of this phenomenon is deferred until such time as the essentials of the theory herein are put to rigorous test.

Some extensions and applications of the basic theoretical deductions contained herein, particularly in regard to the tactical utility of meteorology and forecasting for sound ranging and artillery fire, are tentatively proposed to form the substance of Volume III of this series of reports.

[*] A slide rule for determining meteorological corrections for sound propagation through space has been devised, and will be discussed in Volume II of this series of reports.

## II. THEORY FOR SPACE DETERMINATION OF THE DIRECTION OF SOUND PROPAGATION.

**1. Direction Cosines of Normal to Sound Wave Front.**—Consider a left-handed coordinate system as in Figure 1, and an element *ABC* of any plane, whose normal *PN* has the direction angles $\alpha$, $\beta$, $\gamma$ with the positive *X*-, *Y*-, and *Z*-axes respectively. If we set $\lambda = \cos\alpha$, $\mu = \cos\beta$, $\nu = \cos\gamma$, then $\lambda$, $\mu$, $\nu$ become the direction cosines of the normal to the plane *ABC*, and the equation of the plane is given by

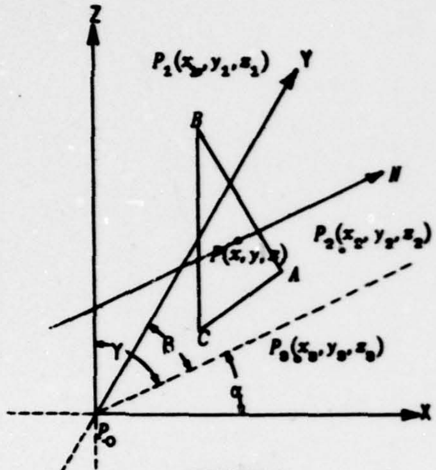


FIGURE 1.

$$\lambda x + \mu y + \nu z = p, \qquad (1)$$

where $p$ is the distance along the normal, passing through the origin $P_0$, to the plane. If now $P_1$, $P_2$, $P_3$ are any three points in space, and if the direction of propagation of the plane is invariant over the dimensions of the tetrahedron formed by $P_0$, $P_1$, $P_2$, $P_3$, then the distances from the origin, $p_1$, $p_2$, $p_3$ of the planes parallel to *ABC* through points $P_1$, $P_2$, $P_3$, are given by equation (1) with the coordinates of the given points substituted for $x$, $y$, $z$. However, if $V$ is the velocity of propagation of the plane *ABC* along the normal, and if we let the arrival time of the plane at $P_0$ be $T_0$, and $T_1$, $T_2$, $T_3$ at $P_1$, $P_2$, $P_3$ respectively, then $p_1 = V(T_1-T_0)$, $p_2 = V(T_2-T_0)$, $p_3 = V(T_3-T_0)$, and we have the three equations

$$\left.\begin{aligned} x_1\lambda + y_1\mu + z_1\nu &= V(T_1-T_0) \\ x_2\lambda + y_2\mu + z_2\nu &= V(T_2-T_0) \\ x_3\lambda + y_3\mu + z_3\nu &= V(T_3-T_0) \end{aligned}\right\} \qquad (2)$$

We can now consider $\lambda$, $\mu$, $\nu$ variable, though subject to the trigonometrical relation $\lambda^2 + \mu^2 + \nu^2 = 1$, and solve the equations (2) for $\lambda$, $\mu$, $\nu$ in terms of the coordinates of $P_1$, $P_2$, $P_3$, arrival times $T_0$, $T_1$, $T_2$, $T_3$, and $V$.

The condition that there exist a unique solution for $\lambda$, $\mu$, $\nu$ of equations (2) above is that

$$D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0. \qquad (3)$$

If $z_1 = z_2 = z_3 = 0$, then $D = 0$, so that the condition that equations (2) have a unique solution reduces to the restriction that any one of the points $P_0$, $P_1$, $P_2$, $P_3$ must not belong to the plane of the other three.

Provided that $P_1$, $P_2$, $P_3$ are so chosen with respect to $P_0$ that the condition $D \neq 0$ is satisfied, we can write

$$\left.\begin{aligned} \lambda &= \frac{V}{D}\begin{vmatrix} T_1-T_0 & y_1 & z_1 \\ T_2-T_0 & y_2 & z_2 \\ T_3-T_0 & y_3 & z_3 \end{vmatrix}, \\ \mu &= \frac{V}{D}\begin{vmatrix} x_1 & T_1-T_0 & z_1 \\ x_2 & T_2-T_0 & z_2 \\ x_3 & T_3-T_0 & z_3 \end{vmatrix}, \\ \text{and} \quad \nu &= \frac{V}{D}\begin{vmatrix} x_1 & y_1 & T_1-T_0 \\ x_2 & y_2 & T_2-T_0 \\ x_3 & y_3 & T_3-T_0 \end{vmatrix}. \end{aligned}\right\} \tag{4}$$

Thus if we can determine $V$, we have a unique solution for $\lambda$, $\mu$, $\nu$. This we can readily do by means of the relation $\lambda^2 + \mu^2 + \nu^2 = 1$, or

$$\frac{V^2}{D^2}\begin{vmatrix} T_1-T_0 & y_1 & z_1 \\ T_2-T_0 & y_2 & z_2 \\ T_3-T_0 & y_3 & z_3 \end{vmatrix}^2 + \frac{V^2}{D^2}\begin{vmatrix} x_1 & T_1-T_0 & z_1 \\ x_2 & T_2-T_0 & z_2 \\ x_3 & T_3-T_0 & z_3 \end{vmatrix}^2 + \frac{V^2}{D^2}\begin{vmatrix} x_1 & y_1 & T_1-T_0 \\ x_2 & y_2 & T_2-T_0 \\ x_3 & y_3 & T_3-T_0 \end{vmatrix}^2 = 1,$$

from which

$$V = \frac{D}{\pm\sqrt{\begin{vmatrix} T_1-T_0 & y_1 & z_1 \\ T_2-T_0 & y_2 & z_2 \\ T_3-T_0 & y_3 & z_3 \end{vmatrix}^2 + \begin{vmatrix} x_1 & T_1-T_0 & z_1 \\ x_2 & T_2-T_0 & z_2 \\ x_3 & T_3-T_0 & z_3 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & T_1-T_0 \\ x_2 & y_2 & T_2-T_0 \\ x_3 & y_3 & T_3-T_0 \end{vmatrix}^2}}. \tag{5}$$

Hence $\lambda$, $\mu$, $\nu$ can be determined uniquely in terms of determinants of order three formed by elements contained in any square array of the system

$$\left\|\begin{matrix} x_1 & y_1 & z_1 & T_1-T_0 \\ x_2 & y_2 & z_2 & T_2-T_0 \\ x_3 & y_3 & z_3 & T_3-T_0 \end{matrix}\right\|,$$

so that the direction of sound approach is independent of the velocity $V$ of the wave front.

**2. Azimuth and Inclination of Sound Wave Front.**—It is possible to convert the direction cosines $\lambda$, $\mu$, $\nu$ defining the position in space of the normal to the wave front, into an elevation angle and an azimuth angle. To this end let $\theta$ be the angle between the positive $Y$-axis and the projection of the normal $PN$

on the $XY$-plane. Consider $\theta$ positive when measured from the $Y$-axis towards the positive $X$-axis. Then from Figure 2 we see that

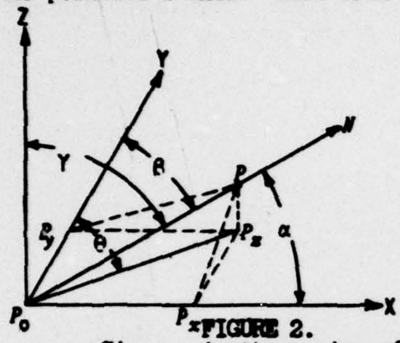


FIGURE 2.

$$\tan\theta = \frac{P_yP_z}{P_oP_y} = \frac{P_oP_x}{P_oP_y} = \frac{P_oP}{P_oP}\frac{\cos\alpha}{\cos\beta} = \frac{\lambda}{\mu}. \quad (6)$$

Hence $\theta = arc\ tan\ \lambda/\mu$, and we have thus determined the azimuth of sound approach as the projection of the normal to the wave front on the plane tangent to the earth at origin $P_o$.

Since $\nu$ is the cosine of the angle between the positive $Z$-axis and the positive direction of the normal to the wave front, it follows that $\cos^{-1}\sqrt{1-\nu^2}$ is the angle of inclination of sound approach with reference to the plane tangent to the earth at the origin $P_o$.

**3. Application to Field Operations.**—The above theoretical deductions are immediately applicable to sound ranging technique provided the restrictions for a unique mathematical solution of the equations are observed. In equations (2) above, we have the four variables $\lambda$, $\mu$, $\nu$, and $V$, but by use of the trigonometrical relation $\lambda^2 + \mu^2 + \nu^2 = 1$, we can reduce the number of variables to three. Hence three detector microphones are required, one at each of the points $P_1$, $P_2$, $P_3$, and an additional microphone at $P_0$, to determine the time $T_0$ that the sound wave plane passed the origin. As pointed out in subsection (1), the positions of these microphones must be such that $D \neq 0$; in other words, all four microphones must not lie in the same plane.

In addition there are a few other limitations on the placement of the microphones. In the analytical treatment above it was assumed that the direction of the normal to the plane remained fixed, and that the velocity $V$ along the normal remained constant, over the dimensions of the tetrahedron formed by $P_0$, $P_1$, $P_2$, $P_3$. Thus, in order that the assumption of a plane wave front be justified and that the velocity not vary over the detector spread, the dimensions of the space array should be small compared with the range over which the sound is propagated. On the other hand, the detector microphones should be placed far enough apart so that the time differences so secured will be large compared with the errors in determining the actual time of arrival of the wave front. Also it is desirable, though admittedly not always feasible, that the vertical dimension of the tetrahedron formed by the four microphones be as nearly comparable as possible with the horizontal distances between detectors, for the same reasons as cited above. It thus follows that for a given range and for a given set of recording equipment there is an *optimum space array* whose minimum dimensions are set by the accuracy of the equipment in recording the time of arrivals and by the accuracy to which the record can be read, and whose maximum dimensions are set by the range over which operations are made, so that the assumption of a plane wave and invariance of its direction and velocity over the dimensions of the space array be a justifiable one.

SOUND RANGING FOR ARTILLERY

## III. THE THEORETICAL VELOCITY OF SOUND

4. **The Velocity of Sound in Dry Air.**—It has been shown by theoretical considerations, and substantiated by experiment, that the velocity of sound in a gas is given by

$$a = \sqrt{\gamma \frac{p}{\rho}}\ , \quad \gamma = \frac{c_p}{c_v}\ ,$$

where $p$ is the pressure, $\rho$ the density of the gas, and $c_p$ and $c_v$ are the specific heats of the gas at constant pressure and at constant volume, respectively. If now the gas is an ideal gas, then its equation of state is given by $p = \rho R T$ , where $T$ is the absolute temperature and $R$ is the gas constant appropriate to the gas. We thus have

$$a = \sqrt{\gamma R T}\ . \tag{7}$$

Air is essentially a mixture of gases, of which oxygen and nitrogen account for about 99%, and apart from variable water vapor content and localized pollution in industrial regions, we can regard air (at least in the lower atmosphere) as a uniform mixture and an ideal gas, having the same equation of state as a single ideal gas, but a gas constant which is a properly weighted arithmetic mean of the individual gas constants of the gases present. Thus we have $\gamma = 1.403$ and $R = 2.8703 \times 10^6$ c.g.s. units for dry air,[1] so that the velocity of sound at 0° C. as given by equation (7) above is 331.57 meters per second, or 362.61 yards per second.

5. **Effect of Water Vapor on the Velocity of Sound.**—Let $e$ be the partial pressure of the water vapor; then the equation of state for water vapor can be written

$$e = \frac{R'}{m_w} \rho_w T, \tag{8}$$

where $m_w = 18$, the molecular weight of water vapor, $\rho_w$ = the density of water vapor, and $T$ the absolute temperature. Similarly, the equation of state of absolutely dry air can be written

$$p_d = \frac{R'}{m_d} \rho_d T, \tag{9}$$

where $m_d = 28.98$, the subscript $d$ referring to air completely devoid of moisture.

From the point of view of the meteorologist, it is convenient to express the amount of moisture in the atmosphere by a quantity $q$, known as the *specific humidity*, and defined by the relation

$$q = \frac{\rho_w}{\rho_w + \rho_d}\ . \tag{10}$$

[1] Values taken from Brunt, *Physical and Dynamical Meteorology*, [illegible] pp. [illegible] and [illegible].

Substituting for $\rho_w$ and $\rho_d$ from equations (8) and (9) we have

$$q = \frac{\rho_w}{\rho_w + \rho_d} = \frac{e\, m_w}{R'\, T} \frac{1}{\frac{e\, m_w}{R'\, T} + \frac{p_d\, m_d}{R'\, T}}$$

$$= \frac{e\, m_w}{e\, m_w + p_d\, m_d} = \frac{e\, m_w}{e\, m_w + (p-e)\, m_d} \quad ,$$

$p$ being the actual atmospheric pressure and equal to $p_d + e$. We can write further

$$q = \frac{m_w}{m_d} \frac{e}{p + e\frac{m_w - m_d}{m_d}} \quad ,$$

and since $\frac{m_w}{m_d} = .6221$ , or $\frac{5}{8}$ approximately,

$$q = \frac{5}{8} \frac{e}{p - \frac{3}{8} e} \quad . \qquad (11)$$

But

$$p = e + p_d = \frac{R'}{m_w} \rho_w\, T + \frac{R'}{m_d} \rho_d\, T$$

$$= \frac{R'\, T}{m_d} \left[\frac{m_d}{m_w} \rho_w + \rho_d\right] \quad .$$

and since $\rho = \rho_d + \rho_w$ , we have

$$p = \frac{R' \rho T}{m_d} \left[1 + \left(\frac{m_d}{m_w} - 1\right) \frac{\rho_w}{\rho}\right]$$

$$= \frac{R' \rho T}{m_d} [1 + 0.6q]. \qquad (12)$$

Thus for air at temperature $T$ having a quantity $q$ of water vapor present we can use the same equation of state as for perfectly dry air at a temperature $T'$, where $T' = T(1 + 0.6q)$. The quantity $T'$ as here defined is the *virtual temperature.*

For an atmosphere not completely devoid of moisture we can therefore write for the velocity of sound,

$$a = \sqrt{\gamma_h RT\, (1 + 0.6q)} \quad , \qquad (13)$$

where $\gamma_h$ is the proper ratio of the specific heats for the moist air mass considered. If the water vapor is uniformly distributed throughout the air mass, we can write

$$\gamma_h = \frac{\gamma_d \frac{\rho_d}{m_d} + \gamma_w \frac{\rho_w}{m_w}}{\frac{\rho_d}{m_d} + \frac{\rho_w}{m_w}}$$

where the subscripts d and w refer to *dry air* and *water vapor*, respectively, at temperature $T$. The ratios $\rho_d/m_d$, $\rho_w/m_w$ express the number of moles of dry air and water vapor, respectively, contained in unit volume of atmosphere. Now by substitution

$$a = \sqrt{\gamma_h \frac{p_d + e}{\rho}} = \sqrt{\left(\gamma_d \frac{\rho_d}{m_d} + \gamma_w \frac{\rho_w}{m_w}\right) \frac{R'T}{\rho}}$$

$$= \sqrt{\left(\gamma_d \rho_d + \gamma_w \frac{m_d}{m_w}\rho_w\right) \frac{1}{\rho} \frac{R'}{m_d}T}$$

$$= \sqrt{\left(\gamma_d \frac{\rho - \rho_w}{\rho} + \gamma_w \frac{m_d}{m_w} \frac{\rho_w}{\rho}\right) \frac{R'}{m_d}T}$$

$$= \sqrt{\gamma_d \left[1 + \left(\frac{\gamma_w}{\gamma_d} \frac{m_d}{m_w} - 1\right) q\right] \frac{R'}{m_d}T} \; . \qquad (14)$$

In the absence of the value of $\gamma_w$ at atmospheric temperatures, we can use its value 1.324 at 100° C. for the purpose of determining the coefficient of $q$ in (14). Thus

$$\frac{\gamma_w}{\gamma_d} \frac{m_d}{m_w} - 1 = \frac{1.324}{1.403} \frac{8}{5} - 1 = 0.5099 \; .$$

Hence

$$a = \sqrt{\gamma_d \frac{R'}{m_d}T \,(1 + 0.5099q)} \; . \qquad (15)$$

Equation (15) expresses the velocity of sound as a function of the constants $\gamma_d$ and $R'/m_d$ for dry air, and the variables atmospheric temperature $T$ and specific humidity $q$. We can re-write (15) as

$$a = \sqrt{\gamma_d \frac{R'}{m_d}T \,[(1 + 0.6q) - 0.0901q]}$$

$$= \sqrt{\gamma_d \frac{R'}{m_d}(T' - 0.0901qT)} \; .$$

where $T'$ is the virtual temperature as defined by (12), and $-0.0901qT$ the correction term due to the slight variation of $\gamma$ with varying quantity of water vapor in the moist air mass.

The specific humidity $q$ in the atmosphere will at most be .04 (this in a saturated atmosphere at temperatures around 100° F.), generally about .015 at normal summer temperatures and humidity, and a decidedly smaller value the lower the temperature, pressure or humidity. The correction term $0.0901qT$ on $T'$ will therefore be very small. For $q = .04$, the most extreme case, the correction is $1.1^{\circ}$ C. for an air mass completely saturated at $38^{\circ}$ C. at sea level pressure. This corresponds to an error of about two feet per second in the velocity of sound, or an error of .18 of 1%. At normal summer temperatures and humidities the error is about $0.4^{\circ}$ C., or about 0.068 of 1% in the velocity of sound. This error steadily decreases with decrease of temperature, humidity and/or pressure.

It is therefore obvious that, to a good approximation, we can use for the equation expressing the velocity of sound as a function of temperature and humidity,

$$a = \sqrt{\gamma RT\ (1 + 0.6q)},$$

where $\gamma$ and $R$ have the values given in sub-section (*) above and $T(1 + 0.6q)$ is the virtual temperature, a quantity much used in meteorological calculations. Since $q = q_{max}\ r$, where $q_{max}$ is the *saturation* specific humidity at temperature $T$ (and existing pressure), and $r$ the relative humidity, we can write

$$a = \sqrt{\gamma RT\ (1 + 0.6q_{max}\ r)}, \quad (16)$$

or

√20.0561   a = 20.0561 √T'

$$a = 21.946\ \sqrt{T'}\ \text{yards per second.} \quad (17)$$

Equation (16) is the most practical expression to use in connection with data secured by atmospheric soundings, inasmuch as the quantities $T$, $q_{max}$, (a function of $T$ and $p$ only), and $r$ are readily computed.[a] At best the correction on the velocity of sound for the presence of water vapor in the atmosphere is a negligible one. It is to be noted, however, that since the gradient of humidity in the atmosphere may be of considerable magnitude in certain instances, the additive effect of the humidity gradient to the temperature gradient is to be considered in connection with the refractive properties of the atmosphere. Equation (16) indicates that the application of the humidity correction is a very simple procedure. As a matter of fact, the proper evaluation of elevations from atmospheric soundings requires the virtual temperature correction, and therefore the data required for equation (16) is immediately at hand. Atmospheric charts are available with the virtual temperature correction indicated thereon.

For convenience of reference, a chart of the virtual temperature correction, $0.6\,T\,q_{max}$, for saturated air at given values of temperature and pressure, is given in Appendix A, page 67, of this report.

[a] It is to be noted that $q = \dfrac{\rho_w}{\rho_w + \rho_d}$ expresses the number of grams of water vapor per gram of moist air, whereas in synoptic meteorology it is customary to express $q$ as the number of grams of water vapor per *kilogram* of moist air.

## IV. EQUATIONS OF MOTION FOR THE PROPAGATION OF SOUND

6. **Differential Equations of a Sound Ray.**—It has been shown in the above that the velocity of sound in still air is essentially a function of temperature. Since temperature may vary over all space, then in general at a point $P: x, y, z$ the velocity of sound is given by $a(x, y, z)$. If the air is in motion there are to be considered in addition the components of the velocity of the medium, given by $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$. If $\lambda$, $\mu$, $\nu$ are the direction cosines of the normal to the wave surface at $P: x, y, z$, then we have for the equations of motion, as indicated in Figure 3,

$$\frac{dx}{dt} = \lambda a + u; \qquad \frac{dy}{dt} = \mu a + v; \qquad \frac{dz}{dt} = \nu a + w . \tag{18}$$

Equations (18) represent, in general, the differential equations of a family of space curves[3] in the atmosphere. We can define a *sound ray* as one member of this family of space curves, where each member has an appropriate "ground value" $\lambda_0$, $\mu_0$, $\nu_0$, and a unique variation of $\lambda$, $\mu$, $\nu$ along the ray.

7. **Variation of Direction of Propagation along the Sound Ray.**—The space variations of $a$, $u$, $v$, and $w$ can be determined by sufficiently dense meteorological soundings of the atmosphere. However, to integrate equations (18) we also need to know the variations of $\lambda$, $\mu$, $\nu$ with $x$, $y$, $z$ along the ray.[4] To this

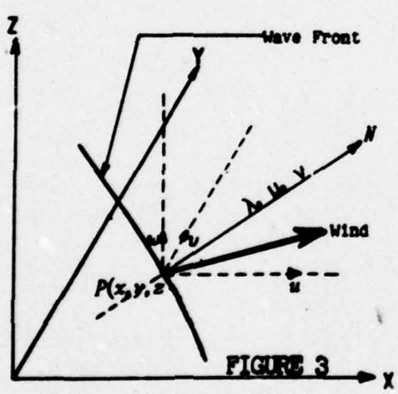


FIGURE 3

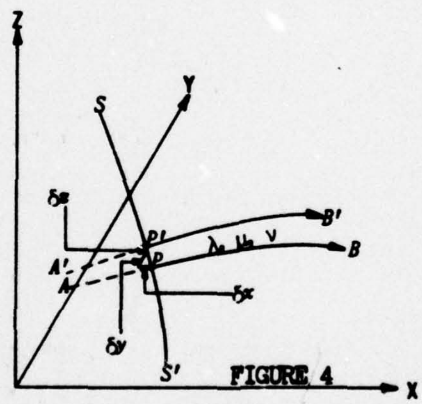


FIGURE 4

end consider two neighboring sound rays, as $AB$ and $A'B'$ in Figure 4, an infinitesimal distance apart. Let $SS'$ represent the surface of the wave front, and $P: x, y, z$ and $P': x + \delta x, y + \delta y, z + \delta z$ two neighboring points on the wave surface at time $t$. Then we have for an infinitesimal variation over the surface,

[3] More accurately stated, these curves are space curves on condition that $a$ and $u$, $v$, $w$ are everywhere continuous over the region traversed by each curve. Wherever $a$ and/or $u$, $v$, $w$ are discontinuous, the space curve will have an angular point; and therefore may, by analogy with the two-dimensional case, be called a broken space curve.

[4] For an analogous treatment of the general equations of sound propagation, see Stewart and Lindsay, *Acoustics*, 1930, pp. 312-316.

$$\left.\begin{aligned}
\frac{d(x+\delta x)}{dt} &= \lambda a + u + \delta(\lambda a + u) = \frac{dx}{dt} + \frac{d}{dt}(\delta x) \\
\frac{d(y+\delta y)}{dt} &= \mu a + v + \delta(\mu a + v) = \frac{dy}{dt} + \frac{d}{dt}(\delta y) \\
\frac{d(z+\delta z)}{dt} &= \nu a + w + \delta(\nu a + w) = \frac{dz}{dt} + \frac{d}{dt}(\delta z)
\end{aligned}\right\} \qquad (19)$$

Subtracting corresponding pairs of equations in (18) and (19), we have

$$\left.\begin{aligned}
\frac{d}{dt}(\delta x) &= \delta(\lambda a + u) = \delta x \frac{\partial}{\partial x}(\lambda a + u) + \delta y \frac{\partial}{\partial y}(\lambda a + u) + \delta z \frac{\partial}{\partial z}(\lambda a + u) \\
\frac{d}{dt}(\delta y) &= \delta(\mu a + v) = \delta y \frac{\partial}{\partial y}(\mu a + v) + \delta z \frac{\partial}{\partial z}(\mu a + v) + \delta x \frac{\partial}{\partial x}(\mu a + v) \\
\frac{d}{dt}(\delta z) &= \delta(\nu a + w) = \delta z \frac{\partial}{\partial z}(\nu a + w) + \delta x \frac{\partial}{\partial x}(\nu a + w) + \delta y \frac{\partial}{\partial y}(\nu a + w)
\end{aligned}\right\} \qquad (20)$$

Differentiating the relation

$$\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z = 0 \qquad (21)$$

with respect to time,

$$\frac{d\lambda}{dt}\delta x + \frac{d\mu}{dt}\delta y + \frac{d\nu}{dt}\delta z + \lambda\frac{d}{dt}(\delta x) + \mu\frac{d}{dt}(\delta y) + \nu\frac{d}{dt}(\delta z) = 0. \qquad (22)$$

Substituting from (20) into (22), we have

$$\begin{aligned}
&\frac{d\lambda}{dt}\delta x + \frac{d\mu}{dt}\delta y + \frac{d\nu}{dt}\delta z + \lambda\,\delta x\frac{\partial}{\partial x}(\lambda a + u) + \lambda\,\delta y\frac{\partial}{\partial y}(\lambda a + u) + \lambda\,\delta z\frac{\partial}{\partial z}(\lambda a + u) \\
&\quad + \mu\,\delta y\frac{\partial}{\partial y}(\mu a + v) + \mu\,\delta z\frac{\partial}{\partial z}(\mu a + v) + \mu\,\delta x\frac{\partial}{\partial x}(\mu a + v) \\
&\qquad + \nu\,\delta z\frac{\partial}{\partial z}(\nu a + w) + \nu\,\delta x\frac{\partial}{\partial x}(\nu a + w) + \nu\,\delta y\frac{\partial}{\partial y}(\nu a + w) = 0. \qquad (23)
\end{aligned}$$

We can write (23) as

$$\begin{aligned}
&\frac{d\lambda}{dt}\delta x + \lambda\,\delta x\frac{\partial}{\partial x}(\lambda a + u) + \mu\,\delta x\frac{\partial}{\partial x}(\mu a + v) + \nu\,\delta x\frac{\partial}{\partial x}(\nu a + w) \\
&\quad + \frac{d\mu}{dt}\delta y + \lambda\,\delta y\frac{\partial}{\partial y}(\lambda a + u) + \mu\,\delta y\frac{\partial}{\partial y}(\mu a + v) + \nu\,\delta y\frac{\partial}{\partial y}(\nu a + w) \\
&\qquad + \frac{d\nu}{dt}\delta z + \lambda\,\delta z\frac{\partial}{\partial z}(\lambda a + u) + \mu\,\delta z\frac{\partial}{\partial z}(\mu a + v) + \nu\,\delta z\frac{\partial}{\partial z}(\nu a + w) = 0. \qquad (24)
\end{aligned}$$

We can write $\frac{\partial}{\partial x}(\lambda a + u) = \frac{\partial}{\partial x}(\lambda a) + \frac{\partial u}{\partial x}$, and similarly for corresponding expressions in (24), so that

$$\frac{d\lambda}{dt}\delta x + \delta x\,[\lambda \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x} + \nu \frac{\partial w}{\partial x} + \lambda \frac{\partial}{\partial x}(\lambda a) + \mu \frac{\partial}{\partial x}(\mu a) + \nu \frac{\partial}{\partial x}(\nu a)]$$

$$+ \frac{d\mu}{dt}\delta y + \delta y\,[\lambda \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial y} + \nu \frac{\partial w}{\partial y} + \lambda \frac{\partial}{\partial y}(\lambda a) + \mu \frac{\partial}{\partial y}(\mu a) + \nu \frac{\partial}{\partial y}(\nu a)]$$

$$+ \frac{d\nu}{dt}\delta z + \delta z\,[\lambda \frac{\partial u}{\partial z} + \mu \frac{\partial v}{\partial z} + \nu \frac{\partial w}{\partial z} + \lambda \frac{\partial}{\partial z}(\lambda a) + \mu \frac{\partial}{\partial z}(\mu a) + \nu \frac{\partial}{\partial z}(\nu a)] = 0. \quad (25)$$

It is possible to write $\lambda \frac{\partial}{\partial x}(\lambda a) + \mu \frac{\partial}{\partial x}(\mu a) + \nu \frac{\partial}{\partial x}(\nu a)$ more simply.
Consider $a = f(x, y, z)$ and $\lambda = F(a)$; then $\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial a}\frac{\partial a}{\partial x}$. Hence

$$\lambda \frac{\partial}{\partial x}(\lambda a) + \mu \frac{\partial}{\partial x}(\mu a) + \nu \frac{\partial}{\partial x}(\nu a)$$

$$= \lambda\,[a \frac{\partial \lambda}{\partial a}\frac{\partial a}{\partial x} + \lambda \frac{\partial a}{\partial x}] + \mu\,[a \frac{\partial \mu}{\partial a}\frac{\partial a}{\partial x} + \mu \frac{\partial a}{\partial x}] + \nu\,[a \frac{\partial \nu}{\partial a}\frac{\partial a}{\partial x} + \nu \frac{\partial a}{\partial x}]$$

$$= \frac{\partial a}{\partial x} + a\,[\lambda \frac{\partial \lambda}{\partial a}\frac{\partial a}{\partial x} + \mu \frac{\partial \mu}{\partial a}\frac{\partial a}{\partial x} + \nu \frac{\partial \nu}{\partial a}\frac{\partial a}{\partial x}]$$

$$= \frac{\partial a}{\partial x}\,[1 + a\,(\lambda \frac{\partial \lambda}{\partial a} + \mu \frac{\partial \mu}{\partial a} + \nu \frac{\partial \nu}{\partial a})] = \frac{\partial a}{\partial x}$$

Substituting in (25),

$$\delta x\,[\frac{d\lambda}{dt} + \frac{\partial a}{\partial x} + \lambda \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x} + \nu \frac{\partial w}{\partial x}]$$

$$+ \delta y\,[\frac{d\mu}{dt} + \frac{\partial a}{\partial y} + \lambda \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial y} + \nu \frac{\partial w}{\partial y}]$$

$$+ \delta z\,[\frac{d\nu}{dt} + \frac{\partial a}{\partial z} + \lambda \frac{\partial u}{\partial z} + \mu \frac{\partial v}{\partial z} + \nu \frac{\partial w}{\partial z}] = 0. \quad (26)$$

Equation (26) must satisfy the relation (21), so that the terms in the brackets in (26) are proportional to λ, μ, ν, respectively; or

$$\left.\begin{aligned} &\frac{1}{\lambda}\,[\frac{d\lambda}{dt} + \frac{\partial a}{\partial x} + \lambda \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x} + \nu \frac{\partial w}{\partial x}] \\ &= \frac{1}{\mu}\,[\frac{d\mu}{dt} + \frac{\partial a}{\partial y} + \lambda \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial y} + \nu \frac{\partial w}{\partial y}] \\ &= \frac{1}{\nu}\,[\frac{d\nu}{dt} + \frac{\partial a}{\partial z} + \lambda \frac{\partial u}{\partial z} + \mu \frac{\partial v}{\partial z} + \nu \frac{\partial w}{\partial z}] \end{aligned}\right\} \quad (27)$$

Equations (27) give the variation of $\lambda$, $\mu$, $\nu$ along the sound ray in terms of the gradients of $a$ and $u$, $v$, $w$, and therefore implicitly in terms of $x$, $y$, $z$. The six equations (18) and (27) are sufficient to determine the position in space of all sound rays emanating from a given source of sound. It follows, then, that the source of sound can be located as the intersection of any two particular sound rays.

In Section II a method was indicated whereby the boundary values $\lambda_0$, $\mu_0$, $\nu_0$ could be determined for the particular sound wave front incident over the detector spread. It is the purpose of the following sections to suggest a technique whereby equations (18) and (27) can be rendered towards an explicit solution, once the "ground values" $\lambda_0$, $\mu_0$, $\nu_0$ are known.

### V. THE PROPAGATION OF SOUND IN A STRATIFIED ATMOSPHERE

It was mentioned in Section IV that the source of sound can be located as the intersection of any two particular sound rays. The nature of equations (27) however, makes it difficult, or at least very tedious, to solve two sets of six equations each, as given by (18) and (27).

A decidedly simpler approach to the problem is to trace each particular sound ray through the atmosphere, making use of the law of refraction and the variation of $a$, $u$, $v$, and $w$ with space (*meteorological corrections*), and thereby determine the distance of the sound source from the space array and the angular correction to be applied to the observed azimuth of sound approach at the space array. A similar procedure for another space array suitably placed will determine a range and corrected azimuth for that array for the same sound source and the intersection of the two corrected azimuths will locate the sound source within narrow limits, *provided that due allowance is made for the difference in elevation of the sound source*[4] *and the origin of the reference coordinate system.*

**8. Equations of Motion in the Stratified Atmosphere.**—Let us assume that as a first approximation the atmosphere is stratified, or in other words horizontally homogeneous, in the lower portion of the atmosphere through which we desire a detailed analysis of sound propagation. Then $w = 0$ and $u$, $v$, and $a$ are functions of $z$ only, so that under these conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial a}{\partial x} = \frac{\partial a}{\partial y} = 0;$$

and accordingly equations (18) and (27) above simplify appreciably to

$$\frac{dx}{dt} = \lambda a + u; \quad \frac{dy}{dt} = \mu a + v; \quad \frac{dz}{dt} = \nu a;$$

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = \frac{1}{\mu}\frac{d\mu}{dt} = \frac{1}{\nu}\left[\frac{d\nu}{dt} + \frac{da}{dz} + \lambda\frac{du}{dz} + \mu\frac{dv}{dz}\right] \quad (28)$$

Integrating $\frac{1}{\lambda}\frac{d\lambda}{dt} = \frac{1}{\mu}\frac{d\mu}{dt}$, we have

$$\log \lambda = \log \mu + k,$$

or

$$\frac{\lambda}{\mu} = constant.$$

Referring to equation (6) in Section II, it is readily seen that the condition $\lambda/\mu = constant$ is simply the condition that *the azimuth of the normals to the wave front surface be the same at every point of the ray.*

[4] As determined from a contour map in the vicinity.

9. **The Law of Refraction.**—It is thus possible to further simplify our equations of motion by choosing a coordinate system such that for a particular ray $\mu = 0$; or in other words, so that the normals to the sound wave surface at every point of a particular sound ray be contained in any of the family of planes parallel to the $XZ$-plane. This involves simply a clockwise rotation[8] about the $Z$-axis of our previous coordinate system, of amount $\frac{\pi}{2} + \theta$, where $\theta = arc\ tan\ \lambda_0/\mu_0$. The direction cosines of the normal to the wave surface with reference to the new axes are $\sqrt{1-\nu^2}$, $0$, $\nu$, since $\nu$ remains unchanged during the specified rotation. Equations (28) can now be written

$$\frac{dx_\theta}{dt} = \sqrt{1-\nu^2}\, a + u_\theta;\quad \frac{dy_\theta}{dt} = v_\theta;\quad \frac{dz}{dt} = \nu a; \tag{29}$$

$$\frac{1}{\sqrt{1-\nu^2}} \frac{d}{dt}(\sqrt{1-\nu^2}) = \frac{1}{\nu}\left[\frac{d\nu}{dt} + \frac{da}{dz} + \sqrt{1-\nu^2}\,\frac{du_\theta}{dz}\right]$$

where the subscript $\theta$ refers to coordinates of, and components along, the new coordinate system.

Consider now the last equation in (29). This can be written as

$$\frac{d}{dt}\sqrt{1-\nu^2} = \frac{\sqrt{1-\nu^2}}{\nu}\frac{d\nu}{dt} + \frac{\sqrt{1-\nu^2}}{\nu}\frac{da}{dz} + \frac{1-\nu^2}{\nu}\frac{du_\theta}{dz}. \tag{30}$$

But

$$\frac{d}{dt}\sqrt{1-\nu^2} = \frac{1}{2}\frac{-2\nu}{\sqrt{1-\nu^2}}\frac{d\nu}{dt},$$

so that

$$\frac{d\nu}{dt} = -\frac{\sqrt{1-\nu^2}}{\nu}\frac{d}{dt}\sqrt{1-\nu^2}.$$

Substituting in (30) we obtain

$$\frac{d}{dt}\sqrt{1-\nu^2} = -\frac{1-\nu^2}{\nu^2}\frac{d}{dt}\sqrt{1-\nu^2} + \frac{\sqrt{1-\nu^2}}{\nu}\frac{da}{dz} + \frac{1-\nu^2}{\nu}\frac{du_\theta}{dz},$$

or

$$\frac{d}{dt}\sqrt{1-\nu^2} = \nu\sqrt{1-\nu^2}\left[\frac{da}{dz} + \sqrt{1-\nu^2}\,\frac{du_\theta}{dz}\right]. \tag{31}$$

[8] We shall adopt the convention that the positive $X$-axis of the coordinate system after rotation point in the direction towards which the sound is being propagated. This convention implies that $\sqrt{1-\nu^2}$ will always be positive independent of the sign of $\nu$.

From (29) we have $\frac{dz}{dt} = \nu a$.

Hence

$$\frac{d}{dz}\sqrt{1-\nu^2} = \frac{\sqrt{1-\nu^2}}{a}\left[\frac{da}{dz} + \sqrt{1-\nu^2}\,\frac{du_\oplus}{dz}\right] . \qquad (32)$$

Clearly

$$\frac{d}{dz}\left(\frac{a}{\sqrt{1-\nu^2}}\right) = \frac{\sqrt{1-\nu^2}\,\frac{da}{dz} - a\,\frac{d}{dz}\sqrt{1-\nu^2}}{1-\nu^2}$$

$$= \frac{1}{\sqrt{1-\nu^2}}\,\frac{da}{dz} - \frac{a}{1-\nu^2}\,\frac{d}{dz}\sqrt{1-\nu^2} . \qquad (33)$$

But from (32) we see that

$$\frac{du_\oplus}{dz} = \frac{a}{1-\nu^2}\,\frac{d}{dz}\sqrt{1-\nu^2} - \frac{1}{\sqrt{1-\nu^2}}\,\frac{da}{dz} ,$$

so that

$$-\frac{d}{dz}\,\frac{a}{\sqrt{1-\nu^2}} = \frac{du_\oplus}{dz} . \qquad (34)$$

Integrating we have

$$-\int_{z_0}^{z} d\left(\frac{a}{\sqrt{1-\nu^2}}\right) = \int_{u_\oplus]_0}^{u_\oplus]_z} du_\oplus ,$$

or

$$\frac{a_0}{\sqrt{1-\nu_0^2}} - \frac{a}{\sqrt{1-\nu^2}} = u_\oplus - [u_\oplus]_0 , \qquad (35)$$

where $a$, $\nu$, and $u_\oplus$ are functions of $z$.

Equation (35) is the general expression of the law of refraction when both the temperature and the veloci[illegible] of the medium are variable.

In Section II there was indicated a means of determining $V$, the *instantaneous velocity of sound at any point of the sound path,* knowing only the space coordinates of the detector microphones and the arrival times of the sound at each detector. If the normal has an angle of inclination to the horizontal $\arccos\sqrt{1-\nu^2}$ , then clearly the apparent velocity of sound along the horizontal will be $\frac{V}{\sqrt{1-\nu^2}}$ . The velocities $V$ and $a$ are thus related by the equation

$$\frac{V}{\sqrt{1-\nu^2}} = \frac{a}{\sqrt{1-\nu^2}} + u_\phi \ . \tag{36}$$

so that the law of refraction can be written

$$\frac{a}{\sqrt{1-\nu^2}} = \frac{V_0}{\sqrt{1-\nu_0^2}} - u_\phi \ . \tag{37}$$

For a particular sound wave front incident over a space array as detailed in Section II, the "ground values" $\lambda_0$, $\mu_0$, $\nu_0$, and $V_0$ are readily determined. Choosing the sound ray which passes through the origin of our coordinate system, we can apply equation (37) in determining the inclination to the horizontal of the normal to the wave surface at every point on the ray, provided that $a$ and $u_\phi$ are known functions of $z$. If there exists a level at which $\sqrt{1-\nu^2} = 1$, then the sound wave is totally refracted* at this level.

**10. General Solution for the Point of Total Refraction.**—Let us consider in detail the case where there is a level $Z_{max}$ such that $\arccos \sqrt{1-\nu^2} = 0$. Then for the particular ray we are tracing, we have the two boundary conditions

$$\lambda = \sqrt{1-\nu_0^2} \ ; \quad \mu = 0 \ ; \quad \nu = \nu_0 \ \text{at} \ z = 0,$$

and

$$\lambda = 1 \ ; \quad \mu = 0 \ ; \quad \nu = 0 \ \text{at} \ z = Z_{max} .$$

Letting $\sqrt{1-\nu^2} = 1$ in equation (37) we obtain

$$a = \frac{V_0}{\sqrt{1-\nu_0^2}} - u_\phi \ , \tag{38}$$

which is the implicit solution for the level of total refraction $Z_{max}$ .

We can now proceed to integrate our equations of motion* (29) as follows:

$$-\frac{dx_\phi}{dz} = \frac{\sqrt{1-\nu^2}}{\nu} + \frac{u_\phi}{\nu a}$$

so that

$$-X_\phi = \lim_{\varepsilon \to 0} \int_0^{Z_{max}-\varepsilon} \frac{\sqrt{1-\nu^2}}{\nu} dz + \lim_{\varepsilon \to 0} \int_0^{Z_{max}-\varepsilon} \frac{u_\phi \, dz}{\nu a} \ . \tag{39}$$

*The level of total refraction is the level at which the sound ray becomes horizontal as a result of bending of the sound ray toward a horizontal plane.

*The negative sign in the left-hand member of (39) is introduced to properly locate the position of the sound source in terms of the reference coordinate system appropriate to the azimuth of sound approach.

Similarly,

$$- Y_{\bullet} = \lim_{\varepsilon \to 0} \int_0^{Z_{max} - \varepsilon} \frac{v_{\bullet}\, dz}{\nu a} . \qquad (40)$$

It is to be noted that the integration with respect to $z$ is performed just up to the level $Z_{max}$ as the integrands in the expressions for $- X_{\bullet}$ and $- Y_{\bullet}$ are not defined for $Z_{max}$. The coordinates $(X_{\bullet}, Y_{\bullet}, Z_{max})$ represent the point of total refraction in the atmosphere of the sound wave surface whose normal has an azimuth θ, and which is incident to the plane $z = 0$ at $(0, 0, 0)$ at an angle $\arccos \nu_0$. It is to be noted that for sound rays incident to the plane $z = 0$ at $(0, 0, 0)$ at angles other than $\arccos \nu_0$, the point of total refraction will not be the same as for $\arccos \nu_0$; *in general, each sound ray will have its own characteristic point of total refraction.* Further discussion of total refraction of sound waves is left to a more complete analysis of the refraction equation in connection with audibility and zones of silence.

Equations (39) and (40) are not immediately integrable, since $a$, $u_{\bullet}$, and $v_{\bullet}$ are complicated functions of $z$, depending on the current state of the atmosphere as regards temperature, humidity, and wind. Moreover, ν is a function of these meteorological elements and hence also of $z$. It is quite obvious that, even if $a$, $u_{\bullet}$, $v_{\bullet}$, and ν were known functions of $z$, the integration of equations (39) and (40) would still be a hopeless task in the majority of cases.

**11. Generalizations Regarding Integration by Successive Layers.**— If we write our differential equations for sound propagation in another form, in particular, differentiating with respect to the cosine of the angle of inclination of the normal to the wave front (which varies with $a$ and $u_{\bullet}$ and therefore depends implicitly on $z$), we obtain a set of equations which are immediately integrable provided only slight and justifiable assumptions are made. We proceed as follows:

From (31) we have

$$\frac{d\sqrt{1-\nu^2}}{dt} = \nu\sqrt{1-\nu^2}\left[\frac{da}{dz} + \sqrt{1-\nu^2}\,\frac{du_{\bullet}}{dz}\right],$$

so that

$$dt = \frac{d\sqrt{1-\nu^2}}{\sqrt{1-(\sqrt{1-\nu^2})^2}\cdot\sqrt{1-\nu^2}\left[\frac{da}{dz} + \sqrt{1-\nu^2}\,\frac{du_{\bullet}}{dz}\right]} . \qquad (41)$$

If we choose our interval of integration so that $\frac{da}{dz}$ and $\frac{du_{\bullet}}{dz}$ are constant for this interval, then equation (41) is immediately integrable. Let $A$ represent the lower boundary, $B$ the upper boundary, of a layer of atmosphere having these properties. Then the time of travel of the sound wave along the unique path determined by the angle of inclination of the wave normal at the lower boundary of the layer and the values of temperature, humidity, and wind throughout the layer, is given by the integration of equation (41) above, between limits determined by the values of the variable of integration at the lower and upper boundaries of the given layer. Before integrating, however, we need to inspect the integrand for singularities within the interval of integration, and test such improper integrals for convergence.

If, further, equations (39) and (40) are expressed in a form similar to (41), the integration for the coordinates $X_{\bullet}$, $Y_{\bullet}$ of the point of total refraction can be accomplished by first integrating individually for each layer of constant $da/dz$ and $du_{\bullet}/dz$, and then summing the increments of these coordinates for the various layers traversed by the sound wave.

Let us first simplify the appearance of equation (41) by the substitution of new variables and parameters. Set $\eta = \sqrt{1-v^2}$, *the cosine of the angle of inclination of the wave front normal to the horizontal.* Furthermore, since the interval of integration is always chosen so that both $\frac{da}{dz}$ and $\frac{du_{\bullet}}{dz}$ are constant, we can set $\alpha \equiv \frac{da/dz}{du_{\bullet}/dz}$ so that $\alpha$ is likewise a constant for any interval of integration chosen in conformity with the above. The term $\frac{da}{dz}$ is the gradient of sound velocity in the vertical in a still atmosphere, and since $a = 21.946\sqrt{T'}$ as given in equation (17), it is clear that

$$\frac{da}{dz} = \frac{10.973}{\sqrt{T'}} \frac{dT'}{dz} , \tag{42}$$

where $T'$ is the virtual temperature, expressed in degrees absolute. The term $dT'/dz$ therefore combines the effect of both the vertical temperature and humidity gradients on the variation of sound velocity with height. Provided we substitute for $\sqrt{T'}$ in equation (42) a properly chosen mean value for the layer, it is evident that the condition $\alpha$ = constant will be very closely approximated by a layer of atmosphere with constant gradients of virtual temperature and wind in the vertical. Thus from the standpoint of the meteorologist the choice of the intervals of integration is greatly simplified; as a matter of fact, the present method of evaluating atmospheric soundings furnishes all data, excepting wind data, requisite to an immediate determination of $\alpha$, and hence also the intervals of integration for purposes of the proposed sound ranging technique contained herein.

We can now write equation (41) as

$$dt = \frac{1}{\frac{du_{\bullet}}{dz}} \frac{d\eta}{\sqrt{1-\eta^2}\;\eta(\alpha + \eta)} ,$$

or

$$\int_{t_A}^{t_B} dt = \frac{1}{\frac{du_{\bullet}}{dz}} \int_{\eta_A}^{\eta_B} \frac{d\eta}{\sqrt{1-\eta^2}\;\eta(\alpha + \eta)} . \tag{43}$$

The integrand is unbounded for $\eta=\pm 1$, $\eta=0$, $\eta=-\alpha$. If $\eta = 0$ at any point of the sound path, then in consideration of the assumption of horizontal homogeneity, it is obvious that the sound path must continue to extend vertically indefinitely. Clearly, then, the condition $\eta = 0$ does not enter into the solution for the sound source. In view of our choice of the reference coordinate system, $\eta$ satisfies the condition $0 < \eta \leq 1$ always. The condition $\eta = 1$ occurs only at a level of total refraction, at which the integrand of the right-hand member of equation (43) becomes unbounded; but on application of the Cauchy "$\mu$-test" for the convergence of an improper integral over a finite interval it can be shown that the integral in (43) converges absolutely, over any interval including $\eta = 1$, provided that at no point of that interval $\eta = -\alpha$. The condition $\eta = -\alpha$ can only exist when $\alpha$ is negative and $|\alpha| \leq 1$. Applying the Cauchy "$\mu$-test" it can be shown that if $\eta = -\alpha$ at a point over the interval of integration, then the integral in (43) does not converge absolutely. The significance of the condition $\eta = -\alpha$ will be discussed in a later section.

**12. Parametric Solution for the Time-Increment of Travel.**—Restricting ourselves to intervals at no point of which $\eta = -\alpha$, we can integrate equation (43). By use of formula #228 in Peirce's *A Short Table of Integrals*, we can expand the integrand in (43), as follows

$$\int \frac{d\eta}{\sqrt{1-\eta^2}\,\eta\,(\alpha+\eta)} = \frac{1}{\alpha}\int \frac{d\eta}{\eta\sqrt{1-\eta^2}} - \frac{1}{\alpha}\int \frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}}\,, \tag{44}$$

so that by formula #129 for the first term of the right-hand member in (44), and formulas #195 and #196 for the second term, we obtain

$$t_B - t_A = [\Delta t]_{AB} = \frac{1}{\frac{d\alpha}{dz}}\left[-\log\frac{1+\sqrt{1-\eta^2}}{\eta} - \begin{cases} \dfrac{1}{\sqrt{1-\alpha^2}}\log 2\,\dfrac{1+\alpha\eta-\sqrt{1-\alpha^2}\,\sqrt{1-\eta^2}}{\eta+\alpha}, & \text{if } |\alpha|<1 \\[2ex] -\dfrac{1}{\alpha}\dfrac{\sqrt{1-\eta^2}}{\eta+\alpha}, & \text{if } |\alpha|=1 \\[2ex] \dfrac{1}{\sqrt{\alpha^2-1}}\arcsin\dfrac{1+\alpha\eta}{\eta+\alpha}, & \text{if } |\alpha|>1. \end{cases}\right]_A^B \tag{45}$$

where $\Delta t_{AB}$ represents the length of time required for the sound wave front to pass through the layer, from the lower boundary *A* to the upper boundary *B*, along that particular path defined in differential form by equations (29). From the nature of the solution for the time of travel it is clear that the length of time is determined uniquely given only (*a*) the values of the angle of inclination of the wave front normal at the boundaries of the layer, (*b*) the values of the vertical gradients of virtual temperature and wind component in the direction $\theta$ throughout the layer, and (*c*) the mean virtual temperature of the layer.

**13. Parametric Solution for the Range-Increment.**—From (29) we have

$$\frac{dx_*}{dt} = \sqrt{1-\nu^2}\;a + u_* \,.$$

With the aid of equation (31) we can write[7]

$$\frac{-dx_\bullet}{d\,\sqrt{1-v^2}} = \frac{\sqrt{1-v^2}\;a + u_\bullet}{v\,\sqrt{1-v^2}\,(\frac{da}{dz} + \sqrt{1-v^2}\;\frac{du_\bullet}{dz})}\,,$$

or more conveniently,

$$-\,\frac{dx_\bullet}{d\eta} = \frac{1}{\frac{du_\bullet}{dz}}\;\frac{\eta a + u_\bullet}{\sqrt{1-\eta^2}\;\eta(\alpha + \eta)}\;. \qquad (46)$$

For the layer $AB$ we have

$$-\int_{X_{\bullet_A}}^{X_{\bullet_B}} dx_\bullet = \frac{1}{\frac{du_\bullet}{dz}} \int_{\eta_A}^{\eta_B} \frac{\eta a + u_\bullet}{\sqrt{1-\eta^2}\,\eta\,(\alpha + \eta)}\;d\eta\;. \qquad (47)$$

Obviously, before we can perform the integration indicated in (47), we need to express $a$ and $u_\bullet$ as functions of $\eta$. These relations are readily secured, since, by choice, $\frac{da}{dz} = C = constant$, and by equations (29),

$$\frac{dz}{dt} = \sqrt{1-\eta^2}\;\;a\;\;.$$

Thus

$$\frac{da}{dt} = C\,\sqrt{1-\eta^2}\;a$$

$$\frac{da}{a} = C\,\sqrt{1-\eta^2}\;dt$$

$$= C\,\sqrt{1-\eta^2}\cdot\frac{d\eta}{\sqrt{1-\eta^2}\;\eta(\frac{da}{dz} + \eta\,\frac{du_\bullet}{dz})}$$

$$= \alpha\,\frac{d\eta}{\eta\,(\alpha + \eta)}\;\;. \qquad (48)$$

For all intervals at no point of which $\eta = -\alpha$, equation (48) integrates readily to

$$log\;a = -\,log\,\frac{\alpha + \eta}{\eta} + log\;k, \qquad (49)$$

where $log\;k$ is the constant of integration. Clearly, (49) can be written

[7]See footnote 6, page (25).

more compactly as

$$a \frac{\alpha + \eta}{\eta} = k \; , \tag{50}$$

which expresses $a$ as a function of $\eta$ and of the constants $\alpha$ for the given layer and $k$ for the particular path in that layer.

Substituting in (50) from the refraction equation (37) we find

$$\left(\frac{V_0}{\eta_0} - u_\Theta\right)(\alpha + \eta) = k \; , \tag{51}$$

so that

$$u_\Theta = \frac{V_0}{\eta_0} - k \frac{1}{\alpha + \eta} \; . \tag{52}$$

Substituting in (47) for $a$ and $u_\Theta$ from (50) and (52) we have

$$\begin{aligned} -\int_{X_{\Theta A}}^{X_{\Theta B}} dx_\Theta &= \frac{1}{\frac{du_\Theta}{dz}} \int_{\eta_A}^{\eta_B} \frac{\frac{k\eta^2}{\alpha + \eta} + \frac{V_0}{\eta_0} - \frac{k}{\alpha + \eta}}{\sqrt{1-\eta^2}\,\eta(\alpha + \eta)}\, d\eta \\ &= \frac{V_0}{\eta_0}\frac{1}{\frac{du_\Theta}{dz}} \int_{\eta_A}^{\eta_B} \frac{d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha + \eta)} - \frac{k}{\frac{du_\Theta}{dz}} \int_{\eta_A}^{\eta_B} \frac{\sqrt{1-\eta^2}}{\eta(\alpha + \eta)^2}\, d\eta \; , \end{aligned} \tag{53}$$

provided the interval of integration does not include $\eta = -\alpha$. The first integral in the right-hand member is proportional to the time of travel along the sound path, and readily integrated, as indicated in (43) and (45). In order to integrate the second term in the right-hand member of (53) with the use of Peirce's Tables, we expand the integrand as the sum of three terms, as follows:

$$\frac{\sqrt{1-\eta^2}}{\eta(\alpha + \eta)^2} = \frac{1}{\alpha}\left[\frac{1}{\alpha}\frac{\sqrt{1-\eta^2}}{\eta} - \frac{1}{\alpha}\frac{\sqrt{1-\eta^2}}{\eta + \alpha} - \frac{\sqrt{1-\eta^2}}{(\eta + \alpha)^2}\right] \; ,$$

so that

$$\frac{-k}{\frac{du_\Theta}{dz}} \int \frac{\sqrt{1-\eta^2}\, d\eta}{\eta(\alpha + \eta)^2} = \frac{-k}{\frac{da}{dz}}\left[\frac{1}{\alpha}\int \frac{\sqrt{1-\eta^2}}{\eta}\, d\eta - \frac{1}{\alpha}\int \frac{\sqrt{1-\eta^2}}{\eta + \alpha}\, d\eta - \int \frac{\sqrt{1-\eta^2}}{(\eta + \alpha)^2}\, d\eta\right] \tag{54}$$

Peirce's *Tables* reduce these integrals to

$$\frac{-k}{\frac{du_e}{dz}}\int\frac{\sqrt{1-\eta^2}}{\eta(\alpha+\eta)^2}\,d\eta = \frac{-k}{\frac{d\alpha}{dz}}\left\{\frac{1}{\alpha}\left[\sqrt{1-\eta^2} - \log\frac{1+\sqrt{1-\eta^2}}{\eta}\right] - \frac{1}{\alpha}\int\frac{\sqrt{1-\eta^2}}{\eta+\alpha}\,d\eta \right.$$

$$\left. - \left[\frac{-\sqrt{1-\eta^2}}{\eta+\alpha} + \alpha\int\frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} - \int\frac{d\eta}{\sqrt{1-\eta^2}}\right]\right\}$$

$$= \frac{-k}{\frac{d\alpha}{dz}}\left\{\frac{\sqrt{1-\eta^2}}{\alpha} - \frac{1}{\alpha}\log\frac{1+\sqrt{1-\eta^2}}{\eta} + \frac{1}{\alpha}\left[-\sqrt{1-\eta^2} - (1-\alpha^2)\int\frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}} - \alpha\int\frac{d\eta}{\sqrt{1-\eta^2}}\right]\right.$$

$$\left. - \left[\frac{-\sqrt{1-\eta^2}}{\eta+\alpha} + \alpha\int\frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}} - \int\frac{d\eta}{\sqrt{1-\eta^2}}\right]\right\} \tag{55}$$

$$= \frac{-k}{\frac{d\alpha}{dz}}\left\{\frac{1}{\alpha}\left[-\log\frac{1+\sqrt{1-\eta^2}}{\eta} - \int\frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}}\right] + \frac{\sqrt{1-\eta^2}}{\eta+\alpha}\right\}.$$

By means of equations (44) and (45) we can re-write equation (55) as

$$\frac{-k}{\frac{du_e}{dz}}\int\frac{\sqrt{1-\eta^2}}{\eta(\alpha+\eta)^2}\,d\eta = \frac{-k}{\frac{d\alpha}{dz}}\left[\frac{1}{\alpha}\frac{d\alpha}{dz}t + \frac{\sqrt{1-\eta^2}}{\eta+\alpha}\right]$$

$$= \frac{-k}{\alpha}t - \frac{k}{\frac{d\alpha}{dz}}\frac{\sqrt{1-\eta^2}}{\eta+\alpha} \tag{56}$$

Hence

$$-X_e = \frac{V_0}{\eta_0}t - \frac{k}{\alpha}t - \frac{k}{\frac{d\alpha}{dz}}\frac{\sqrt{1-\eta^2}}{\eta+\alpha}. \tag{57}$$

Substituting the limits of $\eta$ for the layer $AB$, we have

$$(\Delta X_e)_{AB} = -\left(\frac{V_0}{\eta_0} - \frac{k}{\alpha}\right)(\Delta t)_{AB} + \frac{k}{\frac{d\alpha}{dz}}\left(\frac{\sqrt{1-\eta_B^2}}{\eta_B+\alpha} - \frac{\sqrt{1-\eta_A^2}}{\eta_A+\alpha}\right), \tag{58}$$

where $(\Delta X_e)_{AB} = X_{eB} - X_{eA}$. Equation (58) is the parametric solution for $\Delta X_e$, in terms of the parameters $V_0/\eta_0$ for the sound ray, $\alpha$ for the layer, and $k$ for the particular path in the layer. We shall refer to $(\Delta X_e)_{AB}$ and its equivalent, given by the right-hand member of (58), as the *range-increment*. Furthermore, we shall define $\alpha$, the ratio of sound velocity gradient to wind velocity gradient, as the *stratification parameter*. Lastly, we shall refer to $du_e/dz$ as the *refractive wind gradient*, which nomenclature is an obvious one.

Equation (57) can be written in another form by use of the solutions given in (50) and (51), from which we have

$$\frac{k}{\eta + \alpha} = \frac{a}{\eta}$$

and

$$\frac{k}{\alpha} = \left(\frac{V_0}{\eta_0} - u_\oplus\right)\left(1 + \frac{\eta}{\alpha}\right)$$

$$= \frac{V_0}{\eta_0} - u_\oplus + \left(\frac{V_0}{\eta_0} - u_\oplus\right)\frac{\eta}{\alpha}$$

$$= \frac{V_0}{\eta_0} - u_\oplus + \frac{a}{\alpha} \;. \qquad (59)$$

Substituting in (57) we obtain

$$- X_\oplus = \left(u_\oplus - \frac{a}{\alpha}\right)t - \frac{a}{\frac{da}{dz}}\frac{\sqrt{1-\eta^2}}{\eta} \;. \qquad (60)$$

Substituting as limits the values of the variables in (60) at the bottom $A$ and top $B$ of the layer, we have

$$[- X_\oplus]_A^B = \left(u_{\oplus_B} - \frac{a_B}{\alpha}\right)t_B - \left(u_{\oplus_A} - \frac{a_A}{\alpha}\right)t_A - \frac{a_B}{\frac{da}{dz}}\frac{\sqrt{1-\eta_B^2}}{\eta_B} + \frac{a_A}{\frac{da}{dz}}\frac{\sqrt{1-\eta_A^2}}{\eta_A} \;. \qquad (61)$$

But

$$a_B = a_A + \frac{da}{dz}\Delta z$$

$$u_{\oplus_B} = u_{\oplus_A} + \frac{du_\oplus}{dz}\Delta z.$$

Substituting in (61) we obtain

$$[- X_\oplus]_A^B = \left(u_{\oplus_A} + \frac{du_\oplus}{dz}\Delta z - \frac{a_A + \frac{da}{dz}\Delta z}{\alpha}\right)t_B$$

$$- \frac{a_A + \frac{da}{dz}\Delta z}{\frac{da}{dz}}\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \left(u_\oplus - \frac{a_A}{\alpha}\right)t_A + \frac{a_A}{\frac{da}{dz}}\frac{\sqrt{1-\eta_A^2}}{\eta_A} \qquad (62)$$

$$= \left(u_{\oplus_A} - \frac{a_A}{\alpha}\right)(t_B - t_A) - \frac{a_A}{\frac{da}{dz}}\left(\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{\sqrt{1-\eta_A^2}}{\eta_A}\right) - \Delta z\frac{\sqrt{1-\eta_B^2}}{\eta_B} \;.$$

But $\frac{da}{dz} = \frac{a_B - a_A}{\Delta z}$, and $\alpha = \frac{a_B - a_A}{u_{\oplus B} - u_{\oplus A}}$. Hence

$$[-X_\oplus]_A^B = (u_{\oplus A} - \frac{a_A}{\alpha})(\Delta t)_{AB} - (\frac{a_B}{a_B - a_A}\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{a_A}{a_B - a_A}\frac{\sqrt{1-\eta_A^2}}{\eta_A})\Delta z$$

$$= [u_{\oplus A} - \frac{a_A}{\frac{a_B - a_A}{u_{\oplus B} - u_{\oplus A}}}](\Delta t)_{AB} - [\frac{a_B}{a_B - a_A} tan(cos^{-1}\eta_B) - \frac{a_A}{a_B - a_A} tan(cos^{-1}\eta_A)]\Delta z, \quad (63)$$

or finally, if we set $(\Delta X_\oplus)_{AB} = X_{\oplus B} - X_{\oplus A}$,

$$(\Delta X_\oplus)_{AB} = -[\frac{a_B u_{\oplus A} - a_A u_{\oplus B}}{a_B - a_A}](\Delta t)_{AB} + [\frac{a_B}{a_B - a_A} tan(cos^{-1}\eta_B) - \frac{a_A}{a_B - a_A} tan(cos^{-1}\eta_A)]\Delta z . \quad (64)$$

Equation (64) expresses the range-increment as a function of the meteorological elements at the boundaries of the layer $AB$ of depth $\Delta z$, of the angle of inclination of the wave normal at the boundaries of the layer, and of the time of travel $\Delta t$ of the sound along this particular path through the layer.

**14. Parametric Solution for the Drift-Increment.**—For the corresponding change in the $y_\oplus$ coordinate, $(\Delta Y_\oplus)_{AB}$, which we can appropriately call the *drift-increment*, we have[8] from (29) and (31)

$$\frac{-dy_\oplus}{d\eta} = \frac{1}{\frac{du_\oplus}{dz}} \frac{v_\oplus}{\sqrt{1-\eta^2}\,\eta(\alpha + \eta)},$$

or

$$-\int_{Y_{\oplus A}}^{Y_{\oplus B}} dy_\oplus = \frac{1}{\frac{du_\oplus}{dz}} \int_{\eta_A}^{\eta_B} \frac{v_\oplus\, d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha + \eta)} . \quad (65)$$

Clearly, we need to know $v_\oplus$, the component of wind velocity normal to the sound azimuth, as a function of $\eta$, the cosine of the angle of inclination of the wave normal. For a layer of constant $\alpha$, we can likewise determine $dv_\oplus/dz$, or the vertical gradient of drift velocity due to the wind. For the sake of simplicity and integrability, we further subdivide the layer of constant $\alpha$ into layers of constant $dv_\oplus/dz$. (It is to be noted, however, that wind direction and velocity determinations aloft are so crude at present, that in general a linear variation of $v_\oplus$ with elevation is perhaps the most refined wind structure analysis that can be made with ordinary means at our disposal.) We shall refer to $dv_\oplus/dz$ as the *drift wind gradient*. We have, then, as in the

[8] See footnote 6, page (25).

case of (48),

$$\frac{dv_e}{dt} = \frac{dv_e}{dz}\frac{dz}{dt}$$

$$= \frac{dv_e}{dz}\sqrt{1-\eta^2}\, a$$

$$= \frac{dv_e}{dz}\sqrt{1-\eta^2}\,\frac{k\eta}{\alpha+\eta} \quad . \tag{66}$$

But from (43) we can write (66) as

$$dv_e = k\,\frac{\frac{dv_e}{dz}}{\frac{du_e}{dz}}\,\frac{d\eta}{(\alpha+\eta)^2} \quad . \tag{67}$$

Integrating we find

$$v_e = \frac{-k\,\frac{dv_e}{dz}}{\frac{du_e}{dz}}\,\frac{1}{\alpha+\eta} + \sigma \; , \tag{68}$$

where $\sigma$ is the constant of integration. Equation (68) expresses the drift velocity $v_e$ as a function of the change of wind with altitude, the refractive wind $u_e$, and the parameter $\sigma$ for the particular path in the layer characterized by stratification parameter $\alpha$ and the constant $V_0/\eta_0$ for the sound ray.

Equation (68) is the desired relation for substitution into (65), so that

$$-\int dy_e = \frac{1}{\frac{du_e}{dz}}\int \frac{\frac{dv_e/dz}{du_e/dz}\,\frac{-k}{\alpha+\eta} + \sigma}{\sqrt{1-\eta^2}\,\eta(\alpha+\eta)}\,d\eta$$

$$= \frac{\sigma}{\frac{du_e}{dz}}\int \frac{d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha+\eta)} - \frac{k\,\frac{dv_e}{dz}}{\left(\frac{du_e}{dz}\right)^2}\int \frac{d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha+\eta)^2}$$

$$= \sigma t - \frac{k\,\frac{dv_e}{dz}}{\left(\frac{du_e}{dz}\right)^2}\int \frac{d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha+\eta)^2} \quad . \tag{69}$$

We expand the integrand in the last term of (69) in order to integrate, as follows:

$$\int \frac{d\eta}{\sqrt{1-\eta^2}\,\eta(\alpha+\eta)^2} = \frac{1}{\alpha^2}\int \frac{d\eta}{\eta\sqrt{1-\eta^2}} - \frac{1}{\alpha^2}\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} - \frac{1}{\alpha}\int \frac{d\eta}{(\eta+\alpha)^2\sqrt{1-\eta^2}}$$

$$= \frac{1}{\alpha^2}\int \frac{d\eta}{\eta\sqrt{1-\eta^2}} - \frac{1}{\alpha^2}\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} - \frac{1}{\alpha}\cdot\left[\begin{array}{l} \dfrac{-\sqrt{1-\eta^2}}{(1-\alpha^2)(\eta+\alpha)} - \dfrac{\alpha}{1-\alpha^2}\displaystyle\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}}, \quad \text{if } |\alpha|\neq 1; \\ \\ -\dfrac{1}{3}\dfrac{\sqrt{1-\eta^2}}{\alpha(\eta+\alpha)^2} + \dfrac{1}{3\alpha}\displaystyle\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}}, \quad \text{if } |\alpha| = 1. \end{array}\right]$$

$$= \frac{1}{\alpha^2}\left[-\log\frac{1+\sqrt{1-\eta^2}}{\eta}\right] - \left[\begin{array}{l} \left(\dfrac{1}{\alpha^2} - \dfrac{1}{1-\alpha^2}\right)\displaystyle\int \frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}} - \dfrac{1}{\alpha}\dfrac{1}{1-\alpha^2}\dfrac{\sqrt{1-\eta^2}}{\eta+\alpha}, \quad \text{if } |\alpha|\neq 1; \\ \\ \dfrac{4}{3\alpha^2}\displaystyle\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} - \dfrac{1}{3\alpha^2}\dfrac{\sqrt{1-\eta^2}}{(\eta+\alpha)^2}, \quad \text{if } |\alpha| = 1. \end{array}\right] \quad (70)$$

Now

$$\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} = \left[\begin{array}{l} \dfrac{1}{\sqrt{1-\alpha^2}}\log 2\,\dfrac{1+\alpha\eta-\sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha}, \quad \text{if } |\alpha|<1, \\ \\ -\dfrac{1}{\alpha}\dfrac{\sqrt{1-\eta^2}}{\eta+\alpha}, \quad \text{if } |\alpha|=1, \\ \\ \dfrac{1}{\sqrt{\alpha^2-1}}\arcsin\dfrac{1+\alpha\eta}{\eta+\alpha}, \quad \text{if } |\alpha|>1. \end{array}\right] \quad (71)$$

Quite compactly, with the aid of (45), we can write

$$\int \frac{d\eta}{(\eta+\alpha)\sqrt{1-\eta^2}} = \frac{-d\alpha}{dz}t - \log\frac{1+\sqrt{1-\eta^2}}{\eta}. \quad (72)$$

Hence for $|\alpha| \neq 1$,

$$\int \frac{d\eta}{\sqrt{1-\eta^2}\ \eta(\alpha+\eta)^2} = \frac{1}{1-\alpha^2}\left[-\log\frac{1+\sqrt{1-\eta^2}}{\eta} + \frac{1-2\alpha^2}{\alpha^2}\frac{d\alpha}{dz}t + \frac{1}{\alpha}\frac{\sqrt{1-\eta^2}}{\eta+\alpha}\right]$$

and for $|\alpha|=1$,

$$\int \frac{d\eta}{\sqrt{1-\eta^2}\ \eta(\alpha+\eta)^2} = \frac{1}{3\alpha^2}\left[\log\frac{1+\sqrt{1-\eta^2}}{\eta} + 4\frac{d\alpha}{dz}t + \frac{\sqrt{1-\eta^2}}{(\eta+\alpha)^2}\right] \qquad (73)$$

Substituting in (69) we find for $|\alpha| \neq 1$.

$$-\int dy_e = \sigma t - \frac{k\frac{dv_e}{dz}}{\left(\frac{du_e}{dz}\right)^2}\frac{1}{1-\alpha^2}\left[-\log\frac{1+\sqrt{1-\eta^2}}{\eta} + \frac{1-2\alpha^2}{\alpha^2}\frac{d\alpha}{dz}t + \frac{1}{\alpha}\frac{\sqrt{1-\eta^2}}{\eta+\alpha}\right]$$

and for $|\alpha|=1$,

$$-\int dy_e = \sigma t - \frac{k\frac{dv_e}{dz}}{\left(\frac{du_e}{dz}\right)^2}\frac{1}{3\alpha^2}\left[\log\frac{1+\sqrt{1-\eta^2}}{\eta} + 4\frac{d\alpha}{dz}t + \frac{\sqrt{1-\eta^2}}{(\eta+\alpha)^2}\right]. \qquad (74)$$

Hence for $|\alpha| \neq 1$,

$$(\Delta Y_e)_{AB} = -\sigma(\Delta t)_{AB} + k\frac{\frac{dv_e}{dz}}{\frac{du_e}{dz}}\frac{1}{1-\alpha^2}\left[\frac{1-2\alpha^2}{\alpha}(\Delta t)_{AB} - \frac{1}{\frac{d\alpha}{dz}}\left(\alpha\log\frac{1+\sqrt{1-\eta_B^2}}{1+\sqrt{1-\eta_A^2}}\frac{\eta_A}{\eta_B} - \frac{\sqrt{1-\eta_B^2}}{\eta_B+\alpha} + \frac{\sqrt{1-\eta_A^2}}{\eta_A+\alpha}\right)\right]$$

and for $|\alpha|=1$,

$$(\Delta Y_e)_{AB} = -\sigma(\Delta t)_{AB} + k\frac{\frac{dv_e}{dz}}{\frac{du_e}{dz}}\frac{1}{3\alpha^2}\left[4\alpha(\Delta t)_{AB} + \frac{1}{\frac{du_e}{dz}}\left(\log\frac{1+\sqrt{1-\eta_B^2}}{1+\sqrt{1-\eta_A^2}}\frac{\eta_A}{\eta_B} + \frac{\sqrt{1-\eta_B^2}}{(\eta_B+\alpha)^2} - \frac{\sqrt{1-\eta_A^2}}{(\eta_A+\alpha)^2}\right)\right]. \qquad (75)$$

It is to be noted that the above integrations hold for all intervals of $\eta$ provided that at no point of the interval $\eta = -\alpha$ .

Equations (75) are the parametric solution for the drift-increment for sound propagation through the layer $AB$. As in the case of the range-increment, we can write equations (74) for the drift-increment $\Delta Y_{\oplus}$ in terms of the meteorological elements at the boundaries of the layer. Thus we have from equations (51) and (68)

$$\sigma = v_{\oplus} + \frac{dv_{\oplus}}{du_{\oplus}}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right) . \tag{76}$$

Utilizing the relations (59) and (76) on substituting in (74) we obtain[9], for the case $|\alpha| \neq 1$,

$$\begin{aligned} -Y_{\oplus} &= v_{\oplus}\, t + \frac{dv_{\oplus}}{du_{\oplus}}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right) t - \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1-2\alpha^2}{1-\alpha^2}\,t \\ &\quad + \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\,\frac{1}{\frac{da}{dz}}\,\mathrm{sech}^{-1}\eta - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\,\frac{a}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta^2}}{\eta} \\ &= v_{\oplus}\, t - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a}{\alpha}\, t + \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\left[t + \frac{\mathrm{sech}^{-1}\eta}{\frac{da}{dz}}\right] \\ &\quad - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\,\frac{a}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta^2}}{\eta} . \end{aligned} \tag{77}$$

Substituting the values of the variables at the boundaries $A$ and $B$ of the layer, we have

$$\begin{aligned} \left[-Y_{\oplus}\right]_A^B &= v_{\oplus_B}\, t_B - v_{\oplus_A}\, t_A - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a_B}{\alpha}\, t_B + \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a_A}{\alpha}\, t_A \\ &\quad + \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\left[\left(\frac{V_0}{\eta_0} - u_{\oplus_B} + \frac{a_B}{\alpha}\right)\left(t_B + \frac{\mathrm{sech}^{-1}\eta_B}{\frac{da}{dz}}\right) - \left(\frac{V_0}{\eta_0} - u_{\oplus_A} + \frac{a_A}{\alpha}\right)\left(t_A + \frac{\mathrm{sech}^{-1}\eta_A}{\frac{da}{dz}}\right)\right] \\ &\quad - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\left[\frac{a_B}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{a_A}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_A^2}}{\eta_A}\right] . \end{aligned} \tag{78}$$

[9] The expression $\log \frac{1+\sqrt{1-\eta^2}}{\eta}$ is identical with $\mathrm{sech}^{-1}\eta$ .

Equations (75) are the parametric solution for the drift-increment for sound propagation through the layer *AB*. As in the case of the range-increment, we can write equations (74) for the drift-increment $\Delta Y_{\oplus}$ in terms of the meteorological elements at the boundaries of the layer. Thus we have from equations (51) and (68)

$$\sigma = v_{\oplus} + \frac{dv_{\oplus}}{du_{\oplus}}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right) . \tag{76}$$

Utilizing the relations (59) and (76) on substituting in (74) we obtain[9], for the case $|\alpha| \neq 1$,

$$\begin{aligned} -Y_{\oplus} &= v_{\oplus}\, t + \frac{dv_{\oplus}}{du_{\oplus}}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right) t - \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1-2\alpha^2}{1-\alpha^2}\,t \\ &\quad + \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\,\frac{1}{\frac{da}{dz}}\,\operatorname{sech}^{-1}\eta - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\,\frac{a}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta^2}}{\eta} \\ &= v_{\oplus}\, t - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a}{\alpha}\,t + \left(\frac{V_0}{\eta_0} - u_{\oplus} + \frac{a}{\alpha}\right)\frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\left[t + \frac{\operatorname{sech}^{-1}\eta}{\frac{da}{dz}}\right] \\ &\quad - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\,\frac{a}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta^2}}{\eta} . \end{aligned} \tag{77}$$

Substituting the values of the variables at the boundaries *A* and *B* of the layer, we have

$$\begin{aligned} [-Y_{\oplus}]_A^B &= v_{\oplus B}\, t_B - v_{\oplus A}\, t_A - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a_B}{\alpha}\,t_B + \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{a_A}{\alpha}\,t_A \\ &\quad + \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{\alpha^2}{1-\alpha^2}\left[\left(\frac{V_0}{\eta_0} - u_{\oplus B} + \frac{a_B}{\alpha}\right)\left(t_B + \frac{\operatorname{sech}^{-1}\eta_B}{\frac{da}{dz}}\right) - \left(\frac{V_0}{\eta_0} - u_{\oplus A} + \frac{a_A}{\alpha}\right)\left(t_A + \frac{\operatorname{sech}^{-1}\eta_A}{\frac{da}{dz}}\right)\right] \\ &\quad - \frac{dv_{\oplus}}{du_{\oplus}}\,\frac{1}{1-\alpha^2}\left[\frac{a_B}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{a_A}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_A^2}}{\eta_A}\right] . \end{aligned} \tag{78}$$

[9] The expression $\log \frac{1+\sqrt{1-\eta^2}}{\eta}$ is identical with $\operatorname{sech}^{-1}\eta$.

and by writing the values of the $a$, $u_\oplus$, and $v_\oplus$ at the boundary $B$ in terms of their values at the boundary $A$ and their gradient between the two boundaries, we obtain

$$[-Y_\oplus]_A^B = \left(v_{\oplus A} + \frac{dv_\oplus}{dz}\Delta z\right) t_B - v_{\oplus A} t_A - \frac{dv_\oplus}{du_\oplus}\,\frac{a_A + \frac{da}{dz}\Delta z}{\alpha}\, t_B + \frac{dv_\oplus}{du_\oplus}\,\frac{a_A}{\alpha}\, t_A$$

$$+ \frac{dv_\oplus}{du_\oplus}\,\frac{\alpha^2}{1-\alpha^2}\left[\left(\frac{V_0}{\eta_0} - u_{\oplus A} - \frac{du_\oplus}{dz}\Delta z + \frac{a_A + \frac{da}{dz}\Delta z}{\alpha}\right)\left(t_B + \frac{\operatorname{sech}^{-1}\eta_B}{\frac{da}{dz}}\right)\right. \tag{79}$$

$$\left. - \left(\frac{V_0}{\eta_0} - u_{\oplus A} + \frac{a_A}{\alpha}\right)\left(t_A + \frac{\operatorname{sech}^{-1}\eta_A}{\frac{da}{dz}}\right)\right] - \frac{dv_\oplus}{du_\oplus}\,\frac{1}{1-\alpha^2}\left[\frac{a_A + \frac{da}{dz}\Delta z}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{a_A}{\frac{da}{dz}}\,\frac{\sqrt{1-\eta_A^2}}{\eta_A}\right]$$

But $\alpha = \dfrac{da/dz}{du_\oplus/dz} = \dfrac{a_B - a_A}{u_{\oplus B} - u_{\oplus A}}$, so that

$$\frac{V_0}{\eta_0} - u_{\oplus A} + \frac{a_A}{\alpha} = \frac{V_0}{\eta_0} - u_{\oplus A} + \frac{a_A(u_{\oplus B} - u_{\oplus A})}{a_B - a_A}$$

$$= \frac{V_0}{\eta_0} - \frac{a_B u_{\oplus A} - a_A u_{\oplus B}}{a_B - a_A}\ . \tag{80}$$

Hence

$$[-Y_\oplus]_A^B = v_{\oplus A}\,(t_B - t_A) + \frac{dv_\oplus}{dz}\Delta z\, t_B - \frac{(v_{\oplus B} - v_{\oplus A})\, a_A}{a_B - a_A}\,(t_B - t_A) - \frac{dv_\oplus}{dz}\Delta z\, t_B$$

$$+ \frac{dv_\oplus}{du_\oplus}\,\frac{\alpha^2}{1-\alpha^2}\left(\frac{V_0}{\eta_0} - \frac{a_B u_{\oplus A} - a_A u_{\oplus B}}{a_B - a_A}\right)\left[(t_B - t_A) + \frac{1}{\frac{da}{dz}}(\operatorname{sech}^{-1}\eta_B - \operatorname{sech}^{-1}\eta_A)\right]$$

$$- \frac{dv_\oplus}{du_\oplus}\,\frac{1}{1-\alpha^2}\left[\frac{a_B}{a_B - a_A}\Delta z\,\frac{\sqrt{1-\eta_B^2}}{\eta_B} - \frac{a_A}{a_B - a_A}\Delta z\,\frac{\sqrt{1-\eta_A^2}}{\eta_A}\right]\ . \tag{81}$$

Setting $Y_{\bullet_B} - Y_{\bullet_A} = (\Delta Y_\bullet)_{AB}$, we have

$$(\Delta Y_\bullet)_{AB} = -\Big[\frac{a_B v_{\bullet_A} - a_A v_{\bullet_B}}{a_B - a_A} + \frac{v_{\bullet_B} - v_{\bullet_A}}{u_{\bullet_B} - u_{\bullet_A}}\frac{\alpha^2}{1-\alpha^2}\Big(\frac{V_0}{\eta_0} - \frac{a_B u_{\bullet_A} - a_A u_{\bullet_B}}{a_B - a_A}\Big)\Big](\Delta t)_{AB}$$

$$- \frac{v_{\bullet_B} - v_{\bullet_A}}{u_{\bullet_B} - u_{\bullet_A}}\frac{\Delta z}{1-\alpha^2}\Big[\frac{a_B - a_A}{(u_{\bullet_B} - u_{\bullet_A})^2}\Big(\frac{V_0}{\eta_0} - \frac{a_B u_{\bullet_A} - a_A u_{\bullet_B}}{a_B - a_A}\Big)(sech^{-1}\eta_B - sech^{-1}\eta_A) \qquad (82)$$

$$- \frac{a_B}{a_B - a_A}\tan(\cos^{-1}\eta_B) + \frac{a_A}{a_B - a_A}\tan(\cos^{-1}\eta_A)\Big] ,$$

where $|\alpha| = \Big|\frac{a_B - a_A}{u_{\bullet_B} - u_{\bullet_A}}\Big| \neq 1.$

Following the same procedure for the case $|\alpha| = 1$, we find

$$- Y_\bullet = \Big(v_\bullet + \frac{\frac{dv_\bullet}{dz}}{\frac{du_\bullet}{dz}}\frac{a}{\eta}\Big)t - \frac{4}{3}\frac{\frac{dv_\bullet}{dz}}{\frac{du_\bullet}{dz}}\frac{a}{\eta}\Big(1 + \frac{\eta}{\alpha}\Big)t - \frac{1}{3}\frac{\frac{dv_\bullet}{dz}}{\frac{du_\bullet}{dz}}\frac{a}{\eta}\Big(1 + \frac{\eta}{\alpha}\Big)\frac{1}{\frac{da}{dz}}\Big[sech^{-1}\eta + \frac{\sqrt{1-\eta^2}}{(\eta+\alpha)^2}\Big]$$

$$= v_\bullet t - \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\frac{a}{\eta}t - \frac{4}{3}\frac{dv_\bullet}{du_\bullet}\frac{a}{\alpha}t - \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\frac{a}{\eta}\frac{sech^{-1}\eta}{\frac{da}{dz}} - \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\frac{a}{\alpha}\frac{sech^{-1}\eta}{\frac{da}{dz}} \qquad (83)$$

$$- \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\frac{a}{\alpha\eta}\frac{1}{\frac{da}{dz}}\frac{\sqrt{1-\eta^2}}{\eta + \alpha} .$$

Substituting the limits,

$$\big[-Y_\bullet\big]_A^B = \Big(v_{\bullet_A} + \frac{dv_\bullet}{dz}\Delta z\Big)t_B - v_{\bullet_A}t_A - \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\Big(\frac{V_0}{\eta_0} - u_{\bullet_A} - \frac{du_\bullet}{dz}\Delta z\Big)t_B + \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\Big(\frac{V_0}{\eta_0} - u_{\bullet_A}\Big)t_A$$

$$- \frac{4}{3}\frac{dv_\bullet}{du_\bullet}\frac{a_A + \frac{da}{dz}\Delta z}{\alpha}t_B + \frac{4}{3}\frac{dv_\bullet}{du_\bullet}\frac{a_A}{\alpha}t_A$$

$$- \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\Big(\frac{V_0}{\eta_0} - u_{\bullet_A} - \frac{du_\bullet}{dz}\Delta z\Big)\frac{sech^{-1}\eta_B}{\frac{da}{dz}} + \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\Big(\frac{V_0}{\eta_0} - u_{\bullet_A}\Big)\frac{sech^{-1}\eta_A}{\frac{da}{dz}}$$

$$- \frac{1}{3}\frac{dv_\bullet}{du_\bullet}\frac{a_A + \frac{da}{dz}\Delta z}{\alpha}\frac{sech^{-1}\eta_B}{\frac{da}{dz}} + \frac{1}{3}\frac{dv_\bullet}{dz}\frac{a_A}{\alpha}\frac{sech^{-1}\eta_A}{\frac{da}{dz}}$$

$$- \frac{1}{3\alpha}\frac{dv_\bullet}{du_\bullet}\frac{1}{\frac{da}{dz}}\Big[\frac{a_B\tan(\cos^{-1}\eta_B)}{\eta_B + \alpha} - \frac{a_A\tan(\cos^{-1}\eta_A)}{\eta_A + \alpha}\Big]$$

$$[-Y_{\phi}]_A^B = v_{\phi_A}(t_B-t_A) - \frac{1}{3}\frac{dv_{\phi}}{du_{\phi}}\left(\frac{V_0}{\eta_0} - u_{\phi_A}\right)(t_B-t_A) - \frac{4}{3}\frac{\frac{dv_{\phi}}{dz}}{\frac{da}{dz}}a_A(t_B-t_A)$$

$$- \frac{1}{3}\frac{dv_{\phi}}{du_{\phi}}\left[\left(\frac{V_0}{\eta_0} - u_{\phi_A}\right) + \frac{a_A(u_{\phi_B}-u_{\phi_A})}{a_B-a_A}\right]\left[\frac{sech^{-1}\eta_B - sech^{-1}\eta_A}{\frac{da}{dz}}\right]$$

$$- \frac{1}{3\alpha}\frac{dv_{\phi}}{du_{\phi}}\frac{1}{\frac{da}{dz}}\left[\frac{a_B \tan(\cos^{-1}\eta_B)}{\eta_B+\alpha} - \frac{a_A \tan(\cos^{-1}\eta_A)}{\eta_A+\alpha}\right] . \qquad (84)$$

Hence

$$(\Delta Y_{\phi})_{AB} = -\left[v_{\phi_A} - \frac{1}{3}\frac{v_{\phi_B}-v_{\phi_A}}{u_{\phi_B}-u_{\phi_A}}\left(\frac{V_0}{\eta_0} - u_{\phi_A}\right) - \frac{4}{3}(v_{\phi_B}-v_{\phi_A})\frac{a_A}{a_B-a_A}\right](\Delta t)_{AB}$$

$$+ \frac{1}{3}\frac{v_{\phi_B}-v_{\phi_A}}{u_{\phi_B}-u_{\phi_A}}\left\{\left[\frac{V_0}{\eta_0} - \frac{a_B u_{\phi_A} - a_A u_{\phi_B}}{a_B-a_A}\right]\left[\frac{sech^{-1}\eta_B - sech^{-1}\eta_A}{a_B-a_A}\right)\right.$$

$$\left. + \frac{1}{\alpha}\left[\frac{a_B}{a_B-a_A}\frac{\tan(\cos^{-1}\eta_B)}{\eta_B+\alpha} - \frac{a_A}{a_B-a_A}\frac{\tan(\cos^{-1}\eta_A)}{\eta_A+\alpha}\right]\right\}\Delta z , \qquad (85)$$

where $|\alpha| = \left|\frac{a_B-a_A}{u_{\phi_B}-u_{\phi_A}}\right| = 1.$

**15. The "Lens-Effect" of Critical Atmosphere Layers.**—Let us investigate the nature of sound propagation if $\eta = -\alpha$ (where $|\alpha| \leq 1$ since $0 \leq \eta \leq 1$ always) at some point of the interval of variation of $\eta$. It has already been shown in the analysis of the integrand in equation (43), which represents the time of travel of the sound through an atmospheric layer along a particular path, that the integral does not converge absolutely if $\eta = -\alpha$ at any point of the interval of integration. If $\eta = -\alpha$ is an end-point of the interval of integration, then since the integrand is of constant sign, the Cauchy "$\mu$-test" indicates that the integral diverges. The significance of this condition is demonstrated by the following analysis.

From equation (32) we have

$$\frac{d\eta}{dz} = \frac{\eta}{a}\left(\frac{da}{dz} + \eta\frac{du_{\phi}}{dz}\right) = \frac{du_{\phi}}{dz}\frac{\eta}{a}(\alpha + \eta) .$$

Therefore, it is at once obvious that $-\alpha$ is the limiting value of $\eta$, since $d\eta/dz = 0$ when $\eta = -\alpha$.

Suppose, now, that $\eta = -\alpha < 1$ at the end point $\eta_B$ of the interval of integration $\eta_A$ to $\eta_B$ of equation (43)

Then

$$(\Delta t)_{AB} = \frac{-1}{\frac{da}{dz}} \left[ \log \frac{1+\sqrt{1-\eta^2}}{\eta} + \frac{1}{\sqrt{1-\alpha^2}} \log 2 \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right]_{\eta_A}^{\eta_B}$$

$$= \frac{1}{\frac{da}{dz}} \left[ \log \frac{1+\sqrt{1-\eta_A^2}}{\eta_A} + \frac{1}{\sqrt{1-\alpha^2}} \log 2 \frac{1+\alpha\eta_A - \sqrt{1-\alpha^2}\sqrt{1-\eta_A^2}}{\eta_A+\alpha} \right]$$

$$- \lim_{\eta\to-\alpha} \left\{ \frac{1}{\frac{da}{dz}} \left[ \log \frac{1+\sqrt{1-\eta^2}}{\eta} + \frac{1}{\sqrt{1-\alpha^2}} \log 2 \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right] \right\}$$

$$= \frac{1}{\frac{da}{dz}} \left[ \log \frac{1+\sqrt{1-\eta_A^2}}{\eta_A} + \frac{1}{\sqrt{1-\alpha^2}} \log 2 \frac{1+\alpha\eta_A - \sqrt{1-\alpha^2}\sqrt{1-\eta_A^2}}{\eta_A+\alpha} \right]$$

$$- \frac{1}{\frac{da}{dz}} \log \frac{1+\sqrt{1-\alpha^2}}{-\alpha} - \frac{1}{\sqrt{1-\alpha^2}} \lim_{\eta\to-\alpha} \left[ \log 2 \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right]. \tag{86}$$

Now

$$\lim_{\eta\to-\alpha} \left[ \log 2 \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right]$$

$$= \log 2 + \lim_{\eta\to-\alpha} \log \left[ \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right]. \tag{87}$$

Differentiating both numerator and denominator with respect to $\eta$ in the last term of equation (87) in order to evaluate the indeterminate form $0/0$, we have

$$\lim_{\eta\to-\alpha} \log \left[ \frac{1+\alpha\eta - \sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \right]$$

$$= \lim_{\eta\to-\alpha} \log \left[ \frac{\alpha + \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\eta^2}}\eta}{\alpha} \right]$$

$$= -\infty$$

Hence $(\Delta t)_{AB}$ diverges to infinity as $\eta$ approaches $-\alpha$.

But we are reckoning with only finite intervals of travel time along sound paths; hence, for an atmospheric layer of stratification parameter $\alpha$, we can conclude that *if* $\eta = -\alpha$ *at any point of the layer, then* $\eta = -\alpha$ at *every point of that layer.** On the other hand if $\eta \neq -\alpha$ at any point of the layer, then $\eta$ can never become $-\alpha$ in finite time *in that particular layer.*

It follows further that since $\eta$ cannot pass through the value $-\alpha$, and since $0 \leq \eta \leq 1$ atmospheric layers of stratification parameters $0 \leq -\alpha \leq 1$ will *reduce* the divergence of the bundle of sound rays about $\eta = -\alpha$ in the case $d\alpha/dz > 0$, and *increase* the divergence in the case $d\alpha/dz < 0$. But it is to be noted that this phenomenon is restricted to those particular rays belonging to a family of azimuth direction $\theta$, where $du_a/dz$ is of a magnitude consistent with $0 \leq -\alpha \leq 1$. Since the refractive wind gradient $du_a/dz$ will necessarily vary with $\theta$, it is at once apparent that for layers of sufficient vertical extent, the normal divergence of sound rays may be disturbed locally about that ray for which $\eta = -\alpha$, with a consequent variation in sound intensity. The discussion of the importance of this analytical deduction is reserved for a subsequent report, but it may be pointed out in passing that the above analysis proves a useful tool in the interpretation of the abnormal propagation of sound to great distances and the phenomena of zones of silence and zones of marked audibility at considerable distances from the sound source.

The solution for the time-, range- and drift-increments for the case $\eta = -\alpha$ will be discussed in Section VI.

**16. Alternate Expressions for the Time-Increment.**—The time-increment of travel for a sound wave through a layer of atmosphere is given by equations (45) in sub-section (12). The three solutions given depend on the range of $\alpha$, so that *only real variables appear in the expressions.*

The time-increment, however, can be expressed in another form by change of the variable of integration, or for that matter, by transformation of the expressions in equations (45). Thus

$$\int \frac{d\eta}{(\alpha+\eta)\sqrt{1-\eta^2}} = -\int \frac{d(\cos^{-1}\eta)}{\alpha+\eta}, \tag{89}$$

since $d(\cos^{-1}\eta) = \dfrac{-d\eta}{\sqrt{1-\eta^2}}$. Utilizing formula #239 in Peirce's *Tables*, and the substitution

$$s = \tan 1/2(\cos^{-1}\eta), \tag{90}$$

we find

$$\int \frac{d(\cos^{-1}\eta)}{\alpha+\eta} = \frac{1}{\sqrt{\frac{1+\alpha}{1-\alpha}}\,(1-\alpha)}\left[\int \frac{ds}{s+\sqrt{\frac{1+\alpha}{1-\alpha}}} - \int \frac{ds}{s-\sqrt{\frac{1+\alpha}{1-\alpha}}}\right]$$

$$= \frac{1}{\sqrt{1-\alpha^2}}\left[\log\left(s+\sqrt{\frac{1+\alpha}{1-\alpha}}\right) - \log\left(s-\sqrt{\frac{1+\alpha}{1-\alpha}}\right)\right]. \tag{91}$$

*However, the projection of the sound ray on a plane parallel to the sound azimuth is not a straight line, since $du_a/dz \neq 0$.

Introducing the imaginary $i = \sqrt{-1}$, we can write

$$\int \frac{d(\cos^{-1}\eta)}{\alpha + \eta} = \frac{1}{\sqrt{1-\alpha^2}} \left[ \log\left(s - i\sqrt{\frac{\alpha+1}{\alpha-1}}\right) - \log\left(s + i\sqrt{\frac{\alpha+1}{\alpha-1}}\right) \right],$$

which by use of formula #651 in Peirce's *Tables* becomes

$$\int \frac{d(\cos^{-1}\eta)}{\alpha + \eta} = \frac{1}{\sqrt{1-\alpha^2}} \left[ i \tan^{-1}\left(\frac{-1}{s}\sqrt{\frac{\alpha+1}{\alpha-1}}\right) - i \tan^{-1}\left(\frac{1}{s}\sqrt{\frac{\alpha+1}{\alpha-1}}\right) \right]$$

$$= \frac{-2i}{\sqrt{1-\alpha^2}} \tan^{-1} \frac{1}{s}\sqrt{\frac{\alpha+1}{\alpha-1}} \quad . \qquad (92)$$

From trigonometry we have

$$s = \tan 1/2(\cos^{-1}\eta) = \sqrt{\frac{1-\eta}{1+\eta}},$$

so that (92) becomes

$$\int \frac{d(\cos^{-1}\eta)}{\alpha + \eta} = \frac{-2i}{\sqrt{1-\alpha^2}} \tan^{-1}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\sqrt{\frac{1+\eta}{1-\eta}}\right)$$

$$= \frac{2}{\sqrt{1-\alpha^2}} \tanh^{-1}\left(-\sqrt{\frac{1+\alpha}{1-\alpha}}\sqrt{\frac{1+\eta}{1-\eta}}\right) \qquad (93)$$

$$= \frac{-2}{\sqrt{1-\alpha^2}} \tanh^{-1}\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\sqrt{\frac{1+\eta}{1-\eta}}\right).$$

If $|\alpha|=1$, we can make use of the simpler integrals #296 and #297 in Peirce's *Tables*, securing

$$\int_{\text{for } |\alpha|=1} \frac{d(\cos^{-1}\eta)}{\alpha + \eta} = \frac{1}{\alpha} \frac{\sqrt{1-\eta^2}}{\eta + \alpha} \quad . \qquad (94)$$

Peirce's *Tables*, formula #300, give other variations of the expressions in (93) above.

By writing

$$\int \frac{d\eta}{\eta\sqrt{1-\eta^2}} = -\int \frac{d(\cos^{-1}\eta)}{\eta} \quad ,$$

we obtain by Peirce's *Tables*, #288,

$$\int \frac{d\eta}{\eta\sqrt{1-\eta^2}} = -\log \tan\left(\frac{\pi}{4} + \tan^{-1}\sqrt{\frac{1-\eta}{1+\eta}}\right) \quad . \qquad (95)$$

Hence there are a wide variety of forms for writing the integrations indicated in equation (44) above. For reference there are listed below a number of these expressions from which selection can be made most convenient for the case in hand:

$$
(\Delta t)_{AB} = \frac{\Delta s}{\alpha_B - \alpha_A}
\left[
\begin{array}{c}
-\log \dfrac{1+\sqrt{1-\eta^2}}{\eta} \\
\text{or} \\
-\operatorname{sech}^{-1}\eta \\
\text{or} \\
-\log \tan\left(\dfrac{\pi}{4} + \tan^{-1}\sqrt{\dfrac{1-\eta}{1+\eta}}\right)
\end{array}
\right]_A^B
+
\left[
\begin{array}{c}
\dfrac{1}{\alpha}\,\dfrac{\sqrt{1-\eta^2}}{\eta+\alpha}, \text{ if } |\alpha|=1. \\
\text{or} \\
\dfrac{-1}{\sqrt{1-\alpha^2}} \log \dfrac{1+\alpha\eta-\sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \\
\text{or} \\
\dfrac{-1}{\sqrt{1-\alpha^2}} \log \dfrac{1+\alpha\eta+\sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{\eta+\alpha} \\
\text{or} \\
\dfrac{1}{\sqrt{1-\alpha^2}} \tanh^{-1} \dfrac{\sqrt{1-\alpha^2}\sqrt{1-\eta^2}}{1+\alpha\eta} \\
\text{or} \\
\dfrac{-2}{\sqrt{1-\alpha^2}} \tanh^{-1} \sqrt{\dfrac{1+\alpha}{1-\alpha}}\sqrt{\dfrac{1+\eta}{1-\eta}} \\
\text{or} \\
\dfrac{-1}{\sqrt{\alpha^2-1}} \sin^{-1} \dfrac{1+\alpha\eta}{\eta+\alpha} \\
\text{or} \\
\dfrac{1}{\sqrt{\alpha^2-1}} \sin^{-1} \dfrac{\sqrt{\alpha^2-1}\sqrt{1-\eta^2}}{\eta+\alpha} \\
\text{or} \\
\dfrac{2}{\sqrt{\alpha^2-1}} \tan^{-1} \sqrt{\dfrac{\alpha-1}{\alpha+1}}\sqrt{\dfrac{1-\eta}{1+\eta}} \\
\text{or} \\
\dfrac{1}{\sqrt{\alpha^2-1}} \tan^{-1} \dfrac{\sqrt{\alpha^2-1}\sqrt{1-\eta^2}}{1+\alpha\eta}, \\
\text{if } |\alpha| \neq 1.
\end{array}
\right]_A^B
\quad (96)
$$

## VI. SOME SIMPLE CASES OF ATMOSPHERIC STRATIFICATION.

In the previous section there were derived expressions for the time-, range-, and drift- increments for the propagation of sound through a horizontally homogeneous layer of atmosphere, characterized by a stratification parameter bearing the ratio of the sound velocity gradient to the refractive wind gradient within the layer. The solutions derived were for the most general cases, with $da/dz$, $du_{\bullet}/dz$, $dv_{\bullet}/dz$ constants not necessarily zero, and the sound path a space curve, having a point of total refraction in the atmosphere at $(X_{\bullet}, Y_{\bullet}, Z_{max})$.

In the present section the solutions for the increments in time, range, and drift will be derived for some particularly simple atmospheric stratifications. They may be considered as degenerate cases of the general solutions of the preceding section; and are given here inasmuch as the expressions of the general solution are indeterminate for some of the particular cases cited here.

**17. The Case of $\eta = -\alpha$ Throughout the Atmospheric Layer.**—Since $\eta$ is constant for the layer, the integration cannot be performed with respect to $\eta$ as in the more general cases above. However, we can perform the required integration readily by using $z$ as the variable of integration.

From equations (29) we have

$$\frac{dz}{dt} = \nu a,$$

so that

$$dt = \frac{dz}{\nu a} = \frac{1}{\sqrt{1-\eta^2}}\frac{dz}{a} \quad . \tag{97}$$

Obviously, we need to express $a$ as a function of $z$;

$$a = a_A + \frac{da}{dz}(z-z_A) \quad , \tag{98}$$

where $a_A$ is the velocity of sound at the lower boundary of a layer. Substituting from (98) in equation (97), we obtain

$$dt = \frac{1}{\sqrt{1-\eta^2}}\frac{dz}{a_A + \frac{da}{dz}(z-z_A)} \quad .$$

Integrating over the layer,

$$t_B - t_A = \frac{1}{\sqrt{1-\eta^2}}\int_{z_A}^{z_B}\frac{dz}{a_A + \frac{da}{dz}(z-z_A)} = \frac{1}{\sqrt{1-\eta^2}}\int_{0}^{\Delta z}\frac{d(z-z_A)}{a_A + \frac{da}{dz}(z-z_A)} \quad .$$

where $\Delta z = z_B - z_A$.

Hence[10]

$$(\Delta t)_{AB} = \left\{ \frac{1}{\frac{da}{dz}} \frac{1}{\sqrt{1-\eta^2}} \log \left[a_A + \frac{da}{dz}(z-z_A)\right] \right\}_0^{\Delta z}$$

$$= \frac{1}{\frac{da}{dz}} \frac{1}{\sqrt{1-\eta^2}} \log \frac{a_B}{a_A}$$

$$= \frac{\Delta z}{a_B - a_A} \frac{1}{\sqrt{1-\eta^2}} \log \frac{a_B}{a_A} \quad . \tag{99}$$

For the range-increment, we have from equations (29)

$$\frac{-dx_e}{dt} = \sqrt{1-v^2}\, a + w_e \quad .$$

so that

$$-\int dx_e = \int (\eta a + w_e)dt$$

$$= \int \frac{\eta\, dz}{\sqrt{1-\eta^2}} + \int \frac{w_e\, dz}{a\sqrt{1-\eta^2}} \quad . \tag{100}$$

It is required that $a$ and $w_e$ be expressed as functions of $z$ in the integrand in (100). For $w_e$ we have

$$w_e = w_{e_A} + \frac{dw_e}{dz}(z-z_A) \quad ,$$

where $w_{e_A}$ is the refractive wind at the lower boundary of the layer. From this relation and equation (98) we have, on substitution in (100)

$$-\int dx_e = \frac{\eta}{\sqrt{1-\eta^2}} \int dz + \int \frac{w_{e_A} + \frac{dw_e}{dz}(z-z_A)}{\sqrt{1-\eta^2}\left[a_A + \frac{da}{dz}(z-z_A)\right]} dz$$

$$= \frac{\eta}{\sqrt{1-\eta^2}} \int dz + w_{e_A} \int \frac{dz}{\sqrt{1-\eta^2}\left[a_A + \frac{da}{dz}(z-z_A)\right]} + \frac{dw_e}{dz} \int \frac{(z-z_A)dz}{\sqrt{1-\eta^2}\left[a_A + \frac{da}{dz}(z-z_A)\right]} \tag{101}$$

[10] The solution in (99) does not hold for $da/dz = 0$, but this latter case is of no concern to us, since $\eta$ would then be zero everywhere; that is, if $da/dz = 0$, then $\alpha = 0$, provided $dw_e/dz \neq 0$.

Integrating between the limits of the lower and upper boundary we obtain

$$-(X_{e_B}-X_{e_A}) = \frac{\eta}{\sqrt{1-\eta^2}}\,\Delta z + u_{e_A}\,(\Delta t)_{AB} + \frac{du_e}{dz}\int_0^{\Delta z}\frac{(z-z_A)\,d(z-z_A)}{\sqrt{1-\eta^2}\,[a_A + \frac{da}{dz}(z-z_A)]}\,, \tag{102}$$

so that

$$\begin{aligned}-(\Delta X_e)_{AB} &= \frac{\eta}{\sqrt{1-\eta^2}}\,\Delta z + u_{e_A}\,(\Delta t)_{AB} + \left\{\frac{du_e}{dz}\,\frac{1}{(\frac{da}{dz})^2}\,\frac{1}{\sqrt{1-\eta^2}}\,[a_A + \frac{da}{dz}(z-z_A) - a_A\,\log(a_A + \frac{da}{dz}(z-z_A))]\right\}_0^{\Delta z}\\ &= \frac{\eta}{\sqrt{1-\eta^2}}\,\Delta z + u_{e_A}\,(\Delta t)_{AB} + \frac{du_e}{dz}\,\frac{1}{(\frac{da}{dz})^2}\,\frac{1}{\sqrt{1-\eta^2}}\,[(a_B-a_A) - a_A\,\log\frac{a_B}{a_A}]\,.\end{aligned} \tag{103}$$

By use of the relations derived in (99) we can write (103) more compactly as

$$\begin{aligned}-(\Delta X_e)_{AB} &= \frac{\eta}{\sqrt{1-\eta^2}}\,\Delta z + u_{e_A}\,(\Delta t)_{AB} + \frac{1}{\alpha}\,\frac{\Delta z}{\sqrt{1-\eta^2}} - \frac{a_A}{\alpha}(\Delta t)_{AB}\\ &= (u_{e_A} - a_A\,\frac{u_{e_B}-u_{e_A}}{a_B-a_A})(\Delta t)_{AB} - (\frac{1}{\eta} - \eta)\,\frac{\Delta z}{\sqrt{1-\eta^2}}\,.\end{aligned}$$

Hence

$$(\Delta X_e)_{AB\atop \eta=-\alpha} = -\,[\frac{a_B u_{e_A} - a_A u_{e_B}}{a_B-a_A}](\Delta t)_{AB} + \tan(\cos^{-1}\eta)\,\Delta z. \tag{104}$$

This result is analagous to the expression in equation (64).

The solution for the drift-increment is similar to that for the range-increment, so that we can immediately write

$$\begin{aligned}-(\Delta Y_e)_{AB} &= v_{e_A}(\Delta t)_{AB} + \frac{1}{\sqrt{1-\eta^2}}\,\frac{dv_e}{dz}\,\frac{1}{(\frac{da}{dz})^2}\,[(a_B-a_A) - a_A\,\log\frac{a_B}{a_A}]\\ &= (v_{e_A} - a_A\,\frac{v_{e_B}-v_{e_A}}{a_B-a_A})(\Delta t)_{AB} + \frac{v_{e_B}-v_{e_A}}{u_{e_B}-u_{e_A}}\,\frac{\Delta z}{\alpha\,\sqrt{1-\eta^2}}\,.\end{aligned} \tag{105}$$

Hence

$$(\Delta Y_{\oplus})_{AB} = -\left(\frac{a_B v_{\oplus A} - a_A v_{\oplus B}}{a_B - a_A}\right)(\Delta t)_{AB} + \frac{v_{\oplus B} - v_{\oplus A}}{u_{\oplus B} - u_{\oplus A}} \frac{\Delta z}{1-\alpha^2} \tan(\cos^{-1}\eta). \qquad (106)$$

This result is analogous to the expression for the drift-increment in equation (82).

**18. The Case of Constant Refractive Wind and Variable Temperature Throughout the Atmospheric Layer.**—In this type of stratification, $du_{\oplus}/dz = 0$ and $da/dz \neq 0$, so that $\alpha \to \infty$. Hence for the time increment we have from equations (96),

$$(\Delta t)_{AB} = \frac{-\Delta z}{a_B - a_A}(\operatorname{sech}^{-1}\eta_B - \operatorname{sech}^{-1}\eta_A). \qquad (107)$$

For the range-increment, we obtain from equation (64),

$$(\Delta X_{\oplus})_{AB} = -u_{\oplus}(\Delta t)_{AB} + \left[\frac{a_B}{a_B - a_A}\tan(\cos^{-1}\eta_B) - \frac{a_A}{a_B - a_A}\tan(\cos^{-1}\eta_A)\right]\Delta z. \qquad (108)$$

For the drift-increment we have, from equations (29) and (31),

$$-\int dy_{\oplus} = \int \frac{v_{\oplus}\, d\eta}{\eta\sqrt{1-\eta^2}\,\frac{da}{dz}}. \qquad (109)$$

Expressing $v_{\oplus}$ as a function of $\eta$,

$$\frac{dv_{\oplus}}{dt} = \frac{dv_{\oplus}}{dz}\sqrt{1-\eta^2}\, a$$

$$= \frac{dv_{\oplus}}{dz}\sqrt{1-\eta^2}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right)\eta. \qquad (110)$$

Thus, since $dt = \dfrac{d\eta}{\eta\sqrt{1-\eta^2}\,\frac{da}{dz}}$,

$$dv_{\oplus} = \frac{\frac{dv_{\oplus}}{dz}}{\frac{da}{dz}}\left(\frac{V_0}{\eta_0} - u_{\oplus}\right)d\eta, \qquad (111)$$

and integrating we obtain

$$v_{\oplus} = \frac{\frac{dv_{\oplus}}{dz}}{\frac{da}{dz}}\, a + \sigma, \qquad (112)$$

where $\sigma$ is the constant of integration, and given by

$$\sigma = \frac{a_B v_{\phi_A} - a_A v_{\phi_B}}{a_B - a_A} \quad . \qquad (113)$$

Substituting in (109) ,

$$-\int dy_\phi = \frac{\frac{dv_\phi}{dz}}{(\frac{da}{dz})^2}(\frac{V_\phi}{\eta_o} - u_\phi)\int \frac{d\eta}{\sqrt{1-\eta^2}} + \sigma \int dt, \qquad (114)$$

so that

$$-Y_\phi\Big]_A^B = \frac{v_{\phi_B} - v_{\phi_A}}{a_B - a_A}\frac{a}{\eta}\frac{\Delta z}{a_B - a_A}[-cos^{-1}\eta_B + cos^{-1}\eta_A] + \frac{a_B v_{\phi_A} - a_A v_{\phi_B}}{a_B - a_A}(\Delta t)_{AB} \quad . \qquad (115)$$

Hence

$$(\Delta Y_\phi)_{AB} = -[\frac{a_B v_{\phi_A} - a_A v_{\phi_B}}{a_B - a_A}](\Delta t)_{AB} + \frac{v_{\phi_B} - v_{\phi_A}}{(a_B - a_A)^2}(\frac{V_\phi}{\eta_o} - u_\phi)\ [cos^{-1}\eta_B - cos^{-1}\eta_A]\ \Delta z. \qquad (116)$$

This term is analogous to that given in (82), but not strictly similar, since (82) contains the term $(u_{\phi_B} - u_{\phi_A})$ and $\alpha$ , which in this case approach zero and infinity, respectively.

**19. *The Case of Constant Temperature and Variable Refractive Wind Throughout the Atmospheric Layer.*** --In this case, $da/dz = 0$ and $du_\phi/dz \neq 0$ . Equations (96) cannot be used in their form to determine the time-increment, since they lead to an indeterminant. But conveniently, equation (43) can be written

$$\int dt = \frac{1}{\frac{du_\phi}{dz}}\int \frac{d\eta}{\eta^2\sqrt{1-\eta^2}}$$

$$= \frac{-1}{\frac{du_\phi}{dz}}\frac{\sqrt{1-\eta^2}}{\eta} \quad . \qquad (117)$$

Hence for the layer $AB$ ,

$$(\Delta t)_{AB} = \frac{-\Delta z}{u_{\phi_B} - u_{\phi_A}}\ [tan(cos^{-1}\eta_B) - tan(cos^{-1}\eta_A)] \quad . \qquad (118)$$

For the range-increment we have from (47)

$$-\int dx_\phi = \frac{1}{\frac{du_\phi}{dz}}\int \frac{\eta a + u_\phi}{\eta^2\sqrt{1-\eta^2}} d\eta \quad . \qquad (119)$$

Since $a$ is constant, $u_\oplus = \frac{V_0}{\eta_0} - \frac{a}{\eta}$ , and (119) becomes

$$-\int dx_\oplus = \frac{V_0}{\eta_0}\frac{1}{\frac{du_\oplus}{dz}}\int\frac{d\eta}{\eta^2\sqrt{1-\eta^2}} - \frac{a}{\frac{du_\oplus}{dz}}\int\frac{\sqrt{1-\eta^2}}{\eta^3}d\eta$$

$$= \frac{V_0}{\eta_0}\,t - \frac{a}{\frac{du_\oplus}{dz}}\left[\frac{-\sqrt{1-\eta^2}}{2\eta^2} - \frac{1}{2}\int\frac{d\eta}{\eta\sqrt{1-\eta^2}}\right]$$

$$= \frac{V_0}{\eta_0}\,t + \frac{a}{2\frac{du_\oplus}{dz}}\left[\frac{\sqrt{1-\eta^2}}{\eta^2} - \operatorname{sech}^{-1}\eta\right]$$

$$= \frac{V_0}{\eta_0}\,t + \frac{1}{2\frac{du_\oplus}{dz}}\left[\left(\frac{V_0}{\eta_0} - u_\oplus\right)\frac{\sqrt{1-\eta^2}}{\eta} - a\operatorname{sech}^{-1}\eta\right]$$

$$= \frac{1}{2}\frac{V_0}{\eta_0}\,t - \frac{1}{2\frac{du_\oplus}{dz}}\left[u_\oplus\frac{\sqrt{1-\eta^2}}{\eta} + a\operatorname{sech}^{-1}\eta\right] . \qquad (120)$$

Thus

$$-X_\oplus\Big]_A^B = \frac{1}{2}\frac{V_0}{\eta_0}(\Delta t)_{AB} - \frac{1}{2}\frac{1}{\frac{du_\oplus}{dz}}\left[u_{\oplus B}\frac{\sqrt{1-\eta_B^2}}{\eta_B} - u_{\oplus A}\frac{\sqrt{1-\eta_A^2}}{\eta_A} + a(\operatorname{sech}^{-1}\eta_B - \operatorname{sech}^{-1}\eta_A)\right] .$$

Hence

$$(\Delta X_\oplus)_{AB} = \frac{-1}{2}\frac{V_0}{\eta_0}(\Delta t)_{AB} + \frac{1}{2}\frac{\Delta z}{u_{\oplus B}-u_{\oplus A}}\left[u_{\oplus B}\tan(\cos^{-1}\eta_B) - u_{\oplus A}\tan(\cos^{-1}\eta_A)\right.$$
$$\left. + a(\operatorname{sech}^{-1}\eta_B - \operatorname{sech}^{-1}\eta_A)\right] . \qquad (121)$$

For the drift-increment we have from (65),

$$-\int dy_\oplus = \frac{1}{\frac{du_\oplus}{dz}}\int\frac{v_\oplus\,d\eta}{\eta^2\sqrt{1-\eta^2}} , \qquad (122)$$

and from (68),

$$v_\oplus = \frac{\frac{dv_\oplus}{dz}}{\frac{du_\oplus}{dz}}\left(\frac{-a}{\eta}\right) + \sigma , \qquad (123)$$

since $k = a$, and $\alpha = 0$. Substituting from (123) in (122) we find

$$-\int dy_\oplus = \sigma t - \frac{\frac{dv_\oplus}{dz}}{\left(\frac{du_\oplus}{dz}\right)^2} \int \frac{d\eta}{\eta^2\sqrt{1-\eta^2}}$$

$$= \sigma t - \frac{\frac{dv_\oplus}{dz}}{\left(\frac{du_\oplus}{dz}\right)^2} a \left[-\frac{\sqrt{1-\eta^2}}{2\eta^2} + \frac{1}{2}\int \frac{d\eta}{\eta\sqrt{1-\eta^2}}\right]$$

$$= \sigma t + \frac{\frac{dv_\oplus}{dz}}{\left(\frac{du_\oplus}{dz}\right)^2} \frac{a}{2} \left[\frac{\sqrt{1-\eta^2}}{\eta^2} + sech^{-1}\eta\right] . \qquad (124)$$

Thus

$$-Y_\oplus = \left[v_\oplus + \frac{dv_\oplus}{du_\oplus}\left(\frac{V_0}{\eta_0} - u_\oplus\right)\right]t + \frac{\frac{dv_\oplus}{dz}}{\left(\frac{du_\oplus}{dz}\right)^2} \frac{1}{2} \left[\left(\frac{V_0}{\eta_0} - u_\oplus\right)\frac{\sqrt{1-\eta^2}}{\eta} + a\ sech^{-1}\eta\right] . \qquad (125)$$

Substituting the limits we obtain

$$-Y_\oplus\Big]_A^B = \left(v_{\oplus_A} - \frac{v_{\oplus_B}-v_{\oplus_A}}{u_{\oplus_B}-u_{\oplus_A}} u_{\oplus_A}\right)(\Delta t)_{AB} - \frac{1}{2}\frac{dv_\oplus}{du_\oplus}\frac{V_0}{\eta_0}(\Delta t)_{AB}$$

$$+ \frac{\frac{dv_\oplus}{dz}}{\left(\frac{du_\oplus}{dz}\right)^2} \frac{1}{2} \left[-u_{\oplus_B}\frac{\sqrt{1-\eta^2}}{\eta_B} + u_{\oplus_A}\frac{\sqrt{1-\eta^2}}{\eta_A} + a(sech^{-1}\eta_B - sech^{-1}\eta_A)\right]. \qquad (126)$$

Hence

$$(\Delta Y_\oplus)_{AB} = -\left[\frac{u_{\oplus_B}v_{\oplus_A} - u_{\oplus_A}v_{\oplus_B}}{u_{\oplus_B}-u_{\oplus_A}} - \frac{1}{2}\frac{v_{\oplus_B}-v_{\oplus_A}}{u_{\oplus_B}-u_{\oplus_A}}\frac{V_0}{\eta_0}\right](\Delta t)_{AB}$$

$$+ \frac{1}{2}\frac{(v_{\oplus_B}-v_{\oplus_A})}{(u_{\oplus_B}-u_{\oplus_A})^2}\Delta z \left[u_{\oplus_B}\tan(\cos^{-1}\eta_B) - u_{\oplus_A}\tan(\cos^{-1}\eta_A)\right.$$

$$\left. - a(sech^{-1}\eta_B - sech^{-1}\eta_A)\right]. \qquad (127)$$

**20. The Case of Constant Temperature and Constant Refractive Wind Throughout the Atmospheric Layer.**—In this case both $da/dz$ and $du_{\oplus}/dz$ are zero, so that $\alpha$ has no meaning. But $\eta$ is constant, so that the integrations can readily be performed with respect to $z$. From equations (29) we have

$$\int dt = \int \frac{dz}{a\sqrt{1-\eta^2}} ,$$

so that for the layer $AB$

$$(\Delta t)_{AB} = \frac{\Delta z}{a\sqrt{1-\eta^2}} . \quad (128)$$

For the range-increment we have

$$-\int dx_{\oplus} = \int (\eta a + u_{\oplus}) dt$$

$$= \frac{\eta}{\sqrt{1-\eta^2}} \Delta z + u_{\oplus}(\Delta t)_{AB} . \quad (129)$$

Hence

$$(\Delta X_{\oplus})_{AB} = - u_{\oplus}(\Delta t)_{AB} - \cot(\cos^{-1}\eta)\, \Delta z . \quad (130)$$

For the drift-increment,

$$-\int dy_{\oplus} = \int v_{\oplus} dt = \int \frac{v_{\oplus}\, dz}{a\sqrt{1-\eta^2}} . \quad (131)$$

We need to express $v_{\oplus}$ as a function of $z$, and this we can do simply since

$$v_{\oplus} = v_{\oplus_A} + \frac{dv_{\oplus}}{dz}(z-z_A) .$$

Substituting in (131) and integrating,

$$- Y_{\oplus}\Big]_A^B = \int_0^{\Delta z} \frac{v_{\oplus_A} + \frac{dv_{\oplus}}{dz}(z-z_A)}{a\sqrt{1-\eta^2}} d(z-z_A)$$

$$= \frac{v_{\oplus_A}\Delta z}{a\sqrt{1-\eta^2}} + \frac{dv_{\oplus}}{dz}\frac{(\Delta z)^2}{2}\frac{1}{a\sqrt{1-\eta^2}} . \quad (132)$$

Hence

$$(\Delta Y_{\phi})_{AB} = \frac{-\Delta z}{a\sqrt{1-\eta^2}}\left(v_{\phi_A} + \frac{v_{\phi_B} - v_{\phi_A}}{2}\right)$$

$$= \frac{-\Delta z}{a\sqrt{1-\eta^2}}\left(\frac{v_{\phi_B} + v_{\phi_A}}{2}\right) ,$$

or

$$(\Delta Y_{\phi})_{AB} = -\left(\frac{v_{\phi_B} + v_{\phi_A}}{2}\right)(\Delta t)_{AB} \quad . \tag{133}$$

## SOUND RANGING FOR ARTILLERY

### VII. A MATHEMATICAL SOLUTION FOR THE SOUND SOURCE.

In the foregoing discussion a technique has been devised whereby a given sound wave can be traced through a stratified atmosphere along a particular path defined by the manifold variation of the meteorological elements. The analysis permits any combination of vertical gradients of temperature, humidity, and wind, provided only that the gradient of each meteorological element be constant within the stratum of atmosphere considered. The sound ray thus determined is a continuous space curve. Expressions have been deduced for the time of travel of the sound along the path traversed, and for the increments in the space coordinates of a given point on the wave front as a result of the propagation. All possible inclinations of the wave normal have been considered in the analysis, up to the level at which the sound is totally refracted.

However, it is conceivable that on occasions a sound wave may be incident over a locality without having undergone total refraction at some point along the path of propagation. An obvious example is the case where the sound propagation is along the horizontal; another, where the source of sound is favorably elevated with respect to the locality over which the sound wave is incident. An important consideration not to be overlooked is that there may be several distinct sound wave surfaces incident over the same locality, but to each and every one is attributed a unique sound path. Using the time of first arrival at each microphone of the space array as representative of the arrival of the identical wave pulse at all microphones, the path determined from these arrival times and microphone positions, and from the existing meteorological conditions, is *the path of least time of travel.*

It is the purpose of the present section to apply the rigor of mathematics to the problem of locating the source of sound, given two sound rays determined by use of the technique developed in the previous sections. It may be pointed out at this juncture that mention was made at the outset in Section V that the sound source can be located as the intersection of any two particular sound rays, involving the solution of twelve simultaneous equations, at best a tedious if not impossible task. Yet with recourse to elementary space geometry we can effect a mathematical solution for the sound source in the case of the stratified atmosphere, provided, however, that for sound waves diffracted by a physical body, the direction of the normal to the undisturbed wave surface be known before incidence on the diffracting body. The inclusion of this reservation in connection with the solution of the eight simultaneous equations required (equations (29) above), is an obvious necessity, since any phenomenon which diverts the direction of the normal to the wave front from that computed by means of the refraction equation will necessarily introduce a cumulative error for the remainder of the sound path considered to be in the unobstructed atmosphere.

We shall therefore limit our solution to those cases where the sound path at any instant is at least 200 to 100 feet distant (depending on the wave length of the sound), from any object capable of producing diffraction of the sound wave. Thus, with sets of equations as (29) , one set each for two space arrays of microphones, we can determine the position in space of the source of sound with an exactness dependent on (*a*) the accuracy to which the instrumental equipment will indicate and record the existent conditions, (*b*) the degree to which the assumption of horizontal homogeneity is justified in the portion of the atmosphere considered, and (*c*) the extent that the theoretical treatment herein presented represents the actual propagation of sound.

**21. Transformation of Coordinates.**—Consider two space arrays of microphones, and choose one vertex of each as the origin of a left-handed coordinate system, with the $Y$-axis directed northward, $X$-axis eastward. (See Figure 5). Express the remaining vertices of each tetrahedron in space coordinates referred to the coordinate system selected for each tetrahedron. For the purpose of ready

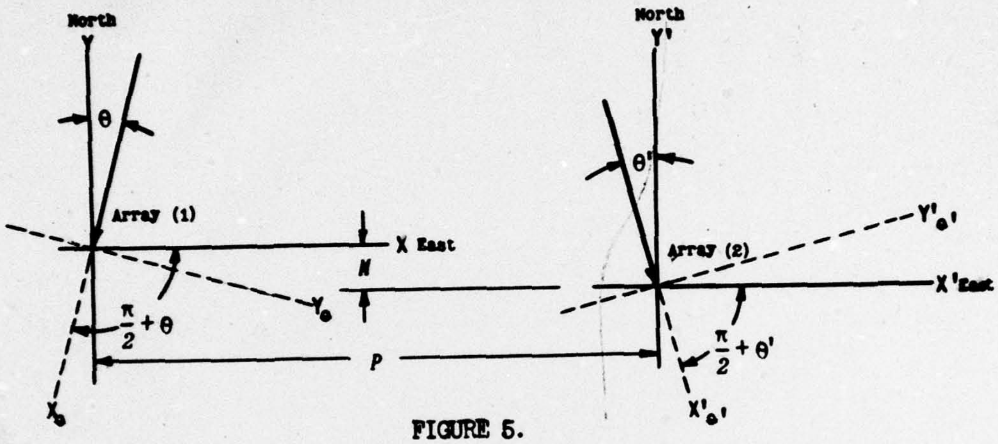


FIGURE 5.

identification, let the generalized coordinates appropriate to space array (1) be $x$, $y$, $z$; those appropriate to array (2) be $x'$, $y'$, $z'$. Let the position of the origin of the $x'$, $y'$, $z'$ coordinate system be known with respect to the $x$, $y$, $z$, system; in particular, let $P$ be the distance in the $x$-direction of the origin of the $x'$, $y'$, $z'$ system, let $N$ be the corresponding distance in the $y$-direction, and let $E$ be the difference in elevation, sign considered.

Further, let $\theta$ and $\theta'$ be the azimuth of sound approach, determined as outlined in Section II, for the corresponding space arrays.

The solutions for the time- , range- , and drift-increments derived in Section V are expressed relative to a coordinate system $x_\phi$ , $y_\phi$ , $z_\phi$ , with orientation such that $X_\phi$ is directed with an azimuth and sense the same as the sound wave normal. The $x_\phi$ , $y_\phi$ , $z_\phi$ system represents a clockwise rotation of amount $\frac{\pi}{2} + \theta$ ,of the $x$, $y$, $z$ coordinate system. For the other space array a rotation of amount $\frac{\pi}{2} + \theta'$ has been performed, and the new axes bear the coordinates $x'_{\phi'}$ , $y'_{\phi'}$ , $z'_{\phi'}$ .

The sound wave which arrives with azimuth $\theta$, elevation $arc\ cos\ \sqrt{1-v^2}$ at the space array (1) has a particular sound path appropriate to it only; likewise for the sound wave incident at space array (2). If we consider individual strata of atmosphere, wherein the vertical gradients of virtual temperature and of wind components $u_{\phi}$, $v_{\phi}$ for array (1), $u_{\phi'}$, $v_{\phi'}$ for array (2), are constant, we can deduce the range- and drift- increments for each stratum by recourse to the appropriate solutions for $\Delta X_{\phi}$ and $\Delta Y_{\phi}$ derived in Section V. To this end we also require the time-increment $\Delta t$, the time of travel of the sound wave through the layer *along the appropriate path*. Inspection of the solutions for these increments reveals that they are functions of various constants for the layer considered, and of the cosine of the angle of inclination of the wave normal *at the boundaries of the layer only*, which we can readily determine by means of the refraction equation (37). And therefore, for a given layer whose meteorological structure is known, and for a given ray along which the variation of $\eta$ is known, *we can determine the position in space of the intersection of the sound ray with horizontal planes representing the boundaries of the various layers traversed by the sound wave.* The coordinates of these points will be the cumulative sums of range- and drift- increments, calculated from the origin of the reference coordinate system, at the appropriate levels.

Since the points along the sound ray appropriate to array (1) are expressed in terms of $x_{\phi}$, $y_{\phi}$, $z_{\phi}$, and those pertaining to array (2) in $x'_{\phi'}$, $y'_{\phi'}$, $z'_{\phi'}$, it is required that a common coordinate system be established for ease of reference, and for the determination of the point of intersection of the two sound rays. For the sake of convenience in position-locating on prepared maps, we shall use as the standard system the $x, y, z$ coordinate system already chosen for tetrahedral array (1). The points expressed in $x_{\phi}$, $y_{\phi}$, $z_{\phi}$ coordinates may be converted into $x, y, z$ coordinates by simple rotation. Thus

$$\begin{aligned} x &= x_{\phi} \cos\left(\frac{\pi}{2}+\theta\right) + y_{\phi} \sin\left(\frac{\pi}{2}+\theta\right) \\ y &= y_{\phi} \cos\left(\frac{\pi}{2}+\theta\right) - x_{\phi} \sin\left(\frac{\pi}{2}+\theta\right) \\ z &= z. \end{aligned} \qquad (134)$$

But $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$, $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$, so that equation (134) can be written more simply as

$$\begin{aligned} x &= -x_{\phi} \sin\theta + y_{\phi} \cos\theta \\ y &= -x_{\phi} \cos\theta - y_{\phi} \sin\theta \\ z &= z. \end{aligned} \qquad (135)$$

Coordinates in the $x'_{\phi'}$, $y'_{\phi'}$, $z'_{\phi'}$ system when expressed in terms of the $x, y, z$ system require both a rotation and translation for conversion.

Thus initially

$$x = x' + P; \quad y = y' + N; \quad z = z' + E. \tag{136}$$

But

$$\begin{aligned} x' &= x'_{\bullet'} \cos\left(\frac{\pi}{2} + \theta'\right) + y'_{\bullet'} \sin\left(\frac{\pi}{2} + \theta'\right) \\ y' &= y'_{\bullet'} \cos\left(\frac{\pi}{2} + \theta'\right) - x'_{\bullet'} \sin\left(\frac{\pi}{2} + \theta'\right) \\ z' &= z'. \end{aligned} \tag{137}$$

Hence

$$\begin{aligned} x &= P - x'_{\bullet'} \sin\theta' + y'_{\bullet'} \cos\theta' \\ y &= N - x'_{\bullet'} \cos\theta' - y'_{\bullet'} \sin\theta' \\ z &= z' + E. \end{aligned} \tag{138}$$

As we proceed with the summation of the time- , range- , and drift- increments along the sound rays originally the paths of their respective sound wave surface, we approach the sound source, and therefore the differences in the $x$- and $y$- coordinates for points on each ray at fixed levels approach a minimum. Mathematically, this condition may be expressed as follows:

$$\begin{aligned} &\lim |- x_{\bullet} \sin\theta + y_{\bullet} \cos\theta - P + x'_{\bullet'} \sin\theta' - y'_{\bullet'} \cos\theta'| < \varepsilon \\ &\lim |- x_{\bullet} \cos\theta - y_{\bullet} \sin\theta - N + x'_{\bullet'} \cos\theta' + y'_{\bullet'} \sin\theta'| < \varepsilon \\ &\lim |z - z' - E| < \varepsilon, \end{aligned} \tag{139}$$

where $\varepsilon$ is a small positive quantity.

Theoretically speaking, the two sound rays intersect, since the sound waves emanate originally from a point source. Practically, however, in the process of tracing the ray through the atmosphere, utilizing for this purpose instrumental equipment subject to inherent errors, it would be a fortunate coincidence were the computed rays to intersect. Therefore we resolve the solution for the sound source into the determination of (*a*) ***the minimum distance between two sound rays,*** and (*b*) ***the position and orientation of the line segment representing this distance with regard to the standard coordinate system.*** The sound source can then be said to be located at some point of this line segment. The length of the line segment is an ***index of reliability*** of the sound ranging procedure involved in each instance of ranging operations.

22. **Minimum Distance between Two Sound Rays.**—For the purpose of ready identification, let the coordinates of general points referred to the $x, y, z,$ system be $x_1, y_1, z_1$ on the sound ray appropriate to space array (1) and $x_2, y_2, z_2$ on the ray appropriate to space array (2). Further, let the additional subscript 1 or 2, be appended to the coordinates of a particular point on each ray to indicate the level (1) or (2), respectively, at which the point on the ray is considered. Thus, $x_{11}, y_{11}, z_{11}; x_{12}, y_{12}, z_{12}$ are

the coordinates of points on ray (1) at the top and bottom of a layer of atmosphere, respectively; and likewise, $x_{21}$, $y_{21}$, $z_{21}$ ; $x_{22}$, $y_{22}$, $z_{22}$ are coordinates of points on ray (2) at the boundaries of the same layer of atmosphere.

For the layer in which $\varepsilon$ in equations (139) above is a minimum (at some point of which, therefore, the sound source is located), substitute for the sound rays (1) and (2) a straight line joining the end points of the ray at the boundaries of the layer. Provided the layer is not excessive in vertical dimension, this approximation is justified, since *within a single shallow layer* the sound ray does not depart appreciably from a straight line[11], so that whatever errors are introduced are inconsequential. It is to be emphasized, however, that such a substitution is not valid, in general, at any part of the sound ray except the last layer involved in the integration, since a slight error introduced elsewhere along the ray becomes a cumulative one in the summation over the remaining portion of the assumed path.

The equations of the lines joining the end points of the rays in the layer, expressed in two-point form, are

$$\left.\begin{aligned} \frac{x - x_{11}}{x_{12}-x_{11}} &= \frac{y - y_{11}}{y_{12}-y_{11}} = \frac{z - z_{11}}{z_{12}-z_{11}} \\ \frac{x - x_{21}}{x_{22}-x_{21}} &= \frac{y - y_{21}}{y_{22}-y_{21}} = \frac{z - z_{21}}{z_{22}-z_{22}} \end{aligned}\right\} . \qquad (140)$$

The denominators are proportional to the direction cosines of the lines, $\lambda_1$, $\mu_1$, $\nu_1$ for line (1), $\lambda_2$, $\mu_2$, $\nu_2$ for line (2), where

$$\lambda_1 = \frac{x_{12}-x_{11}}{r_1} ; \quad \mu_1 = \frac{y_{12}-y_{11}}{r_1} ; \quad \nu_1 = \frac{z_{12}-z_{11}}{r_1} , \qquad (141)$$

where $r_1 = \pm\sqrt{(x_{12}-x_{11})^2 + (y_{12}-y_{11})^2 + (z_{12}-z_{11})^2}$ ,

with corresponding expressions for $\lambda_2$, $\mu_2$, $\nu_2$, Hence equations (140) above can be written in the symmetric form of space geometry,

$$\frac{x - x_{11}}{\lambda_1} = \frac{y - y_{11}}{\mu_1} = \frac{z - z_{11}}{\nu_1} ; \quad \frac{x - x_{21}}{\lambda_2} = \frac{y - y_{21}}{\mu_2} = \frac{z - z_{21}}{\nu_2} . \qquad (142)$$

The angle $\Psi$ between the lines is given by

$$\cos \psi = \lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2 ,$$

[11] Moreover, we are at liberty to sub-divide any layer of constant $\alpha$ and $\frac{dv_s}{dz}$ into as many layers as desired or required; therefore a sufficiently shallow layer can always be secured.

or

$$\sin^2 \psi = (\lambda_1 \mu_2 - \lambda_2 \mu_1)^2 + (\mu_1 \nu_2 - \mu_2 \nu_1)^2 + (\nu_1 \lambda_2 - \nu_2 \lambda_1)^2 \quad . \tag{143}$$

If the two lines given by equations (142) intersect, then their point of intersection can be readily determined by solving the equations simultaneously. We have, then,

$$\frac{x - x_{11}}{\lambda_1} = \frac{y - y_{11}}{\mu_1} \text{ and } \frac{x - x_{21}}{\lambda_2} = \frac{y - y_{21}}{\mu_2} \quad .$$

so that

$$x = x_{11} + \frac{\lambda_1}{\mu_1}(y - y_{11}) \text{ and } x = x_{21} + \frac{\lambda_2}{\mu_2}(y - y_{21}) \quad .$$

On eliminating $x$ and solving for $y$, and subsequently for $x$ and $z$, we finally obtain

$$\left.\begin{aligned} y_{L1-VP2} &= y_{11} + \frac{\mu_1 \mu_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(x_{21} - x_{11}) - \frac{\lambda_2 \mu_1}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(y_{21} - y_{11}) \\ x_{L1-VP2} &= x_{11} - \frac{\lambda_1 \lambda_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(y_{21} - y_{11}) + \frac{\lambda_1 \mu_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(x_{21} - x_{11}) \\ z_{L1-VP2} &= z_{11} + \frac{\mu_2 \nu_1}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(x_{21} - x_{11}) - \frac{\lambda_2 \nu_1}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(y_{21} - y_{11}) \end{aligned}\right\} \quad . \tag{144}$$

The point determined by relations (144) above is the intersection of line (1) with the vertical plane containing the line (2); and hence the designated subscript. For the values $x$, $y$, given in (144) for this point of intersection, we find on substitution in the equation

$$\frac{y - y_{21}}{\mu_2} = \frac{z - z_{21}}{\nu_2}$$

the vertical coordinate $z_{L2}$ on line (2) when $x$ and $y$ have these values. Thus

$$\begin{aligned} z_{L2} &= z_{21} + \frac{\nu_2}{\mu_2} \left[y_{11} + \frac{\mu_1 \mu_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(x_{21} - x_{11}) - \frac{\lambda_2 \mu_1}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(y_{21} - y_{11}) - y_{21}\right] \\ &= z_{21} + \frac{\mu_1 \nu_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(x_{21} - x_{11}) - \frac{\lambda_1 \nu_2}{\lambda_1 \mu_2 - \lambda_2 \mu_1}(y_{21} - y_{11}) \quad . \end{aligned} \tag{145}$$

If the two given lines intersect, then $z_{L1-VP2} = z_{L2}$, and the point $x_{L1-VP2}$, $y_{L1-VP2}$, $z_{L1-VP2}$ determined above is the position of the sound source.

If the two lines do not intersect, we can determine the shortest distance between them. This distance will be a line segment perpendicular to each of the lines. The direction cosines $\lambda$, $\mu$, $\nu$ of the line segment which represents the shortest distance between the lines, can be determined in terms of the direction cosines $\lambda_1$, $\mu_1$, $\nu_1$, and $\lambda_2$, $\mu_2$, $\nu_2$. Thus from (143) we have

$$\lambda_1 \lambda + \mu_1 \mu + \nu_1 \nu = 0 \qquad (146)$$
$$\lambda_2 \lambda + \mu_2 \mu + \nu_2 \nu = 0 .$$

Eliminating $\lambda$ we obtain

$$\mu(\lambda_2\mu_1 - \lambda_1\mu_2) + \nu(\nu_1\lambda_2 - \nu_2\lambda_1) = 0 ;$$

and eliminating $\mu$

$$\lambda(\lambda_1\mu_2 - \lambda_2\mu_1) + \nu(\mu_2\nu_1 - \mu_1\nu_2) = 0 ;$$

so that

$$\frac{\lambda}{\mu_1\nu_2 - \mu_2\nu_1} = \frac{\mu}{\nu_1\lambda_2 - \nu_2\lambda_1} = \frac{\nu}{\lambda_1\mu_2 - \lambda_2\mu_1} . \qquad (147)$$

Substituting in (143) we have

$$\sin^2 \Psi = \frac{\nu^2}{\lambda^2}(\mu_1\nu_2 - \mu_2\nu_1)^2 + (\mu_1\nu_2 - \mu_2\nu_1)^2 + \frac{\mu^2}{\lambda^2}(\mu_1\nu_2 - \mu_2\nu_1)^2$$
$$= \frac{\nu^2 + \lambda^2 + \mu^2}{\lambda^2}(\mu_1\nu_2 - \mu_2\nu_1)^2 ,$$

so that we can further write

$$\frac{\lambda}{\mu_1\nu_2 - \mu_2\nu_1} = \frac{\mu}{\nu_1\lambda_2 - \nu_2\lambda_1} = \frac{\nu}{\lambda_1\mu_2 - \lambda_2\mu_1} = \frac{\pm 1}{\sin \Psi} , \qquad (148)$$

where $\Psi$ is the angle between the given lines.

The length $d$ of the line segment which is the shortest distance between the given lines* is equal to the projection on it of any other segment joining a point on one of the given lines with any point on the other. Using points $x_{11}$, $y_{11}$, $z_{11}$ and $x_{21}$, $y_{21}$, $z_{21}$, we have

$$d = \lambda(x_{11} - x_{21}) + \mu(y_{11} - y_{21}) + \nu(z_{11} - z_{21}) . \qquad (149)$$

By use of relations (148) we can write,

$$d = \frac{\pm 1}{\sin \Psi} [(\mu_1\nu_2 - \mu_2\nu_1)(x_{11} - x_{21}) + (\nu_1\lambda_2 - \nu_2\lambda_1)(y_{11} - y_{21})$$
$$+ (\lambda_1\mu_2 - \lambda_2\mu_1)(z_{11} - z_{21})] ,$$

*Hence perpendicular to both.

or in determinant form,

$$d = \frac{\pm 1}{\sin \psi} \begin{vmatrix} (x_{11} - x_{21}) & \lambda_1 & \lambda_2 \\ (y_{11} - y_{21}) & \mu_1 & \mu_2 \\ (z_{11} - z_{21}) & \nu_1 & \nu_2 \end{vmatrix} . \qquad (150)$$

The length of the segment, then, is a function of but only the endpoints of the given lines at the boundaries of the layer.

**23. Locus of the Sound Source.**—Solving (149) simultaneously with equations (146) for $\lambda$, $\mu$, $\nu$, we have

$$\lambda = \frac{d}{\Delta}\begin{vmatrix} 0 & \mu_1 & \nu_1 \\ 0 & \mu_2 & \nu_2 \\ 1 & y_{11}-y_{21} & z_{11}-z_{21} \end{vmatrix} ; \mu = \frac{d}{\Delta}\begin{vmatrix} \lambda_1 & 0 & \nu_1 \\ \lambda_2 & 0 & \nu_2 \\ x_{11}-x_{21} & 1 & z_{11}-z_{21} \end{vmatrix} ; \nu = \frac{d}{\Delta}\begin{vmatrix} \lambda_1 & \mu_1 & 0 \\ \lambda_2 & \mu_2 & 0 \\ x_{11}-x_{21} & y_{11}-y_{21} & 1 \end{vmatrix} , \qquad (151)$$

$$\Delta \equiv \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ x_{11}-x_{21} & y_{11}-y_{21} & z_{11}-z_{21} \end{vmatrix} .$$

Substituting from (150) we have the simpler expressions

$$\lambda = \pm \frac{\mu_1 \nu_2 - \mu_2 \nu_1}{\sin \psi} ; \quad \mu = \pm \frac{\lambda_1 \nu_2 - \lambda_2 \nu_1}{\sin \psi} ; \quad \nu = \pm \frac{\lambda_1 \mu_2 - \lambda_2 \mu_1}{\sin \psi} , \qquad (152)$$

where the proper sign is chosen so that $d$ is measured in positive sense from the given line (1) to line (2). Comparison of $z_{L1} - _{VP2}$ and $z_{L2}$ in (144) and (145) above will indicate the direction of the perpendicular required.

Having so determined $d$, the minimum distance between the two lines, and the direction cosines $\lambda$, $\mu$, $\nu$ of this common perpendicular, we can regard this line segment as the locus of the sound source. However, we further require the space coordinates of the end-points of the common perpendicular, so that the locus of the sound source may be definitely known with respect to the origin of the reference coordinate system.

To determine the coordinates of the point of intersection of the common perpendicular with line (1), translate line (2) parallel to itself the distance $d$ in the plane containing both line (2) and the common perpendicular. The $x_2$, $y_2$, $z_2$ coordinates of fixed points on line (2) then become $x_2 - \lambda d$, $y_2 - \mu d$, $z_2 - \nu d$, and the equation of line (2) in its position after the specified translation is

$$\frac{x - (x_{21} - \lambda d)}{\lambda_2} = \frac{y - (y_{21} - \mu d)}{\mu_2} = \frac{z - (z_{21} - \nu d)}{\nu_2} \qquad (153)$$

Solving equation (153) simultaneously with the equation for line (1) given in (142),

$$x = (x_{21} - \lambda d) + \frac{\lambda_2}{\mu_2} y - \frac{\lambda_2}{\mu_2}(y_{21} - \mu d)$$

$$x = x_{11} + \frac{\lambda_1}{\mu_1}(y - y_{11}) \quad ,$$

so that

$$\left(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}\right) y = x_{11} - (x_{21} - \lambda d) + \frac{\lambda_2}{\mu_2}(y_{21} - \mu d) - \frac{\lambda_1}{\mu_1} y_{11} \quad ,$$

or

$$\left.\begin{aligned} y_{p=o} &= y_{11} + \frac{\mu_1\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1}[(x_{21} - \lambda d) - x_{11}] - \frac{\lambda_2\mu_1}{\lambda_1\mu_2 - \lambda_2\mu_1}[(y_{21} - \mu d) - y_{11}] \\ x_{p=o} &= x_{11} + \frac{\lambda_1\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1}[(x_{21} - \lambda d) - x_{11}] - \frac{\lambda_1\lambda_2}{\lambda_1\mu_2 - \lambda_2\mu_1}[(y_{21} - \mu d) - y_{11}] \\ z_{p=o} &= z_{11} + \frac{\mu_2\nu_1}{\lambda_1\mu_2 - \lambda_2\mu_1}[(x_{21} - \lambda d) - x_{11}] - \frac{\lambda_2\nu_1}{\lambda_1\mu_2 - \lambda_2\mu_1}[(y_{21} - \mu d) - y_{11}] \quad , \end{aligned}\right\} \quad (154)$$

where $x_{p=o}$, $y_{p=o}$, $z_{p=o}$ are the coordinates of the intersection of the perpendicular with line (1).

Lastly, the intersection of the common perpendicular with line (2) is given by

$$\left.\begin{aligned} x_{p=p} &= x_{p=o} + \lambda d \\ y_{p=p} &= y_{p=o} + \mu d \\ z_{p=p} &= z_{p=o} + \nu d \quad . \end{aligned}\right\} \quad (155)$$

**24. Expressions of Position of Sound Source for Purpose of Map-Plotting.—** The mathematical solution for the sound source is, therefore,

1. Equation (144) if the sound rays intersect, using as a criterion relation (145) in connection with (144) ; or
2. The common perpendicular *d*, representing the shortest distance between the rays, positioned as follows:

   *a*. Direction cosines $\lambda$, $\mu$, $\nu$ by equation (152)

   *b* Length *d* by equation (150)

   *c*. End-points as given in equations (154) and (155) .

   We shall choose the mid-point of the line segment *d* as the point source of sound, and utilize the magnitude of *d* as an *index of reliability*.

The evaluation of the position of the sound source in terms of space coordinates permits the immediate adaptation of this determination to expression of the location of the source in terms of *grid-coordinates* for purpose of map-plotting, provided of course, that the grid-coordinates of the origin of the reference coordinate system are known. The elevation of the source relative to the origin of the reference system is determined in addition.

The position of the sound source may also be expressed as (*a*) an angular deflection (*true azimuth*) of the source from a fixed position, and (*b*) the distance along this azimuth direction from the fixed position to source (*azimuth-range*). Letting $\zeta$ represent the true azimuth angle from north, $R$ the range along the azimuth, we have

$$\zeta = arc\ tan\ \frac{X_s - X_0}{Y_s - Y_0}\ ,$$

$$R = \sqrt{(X_s - X_0)^2 + (Y_s - Y_0)^2}\ , \qquad (154)$$

where $(X_s, Y_s)$ and $(X_0, Y_0)$ are the grid-coordinates of the sound source and fixed position, respectively.

**25. Solution for Horizontal Propagation of Sound.**—The solutions given above are applicable whenever the paths of the sound waves considered are space curves. The case where the sound propagation is along the horizontal is less suited to rigorous analysis, since irregularities of terrain and local micro-meteorological influences both operate in the direction of non-homogeneity of the medium about the sound path. Diffraction over local obstacles adds to complexity in the interpretation of the net effect of the influences operating. Perhaps the best solution in this case is to assume the net effect of the disturbing influences zero, and correct for the observed azimuth of sound approach at each array by a factor proportional to the ratio of drift wind at the ground level to the actual velocity of the sound along the horizontal. The intersection of the two lines representing the corrected azimuths will then indicate the position of the sound source.

It is to be noted, however, that if the angle of inclination of the wave normal at a space array is zero, this observation need not necessarily be interpreted as the horizontal propagation of sound. It is conceivable that, on occasions, a particular sound wave, though propagated along a space curve in the atmosphere and totally refracted at some level, may arrive at the microphone array at zero angle with respect to the horizontal. The sound wave may thus be considered to be totally refracted at the microphone array by the shallow layer of atmosphere next to the surface of the ground. However, an adequate criterion is available in the refraction equation (37), since (*a*) if the ground surface layer of atmosphere is the medium resulting in total refraction of the sound wave, the inclination of the wave normal at the upper boundary of the atmospheric layer adjacent to the ground will be other than $\eta = 1$; whereas (*b*) if the direction of sound propagation is along the horizontal for the entire path, then $\eta = 1$ both at the array and at the upper boundary of the surface layer of atmosphere.

The solution for the sound source for case (*a*) is no different from that for the general case above. The solution for case (*b*) is, on the basis of the assumptions above, the angular correction $\varepsilon$ to be applied to $\theta$ where

$$\varepsilon = arc\ tan \frac{-v_0}{V_0} \quad , \tag{157}$$

$V_0$ being the velocity of sound along the wave front normal. The true azimuth of the sound source from the fixed position is $\theta + \varepsilon$ .

Using the coordinates indicated for the two arrays illustrated in Figure 5, we have for the case of horizontal sound propagation,

$$\left.\begin{aligned} x &= y\ tan\ (\theta + \varepsilon) \text{ for ray (1)} \\ \frac{x'}{y'} &= tan\ (\theta' + \varepsilon') \text{ for ray (2)} \end{aligned}\right\} \quad . \tag{158}$$

But $\qquad x = P + x'\ , \quad y = N + y',$

so that $\qquad x = y\ tan\ (\theta + \varepsilon)$

$$\frac{x - P}{y - N} = tan\ (\theta' + \varepsilon'),$$

so that finally

$$\left.\begin{aligned} y_s &= \frac{P - N\ tan\ (\theta' + \varepsilon')}{tan\ (\theta + \varepsilon) - tan\ (\theta' + \varepsilon')} \\ x_s &= \frac{P - N\ tan\ (\theta' + \varepsilon')}{tan\ (\theta + \varepsilon) - tan\ (\theta' + \varepsilon')} \cdot tan\ (\theta + \varepsilon) \ . \end{aligned}\right\} \tag{159}$$

where $x_s$ , $y_s$ are the coordinates of the source referred to the $x$, $y$, $z$, coordinate system.

Knowing $V_0$, the velocity of sound along the normal and therefore in this case the same for the horizontal, and knowing the azimuth range $R$, where $R = \sqrt{x_s^2 + y_s^2}$ , the time of travel is given by

$$t = \frac{R}{V_0} \quad . \tag{160}$$

If the sound propagation from a given source to one array is along the horizontal and along a space curve to another array, the solution for the source is determined as in the general case above.

## SUMMARY

In the previous sections equations have been derived which can be utilized in tracing the sound ray through space from the microphone array back to the sound source. For this computation a detailed knowledge of the meteorological structure of the atmosphere is required, in particular, the variation of temperature, humidity, wind direction, and wind velocity with elevation.

Equations (4), (5), and (6) of Section II afford a means of determining the direction of sound arrival at the microphone array, both in azimuth and in inclination to the horizontal.

Equations (16) and (17) of Section III express the velocity of sound as a function of temperature and humidity. By this means the variation of sound velocity with height can be determined from a measurement of these meteorological elements.

Equation (37) of Section V is the refraction equation which relates the inclination of the sound wave normal at one point with its inclination at another, provided the meteorological elements at each point are known. By means of this equation it is possible to trace the sound ray through the atmosphere to its point of total refraction. Equation (45) expresses the time of travel of the sound ray through a layer of atmosphere, and equations (64) and (82) or (85) afford the means of computing the "meteorological corrections" along the path of propagation.

Equations (144), (145), (150), (152), (154), and (155) of Section VII afford a means of computing the position of the sound source mathematically from the space curves representing sound paths to two microphone arrays.

In a subsequent report a description of a practical field procedure for rapid calculation of the sound path will be presented. A special slide rule, with scales incorporating the features of the above equations, will be described, and its operation for computation of meteorological corrections for sound ranging will be discussed.

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APPENDIX A

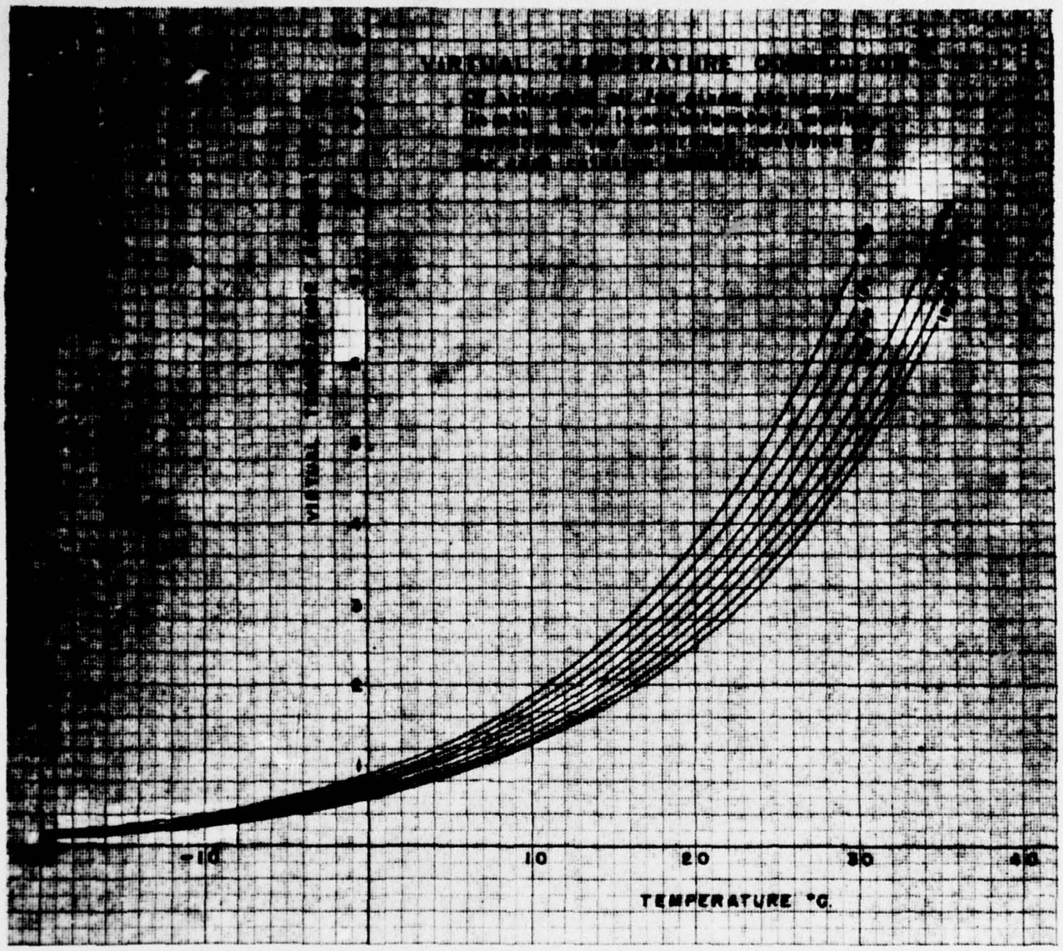


Chart of Virtual Temperature Correction



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