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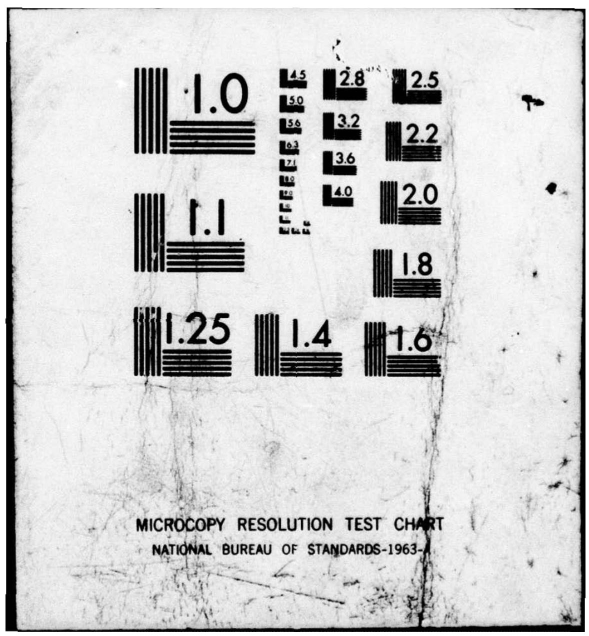
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UNIFIED TREATMENT OF INEQUALITIES OF THE WEIERSTRASS PRODUCT TYPE

by

Emad El-Neweihi, University of Kentucky and

Frank Proschan[1], Florida State University

FSU Statistics Report M396
AFOSR Technical Report No. 67

December, 1976
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Tallahassee, Florida 32306

See Form 1473

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ABSTRACT

A typical inequality of the Weierstrass product type states that $\prod_{i=1}^{n} (1+A_i) \ge (n+1)^n \prod_{i=1}^{n} A_i$, where $A_i \ge 0$, $i=1,\dots,n$, and $\sum_{i=1}^{n} A_i = 1$. This short note shows that the powerful tools of majorization and Schur functions provide a unified method for deriving a variety of Weierstrass type inequalities.

UNIFIED TREATMENT OF INEQUALITIES OF THE WEIERSTRASS PRODUCT TYPE

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Emad El-Neweihi, University of Kentucky and

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In a note by Klamkin and Newman [1], it was shown that

$$(1) \quad \prod_{i=1}^{n} (1+A_i) \geq (n+1)^n \prod_{i=1}^{n} A_i$$

$$(2) \quad \prod_{i=1}^{n} (1-A_i) \geq (n-1)^n \prod_{i=1}^{n} A_i,$$

where $A_i \geq 0$ $(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} A_i = 1$. Under the same conditions it was shown by Klamkin in [2] that

$$(3) \quad \frac{\prod_{i=1}^{n} (1+A_i)}{(n+1)^n} \geq \frac{\prod_{i=1}^{n} (1-A_i)}{(n-1)^n}$$

with equality if $A_i = \frac{1}{n}$. In [2] the author refers to a similar inequality to (2) Ky Fan [[4], p. 363] under tighter conditions on A_i but more relaxed condition on ΣA_i, namely,

$$(4) \quad \prod_{i=1}^{n} (1-A_i) \geq \left\{\frac{n - \Sigma A_i}{\Sigma A_i}\right\}^n \prod_{i=1}^{n} A_i \qquad \text{for } 0 < A_i \leq \tfrac{1}{2}.$$

Inequalities (1), (2) and (3) are proved by the authors as extensions of the Weierstrass product inequalities [see [1]].

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The main purpose of this note is to show that the inequalities (1), (2), (3) and (4) can be obtained by a uniform approach using the powerful tools of majorization and Schur-functions. Majorization (defined in Section 2) is a partial ordering in R_n, the n-dimensional Euclidean space. A Schur-function is a function that is monotone with respect to this partial ordering. In Section 2 we show that inequalities (1), (2), (3) and (4) follow immediately by observing that certain functions are Schur-functions, thus providing a unifying and more transparent proof of these inequalities.

Section 2. We now give the standard definitions and results of majorization and Schur-functions needed for proving inequalities (1), (2), (3) and (4).

Given a vector $\underset{\sim}{x} = (x_1,\ldots,x_n)$, let $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$ denote a decreasing rearrangement of $x_1,\ldots x_n$.

Definition 1. *A vector $\underset{\sim}{x}$ is said to majorize a vector $\underset{\sim}{x}'$ if*

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} x'_{[i]}, \quad j=1,\ldots,n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]};$$

in symbols $\underset{\sim}{x} \overset{m}{\geq} \underset{\sim}{x}'$.

Definition 2. A function $f:R_n \to R$ is said to be a Schur-convex (Schur-concave) function if $\underset{\sim}{x} \overset{m}{\geq} \underset{\sim}{x}'$ implies that $f(\underset{\sim}{x}) \geq (\leq) f(\underset{\sim}{x}')$. Functions which are either Schur-convex or Schur-concave are called Schur-functions.

Theorem 1. Let ϕ: $R \to R$ be a log-convex (log-concave) real-valued function. Let g: $R_n \to R$ defined by $g(\underset{\sim}{x}) = \prod_{i=1}^{n} \phi(x_i)$, where $\underset{\sim}{x} = (x_1,\ldots,x_n)$. Then g is Schur-convex (Schur-concave).

Theorem 1 follows immediately from a theorem by Ostrowski, (1952) [3].

We are now ready to prove inequalities (1), (2), (3) and (4). Let $\underset{\sim}{A} = (A_1,\ldots,A_n)$, where $A_i \geq 0$, $\sum_{i=1}^{n} A_i = 1$. Clearly $\underset{\sim}{A} \overset{m}{\geq} (\frac{1}{n},\ldots,\frac{1}{n})$. Now let $\phi_1(x) = \frac{1+x}{x}$, where $x \geq 0$. Then $\phi_1(x)$ is log-convex. Therefore by Th. 1, $\prod_{i=1}^{n} \phi_1(x_i)$ is Schur-convex. Hence $\prod_{i=1}^{n} \frac{(1+A_i)}{A_i} \geq (n+1)^n$ which establishes (1).

Now let $\phi_2(x) = \frac{1+x}{1-x}$, $0 \leq x < 1$. Then $\phi_2(x)$ is log-convex and so by Theorem 1, $\prod_{i=1}^{n} \phi_2(x_i)$ is Schur-convex. It then follows that $\prod_{i=1}^{n} \frac{(1+A_i)}{(1-A_i)} \geq \frac{(n+1)^n}{(n-1)^n}$, which establishes (3). Equality holds if $A_i = \frac{1}{n}$ since $\phi_2(x)$ is strictly log convex.

Now let $\phi_3(x) = \frac{1-x}{x}$; then $\phi_3(x)$ is log-convex for $0 < x \leq \frac{1}{2}$ and log-concave for $\frac{1}{2} \leq x < 1$. Also let $\underset{\sim}{A} = (A_1,\ldots,A_n)$, where $0 \leq A_i \leq \frac{1}{2}$. Clearly $\underset{\sim}{A} \overset{m}{\geq} (\frac{\Sigma A_i}{n},\ldots,\frac{\Sigma A_i}{n})$, so that $\prod_{i=1}^{n} (\frac{1-A_i}{A_i}) \geq [\frac{1-\Sigma A_i/n}{\Sigma A_i/n}]^n = [\frac{n-\Sigma A_i}{\Sigma A_i}]^n$, establishing (4).

Finally, let $\underset{\sim}{A}^1 = ((n-1)A_1,\ldots,(n-1)A_n)$ and let $\underset{\sim}{A}^2 = (1-A_1,\ldots,1-A_n)$ where $A_i \geq 0$, $\Sigma A_i = 1$. It can be easily verified that $\underset{\sim}{A}^1 \overset{m}{\geq} \underset{\sim}{A}^2$. By Theorem 1, $g(\underset{\sim}{x}) = \prod_{i=1}^{n} x_i$ is a Schur-concave function. It then follows that

$$(n-1)^n \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} (n-1)A_i \leq \prod_{i=1}^{n} (1-A_i),$$

proving inequality (2).

It is apparent that many additional inequalities of the Weierstrass product type can be formulated and proved by choosing the appropriate log-concave function, forming products to obtain a Schur-convex function, and then using Definition 2 above.

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18 AFOSR 19

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1. REPORT NUMBER: AFOSR - TR - 77 - 0034 TR-67
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): 6 UNIFIED TREATMENT OF INEQUALITIES OF THE WEIERSTRASS PRODUCT TYPE.
5. TYPE OF REPORT & PERIOD COVERED: 9 Technical [redacted] rept.
6. PERFORMING ORG. REPORT NUMBER: 14 FSU Statistic [redacted] -M390
7. AUTHOR(s): 10 Emad El-Neweihi [redacted] Frank Proschan
8. CONTRACT OR GRANT NUMBER(s): 15 AF- AFOSR [redacted] -2581-74
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Florida State University, Department of Statistics, Tallahassee, Florida 32306

400 277

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304 A5 16 17
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332
12. REPORT DATE: 11 Dec [redacted] 76
13. NUMBER OF PAGES: 4
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 7p
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Weierstrass Product Inequalities, Schur functions, Majorization
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A typical inequality of the Weierstrass product type states that $\prod_{i=1}^{n} (1+A_i) \ge (n+1)^n \prod_{i=1}^{n} A_i$, where $A_i \ge 0$, $i=1,\dots, n$, and $\sum_{i=1}^{n} A_i = 1$. This short note shows that the powerful tools of majorization and Schur functions provide a unified method for deriving a variety of Weierstrass type inequalities.

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