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ABSTRACT

It is shown that a random process is spherically invariant if and only if it is equivalent to a zero mean Gaussian process multiplied by an independent random variable. Several properties of spherically invariant random processes follow in a simple and direct fashion from this representation.

INTRODUCTION

Spherically invariant random processes (SIRP's) were introduced by Vershik [1] when he was investigating a class of random processes which shared some properties characteristic of Gaussian processes. In particular, he found that, for second order SIRP's, all mean square estimation problems have linear solutions, and this class of processes is closed under linear operations. In an interesting paper, Blake and Thomas [2] explored some important properties of SIRP's. Then, in a recent paper [3] Yao presented some very significant results concerning SIRP's. In particular, he presented a representation theorem for the family of finite dimensional distributions of SIRP's. The references in [3] provide a summary of other work done in this area.

By the well-known theorem of Kolmogorov [4; 5, pp.32-37], the consistent and symmetric family of all finite dimensional distributions of a random process is necessary and sufficient to give a statistical characterization of the process (i.e. to uniquely define the probability distribution in the sample function space for all Borel sets of the sample function space). In practice, few processes are characterized in this way; the major difficulty being the mathematical intractability of general n dimensional distributions. However, a SIRP can be conveniently characterized in terms of its family of finite dimensional distributions (see [2] and [3]). In this paper we will develop a representation theorem for SIRP's by using the family of finite dimensional distributions of a SIRP. It will be shown that, loosely speaking, at the heart of any SIRP is a Gaussian process. This will enable us to extend many results known for Gaussian processes to SIRP's.

DEVELOPMENT

We will say that a random process is singular if one random variable in the process can be written as a finite linear combination of the other random variables in the process. It will be seen later in this paper that (except for the identically zero process) the case considered by Yao is the nonsingular

Let $\mathcal{C}$ denote the collection of all classes of consistent finite order characteristic functions of SIRP's whose n -th order characteristic function is a function of an n -th order positive definite quadratic form. Obviously, not just any function of the quadratic form will work. Let u_n represent an n dimensional row vector. Let R_n denote an $n \times n$ positive definite matrix. Let a prime denote the transpose. The following theorem is due to Yao [3].

$$\{\varphi_n(u_n) = \psi_n(u_n R u'_n), n \geq 1\}$$
$$\psi_n(r) = \psi(r) = \int_0^\infty \exp(-\frac{1}{2}rv^2) dF(v) ,$$

This theorem can be extended to include the singular case in the following straightforward manner. Consider a singular SIRP. Select N random variables from this process. Denote them by the row vector X_N . Assume that this set of random variables is singular. Let X_r denote the row vector of dimension r such that each random variable in X_r is also in X_N , X_r is nonsingular, and $X_N = X_r A$, where A is an $r \times N$ transformation matrix. Then from Theorem 1, we know that the characteristic function of X_r , denoted by

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$$C_r(u_r) = \int_0^\infty \exp\left(-\frac{v^2}{2} u_r' R_r u_r\right) dF(v)$$

where R_r is some positive definite $r \times r$ matrix and $F(\cdot)$ is some probability distribution function supported on the non negative half line. The characteristic function of X_N is given by

$$\begin{aligned} C_N(u_N) &= E\{\exp(i u_N' X_N')\} \\ &= E\{\exp(i u_N' A' X_r')\} \\ &= C_r(u_N' A') \\ &= \int_0^\infty \exp\left(-\frac{v^2}{2} u_N' A' R_r A u_N'\right) dF(v) . \end{aligned}$$

Thus we see that the characteristic function of X_N is a function of a non negative definite quadratic form.

Conversely, consider the class of characteristic functions

$$\left\{ \varphi_n(u_n) = \int_0^\infty \exp\left(-\frac{v^2}{2} u_n' \bar{R}_n u_n'\right) dF(v) , n \geq 1 \right\}$$

where $\bar{R}_n$ is an $n \times n$ non negative definite matrix. It is readily seen that this is the class of characteristic functions of a random process which can be represented as a Gaussian random process, with zero mean and covariance matrix $\bar{R}_n$, multiplied by a random variable whose probability distribution function is given by $F(\cdot)$. Thus it follows from Kolmogorov's theorem that this class of characteristic functions is consistent. Also, from the definition of SIRP's, it follows that the corresponding random process is spherically invariant.

Let $\bar{\mathcal{C}}$ denote the collection of all classes of consistent finite order characteristic functions of SIRP's. Then we have the following extension of Theorem 1.

Theorem 2: Theorem 1 is true if $\mathcal{C}$ is replaced by $\bar{\mathcal{C}}$ and R_n is replaced by $\bar{R}_n$.

It follows from Theorem 2 and Kolmogorov's theorem that a random process is a SIRP if and only if it is of the form $AY(t)$, where A is a random variable whose distribution function is supported on the non negative half line and

$Y(t)$ is a zero mean Gaussian process independent of A . Consider, for the moment, the case where the distribution function of A is not necessarily supported on the non negative half line. The n -th order characteristic function of any n random variables in the process is of the form

$$C_n(u_n) = \int_{-\infty}^{\infty} \exp(-\frac{v^2}{2} u_n \bar{R}_n u_n') dF(v),$$

where $\bar{R}_n$ is the covariance matrix of the n mutually Gaussian random variables $Y(t_1), \dots, Y(t_n)$. It is readily seen that this integral can be expressed as

$$C_n(u_n) = \int_0^{\infty} \exp(-\frac{v^2}{2} u_n \bar{R}_n u_n') d\tilde{F}(v),$$

where $\tilde{F}(\cdot)$ is a distribution function supported on the non negative half line. That is, if F is the distribution function of A , then $\tilde{F}$ is the distribution function of $|A|$. In other words, the random processes $AY(t)$ and $|A|Y(t)$ have the same family of finite dimensional distributions. We summarize these results in the following representation theorem for SIRP's.

Theorem 3: A random process is a spherically invariant random process if and only if it is equivalent to a random process of the form $AY(t)$, where A is an arbitrary random variable and $Y(t)$ is a zero mean Gaussian process independent of A .

Theorem 3 characterizes SIRP's as zero mean Gaussian processes with random amplitudes. It is readily seen that a SIRP is Gaussian if and only if $F(\cdot)$ is atomic with only one atom (i.e. A is a constant).

In the sequel we will exclude the identically zero process from consideration. Consider the n -th order characteristic function associated with a SIRP. It follows in a straightforward fashion that the non negative definite matrix R_n in the quadratic form is the covariance matrix associated with the underlying Gaussian random variables, given in Theorem 3. If this matrix is positive definite, then it follows that the corresponding n Gaussian random variables are nonsingular. From this it follows that the n random variables from the SIRP are also nonsingular. In the case considered by Yao [3], R_n was positive definite for all n . Thus he considered nonsingular SIRP's. Conversely, if n random variables from a SIRP are singular, then the corresponding Gaussian random variables are singular, and thus the corresponding covariance matrix is not positive definite.

It is well known [1] that if a SIRP is ergodic, then it is a Gaussian process. Theorem 3 provides a simple illustration of this fact; that is, one

sample function will not yield any information concerning the statistics of the random amplitude, and the amplitude is nonrandom if and only if the process is Gaussian.

Theorem 3 also provides a simple illustration of the fact [1] that a (deterministic) linear transformation of a SIRP results in a SIRP.

Theorem 3 furnishes a convenient method for studying the sample function properties of SIRP's. Sample function properties of Gaussian processes have been widely studied (see, for example, [5, Chapters 9-13]). We will consider a few special cases of interest, and we will state the results in terms of SIRP's. In the sequel, the zero mean SIRP $X(t)$ will be assumed to be separable, stationary, second order, and mean square continuous. Also, we will assume that $P\{X(t) = 0\} = 0$. The next theorem follows from the work of Dobrushin [6].

Theorem 4: Either

1. the sample functions of $X(t)$ are continuous with probability one
- or
2. with probability one, the sample functions of $X(t)$ have discontinuities of the second kind at every point.

The following corollary is a result of the further work of Belyaev [7].

Corollary 1: If the sample functions of $X(t)$ are not almost surely continuous, then they are almost surely unbounded in any interval.

The next theorem follows from the work of Hunt [8].

Theorem 5: Let $S(\omega)$ denote the spectral distribution function of $X(t)$. If

$$\int_0^{\infty} [\log(1+\omega)]^a dS(\omega) < \infty ,$$

for some $a > 1$, then $X(t)$ has continuous sample functions with probability one. Also, if

$$\int_0^{\infty} \omega^2 [\log(1+\omega)]^a dS(\omega) < \infty ,$$

for some $a > 1$, then $X(t)$ has continuous sample function derivatives with probability one.

The expected rate of zero crossings of a Gaussian random process has been studied by many investigators. The following theorem is based on the work of Ylvisaker [9].

Theorem 6: Let $R(\tau)$ denote the autocorrelation function of $X(t)$. The expected number of zero crossings of $X(t)$ in an interval of length T is given by

$$\frac{T}{\pi} \left[\frac{R''(0)}{R(0)} \right]^{1/2}$$

if $R(\tau)$ has a finite second derivative at the origin. If $R(\tau)$ does not have a finite second derivative at the origin, then the expected number of zero crossings in any interval is infinite.

Now we will consider some common zero memory nonlinearities and we will study their effects upon the second moment properties of $X(t)$. Represent $X(t)$ as $|A|Y(t)$, where $Y(t)$ is a Gaussian process independent of the random variable A . Also, take $Y(t)$ to have unit variance. Let $R(\tau)$ denote the autocorrelation function of $X(t)$, and let $\rho(\tau)$ denote the normalized autocorrelation function,

$$\rho(\tau) = \frac{R(\tau)}{R(0)}.$$

Notice that $\rho(\tau)$ is also the autocorrelation function of $Y(t)$.

Let g_1 denote a hard limiter; that is,

$$g_1(u) = \begin{cases} 1 & \text{if } u \geq 0 \\ -1 & \text{if } u < 0 \end{cases}.$$

It is clear that

$$E\{g_1[X(t+\tau)]g_1[X(t)]\} = E\{g_1[Y(t+\tau)]g_1[Y(t)]\}.$$

Then it follows from the well known result for Gaussian processes [10, pp.280-284] that

$$E\{g_1[X(t+\tau)]g_1[X(t)]\} = \frac{2}{\pi} \arcsin \rho(\tau).$$

Let g_2 denote a half wave linear rectifier; that is

$$g_2(u) = \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} .$$

Then we have

$$E\{g_2[X(t+\tau)]g_2[X(t)]\} = E\{E\{g_2[A|Y(t+\tau)]g_2[A|Y(t)]|A\}\} .$$

It follows from [10, pp.294-295] that

$$E\{g_2[A|Y(t+\tau)]g_2[A|Y(t)]|A\} = \frac{A^2}{4} \rho(\tau) + \frac{A^2}{2\pi} \left[\rho(\tau) \arcsin \rho(\tau) + \sqrt{1 - [\rho(\tau)]^2} \right] .$$

Therefore, we have that

$$E\{g_2[X(t+\tau)]g_2[X(t)]\} = \frac{R(\tau)}{4} + \frac{R(0)}{2\pi} \left[\rho(\tau) \arcsin \rho(\tau) + \sqrt{1 - [\rho(\tau)]^2} \right] .$$

Let g_3 denote the square law device; that is,

$$g_3(u) = u^2 .$$

Assume that the fourth moment of $X(t)$ exists. From the well known result for Gaussian processes [10, p.264], we have

$$E\{g_3[A|Y(t+\tau)]g_3[A|Y(t)]|A\} = A^4 + 2A^4[\rho(\tau)]^2 .$$

Then it follows that

$$E\{g_3[X(t+\tau)]g_3[X(t)]\} = [R(0)]^2 + 2[R(\tau)]^2 .$$

Notice that for each of the above three zero memory nonlinearities, the transformation upon the second moment properties is exactly the same as for a Gaussian process. This result is not true in general. For example, consider Baum's limiter

$$g_4(u) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^u \exp\left(-\frac{v^2}{2}\right) dv .$$

We have [10, pp.298-302]

$$E\{g_4[|A|Y(t+\tau)]g_4[|A|Y(t)]|A\} = \frac{1}{2\pi} \arcsin \left[\frac{|A|\rho(\tau)}{|A|+1} \right] .$$

Thus

$$E\{g_4[|A|Y(t+\tau)]g_4[Y(t)]\} = \frac{1}{2\pi} \int_0^\infty \arcsin \left[\frac{v\rho(\tau)}{v+1} \right] dF(v) .$$

If

$$F(v) = \begin{cases} 0, & v < \sigma \\ 1, & v \geq \sigma \end{cases} ,$$

then $X(t)$ is a Gaussian process with autocorrelation $\sigma^2 \rho(\tau)$, and the above integral is equal to

$$\frac{1}{2\pi} \arcsin \left[\frac{\sigma \rho(\tau)}{\sigma+1} \right] .$$

If

$$F(v) = \begin{cases} 0, & v < \sigma/2 \\ 0.8, & \sigma/2 \leq v < 2\sigma \\ 1, & v \geq 2\sigma \end{cases} ,$$

then $X(t)$ is a non-Gaussian SIRP with autocorrelation $\sigma^2 \rho(\tau)$, and the above integral is equal to

$$\frac{2}{5\pi} \arcsin \left[\frac{\sigma \rho(\tau)}{\sigma+2} \right] + \frac{1}{10\pi} \arcsin \left[\frac{2\sigma \rho(\tau)}{2\sigma+1} \right] .$$

Thus the transformation upon the second moment properties induced by Baum's limiter when the input is a SIRP is not necessarily the same as when the input is a Gaussian process.

CONCLUSION

In this paper we have established a useful representation for SIRP's, which is given in Theorem 3. This theorem was then employed to establish several properties of SIRP's.

Consider a random process of the form $AY(t)$, where $Y(t)$ is a zero mean Gaussian process and A is an independent random variable. It is easy to show that such a random process is spherically invariant. However, the essential result of Theorem 3 is that any SIRP can always be represented by an equivalent random process having this form. This result illustrates the essential role played by Gaussian processes in the representation of SIRP's.

ACKNOWLEDGEMENTS

This research is supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR-76-3062 and by the National Science Foundation under Grant ENG76-09443. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

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19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER 18 AFOSR-TR-77-0024	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) A REPRESENTATION FOR SPHERICALLY INVARIANT RANDOM PROCESSES		5. TYPE OF REPORT & PERIOD COVERED Interim	
7. AUTHOR(s) 10 Gary L. Wise Neal C. Gallagher		6. PERFORMING ORG. REPORT NUMBER	
9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Texas at Austin Department of Electrical Engineering Austin, Texas 78712		8. CONTRACT OR GRANT NUMBER(s) 15 AF- AFOSR-3062-76	
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 17A5 16 2304	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 12 P.		12. REPORT DATE 11 1976	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		13. NUMBER OF PAGES 10	
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		15. SECURITY CLASS. (of this report) UNCLASSIFIED	
18. SUPPLEMENTARY NOTES Presented at the Fourteenth Annual Allerton Conference on Circuit and System Theory, 1976; to be published in the Proceedings of the Conference.		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Spherically invariant random processes			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) It is shown that a random process is spherically invariant if and only if it is equivalent to a zero mean Gaussian process multiplied by an independent random variable. Several properties of spherically invariant random processes follow in a simple and direct fashion from this representation.			