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OPTIMAL RADIUS OF CONVERGENCE OF INTERPOLATORY ITERATIONS FOR 0--ETC(U)
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OPTIMAL RADIUS OF CONVERGENCE OF INTERPOLATORY
ITERATIONS FOR OPERATOR EQUATIONS

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ABSTRACT

The convergence of the class of direct interpolatory iterations I_n for a simple zero of a non-linear operator F in a Banach space of finite or infinite dimension is studied.

A general convergence result is established and used to show that if F is entire the "radius of convergence" goes to infinity with n while if F is analytic in a ball of radius R the radius of convergence increases to at least $R/2$ with n .

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1. INTRODUCTION

We study the convergence of the class of direct interpolatory iterations I_n , where I_n is of order n and $n \geq 3$, for a simple zero of a nonlinear operator F in a Banach space of finite or infinite dimension. For $n = 2$, see Traub and Woźniakowski [77a].

We establish a basic convergence theorem for I_n and apply it to two classes of problems. If F is an entire function of growth order ρ , then the "radius of convergence" goes to infinity at least as fast as $n^{1/\rho}$. We show this result is sharp for all odd n and $\rho = 1$. If F is analytic in a disk of radius R , then the "radius of convergence" increases to at least $R/2$ with n . This result is also sharp for all odd n .

The calculation of the next iterate, $x_{i+1} = I_n(x_i)$, requires the solution of a polynomial operator equation of degree $n-1$. In Traub and Woźniakowski [77b] we consider one way in which this can be done.

2. INTERPOLATORY ITERATION I_n

We consider the solution of the non-linear equation

$$(2.1) \quad F(x) = 0$$

where $F: D \subset B_1 \rightarrow B_2$ and B_1, B_2 are real or complex Banach spaces of dimension N , $N = \dim(B_1) = \dim(B_2)$, $1 \leq N \leq +\infty$. We solve (2.1) by one-point stationary iterations without memory in the sense of Traub [64]. Let x_1 be a sufficiently close approximation to a simple zero α , i.e., $F(\alpha) = 0$ and $F'(\alpha)^{-1}$ exists and is bounded. Suppose that the next approximation x_{i+1} depends on the "standard information" $\mathfrak{N}$ on F , i.e.,

$$(2.2) \quad x_{i+1} = \varphi(x_1, \mathfrak{N}(x_1; F))$$

where φ is an iteration operator and

$$(2.3) \quad \mathfrak{N}(x_1; F) = \{F(x), F'(x), \dots, F^{(n-1)}(x)\} \quad \text{for } n \geq 2.$$

In Traub and Woźniakowski [76b] we showed that standard information is optimal (in a sense made precise in that paper) among all linear information sets which use n evaluations. Any iteration based on (2.3) has order of convergence no greater than n . This was first proved by Traub [64] and Kung and Traub [74] for scalar problems and by Woźniakowski [74] for multivariate and abstract problems, see also Traub and Woźniakowski [76a] who use a non-asymptotic definition of order and derive strict lower and upper bounds on complexity. In this paper we study the convergence of a class of iterations which use standard information. The optimal order is achievable by an interpolatory iteration I_n defined as follows:

- (i) construct a polynomial w_i of degree $\leq n-1$ which agrees with the standard information on F , i.e.,

$$(2.4) \quad w_i^{(j)}(x_i) = F^{(j)}(x_i), \quad j = 0, 1, \dots, n-1,$$

- (ii) define the next approximation x_{i+1} as a zero of w_i , $w_i(x_{i+1}) = 0$, with a certain criterion of its choice.

We shall write $x_{i+1} = I_n(x_i; F)$. From (2.4) we get

$$(2.5) \quad w_i(x) = F(x_i) + F'(x_i)(x-x_i) + \dots + \frac{1}{(n-1)!} F^{(n-1)}(x_i)(x-x_i)^{n-1}.$$

For $n = 2$ we obtain Newton iteration since the unique zero of w_i is given by $x_{i+1} = x_i - F'(x_i)^{-1} F(x_i)$. Throughout this paper we assume $n \geq 3$. For $n = 2$, see Traub and Woźniakowski [77a].

Remark 2.1

Inverse interpolation can also be used in which case (2.4) is replaced by

$$w_i^{(j)}(F(x_i)) = g^{(j)}(F(x_i)), \quad j = 0, 1, \dots, n-1,$$

where $g(x) = F^{-1}(x)$ is the inverse operator to F . Define $x_{i+1} = w_i(0)$. The problem of solving the "polynomial operator equation" (2.5) is then eliminated but the derivatives of the inverse function must be computed. See Traub [64, Appendix B and Section 11.2] for the case $N < \infty$ and Brent and Kung [76] for $N = 1$ and n large. ■

We deal with the character of convergence of the interpolatory iteration I_n . Let α be a simple zero of F , $e_i = \|x_i - \alpha\|$ and $J = \{x: \|x - \alpha\| \leq \Gamma\}$. Define

$$(2.6) \quad A_j = A_j(\Gamma) = \sup_{x \in J} \|F'(\alpha)^{-1} \frac{F^{(j)}(x)}{j!}\|, \quad j = 2, 3, \dots$$

whenever $F^{(j)}(x)$ exists. Let q be any number such that $0 < q < 1$ and let

$$\delta_n = \begin{cases} \max(\frac{nq}{1+q}, 1) & \text{if } N = +\infty \\ 1 & \text{if } N < +\infty, \end{cases}$$

$$\delta = \begin{cases} 2 & \text{if } N = +\infty \\ 1 & \text{if } N < +\infty. \end{cases}$$

Theorem 2.1

If F is n -times differentiable in J , $n \geq 3$, and

$$(2.7) \quad \frac{A_n(1+q)^{n-1}}{q} \delta_n + A_2 q \Gamma \delta < 1,$$

$$(2.8) \quad x_0 \in J$$

then the polynomial w_i has a zero in $J_i = \{x: \|x - \alpha\| \leq q^i \Gamma\}$ for any i . Define x_{i+1} as any zero of w_i in J_i .

Then

$$(2.9) \quad \lim_i x_i = \alpha, \quad e_{i+1} \leq \bar{q}_i e_i \quad \text{where}$$

$$\bar{q}_i = \frac{A_n(1+q)^{n-1} \Gamma^{n-1} q^{i(n-1)}}{1 - A_n(1+q)^{n-1} \Gamma^{n-1} q^{i(n-1)} - A_2 q \Gamma q^i} \leq q, \quad \forall i.$$

$$(2.10) \quad e_{i+1} \leq C_{i,n} e_i^n \quad \text{where}$$

$$C_{i,n} = A_n \left(1 + \frac{e_{i+1}}{e_i}\right)^n / (1 - A_2 e_{i+1}), \quad \lim_i C_{i,n} = A_n,$$

$$(2.11) \quad x_{i+1}^{-\alpha} = (-1)^n \frac{F'(\alpha)^{-1}}{n!} F^{(n)}(\alpha) (x_i - \alpha)^n + o(\|x_i - \alpha\|^n).$$

Proof

Define

$$(2.12) \quad R_j(x; y) = \int_0^1 F^{(j)}(y + t(x-y)) (x-y)^j \frac{(1-t)^{j-1}}{(j-1)!} dt$$

for $x, y \in J$ and $j = 2$ and n . Assume by induction that $x_i \in J_i$. This holds for $i = 0$ since $J_0 = J$. From (2.4) we get the error formula

$$(2.13) \quad F(x) = w_i(x) + R_n(x; x_i),$$

see Rall [69, p.124]. We want to show that w_i has a zero in J_{i+1} . The equation $w_i(x) = 0$ in J_{i+1} is equivalent to $F(x) = R_n(x; x_i)$. Since $F(x) = F'(\alpha)(x-\alpha) + R_2(x; \alpha)$ we get the equation

$$(2.14) \quad x = H(x) \stackrel{\text{df}}{=} \alpha + F'(\alpha)^{-1} \{R_n(x; x_i) - R_2(x; \alpha)\}.$$

We verify $H(J_{i+1}) \subset J_{i+1}$. Let $x \in J_{i+1}$. Then from (2.6) and (2.12) we get

$$\begin{aligned} \|H(x) - \alpha\| &\leq A_n \|x - x_i\|^n + A_2 \|x - \alpha\|^2 \leq A_n (q^{i+1} + q^i)^n \Gamma^n + A_2 q^{2(i+1)} \Gamma^2 = \\ &= q^{i+1} \Gamma \left(\frac{A_n (1+q)^{n-1} \Gamma^{n-1}}{q} q^{i(n-1)} + A_2 q \Gamma q^i \right) \leq q^{i+1} \Gamma \end{aligned}$$

due to (2.7). Thus $H(J_{i+1}) \subset J_{i+1}$.

If $N < +\infty$ then from the Brouwer fixed point theorem (see e.g., Ortega and Rheinboldt [70, p.161]) there follows the existence of the solution of (2.14), $x_{i+1} \in J_{i+1}$. If $N = +\infty$ we shall use the contraction mapping theorem (see e.g., Ortega and Rheinboldt [70, p.120]). From (2.13) and (2.14) we get for $x \in J_{i+1}$

$$H'(x) = F'(\alpha)^{-1} \{ (F'(x) - w'_i(x)) - (F'(x) - F'(\alpha)) \}.$$

From (2.7) we get

$$\|H'(x)\| \leq nA_n \|x - x_i\|^{n-1} + 2A_2 \|x - \alpha\| \leq nA_n (1+q)^{n-1} \Gamma^{n-1} + 2A_2 q \Gamma < 1.$$

Hence H is contractive on J_{i+1} and there exists a unique solution x_{i+1} of (2.14) in J_{i+1} for any i .

Now let $N \leq +\infty$. Setting $x = x_{i+1}$ in (2.14) we get

$$e_{i+1} \leq A_n \|x_{i+1} - x_i\|^n + A_2 e_{i+1}^2 \leq A_n (1+q)^{n-1} \Gamma^{n-1} q^{i(n-1)} (e_{i+1} + e_i) + A_2 q^{i+1} \Gamma e_{i+1}.$$

Thus $e_{i+1} \leq \bar{q}_i e_i$ where $\bar{q}_i$ is given in (2.9) and $\bar{q}_i \leq q$ due to (2.7). Hence

$\lim_i e_i = 0$. Returning to (2.14) we have

$$e_{i+1} \leq C_{i,n} e_i^n \text{ where } C_{i,n} = A_n (1 + \frac{e_{i+1}}{e_i})^n / (1 - A_2 e_{i+1}).$$

Since $C_{i,n}$ is bounded, $e_{i+1}/e_i \leq C_{i,n} e_i^{n-1} \rightarrow 0$. Hence $\lim_i C_{i,n} = A_n$ which proves (2.10). Next observe that

$$R_j(x_{i+1}; x_i) = \frac{(-1)^j}{j!} F^{(j)}(\alpha) (x_i - \alpha)^j + o(\|x_i - \alpha\|^j).$$

Thus (2.14) for $x = x_{i+1}$ gives

$$x_{i+1} - \alpha = (-1)^n F'(\alpha)^{-1} \frac{F^{(n)}(\alpha)}{n!} (x_i - \alpha)^n + o(\|x_{i+1} - \alpha\| + \|x_i - \alpha\|^n)$$

which proves (2.11) and theorem 2.1. ■

Remark 2.2

For $N = +\infty$ we proved that there exists a unique zero of w_i in J_i for any i . For the multivariate case, $N < +\infty$, this need not be true. The next approximation x_{i+1} can be defined as any zero of w_i in J_i . However, it is easy to show that for large i the polynomial w_i has a unique zero in J_i . ■

Remark 2.3

The computation of x_{i+1} requires the solution of the "polynomial operator equation" $w_i(x) = 0$. There are a number of ways for dealing with the problem of solving this equation numerically. One is to apply a number of Newton iterations (say k) starting with x_i and taking the k th Newton iterate as x_{i+1} . Traub and Woźniakowski [77b] study the convergence and complexity of such "Interpolatory-Newton" iterations. ■

Remark 2.4

If we additionally assume that F is $(n+1)$ -times differentiable in J then using the same technique it may be shown that

$$x_{i+1} - \alpha = (-1)^n F'(\alpha) \frac{1}{n!} F^{(n)}(\alpha) (x_i - \alpha)^n + \delta_{i,n}$$

where $\|\delta_{i,n}\| \leq C \|x_i - \alpha\|^{n+1}$ and $C = C(A_2, A_n, A_{n+1})$. ■

In theorem 2.1 we assume that $x_0 \in J$. Note that the radius of J depends on F and q . For a given problem F one can ask what is the optimal value of q which maximizes r . In the next section we deal with that problem for any value of n .

3. BALL OF CONVERGENCE OF I_n

In the previous section we proved that if F is sufficiently regular and if an initial approximation x_0 belongs to the ball J then the interpolatory iteration I_n converges. Of course we would like to have J as large as possible. To make this idea more precise we define a ball of convergence as follows.

Definition 3.1

A ball $J_n = J_n(F) = \{x: \|x - \alpha\| < \Gamma_n^*\}$ where $\Gamma_n^* = \Gamma_n^*(F)$ is said to be a ball of convergence of the interpolatory iteration I_n for a function F if for any $x_0 \in J_n$ the sequence $x_{i+1} = I_n(x_i; F)$ converges to the solution α and for every $\epsilon > 0$, there exists x_0 such that $\|x_0 - \alpha\| \leq \Gamma_n^* + \epsilon$ and $x_{i+1} = I_n(x_i; F)$ is not convergent to α . Γ_n^* is called the radius of the ball of convergence or the radius of convergence for short. ■

Of course the ball of convergence can be defined for any iteration Φ in the same way. In fact it is sometimes more convenient to have the concept of domain of convergence D where $\alpha \in D$ and starting from any point $x_0 \in D$ the sequence $x_{i+1} = \Phi(x_i; F)$ converges to α . In general it is very hard to find D and therefore we restrict ourselves only to the case when D is a ball with center at α ; compare Ortega and Rheinboldt [70, p.236].

Note that if F is a polynomial of degree $\leq n-1$ then the interpolatory polynomial $w_1 \equiv F$ and we get convergence for any x_0 . This means $\Gamma_n^* = +\infty$ and $J_n(F) = B_1$. To exclude this exceptional case we shall assume throughout this section that F is not a polynomial.

Recall that $F: D \rightarrow B_2$, $D \subset B_1$ and

$$(3.1) \quad A_n = A_n(r) = \sup_{x \in J(r)} \|F'(\alpha)^{-1} \frac{F^{(n)}(x)}{n!}\|, \quad n = 2, 3, \dots$$

for r such that $J(r) = \{x: \|x - \alpha\| \leq r\} \subset D$.

Define a function $g(r) = g(r, q)$ as

$$(3.2) \quad g(r) = \frac{A_n(r)(1+q)^n r^{n-1}}{q} \delta_n + A_2(r)qr\delta$$

where δ_n and δ are defined in Section 2. Note that g is strictly increasing and $g(0) = 0$, $g(+\infty) = +\infty$. Thus there exists a unique $r^* = r^*(q) > 0$ such that $g(r^*) = 1$. Let

$$(3.3) \quad r_n^* = \max_{0 < q < 1} r^*(q).$$

Applying theorem 2.1 with the q which maximizes r^* we can define the radius Γ in (2.7) as $\Gamma = r_n^* - \epsilon$ for any sufficiently small positive ϵ . From theorem 2.1 the radius Γ_n^* of the ball of convergence $J_n(F)$ can be bounded below by

$$(3.4) \quad \Gamma_n^* \geq r_n^*.$$

We shall see that for some problems $\Gamma_n^* \cong r_n^*$ for large n which indicates that theorem 2.1 is asymptotically sharp. Furthermore we shall show that for some problems r_n^* increases with n or even tends to infinity with n .

We consider two cases depending on the domain of F as follows:

- (i) F is an entire function, i.e., $D = B_1$ and B_1, B_2 are Banach spaces over the complex or real field.
- (ii) F is an analytic function in a finite domain $D = \{x: \|x - \alpha\| < R\}$ where $0 < R < +\infty$.

Case (i). Let F be an entire function. Thus

$$(3.5) \quad F(x) = \sum_{i=1}^{\infty} \frac{1}{i!} F^{(i)}(\alpha) (x-\alpha)^i, \quad \forall x \in B_1.$$

Definition 3.2

We say F has the growth order ρ , $0 < \rho < +\infty$ and the type τ , $\tau > 0$, of its order if

$$\frac{\|F^{(i)}(\alpha)\|}{i!} \leq M \left(\frac{\tau^i}{i!} \right)^{\frac{1}{\rho}}$$

for a constant M and $i = 0, 1, \dots$.

Compare with the definition for the scalar case in Hille [62]. For the sake of simplicity we do not consider the growth order $\rho = 0$ or $\rho = +\infty$. However, it is possible to analyze such cases as well.

Define

$$(3.6) \quad f(z) = M \sum_{i=1}^{\infty} \left(\frac{\tau^i}{i!} \right)^{\frac{1}{\rho}} z^i, \quad z \in \mathbb{C}.$$

Then

$$(3.7) \quad \|F^{(j)}(x)\| \leq f^{(j)}(\|x-\alpha\|)$$

for $x \in B_1$ and $j = 0, 1, \dots$.

We need bounds on the growth of $f^{(j)}$. From Hille [62, p.183] follows that there exists a constant $c_1 > 0$ such that

$$(3.8) \quad \max_{|z| \leq r} |f(z)| = f(r) \leq c_1 r^{\rho/2} \exp\left(\frac{\tau}{\rho} r^{\rho}\right), \quad \forall r > 0.$$

Lemma 3.1

Let $\mu = \max(0, 1-1/\rho)$. Then

$$(3.9) \quad \frac{f^{(n)}(x)}{n!} \leq \frac{(2^{\mu} \tau^{\rho})^n}{(n!)^{1/\rho}} f(2^{\mu} x).$$

Proof

Note that

$$f^{(n)}(x) = MC^n \sum_{i=0}^{\infty} c_{i,n} \frac{(Cx)^i}{(i!)^{\nu}}$$

where $\nu = 1/\rho$, $C = \tau^{\nu}$ and $c_{i,n} = \left[\frac{(n+i)!}{n!} \right]^{1-\nu}$.

It may be shown that

$$n! \leq \binom{n+i}{n} n! \leq 2^{n+i} n!, \quad \forall n, i.$$

If $\rho < 1$ then $c_{i,n} \leq (n!)^{1-\nu}$ and

$$\frac{f^{(n)}(x)}{n!} \leq \frac{C^n}{(n!)^{1/\rho}} f(x)$$

which proves (3.9) with $\mu = 0$. Assume $\rho \geq 1$. Then $\mu = 1-\nu > 0$ and

$c_{i,n} \leq [n! 2^{n+i}]^{\nu}$. Hence

$$\frac{f^{(n)}(x)}{n!} \leq \frac{(2^{\mu} C)^n}{(n!)^{1/\rho}} f(2^{\mu} x)$$

which proves (3.9).

We are ready to prove the following theorem.

Theorem 3.1

If F has the growth order ρ and the type τ of its order and if

$$(3.10) \quad q = q_n = \{(c_2 n)^{\frac{1+\rho/2}{\rho}} \exp(2^{\mu\rho} \tau c_2 n / \rho)\}^{-1} c_3$$

where c_2 and c_3 are any positive numbers such that

$$(3.11) \quad c_2 2^{\mu\rho} \tau \exp(2c_2 2^{\mu\rho} \tau + 1) < 1, \quad c_3 < 2^{\frac{1}{\rho} - 2\mu} / (c_1 \delta \tau^{2/\rho})$$

and μ , δ and c_1 are defined as above then theorem 2.1 holds with

$$(3.12) \quad \Gamma = \Gamma_n = (c_2 n)^{1/\rho(1+o(1))}, \quad \forall n.$$

Proof

We want to estimate g defined by (3.2). Since $q_n = o(n)$, $\delta_n = 1$ in (2.7) and (3.2) for large n . After some algebraic manipulations we get from Lemma 3.2

$$g((c_2 n)^{\frac{1}{\rho}}) = 0 \left(\{c_2 2^{\mu\rho} \tau \exp(2c_2 2^{\mu\rho} \tau + 1)\}^{\frac{n}{\rho}} \frac{n^{\frac{n}{\rho}+1}}{(n! e^n)^{1/\rho}} \right) + \frac{(2^{\mu} \tau^{1/\rho})^2}{2^{1/\rho}} \delta c_1 c_3.$$

By Stirling's formula

$$n! e^n = n^{n+1/2} \sqrt{2\pi} (1+o(1))$$

and due to (3.11), $g((c_2 n)^{1/\rho}) < 1$ for large n . This means that (2.7) holds for $\Gamma = \Gamma_n = (c_2 n)^{1/\rho(1+o(1))}$ for every n .

Theorem 3.1 states a "type of global convergence" of the interpolatory iteration I_n . The iteration I_n is convergent for $\Gamma_n \cong (c_2 n)^{1/\rho}$ which tends to infinity with n and the growth of the radius depends on the growth order ρ .

From (3.3) and (3.4) we get

Corollary 3.1

If F has the growth order ρ then the radius Γ_n^* of the ball of convergence of the interpolatory iteration I_n satisfies

$$(3.13) \quad \Gamma_n^* \geq (c_2 n)^{1/\rho} (1+o(1)), \quad \forall n.$$

We want to prove that (3.13) is sharp for n odd and $\rho = 1$.

Theorem 3.2

There exists a problem F of growth order $\rho = 1$ for which (3.13) is sharp for all odd n .

Proof

Let

$$(3.14) \quad F(x) = e^x - a, \quad a > 0,$$

for real x . The growth order ρ and the type τ are now equal to unity. From Corollary (3.1) we get

$$\Gamma_n^*(F) \geq c_2 n (1+o(1))$$

where $c_2 \exp(2c_2+1) < 1$ which means $c_2 < 0.23$. We shall show that

$$(3.15) \quad \Gamma_n^*(F) \leq c_4 n (1+o(1))$$

for n odd where $c_4 \exp(1+c_4) > 1$ which means $c_4 > 0.28$.

Let w be the interpolatory polynomial of degree $\leq n-1$ such that $w^{(j)}(x_0) = F^{(j)}(x_0)$ for $j = 0, 1, \dots, n-1$. Then

$$w(x) = e^{x_0} S_{n-1}(x-x_0) - a$$

where

$$S_k(x) \stackrel{\text{df}}{=} \sum_{i=0}^k \frac{x^i}{i!} = \int_0^\infty \frac{(x+t)^k}{k!} e^{-t} dt,$$

see Newman and Rivlin [72]. From this it follows that S_{2k} does not have real zeros and S_{2k-1} does have a unique real zero which we label by z_{2k-1} . It is known that

$$z_{2k-1} = -c_5(2k-1)(1+o(1)), \quad \forall k,$$

where $c_5 \exp(1+c_5) = 1$, $c_5 \doteq 0.28$, see Szegő [24] and Rosenblum [57]. Observe that $S'_{2k}(x) = S_{2k-1}(x)$ which gives

$$\min_x S_{2k}(x) = S_{2k}(z_{2k-1}) = \frac{z_{2k-1}^{2k}}{(2k)!}.$$

We want to find x_0 such that the polynomial w does not have real zeros which means that the interpolatory iteration is not defined at x_0 and $x_0 \notin J_n(F)$. The equation $w(x) = 0$ is equivalent to

$$S_{n-1}(x-x_0) = ae^{-x_0}.$$

Let $x_0 = c_4 n$. Then

$$(3.16) \quad S_{n-1}(x-x_0) = ae^{-x_0} \geq \frac{z_{n-2}^{n-1}}{(n-1)!} = ae^{-c_4 n}.$$

Note that $\frac{|z_{n-2}|^{n-1}}{\sqrt{(n-1)!}} = c_5 e(1+o(1))$ and $\sqrt[n-1]{ae^{-c_4 n}} = e^{-c_4}(1+o(1))$. Since $c_5 e = \exp(-c_5)$ and $c_5 < c_4$ then (3.16) is always positive for large n . Hence (3.15) and theorem 3.2 are proven. ■

However, it may be shown that $\Gamma_n^*(F) = +\infty$ for n even for the problem (3.14). The sharpness of (3.13) for n even or $\rho \neq 1$ is open.

Case (ii). Let F be analytic in D where

$$D = \{x: \|x-\alpha\| < R\}, \quad 0 < R < +\infty.$$

This means that for any sufficiently small $\epsilon > 0$ there exists $M = M(F; \epsilon)$ such that

$$(3.17) \quad \frac{\|F^{(i)}(\alpha)\|}{i!} \leq M C^i \quad \text{where } C = \frac{1}{R-\epsilon}, \quad i = 2, 3, \dots$$

Theorem 3.3

If F satisfies (3.17) and if

$$(3.18) \quad q = q_n = \frac{1}{MCn}$$

then theorem 2.1 holds with

$$(3.19) \quad \Gamma = \Gamma_n = \frac{1-\epsilon}{2}(R-\epsilon)(1+o(1)).$$

Proof

Define

$$f(z) = M \sum_{i=0}^{\infty} (Cz)^i = \frac{M}{1-Cz} \quad \text{for } |z| \leq \frac{1}{C} = R-\epsilon.$$

From (3.17) we get

$$(3.20) \quad \|F^{(i)}(x)\| \leq f^{(i)}(\|x-\alpha\|) = \frac{i! MC^i}{(1-C\|x-\alpha\|)^{i+1}}$$

for any x such that $\|x-\alpha\| < R-\epsilon$ and $i = 2, 3, \dots$

Let $r = r_n = \frac{1-\epsilon}{2C} = \frac{1-\epsilon}{2}(R-\epsilon)$. From (3.2) and (3.20) we have

$$g(r) = O\left(n\left(\frac{1-\epsilon}{1+\epsilon}\right)^n + \frac{1}{n}\right) < 1$$

for large n . This means that (2.7) holds for $\Gamma = \Gamma_n = \frac{1-\epsilon}{2}(R-\epsilon)(1+o(1))$ for all n . ■

Remark 3.1

It is possible to get a slightly sharper estimate of Γ_n in (3.13). It may be shown that

$$\Gamma_n = \frac{R-\epsilon}{2} \left(1 - \frac{\ln(nc_6)}{2n} (1+o(1)) \right)$$

where $c_6 = c_6(M, C)$ is a positive constant. ■

Since ϵ can be arbitrarily small, theorem 3.3 states that $\Gamma_n \cong \frac{1}{2}R$ for large n . This means that the radius Γ_n is about one half of the domain radius R . Once more this gives a "type of global convergence". From (3.4) we get

Corollary 3.2

If F satisfies (3.17) then the radius Γ_n^* of the ball of convergence of the interpolatory iteration I_n satisfies

$$(3.21) \quad \Gamma_n^* \geq \frac{1}{2}R(1+o(1)), \quad \forall n. \quad \blacksquare$$

We now show that, in general, (3.21) is sharp for n odd.

Theorem 3.4

There exists a problem F for which (3.21) is sharp for all odd n .

Proof

Let

$$(3.22) \quad F(x) = \frac{1}{1-Cx} - 1, \quad 0 < C < 1$$

for real x such that $|x| < R = \frac{1}{C}$. We shall show that

$$\Gamma_n^*(F) \leq \frac{1}{2}R, \quad \forall n \text{ odd.}$$

The interpolatory polynomial w is given by

$$w(x) = \sum_{i=0}^{n-1} \frac{C^i}{(1-Cx_0)^{i+1}} (x-x_0)^i - 1, \quad n \text{ odd.}$$

The equation $w(x) = 0$ is equivalent to

$$(3.23) \quad \left[\frac{C(x-x_0)}{1-Cx_0} \right]^n = Cx.$$

Let $x_0 = \frac{1}{2C} = \frac{1}{2}R$. Then (3.23) becomes

$$(3.24) \quad (2Cx-1)^n = Cx.$$

It is straightforward to verify that (3.24) does not have zeros in $[-R, R]$.

This implies $x_0 \notin J_n(F)$ and

$$\Gamma_n^*(F) \leq \frac{1}{2}R, \quad \forall n \text{ odd.}$$

However, it may be shown that $\Gamma_n^*(F) = R$ for n even. Thus the sharpness of (3.21) is open for n even.

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REFERENCES

- Brent and Kung [76] Brent, R. P. and Kung, H. T., "Fast Algorithms for Manipulating Formal Power Series," Dept. of Computer Science Report, Carnegie-Mellon University, 1976.
- Hille [62] Hille, E., Analytic Function Theory, Vol. II, Ginn, Boston, 1962.
- Kung and Traub [74] Kung, H. T. and Traub, J. F., "Optimal Order of One-Point and Multipoint Iteration," Journal of ACM, 21, 1974, 643-651.
- Newman and Rivlin [72] Newman, D. J. and Rivlin, T. J., "The Zeros of the Partial Sums of the Exponential Function," J. App. Theory, 5, (1972), 405-412.
- Ortega and Rheinboldt [70] Ortega, J. M. and Rheinboldt, W. C., Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
- Rall [69] Rall, L. B., Computational Solution of Nonlinear Operator Equations, John Wiley and Sons, New York, 1969.
- Rosenbloom [55] Rosenbloom, P. C., "Distribution of Zeros of Polynomials," in Lectures on Functions of a Complex Variable, edited by W. Kaplan, Ann Arbor, 1955, 265-285.
- Szegő [24] Szegő, G., "Über eine Eigenschaft der Exponentialreihe," Berlin Math. Ges. Sitzungsber., 23 (1924), 50-64.
- Traub [64] Traub, J. F., Iterative Methods for the Solution of Equations, Prentice Hall, 1964.
- Traub and Woźniakowski [76a] Traub, J. F. and Woźniakowski, H., "Strict Lower and Upper Bounds on Iterative Computational Complexity," in Analytic Computation Complexity, edited by J. F. Traub, Academic Press, 1976, 15-34.
- Traub and Woźniakowski [76b] Traub, J. F. and Woźniakowski, H., "Optimal Linear Information for the Solution of Nonlinear Equations," in Algorithms and Complexity: New Directions and Recent Results, edited by J. F. Traub, Academic Press, 1976.

Traub and Woźniakowski [77a]

Traub, J. F. and Woźniakowski, H., "Convergence and Complexity of Newton Iteration for Operator Equations," Dept. of Computer Science Report, Carnegie-Mellon University, 1977.

Traub and Woźniakowski [77b]

Traub, J. F. and Woźniakowski, H., "Convergence and Complexity of Interpolatory-Newton Iterations in a Banach Space," Dept. of Computer Science Report, Carnegie-Mellon University, 1977.

Woźniakowski [74]

Woźniakowski, H., "Maximal Stationary Iterative Methods for the Solution of Operator Equations," SIAM J. Numer. Anal., 11, 5, (October, 1974) 934 - 949.

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