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ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF SOME CLASSIFICATI--ETC(U)
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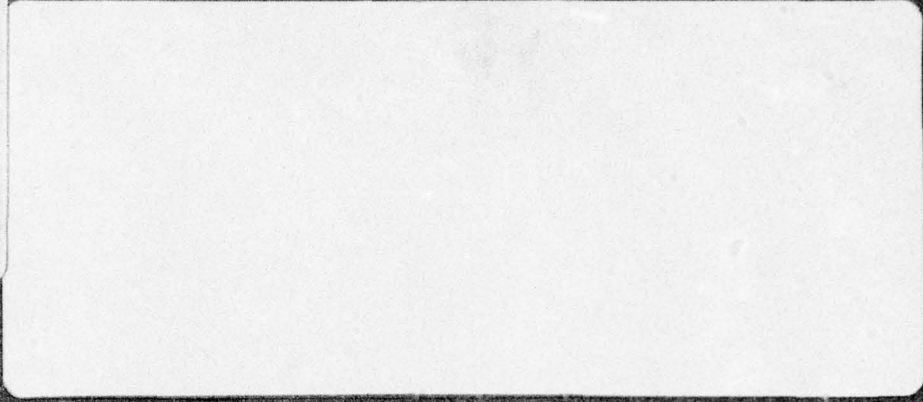
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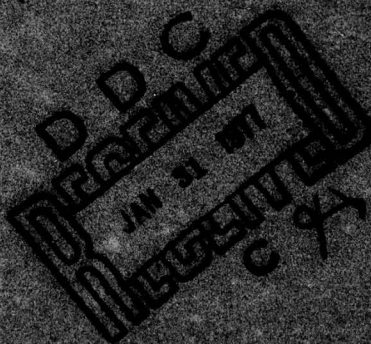
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ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF SOME
CLASSIFICATION STATISTICS AND THE PROBABILITIES
OF MISCLASSIFICATION WHEN THE TRAINING SAMPLES ARE
DEPENDENT

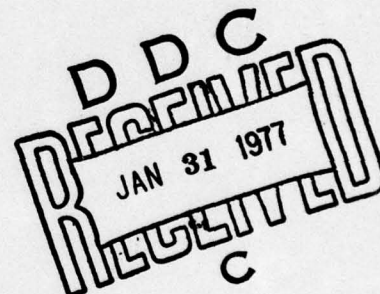
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SUMMARY

Asymptotic expansions of the distributions of some classification statistics and associated probabilities of misclassification are considered for a two population classification problem when the population distributions follow a stationary Gaussian process. Special cases have been considered when the population distributions follow a first order autoregressive process and, in particular, the probabilities of misclassification is studied as a function of the measure of dependence between the two populations.

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1. Introduction: Let ω_0 be an experimental unit which is a random outcome from a population π . It is known that π is identical to one of the two specified populations π_1 and π_2 , where π_1 and π_2 denote the same population π^* at two different points of time t_1 and t_2 , respectively. Let $X_0 = X(\omega_0)$ be a $p \times 1$ vector of measurement on the unit ω_0 . The problem is to identify π with one of π_1 and π_2 based on X_0 and the knowledge of the distributions of X_0 in π_1 and π_2 , which are not completely known. Information about these distributions is obtained from a sample of N units $\omega_1, \dots, \omega_N$ (called training sample) from π^* with $X_{i\alpha}$ as the X -observation on the unit ω_α observed at time t_i , $\alpha=1, \dots, N$; $i=1, 2$.

Let X_t denote the X -observation at time t . We shall assume that

$$(1.1) \quad X_t = m_t + U_t,$$

where U_t follows a stationary Gaussian process, and in particular,

$$(1.2) \quad \begin{pmatrix} U_{t_1} \\ U_{t_2} \end{pmatrix} \sim N_{2p} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau & \Sigma \end{pmatrix} \right],$$

where Σ is a nonsingular matrix and τ is a $p \times p$ symmetric matrix (see Anderson (1971)). Then $(X_{1\alpha}^1 X_{2\alpha}^1)$, $(\alpha=1, \dots, N)$ are i.i.d. and

$$(1.3) \quad \begin{pmatrix} X_{1\alpha} \\ X_{2\alpha} \end{pmatrix} \sim N_{2p} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau & \Sigma \end{pmatrix} \right],$$

where

$$(1.4) \quad \mu_i = m_{t_i}, \quad i=1, 2.$$

A special case of (1.2) will also be treated when U_t follows a first order autoregressive process, i.e.,

$$(1.5) \quad U_t = \lambda U_{t-1} + \epsilon_t, \quad t=0, \pm 1, \pm 2, \dots,$$

where $|\lambda| < 1$ and ϵ_t 's are i.i.d. $N_p(0, \Lambda)$. Then we can take (see Anderson (1971))

$$(1.6) \quad \tau = \rho \Sigma, \quad |\rho| < 1.$$

Let H_1 denote the hypothesis that ω_0 is from π_1 ($i=1,2$). When μ_1, μ_2 and Σ are known the form of a likelihood-ratio rule (see Anderson (1958)) is given by the following: Accept H_1 iff

$$(1.7) \quad W^* = ((X_0 - \mu_2; \Sigma)) - ((X_0 - \mu_1; \Sigma)) > k,$$

where k is a constant, and for a $p \times 1$ vector Y and a $p \times p$ nonsingular matrix B

$$(1.8) \quad ((Y; B)) = Y' B^{-1} Y.$$

When some of the parameters μ_1, μ_2 and Σ in (1.7) are unknown we replace them by their respective estimates, chosen suitably, based on the samples $(X'_{1\alpha}, X'_{2\alpha})$, $\alpha=1, \dots, N$. Such rules will be called plug-in likelihood-ratio (PLR) rules. The notion of PLR rules was first introduced by Wald (1944).

In this paper we assume that μ_1 and μ_2 are not known and we consider several cases depending on the available knowledge about Σ and τ .

The estimates of μ_1 and μ_2 will be taken, respectively, as \bar{X}_1 and \bar{X}_2 , where

$$(1.9) \quad \bar{X}_i = \frac{1}{N} \sum_{\alpha=1}^N X_{i\alpha}, \quad i=1,2.$$

Define

$$(1.10) \quad S_{ij} = \sum_{\alpha=1}^N (X_{i\alpha} - \bar{X}_i)(X_{j\alpha} - \bar{X}_j)'; \quad i,j=1,2.$$

The plug-in version of W^* is given in general, by

$$(1.11) \quad W_N = (\bar{X}_1 - \bar{X}_2)' B^{-1} (X_0 - \frac{1}{2}(\bar{X}_1 + \bar{X}_2)),$$

where the matrix B is defined below.

Case (a): $B = \Sigma$, when Σ is known.

Case (b): When Σ and τ are unknown,

$$(1.12) \quad B = (S_{11} + S_{22}) / (2N-2).$$

Case (c): When Σ is unknown and $\tau = \rho\Sigma$ with known ρ ,

$$(1.13) \quad B = (S_{11} - \rho S_{12} - \rho S_{21} + S_{22}) / (1 - \rho^2)(2N-2).$$

Case (d): When Σ is unknown and $\tau = \rho\Sigma$ with unknown ρ ,

$$(1.14) \quad B = (S_{11} + S_{22}) / (2N-2).$$

Note that in each of the above cases B is an unbiased consistent estimate of Σ . In cases (b) and (c), B is an asymptotically efficient estimate of Σ ; however, an asymptotically efficient estimate of Σ in case (d) is difficult to obtain (when $p > 1$). The limiting distribution of W_N as $N \rightarrow \infty$ is normal with variance α and mean $\frac{1}{2}\alpha$ if X_0 is from π_1 and mean $-\frac{1}{2}\alpha$ if X_0 is from π_2 , where

$$(1.15) \quad \alpha = ((\mu_1 - \mu_2; \Sigma)).$$

We shall assume $\alpha > 0$.

In this paper we derive asymptotic expansions of the distributions of $(W_N - \frac{1}{2}Q_N)/Q_N^{1/2}$ and $(W_N + \frac{1}{2}Q_N)/Q_N^{1/2}$ when $E X_0 = \mu_1$ and $E X_0 = \mu_2$, respectively, in powers of $1/N$, where

$$(1.16) \quad Q_N = ((\bar{X}_1 - \bar{X}_2; B)).$$

Anderson (1973) obtained the asymptotic expansions of these distributions when $\tau = 0$ and B is given by (1.11). We shall closely follow Anderson's method; however we have to modify and extend Anderson's results because \bar{X}_1 and \bar{X}_2 are correlated and the presence of τ leads to a different estimate of Σ .

We shall also derive asymptotic expansions of the distributions of $(W_N - \frac{1}{2}\alpha)/\alpha^{1/2}$ and $(W_N + \frac{1}{2}\alpha)/\alpha^{1/2}$ when $E X_0 = \mu_1$ and $E X_0 = \mu_2$, respectively. This would extend the results of Okamoto (1963) who derived these asymptotic distributions when $\tau = 0$ and the usual estimate is taken for Σ . For deriving these results we shall again follow Anderson's method which is much simpler than Okamoto's method.

Finally, asymptotic expansions for $P(W_N \leq 0)$ will be obtained under both the hypotheses H_1 and H_2 .

In classification theory, usually the training samples from π_1 and π_2 are drawn independently, i.e., on different sets of units. This is the first paper where the classification problem is treated with a dependent training sample which occurs quite often in practice. The results in this paper also reveal the influence of the covariance matrix τ on the distributions and the probabilities of misclassification.

2. Asymptotic Expansions of the Distributions of Studentized W_N

There exists a nonsingular $p \times p$ matrix L such that

$$(2.1) \quad \begin{aligned} L \Sigma L' &= I_p, \\ L \tau L' &= D = \text{diag}(\rho_1, \dots, \rho_p). \end{aligned}$$

It can be seen that W_N and Q_N (in all the cases (a)-(d)) are invariant under the transformations

$$(2.2) \quad \begin{aligned} X_0 &\rightarrow L(X_0 - \mu_1) \\ \bar{X}_1 &\rightarrow L(\bar{X}_1 - \mu_1) \\ \bar{X}_2 &\rightarrow L(\bar{X}_2 - \mu_1) \\ S_{ij} &\rightarrow L S_{ij} L'; \quad i, j=1, 2. \end{aligned}$$

Hence, without loss of generality, we shall assume that

$$(2.3) \quad \begin{aligned} \Sigma &= I_p, \quad \tau = D = \text{diag}(\rho_1, \dots, \rho_p), \\ \mu_1 &= 0, \quad \mu_2 = -\delta, \end{aligned}$$

where

$$(2.4) \quad \delta' \delta = \alpha.$$

As in Anderson (1973), we define Y , Z and V as follows.

$$(2.5) \quad \begin{aligned} \bar{X}_1 - \bar{X}_2 &= \delta + Y/n^{1/2}, & \bar{X}_1 &= Z/n^{1/2}, \\ B &= I_p + V/n^{1/2}, & n &= 2(N-1). \end{aligned}$$

Then

$$(2.6) \quad \begin{pmatrix} Y \\ Z \end{pmatrix} \sim N_{2p} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2(n/N)(I_p - D) & (n/N)(I_p - D) \\ (n/N)(I_p - D) & (n/N)I_p \end{pmatrix} \right].$$

Let $Y = (Y_1, \dots, Y_p)'$, $Z = (Z_1, \dots, Z_p)'$, $V = [V_{ij}]$. Let J_n be the subset of the sample space defined by

$$(2.7) \quad J_n = \{ |Y_i| < 4(\log n)^{1/2}, \quad |Z_i| < 2(\log n)^{1/2}, \\ |V_{ij}| < 2 \log n; \quad i, j=1, 2, \dots, p \}.$$

The proof of the following lemma is given in the appendix.

Lemma 2.1: $P(J_n) = 1 - o(n^{-2}).$

Now

$$(2.8) \quad Q_N = ((\delta + Y/n^{1/2}; I_p + V/n^{1/2})),$$

and let

$$(2.9) \quad G_{1N} = (\delta + Y/n^{1/2})' (I_p + V/n^{1/2})^{-1} (Z/n^{1/2}),$$

$$(2.10) \quad G_{2N} = ((\delta + Y/n^{1/2}; (I_p + V/n^{1/2})^2)).$$

Assume now $\mathcal{E}X_0 = 0$ (i.e., under H_1). Then

$$(2.11) \quad P(W_N \leq u) = \mathcal{E} \Phi[(uQ_N^{1/2} + G_{1N})G_{2N}^{-1/2}],$$

where Φ is the c.d.f. of $N(0, 1)$. Now

$$(2.12) \quad | \mathcal{E} \Phi[(uQ_N^{1/2} + G_{1N})/G_{2N}^{1/2}] \chi(J_n) \\ - \mathcal{E} \Phi[(uQ_N^{1/2} + G_{1N})/G_{2N}^{1/2}] | \\ \leq \mathcal{E} \chi(J_n^c) = o(n^{-2}),$$

where χ stands for the indicator function of a set.

Following Anderson (1973), it can be shown, for sufficiently large n and Y, Z, V in J_n , that

$$(2.13) \quad \begin{aligned} \mathfrak{F}[(uG_N^{1/2} + G_{1N})/G_{2N}^{1/2}] &= \mathfrak{F}(u) + n^{-1/2}\varphi(u)C(Z,V) \\ &+ n^{-1}\varphi(u)[D(Y,Z,V) - (u/2)C^2(Z,V)] + R_n, \end{aligned}$$

where $\varphi(u) = d\mathfrak{F}(u)/du$,

$$(2.14) \quad C(Z,V) = (u/2\Delta^2)\delta'V\delta + (\delta'Z)/\Delta,$$

$$(2.15) \quad \begin{aligned} D(Y,Z,V) &= \Delta^{-1}(Y'Z - \delta'VZ) \\ &+ \Delta^{-2}u(\delta'VY - \delta'V^2\delta) \\ &- \Delta^{-3}(\delta'Y\delta'Z - \delta'Z\delta'V\delta) \\ &+ u\Delta^{-4}[(7/8)(\delta'V\delta)^2 - \delta'Y\delta'V\delta], \end{aligned}$$

and

$$\Delta = \alpha^{1/2}.$$

R_n is the sum of three terms R_{1n} , R_{2n} and R_{3n} , where R_{1n} is a homogeneous polynomial (not depending on n) of degree 3 in Y, Z and V multiplied by $n^{-3/2}$, R_{2n} is a homogeneous polynomial (not depending on n) of degree 4 in Y, Z and V multiplied by n^{-2} , and R_{3n} is $O((\log n)^5/n^{5/2})$.

Next, we follow Anderson's method, although we clarify his results and fill in some gaps. First note that the expectation of a polynomial in Y_i 's and Z_j 's is $O(1)$, and

$$(2.16) \quad \varepsilon_{V_{ij}}^{2k-1} = O(n^{-1/2}), \quad \varepsilon_{V_{ij}}^{2k} = O(1)$$

for any positive integer k . We shall now obtain the expectation of the right-hand side of (2.13) multiplied by $\chi(J_n)$.

$E[\delta'Z\chi(J_n)] = 0$ since $EZ = 0$ and J_n is symmetric about 0 in Z_i and the density of Z_i is symmetric about 0. Moreover, $E[\delta'V\delta\chi(J_n)] = O(n^{-3/2})$, since $EV = 0$,

$$|E(\delta'V\delta)\chi(J_n)| = |\delta'EV\chi(J_n^C)\delta|,$$

and

$$(2.17) \quad E[|v_{ij}|\chi(|v_{ij}| > 2 \log n)] \\ \leq [Ev_{ij}^2]^{1/2}P[\chi(|v_{ij}| > 2 \log n)] = O(n^{-3/2})$$

by taking $k = 3$ in (A.3). Thus

$$(2.18) \quad E[C(Z,V)\chi(J_n)] = O(n^{-3/2}).$$

Next, by Cauchy-Schwarz' inequality

$$(2.19) \quad |E[D(Y,Z,V) - (u/2)C^2(Z,V)] - E[D(Y,Z,V) - (u/2)C^2(Z,V)]\chi(J_n)| \\ \leq [E\{D(Y,Z,V) - (u/2)C^2(Z,V)\}^2]^{1/2}[P(J_n^C)]^{1/2} = o(n^{-1}).$$

Similarly

$$(2.20) \quad |ER_{1n} - ER_{1n}\chi(J_n)| = O(n^{-2}).$$

Since the third-order moments of the elements of Y , Z and V are either 0 or $O(n^{-1/2})$, $ER_{1n} = O(n^{-1/2})$. Note now

$$(2.21) \quad |ER_{2n}\chi(J_n)| \leq E|R_{2n}| = O(1),$$

$$(2.22) \quad |\mathcal{E}R_{3n}\chi(J_n)| = O((\log n)^5/n^{5/2}) = o(n^{-2}).$$

Combining the above results, we get

$$(2.23) \quad \begin{aligned} & \mathcal{E}\hat{\Phi}[(uQ_N^{1/2} + G_{1N})G_{2N}^{-1/2}] \\ &= \hat{\Phi}(u) + n^{-1}\varphi(u)\mathcal{E}[D(Y,Z,V) - (u/2)C^2(Z,V)] + O(n^{-2}). \end{aligned}$$

The following theorem is thus proved.

Theorem 2.1: When H_1 obtains

$$\begin{aligned} & P[(W_N - \frac{1}{2}Q_N)/Q_N^{1/2} \leq u] \\ &= \hat{\Phi}(u) + n^{-1}\varphi(u)\mathcal{E}[D(Y,Z,V) - (u/2)C^2(Z,V)] + O(n^{-2}). \end{aligned}$$

Note now

$$(2.24) \quad \begin{aligned} & P_1[(W_N - \frac{1}{2}Q_N)/Q_N^{1/2} \leq u] \\ &= P_2[(W_N + \frac{1}{2}Q_N)/Q_N^{1/2} \geq -u], \end{aligned}$$

where P_i denotes the probability under H_i , $i=1,2$. Thus we get the following corollary.

Corollary 2.1.1: When H_2 obtains

$$\begin{aligned} & P[(W_N + \frac{1}{2}Q_N)/Q_N^{1/2} \leq u] \\ &= \hat{\Phi}(u) - n^{-1}\varphi(u)\mathcal{E}[D(Y,Z,V) - (u/2)C^2(Z,V)] + O(n^{-2}). \end{aligned}$$

The above two results are similar to those of Anderson's (1973) except that we needed a modified proof to treat our case. The presence of τ now

leads to new problems regarding the evaluation of $\mathcal{E}D(Y,Z,V)$ and $\mathcal{E}C^2(Z,V)$. We shall consider this now for different cases.

Case (a): $B = I_p$. In this case $V = 0$ and $|v_{ij}| < 2 \log n$ for all i, j trivially holds.

$$\begin{aligned} \mathcal{E}D(Y,Z,V) &= \Delta^{-1} \mathcal{E}(Y'Z) - \Delta^{-3} \mathcal{E}(\delta'Y\delta'Z) \\ (2.25) \qquad &= \Delta^{-1} (n/N) \sum_{i=1}^p (1-\rho_i) - \Delta^{-3} (n/N) \delta'(I_p - D)\delta. \end{aligned}$$

In terms of the original parameters,

$$\begin{aligned} \mathcal{E}D(Y,Z,V) &= \Delta^{-1} (n/N) \text{tr}(I_p - \Sigma^{-1}\tau) \\ (2.26) \qquad &- \Delta^{-3} (n/N) [\Delta^2 - (\mu_1 - \mu_2)' \Sigma^{-1} \tau \Sigma^{-1} (\mu_1 - \mu_2)]. \end{aligned}$$

For the special case $\tau = \rho \Sigma$,

$$(2.27) \quad \mathcal{E}D(Y,Z,V) = \Delta^{-1} (n/N) (p-1)(1-\rho).$$

Next

$$(2.28) \quad \mathcal{E}C^2(Z,V) = \Delta^{-2} \mathcal{E}(\delta'Z)^2 = (n/N).$$

For what follows, (n/N) is replaced by 2. Theorem 2.1 now yields the following:

Theorem 2.2: When $B = \Sigma$ and H_1 obtains

$$\begin{aligned} P\left[\left(\frac{W_N}{N} - \frac{1}{2} \frac{Q_N}{N}\right) / Q_N^{1/2} \leq u\right] &= \Phi(u) + n^{-1} \varphi(u) [2\Delta^{-1}(p-1) - 2\Delta^{-1} \text{tr}(\Sigma^{-1}\tau) \\ &+ 2\Delta^{-3} (\mu_1 - \mu_2)' \Sigma^{-1} \tau \Sigma^{-1} (\mu_1 - \mu_2) - u] + O(n^{-2}), \end{aligned}$$

which reduces to

$$\hat{\Phi}(u) + n^{-1}\varphi(u)[2\Delta^{-1}(p-1)(1-\rho) - u] + O(n^{-2}),$$

when $\tau = \rho\Sigma$.

Case (b): $B = (S_{11} + S_{22}) / (2N - 2).$

Note that

$$\begin{aligned} & \mathcal{E}[D(Y, Z, V) - (u/2)C^2(Z, V)] \\ &= 2\Delta^{-1}\text{tr}(I_p - D) - 2\Delta^{-3}\delta'(I_p - D)\delta - u - u\Delta^{-2}\mathcal{E}(\delta'V^2\delta) \\ (2.29) \quad &+ [(7/8)u\Delta^{-4} - (u^3/8\Delta^4)]\mathcal{E}(\delta'V\delta)^2 + O(n^{-1}). \end{aligned}$$

In this case

$$(2.30) \quad V = (4n)^{-1/2}[B_1 + B_2 - 2nI_p],$$

where

$$(2.31) \quad B_1 = S_{11} + S_{22} + S_{12} + S_{21} \sim W_p[2(I_p + D), n/2],$$

$$(2.32) \quad B_2 = S_{11} + S_{22} - S_{12} - S_{21} \sim W_p[2(I_p - D), n/2],$$

and B_1 and B_2 are independently distributed. Hence

$$(2.33) \quad \mathcal{E}V^2 = (4n)^{-1}[\mathcal{E}(B_1 - \mathcal{E}B_1)^2 + \mathcal{E}(B_2 - \mathcal{E}B_2)^2].$$

$$(2.34) \quad \mathcal{E}(\delta'V\delta)^2 = (4n)^{-1}[\text{Var}(\delta'B_1\delta) + \text{Var}(\delta'B_2\delta)].$$

We shall reduce the above expressions using the following lemma.

Lemma 2.2: Let A be a $p \times p$ random matrix distributed as $W_p(\Lambda, m)$, where Λ is a nonsingular diagonal matrix. Then

$$(a) \quad E(A - EA)^2 = m[\Lambda^2 + \Lambda(\text{tr } \Lambda)],$$

and for any vector $\delta \neq 0$

$$(b) \quad \text{Var}(\delta' A \delta) = 2m(\delta' \Lambda \delta)^2.$$

The proof of this lemma is omitted.

Thus

$$\begin{aligned} EV^2 &= (4n)^{-1} [4(I_p + D)^2 + 4(I_p + D)\{\text{tr}(I_p + D)\} \\ &\quad + 4(I_p - D)^2 + 4(I_p - D)\{\text{tr}(I_p - D)\}] (n/2) \\ (2.35) \quad &= [(p+1)I_p + (\text{tr } D)D + D^2], \end{aligned}$$

$$(2.36) \quad E(\delta' V^2 \delta) = (p+1)\Delta^2 + \delta' D^2 \delta + \delta' D \delta (\text{tr } D),$$

$$(2.37) \quad E(\delta' V \delta)^2 = 2[\Delta^4 + (\delta' D \delta)^2].$$

From Theorem 2.1 and the above results we get the following theorem.

Theorem 2.3: When $B = (S_{11} + S_{22}) / (2N - 2)$ and H_1 obtains

$$\begin{aligned} P[(W_N - \frac{1}{2}Q_N) / Q_N^{1/2} \leq u] &= \Phi(u) + n^{-1} \phi(u) [2\Delta^{-1} \text{tr}(I_p - \Sigma^{-1} \tau) - u \\ &\quad - 2\Delta^{-3} (\mu_1 - \mu_2)' (\Sigma^{-1} - \Sigma^{-1} \tau \Sigma^{-1}) (\mu_1 - \mu_2) - u(p+1) \\ &\quad - u\Delta^{-2} (\mu_1 - \mu_2)' \Sigma^{-1} \tau \Sigma^{-1} \tau \Sigma^{-1} (\mu_1 - \mu_2) \end{aligned}$$

$$\begin{aligned}
& - u\Delta^{-2}(\mu_1 - \mu_2)' \Sigma^{-1} \tau \Sigma^{-1} (\mu_1 - \mu_2) \text{tr}(\tau \Sigma^{-1}) + ((7/4)u - u^3/4) \\
& + [(7/4)u\Delta^{-4} - (u^3/4)\Delta^4][(\mu_1 - \mu_2)' \Sigma^{-1} \tau \Sigma^{-1} (\mu_1 - \mu_2)]^2 + O(n^{-2}),
\end{aligned}$$

which reduces to

$$\begin{aligned}
\mathfrak{g}(u) + n^{-1}\varphi(u)[(2/\Delta)(p-1)(1-\rho) - u\{p + 1/4 + (p-3/4)\rho^2\} \\
- (u^3/4)(1+\rho^2)] + O(n^{-2}),
\end{aligned}$$

when $\tau = \rho\Sigma$.

Case (c): $B = (S_{11} - \rho S_{12} - \rho S_{21} + S_{22}) / (1 - \rho^2)(2N - 2)$, and $D = \rho I_p$.

In this case $2(N-1)B \sim W_p(I_p, n)$, and

$$V = n^{1/2}[B - \mathcal{E}(B)].$$

Hence, from Lemma 2.2

$$(2.38) \quad \mathcal{E}(\delta' V^2 \delta) = (p+1)\Delta^2,$$

$$(2.39) \quad \mathcal{E}(\delta' V \delta)^2 = 2\Delta^4.$$

Theorem 2.1 now yields the following.

Theorem 2.4: When $\tau = \rho\Sigma$, $B = (S_{11} - \rho S_{12} - \rho S_{21} + S_{22}) / (1 - \rho^2)(2N - 2)$ and H_1 obtains

$$\begin{aligned}
& P[(W_n - \frac{1}{2}Q_N) / Q_N^{1/2} \leq u] \\
& = \mathfrak{g}(u) + n^{-1}\varphi(u)[2\Delta^{-1}(1-\rho)(p-1) - u(p+1/4) - u^3/4] + O(n^{-2}).
\end{aligned}$$

3. Asymptotic Expansions of the Distributions of Normalized W_n .

First note that under H_1

$$(3.1) \quad P[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u] = \mathcal{E}\{[w + \frac{1}{2}G_{3n}]G_{2N}^{-1/2}\},$$

where

$$(3.2) \quad w = u\Delta + \Delta^2/2,$$

$$(3.3) \quad G_{3n} = (\bar{X}_1 + \bar{X}_2)' B^{-1} (\bar{X}_1 - \bar{X}_2),$$

and G_{2n} is given by (2.10). Define $U = (U_1, \dots, U_p)'$ by

$$(3.4) \quad (\bar{X}_1 + \bar{X}_2) = U/n^{1/2} - \delta,$$

and Y and V by (2.5). Then

$$(3.5) \quad \begin{pmatrix} Y \\ U \end{pmatrix} \sim N_{2p} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2(n/N)(I_p - D) & 0 \\ 0 & 2(n/N)(I_p + D) \end{pmatrix} \right].$$

Redefine J_n by J_n^* , where

$$(3.6) \quad J_n^* = \{ |Y_i| < 4(\log n)^{1/2}, |U_i| < 4(\log n)^{1/2}, \\ |V_{ij}| < 2 \log n; i, j=1, \dots, p \}.$$

Then it can be shown as in Lemma 2.1 that

$$(3.7) \quad P(J_n^*) = 1 - o(n^{-2}).$$

Replacing J_n by J_n^* and proceeding as in Section 2 it can be shown that under H_1

$$\begin{aligned}
 & P[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u] \\
 (3.8) \quad & = \Phi(u) + n^{-1}\varphi(u)\mathcal{E}[D^*(Y,U,V) - (u/2)C^{*2}(Y,U,V)] + O(n^{-2}),
 \end{aligned}$$

where

$$\begin{aligned}
 D^*(Y,U,V) &= (2\Delta)^{-1}[U'Y + \delta'VY - \delta'V^2\delta - U'V\delta] \\
 &\quad - (2\Delta^3)^{-1}[(\delta'Y - \delta'V\delta)(U'\delta + \delta'V\delta - \delta'Y)] \\
 (3.9) \quad &\quad - u[(2\Delta^2)^{-1}(3\delta'V^2\delta + Y'Y - 4\delta'VY) - 3(2\Delta^4)^{-1}(\delta'Y - \delta'V\delta)^2],
 \end{aligned}$$

$$(3.10) \quad C^*(Y,U,V) = (2\Delta)^{-1}(U'\delta + \delta'V\delta - \delta'Y) - u\Delta^{-2}/\delta'Y - \delta'V\delta.$$

Now, in terms of the original parameters

$$\begin{aligned}
 \mathcal{E}D^*(Y,U,V) &= \mathcal{E}(\delta'V\delta)^2[(2\Delta^3)^{-1} + 3u(2\Delta^4)^{-1}] \\
 &\quad - \mathcal{E}(\delta'V^2\delta)[(2\Delta)^{-1} + 3u(2\Delta^2)^{-1}] \\
 &\quad + (2/\Delta^3)(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2) \\
 &\quad - (2u/\Delta^2)\text{tr}(I_p - \Sigma^{-1}\tau) \\
 (3.11) \quad &\quad + (6u/\Delta^4)(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2) + O(1/n),
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{E}C^{*2}(Y,U,V) &= 2 + 4(u^2/\Delta^4 + u/\Delta^3)(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2) \\
 (3.12) \quad &\quad + [(1/2\Delta)^2 + (u^2/\Delta^4) + (u/\Delta^3)]\mathcal{E}(\delta'V\delta)^2 + O(1/n).
 \end{aligned}$$

Case (a): $B = I_p$.

Here $V = 0$ and we get the following theorem from (3.8) to (3.12).

Theorem 3.1: When Σ is known, $B = \Sigma$, and H_1 obtains

$$\begin{aligned} P[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u] &= \Phi(u) + n^{-1}\varphi(u)[2\Delta^{-3}(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2) \\ &\quad - u - 2u\Delta^{-2}\text{tr}(I_p - \Sigma^{-1}\tau) \\ &\quad - 2(u^3\Delta^{-4} + u^2\Delta^{-3})(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2) \\ &\quad + 6u\Delta^{-4}(\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2)] + O(n^{-2}), \end{aligned}$$

which reduces to

$$\begin{aligned} \Phi(u) + n^{-1}\varphi(u)[2(1-\rho)\Delta^{-1} - u - 2(p-3)(1-\rho)u\Delta^{-2} \\ - 2(1-\rho)(u^3\Delta^{-2} + u^2\Delta^{-1})] + O(n^{-2}), \end{aligned}$$

when $\tau = \rho\Sigma$.

Case (b): Σ and τ are unknown, and B is given by (1.11).

In this case, $P[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u]$ under H_1 can be obtained from (3.8), (3.11), (3.12), (2.36) and (2.37); due to its lengthy form the explicit expression is omitted.

Case (c): Σ is unknown, $\tau = \rho\Sigma$ with known ρ , and B is given by (1.12)

From (3.8), (3.11), (3.12), (2.38) and (2.39) we get the following.

Theorem 3.2: When H_1 obtains and B is given by (1.12)

$$\begin{aligned} P[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u] &= \Phi(u) + n^{-1}\varphi(u)[2(1-\rho)\Delta^{-1} - u(3p-1)/2 \\ &\quad - \Delta(p-1)/2 - 2(p-3)(1-\rho)u\Delta^{-2} - 2(1-\rho)(u^3\Delta^{-2} + u^2\Delta^{-1}) \\ &\quad - u(\Delta^2/4 + u^2 + u\Delta)] + O(n^{-2}). \end{aligned}$$

Similar results can be obtained when H_2 obtains using the following relation:

$$(3.13) \quad P_1[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} \leq u] = P_2[(W_N + \frac{1}{2}\alpha)/\alpha^{1/2} \geq -u],$$

where P_i is the probability under H_i ($i=1,2$).

4. Asymptotic Expansions of Probabilities of Misclassification.

Let us consider the rule which classifies X_0 into π_1 iff

$$(4.1) \quad W_N \geq 0,$$

where W_N is given by (1.11). Then the probability of the misclassification (PMC) for this rule under H_1 is given by

$$(4.2) \quad P_1[(W_N - \frac{1}{2}\alpha)/\alpha^{1/2} < -\Delta/2],$$

and its asymptotic expansion can be obtained from (3.8) with $u = \Delta/2$. It follows from (3.13) that the PMC for this rule under H_2 is the same as (4.2). Using results in Section 3 we shall now state the asymptotic expansions of (4.2) for different cases under consideration.

When Σ is known and B is taken as Σ , (4.2) reduces to

$$(4.3) \quad \begin{aligned} & \Phi(-\Delta/2) + n^{-1}\phi(\Delta/2)[\Delta/2 + \Delta^{-1}\text{tr}(I_p - \Sigma^{-1}\tau)] \\ & - (\mu_1 - \mu_2)'(\Sigma^{-1} - \Sigma^{-1}\tau\Sigma^{-1})(\mu_1 - \mu_2)[\Delta^{-3} + (4\Delta)^{-1}] + O(n^{-2}). \end{aligned}$$

In particular, when $\tau = \rho\Sigma$, (4.3) reduces to

$$(4.4) \quad \Phi(-\Delta/2) + n^{-1}\phi(\Delta/2)[\Delta(1+\rho)/4 + \Delta^{-1}(p-1)(1-\rho)] + O(n^{-2}).$$

When $\tau = \rho\Sigma$ with known ρ but unknown Σ , and B is given by (1.13), (4.2) becomes

$$(4.5) \quad \bar{\Phi}(-\Delta/2) + n^{-1}\varphi(\Delta/2)[\Delta(p-1)/4 + \Delta(1+\rho)/4 + (p-1)(1-\rho)\Delta^{-1}] + O(n^{-2}).$$

When $\tau = \rho\Sigma$ with both ρ and Σ unknown, and B is given by (1.15), (4.2) reduces to

$$(4.6) \quad \bar{\Phi}(-\Delta/2) + n^{-1}\varphi(\Delta/2)[\Delta(1+\rho)/4 + \Delta(1+\rho^2)(p-1)/4 + (p-1)(1-\rho)\Delta^{-1}] + O(n^{-2}).$$

Remarks:

(i) Assume $p > 1$. Then the PMC of the rule given by (4.1) can be studied as a function of ρ , to the order of approximations indicated, when $\tau = \rho\Sigma$. It follows from (4.4) and (4.5) that the PMC's in these two cases increase (or decrease) with ρ when $\Delta^2 > (\text{or } <) 4(p-1)$. It follows from (4.6) that the PMC in this case increases (or decreases) with ρ according as $\rho > (\text{or } <) [2\Delta^{-2} - (2p-2)^{-1}]$.

(ii) When $p = 1$, the PMC's in all the cases are increasing functions of ρ . However, it is shown in Bandyopadhyay (1974) that the exact PMC decreases as ρ decreases when Δ is large and it increases as ρ decreases when Δ is small.

(iii) When $\tau = 0$, the results in Section 2 yield Anderson's results (1973) and the results in Sections 3 and 4 yield Okamoto's results (1963).

APPENDIX

Proof of Lemma 2.1:

$$P[|Y_1| > 4(\log n)^{1/2}] = \left(\frac{2}{\pi}\right)^{1/2} \int_{t_n}^{\infty} e^{-1/2x^2} dx$$

where

$$t_n = 4(\log n)^{1/2} [2(n/N)(1-\rho_1)]^{-1/2}.$$

By Mill's ratio inequality

$$P[|Y_1| > 4(\log n)^{1/2}] \leq \left(\frac{2}{\pi}\right)^{1/2} e^{-1/2t_n^2} t_n^{-1}.$$

Now

$$\begin{aligned} & e^{-t_n^2/2} t_n^{-1} \\ &= e^{-8 \log n / (n-N)(1-\rho_1)} [4(\log n)^{1/2} \{2(n/N)(1-\rho_1)\}^{-1/2}]^{-1} \\ &= [n^{8N/n(1-\rho_1)} (\log n)^{1/2}]^{-1} [2(1-\rho_1)(n/N)]^{1/2} (1/4) \\ &\leq [n^2 (\log n)^{1/2}]^{-1} (2)^{-1/2} = o(n^{-2}). \end{aligned}$$

Hence

$$(A.1) \quad P[|Y_1| > 4(\log n)^{1/2}] \leq o(n^{-2}).$$

Similarly

$$(A.2) \quad P[|Z_1| > 2(\log n)^{1/2}] \leq o(n^{-2}).$$

From (1.12) and (1.13) it follows that $B = B_1 + B_2$,
 $(N-1)B_k \sim W_p[I_p, N-1]$, ($k=1,2$) but B_1 and B_2 are not necessarily
independent. Let $B_k = ((B_{ijk}))$. Now

$$V_{ii} > 2 \log n$$

$$\Leftrightarrow (\sqrt{n}/2)[B_{1ii} + B_{2ii} - 2] > 2 \log n$$

$$\Leftrightarrow (n/2)[B_{1ii} + B_{2ii}] > 2\sqrt{n} \log n + n.$$

Hence, for $\theta > 0$

$$\begin{aligned} P[V_{ii} > 2 \log n] &\leq e^{-\theta(\sqrt{n} \log n + n/2)} \\ &\quad \cdot E e^{(\theta/2)(B_{1ii} + B_{2ii})(N-1)} \\ &\leq e^{-\theta(\sqrt{n} \log n + n/2)} \\ &\quad \cdot E \left[e^{\theta(N-1)B_{1ii}} + e^{\theta(N-1)B_{2ii}} \right] (1/2) \\ &= e^{-\theta(\sqrt{n} \log n + n/2)} (1-2\theta)^{-n/4}, \quad 0 < \theta < 1/2. \end{aligned}$$

Let

$$(A.3) \quad \theta = k/\sqrt{n}.$$

Fix k and let n be sufficiently large such that $k/\sqrt{n} = \theta < 1/2$. Then

$$P[V_{ii} > 2 \log n] \leq e^{-k\sqrt{n}/2} e^{-k \log n} (1-2k/\sqrt{n})^{-n/4}.$$

But, since $n/4 = (\sqrt{n}/2k)k\sqrt{n}/2$, we have

$$P[V_{ii} > 2 \log n] \leq \text{Constant } e^{-k \log n} = O(n^{-k}).$$

Similarly

$$P[-V_{ii} > 2 \log n] \leq O(n^{-k}).$$

Hence

$$(A.4) \quad P[|V_{ii}| > 2 \log n] \leq O(n^{-k}).$$

Now let $i \neq j$.

$$V_{ij} > 2 \log n$$

$$\Leftrightarrow (n/4)(B_{1ij} + B_{2ij}) > \sqrt{n} \log n.$$

Hence

$$\begin{aligned} P[V_{ij} > 2 \log n] &\leq e^{-\theta \sqrt{n} \log n} E e^{(n/4)\theta(B_{1ij} + B_{2ij})} \\ &\leq e^{-\theta \sqrt{n} \log n} E \left[e^{(n/2)\theta B_{1ij}} + e^{(n/2)\theta B_{2ij}} \right] (1/2) \\ &= e^{-\theta \sqrt{n} \log n} (1-\theta^2)^{n/4}, \quad 0 < \theta < 1/2. \end{aligned}$$

Again, for fixed k and for θ as in (A.3) and for sufficiently large n such that $\theta = k/\sqrt{n} < 1/2$, we have

$$P[V_{ij} > 2 \log n] \leq e^{-k \log n} (1-k^2/n)^{-n/4} = O(n^{-k}).$$

Similarly,

$$P[-V_{ij} > 2 \log n] \leq O(n^{-k}).$$

Hence,

$$(A.5) \quad P[|v_{ij}| > 2 \log n] \leq O(n^{-k}).$$

Lemma 2.1 follows from (A.1), (A.2), (A.4) and (A.5) for $k > 2$.

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Asymptotic expansions of the distributions of some classification statistics and associated probabilities of misclassification are considered for a two population classification problem when the population distributions follow a stationary Gaussian process. Special cases have been considered when the population distributions follow a first order autoregressive process and, in particular, the probabilities of misclassification is studied as a function of the measure of dependence between the two populations.