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REPEATED CASCADE THEORY OF HOMOGENEOUS TURBULENCE

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13. ABSTRACT

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The second hierarchy governs the eddy viscosities of different ranks, related to relaxation frequencies of such ranks, in the form of a memory chain. It is cut off by an implicit viscous mechanism.

For zero wind gradient, the spectrum in the inertial subrange recovers the Kolmogoroff $k^{-5/3}$ law with a numerical constant 1.58, in good agreement with experiments. For a strong wind gradient, the spectrum in the production subrange has a k^{-1} law. In the viscous subrange, a law of approach confirming the Heisenberg k^{-7} power spectrum, and a viscous cutoff in the form of an exponential tail are obtained, insuring the convergence of high order spectral moments. The critical wave numbers characteristic of the production, inertia, viscous subranges, and the cutoff are determined, together with their numerical coefficients.

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REPEATED CASCADE THEORY OF HOMOGENEOUS TURBULENCE

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Repeated Cascade Theory of Homogeneous Turbulence*

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ABSTRACT

The problem of turbulent spectrum engenders two coupled hierarchies: one originates from the development of stress, leading to a transfer function, and the other from the development of an eddy viscosity. In order to incorporate physical roles among scales, the turbulent velocity fluctuation is decomposed into a series of ranks in the increasing order of randomness, contributing successively to: energy or stress, eddy viscosity, relaxation frequency, and higher rank frequencies in the memory chain. As a result, the first hierarchy mentioned above becomes closed at the quadruple correlation.

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I. INTRODUCTION

The analytical study of the spectral structure of turbulence by means of hydrodynamic equations has recently received much attention. Since they are nonlinear in the second-order, the evolution of a second order correlation will involve a third-order correlation, called transfer function in Fourier space, responsible for the transfer of energy across the spectrum; in its turn the evolution of the transfer function will call for a quadruple correlation, and the sequence continues.

In analogy with the molecular viscosity in theories of gases, the concept of eddy viscosity was introduced in single cascade theories[1,2,3,4].

In representing the transfer function, they divide the spectrum into a vorticity portion and a portion of smaller scales representing eddy viscosity. The two portions interact in a cascade mechanism. The dividing wave number k was considered as an independent variable in an integral equation. It was then necessary to empirically postulate the functional structure of eddy viscosity[2], in order to reduce the transfer function which has the form of a triple correlation explicitly in terms of the spectral distribution, rendering the hierarchy closed. As a solution, they predicted a power law spectrum, in agreement with the Kolmogoroff law[5] founded on dimensional grounds.

The dynamical problem of the eddy transfer, and of the approach to equilibrium of the eddy viscosity, or eddy relaxation, requires a repeated cascade of many ranks. A phenomenological theory has been proposed by Tchen[6]. It helped to eliminate several ambiguities of deriving eddy viscosities in a variety of applied problems of turbulence: shear driven turbulence[7], buoyancy driven turbulence[6], and plasma turbulence[8]. On the other hand, a self-consistent dynamical theory involves pushing the hierarchy to its fourth order mode-coupling. To that end, we should recall that the closure method of cumulant degeneration by quasi-normal approximation[9], and the degeneration by direction interaction[10,11] have not been successful in reproducing a Kolmogoroff law[5] of turbulence, due to a lack or excess of their account of nonlinear randomnization of phases for the cutoff of memory chain. Various attempts[12-16] dealing with the eddy relaxation have been discussed in literature. A review of the above efforts and difficulties has been given by Orszag[17].

The principal difficulty of closure lies in the homogeneous structure of the four waves in the quadruple correlation, with four velocities at the same time instant, without a clear pattern indicating a basis for decoupling. The added condition of having a vanishing sum of modes among the four waves does not lend any help. In order to overcome this difficulty, we shall impress a certain self-selectivity in the coupling of modes, and thereby destroy the above apparent homogeneity in the structure of correlation. When we deal ranks to the velocity field, it will be seen that the evolution of a double correlation at a given rank is controlled by its coupling with its neighbor ranks. Depending upon the time arguments, it may play either the role of a stress gaining momentum from its neighbor lower rank, or the role of an eddy viscosity dissipating momentum to its next higher rank. In particular, the quadruple correlation will turn out to take the form of two correlations of different ranks with such different roles. This provides a clear basis for decoupling of modes and hence a closure of the hierarchy. As the coupling may occur between a mode and its two neighbor ranks, the statistics of weak coupling permits the transfer of energy in both directions of wave numbers. However, the statistics of strong coupling allow the formation of eddy viscosity from larger wave numbers only, thus securing a transfer toward that direction.

II. STRUCTURE OF REPEATED CASCADE

A turbulent motion has a continuous spectrum of scales which are coupled. As a quasi-stationary process, the large scales form a "macroscopic" background, prescribing the initial conditions for the motion of smaller scales. The smaller scales move more "randomly" in the framework of the macroscopic background, and as a result of the statistical effect of fluctuations, offer and shape up certain transport properties in the background medium. An ensemble average is prepared to screen between the two entities.

The above macroscopic and random conditions are considered relative with respect to any scale in the spectrum. A "rank" labeling the degree of randomness will be introduced and will be denoted by a superscript $\alpha = 1, 2, \ldots N$, with a higher rank having a higher degree of randomness. The total number N of ranks depends on the refinement of our representation. Thus, in the increasing degree of randomness, we write

$$\underset{\sim}{u} = \sum_{\alpha=0}^{N} \underset{\sim}{u}^{\alpha} \tag{1}$$

for velocities $u(t,x)$ and $u^{\alpha}(t,x)$ in the position x space, or for velocities $u(t,k)$ and $u^{\alpha}(t,k)$ in the wave number k space.

When a permanent wind speed $\underset{\sim}{U}$ is present, we can add $\underset{\sim}{U}$ to $\underset{\sim}{u}$ by writing

$$\underset{\sim}{\tilde{u}} = \underset{\sim}{U} + \underset{\sim}{u}$$

or considering $U = u^{(-1)}$ as the (-1) rank to be added to the cascade decomposition (1).

In the following we shall call

$$\underset{\sim}{V}^{\alpha} = \sum_{\beta=0}^{\alpha-1} \underset{\sim}{u}^{\beta} \quad , \quad \underset{\sim}{v}^{\alpha} = \sum_{\beta=\alpha}^{N} \underset{\sim}{u}^{\beta} \tag{2}$$

the "macroscopic" and "random" components, respectively, with reference to the rank $\alpha = 1, 2, \ldots N$, and write

$$\underset{\sim}{u} = \underset{\sim}{V}^{\alpha} + \underset{\sim}{v}^{\alpha}$$

$$\tilde{\underset{\sim}{V}}^{\alpha} = \underset{\sim}{U} + \underset{\sim}{V}^{\alpha} \qquad (3)$$

It is expected that a "cascade ensemble average" of rank α, or briefly "rank average", denoted by

$$\langle \ldots \rangle^{\alpha}$$

could separate the above two components, by averaging over realizations under identical macroscopic background V^{α}; after such averaging procedure, the random component v^{α}, or fluctuation, becomes macroscopically negligible. Thus, we have

$$\langle \underset{\sim}{u} \rangle^{\alpha} = V^{\alpha} \qquad (4)$$

$$\langle \underset{\sim}{v}^{\alpha} \rangle = 0 \qquad (5)$$

$$\langle \underset{\sim}{V}^{\alpha} \rangle = \underset{\sim}{V}^{\alpha} \qquad (6)$$

In Sec. III we shall determine the cascade ensemble average by means of a distribution function, and find the relation between the cascade ensemble average applicable to a quasi-homogeneous system and the global ensemble average applicable to a homogeneous system. In addition, the above average rules (4) - (6) and other rules will be derived on the basis of that distribution function.

In the form (1) the series is purely formal. Now it is necessary to make precise its physical significance and assign roles to the ranks. It should be emphasized that (1) is not a series of perturbations of decreasing amplitude as in an iteration procedure, but is a series of physical processes. The rank u^0 represents a velocity in the portion of spectrum between wave numbers o and k^0 which contributes to a kinetic energy. As the cascade transfer of energy across a spectrum is minifested with the aid of an eddy viscosity, we let $u' \equiv u^{(1)}$ contribute to such an eddy viscosity. Since the approach to equilibrium of the eddy viscosity as a transport property requires a relaxation time or frequency, we let $u'' \equiv u^{(2)}$ determine a relaxation frequency. The pursuance to higher orders requires a higher rank eddy viscosity and relaxation frequency, and the sequence of repeated cascade continues. Such a dynamical cascade portrays a hierarchy. While its closure will form the main task in Secs. VI and VIII, here we only mention that the number of ranks kept in the series (1) will equal the number of dynamical processes involved in the repeated cascade, and, therefore, will determine the order of the highest correlation to be degenerated in the closure of the dynamical hierarchy.

In the context of spectral structure, the turbulent motion belongs to a quasi-stationary process, in which the large scale motions are considered relatively macroscopic, bulky, and inert, while the smaller scales are more random and swift. Such a picture has already been incorporated in early treatments of turbulence by Boussinesq[18], Richardson[19], Reynolds and Lorentz[20]. In the same way, here we associate a high degree of randomness to high wave numbers, by writing

$$\underset{\sim}{u}^{\alpha}(t, \underset{\sim}{x}) = \int_{k^{\alpha-1}}^{k^{\alpha}} d\underset{\sim}{k}\ \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\ \underset{\sim}{u}(t, \underset{\sim}{k}) \tag{7}$$

$$\equiv \int_{|\underset{\sim}{k}|=0}^{\infty} d\underset{\sim}{k}\ \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\ \underset{\sim}{u}^{\alpha}(t, \underset{\sim}{k}) \tag{8}$$

In the notation (8), $\underset{\sim}{u}^{\alpha}(t,k)$ is the Fourier component truncated between $k^{\alpha-1}$ and k^{α}. The truncation (7) need not be sharp, and if necessary can be relied upon a scaling distribution

$$H^{\alpha}(k) \cong 1 \quad \text{for} \quad k^{\alpha-1} < k < k^{\alpha}$$

$$\cong 0 \quad \text{for} \quad k < k^{\alpha-1} \quad \text{or} \quad k > k^{\alpha}$$

defining

$$\underset{\sim}{u}^{\alpha}(t, \underset{\sim}{k}) = H^{\alpha}(k)\ \underset{\sim}{u}(t, \underset{\sim}{k})$$

and, by inversion,

$$\underset{\sim}{u}^{\alpha}(t, \underset{\sim}{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\underset{\sim}{x}'\ \underset{\sim}{u}(t, \underset{\sim}{x}')\ H^{\alpha}(|\underset{\sim}{x}-\underset{\sim}{x}'|)$$

III. KINETIC DESCRIPTION OF REPEATED CASCADE

A. Kinetic Description of Locally Homogeneous Turbulence

In a stationary and homogeneous turbulence, the random function $\underset{\sim}{u}(t,\underset{\sim}{x})$ in (1) can be described statistically as a stationary and homogeneous process. However, the random function $\underset{\sim}{v}^{\alpha}(t,\underset{\sim}{x})$ which varies in a macroscopic, inhomogeneous background $\underset{\sim}{V}^{\alpha}(t,\underset{\sim}{x})$ is non-stationary and non-homogeneous.

We shall first discuss the kinetic description of turbulent field $\underset{\sim}{u}(t,\underset{\sim}{x})$ in a homogeneous system. The kinetic description of turbulence has been approached[21] in analogy with the kinetic theory of gases, and has yielded a kinetic hierarchy similar to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations. The kinetic hierarchy begins with the one-point distribution function

$$\varphi(t, \underset{\sim}{x}; \underset{\sim}{a})\, d\underset{\sim}{a}$$

for the velocity $\underset{\sim}{u}(t,\underset{\sim}{x})$ to take values between

$$\underset{\sim}{a} \leq \underset{\sim}{u}(t, \underset{\sim}{x}) \leq \underset{\sim}{a} + d\underset{\sim}{a}$$

with the normalization

$$\int_{-\infty}^{\infty} d\underset{\sim}{a}\, \varphi = 1$$

An ensemble average on the global scale, denoted by a bar, can be defined by

$$Pf \equiv \overline{f(\underset{\sim}{u})} = \int_{-\infty}^{\infty} d\underset{\sim}{a}\, f(\underset{\sim}{a})\, \varphi(t, \underset{\sim}{x}; \underset{\sim}{a}) \qquad (9)$$

where P is a global projection operator. In particular, we find the first moment:

$$\overline{\tilde{\underset{\sim}{u}}} = \int_{-\infty}^{\infty} d\underset{\sim}{a}\, (\underset{\sim}{U} + \underset{\sim}{a})\, \varphi$$

$$= \underset{\sim}{U}$$

or equivalently

$$\overline{\underset{\sim}{u}} = \int_{-\infty}^{\infty} d\underset{\sim}{a}\, \underset{\sim}{a}\, \varphi$$

$$= 0$$

By taking the first order moment of the first equation of the hierarchy, we find the first moment equation, which is identical to the Reynolds equation, governing the development of $\underset{\sim}{U}$ in the presence of a Reynolds stress,

$$\overline{-u_i u_j}$$

obtainable in a simpler way, as we know, by taking the global average, denoted by a bar, of the Navier-Stokes equation for $\underset{\sim}{u}$ directly. Thus, we feel that the formal development of the moment hierarchy, without introducing assumptions as to the physical processes, will not yield much new information. Therefore, in the following we shall consider a new set of distributions which incorporate ranks representative of physical processes, to be governed by a new kinetic hierarchy.

B. Kinetic Description of Quasi-stationary and Quasi-homogeneous Turbulence

We introduce a distribution

$$D^N(t,\underset{\sim}{x};\underset{\sim}{a}^0,\ldots \underset{\sim}{a}^\alpha,\ldots \underset{\sim}{a}^N)d\underset{\sim}{a}^0\ldots d\underset{\sim}{a}^N$$

describing the dynamical state of all the ranks $0 \leq \alpha \leq N$, at time t and position $\underset{\sim}{x}$, so that the velocity $\underset{\sim}{u}^\alpha(t,\underset{\sim}{x})$ of rank will take values between $\underset{\sim}{a}^\alpha$ and $\underset{\sim}{a}^\alpha + d\underset{\sim}{a}^\alpha$. We normalize D^N to unity

$$\int_{-\infty}^{\infty} d\underset{\sim}{a}^0 \cdots d\underset{\sim}{a}^N \, D^N = 1 \tag{10}$$

The relation between φ and D^N is governed by the following integration

$$\varphi(t,\underset{\sim}{x};\underset{\sim}{a})\,d\underset{\sim}{a} = d\underset{\sim}{a} \int_{\sum_{\alpha=0}^{N} \underset{\sim}{a}^\alpha = \underset{\sim}{a}} da^{(1)} \cdots da^N \, D^N \tag{11}$$

For the sake of brevity in (10) we have represented the (N+1) fold integration with respect to all variables $\underset{\sim}{a}^0\ldots\underset{\sim}{a}^N$ covering the whole domain by

$$\int_{-\infty}^{\infty} d\underset{\sim}{a}^0 \cdots d\underset{\sim}{a}^N$$

and we have represented by

$$d\underset{\sim}{a} \int_{\sum_{\alpha=0}^{N} \underset{\sim}{a}^{\alpha} = \underset{\sim}{a}} da^{(1)} \dots da^{N}$$

the N-fold integration with respect to $\underset{\sim}{a}^{(1)} \dots \underset{\sim}{a}^{N}$, while $\underset{\sim}{a}^{0}$ is replaced by

$$\underset{\sim}{a}^{0} = \underset{\sim}{a} - \left(\underset{\sim}{a}^{(1)} + \dots + \underset{\sim}{a}^{N}\right)$$

As a consequence, we have

$$\int_{-\infty}^{\infty} d\underset{\sim}{a}^{0} \dots d\underset{\sim}{a}^{N} \, \underset{\sim}{a}^{\alpha} D^{N} = 0 \qquad \text{for} \qquad 0 \leq \alpha \leq N$$

By integrating over all the low ranks covering the macroscopic component V^{α} in (2), we find the distribution governing the velocity components of the random variable $\underset{\sim}{v}^{\alpha}$

$$F^{\alpha}\left(t, \underset{\sim}{x}; \underset{\sim}{a}^{\alpha} \dots \underset{\sim}{a}^{N}\right) = \int_{-\infty}^{\infty} d\underset{\sim}{a}^{0} \dots d\underset{\sim}{a}^{\alpha-1} D^{N}, \quad \text{for} \quad \alpha \geq 1$$

$$F^{0} = D^{N} \qquad \text{for} \qquad \alpha = 0 \tag{12}$$

It is also normalized to unity, as a consequence of (10)

$$\int_{-\infty}^{\infty} d\underset{\sim}{a}^{\alpha} \dots d\underset{\sim}{a}^{N} F^{\alpha} = 1 \tag{13}$$

The distribution $F^{\alpha}(t,\underset{\sim}{x};\underset{\sim}{a} \ldots \underset{\sim}{a}^N)$ gives the probability of the velocity components $\underset{\sim}{u}^{\alpha},\ldots\ldots, \underset{\sim}{u}^N$ composing the random variable $\underset{\sim}{v}^{\alpha}$, in realizations of an identically given set of low rank velocity components $\underset{\sim}{u}^{o},\ldots\underset{\sim}{u}^{\alpha-1}$ composing the macroscopic variable $\underset{\sim}{V}^{\alpha}$; therefore, it is expected that the distribution F^{α} will be prepared to screen out any component in the repeated cascade. When that distribution is used to define an average, called a "cascade ensemble average", or rank average, and denoted by

$$P^{\alpha} f(\underset{\sim}{u}) \equiv \left\langle f(\underset{\sim}{u}) \right\rangle^{\alpha} \equiv \int_{-\infty}^{\infty} d\underset{\sim}{a}^{\alpha} \cdots d\underset{\sim}{a}^{N}\, F^{\alpha} f(\underset{\sim}{a}) \tag{14}$$

it will smooth out all random ranks of $\underset{\sim}{v}^{\alpha}$, and produce a stationary macroscopic component

$$\langle \underset{\sim}{u} \rangle^{\alpha} = \int_{-\infty}^{\infty} d\underset{\sim}{a}^{\alpha} \cdots d\underset{\sim}{a}^{N}\, F^{\alpha} \underset{\sim}{a} = V^{\alpha}$$

in justification of the rank average rule stated qualitatively in (4). If we apply the operator P^{α} to itself, we have

$$P^{\alpha} P^{\alpha} f(\underset{\sim}{u}) = \int_{-\infty}^{\infty} d\underset{\sim}{a}^{\alpha} \cdots d\underset{\sim}{a}^{N}\, F^{\alpha} \left[\int_{-\infty}^{\infty} d\underset{\sim}{a}_1^{\alpha} \cdots d\underset{\sim}{a}_1^{N}\, F^{\alpha} f(\underset{\sim}{a}_1) \right]$$

since the integral within the brackets does not depend on $\underset{\sim}{a}^{\alpha},\ldots\underset{\sim}{a}^{N}$, we can use the normalization relation (13) and obtain

$$(P^{\alpha})^2 = P^{\alpha}, \quad \text{for} \quad 0 \le \alpha \le N$$

a necessary and sufficient condition for a projection operator. Therefore, the ensemble average (14), can be regarded as a projection with the projection operator P^α specified for rank α in the cascade description.

It is to be remarked that, with $\alpha = 0$, the projection operator P^0, as defined by (14), reduces to

$$P^0 = P \quad ,$$

the projection operator P, as defined by (9). The proof, which uses the relations (9), (11), and (12), has been omitted. The projection operator P has customarily been used in statistical mechanics of a homogeneous system[23], while the cascade projector operator P^α has the advantage of incorporating the distinctive roles of macroscopic and random scales in a non-homogeneous system. Once the macroscopic component is obtained by the ensemble average (14), the random component follows from the difference

$$\underset{\sim}{v}^\alpha = \underset{\sim}{u} - \underset{\sim}{V}^\alpha$$

according to definition (3).

The above macroscopic regularity and the finer randomness, as represented by $\underset{\sim}{V}^\alpha$ and $\underset{\sim}{v}^\alpha$, respectively, are relative, as the bounding wave numbers $k^\alpha (o<\alpha<N)$ are permitted to vary as independent variables.

From the averaging procedure by means of the cascade ensemble average (14), we derive the following relations, the proof of which is elementary and, therefore, will be omitted:

$$\langle \underset{\sim}{V}^{\beta} \rangle^{\alpha} = \underset{\sim}{V}^{\beta}, \quad \langle \underset{\sim}{v}^{\beta} \rangle^{\alpha} = \underset{\sim}{v}^{\beta} - \underset{\sim}{v}^{\alpha}, \quad \text{for} \quad \alpha \geq \beta$$

$$\langle \underset{\sim}{V}^{\beta} \rangle^{\alpha} = V^{\alpha}, \quad \langle \underset{\sim}{v}^{\beta} \rangle^{\alpha} = 0, \quad \text{for} \quad \beta \geq \alpha$$

$$\langle \underset{\sim}{u}^{\beta} \rangle^{\alpha} = 0 \quad \text{for} \quad \alpha \leq \beta$$

$$\langle \underset{\sim}{u}^{\beta} \rangle^{\alpha} = \underset{\sim}{u}^{\beta} \quad \text{for} \quad \alpha > \beta$$

$$\underset{\sim}{u}^{\alpha} = \underset{\sim}{V}^{\alpha+1} - \underset{\sim}{V}^{\alpha}, \quad \underset{\sim}{u}^{\alpha} = \langle \underset{\sim}{v}^{\alpha} \rangle^{\alpha+1} \tag{15}$$

Since we are not interested in the cascade kinetic hierarchy at present, we shall develop the cascade hierarchy of moments by a direct application of the cascade ensemble average (14) on the Navier-Stokes equation of motion. For this purpose, we can transform the Navier-Stokes equation governing $\underset{\sim}{u}$ into a system of equations governing $\underset{\sim}{V}^{\alpha}$, $\underset{\sim}{v}^{\alpha}$, and $\underset{\sim}{u}^{\alpha}$ through the use of relations (15), (see Sec. V). The hierarchy of moments thus generated will be discussed in Sec. VI. The discussions given here are not intended to develop a kinetic hierarchy, but to help make precise the cascade ensemble average (15) which are necessary for generating the moments, and at the same time to clarify the distinction between the global ensemble average (9) and the cascade ensemble average (14).

IV. PROPERTIES OF CASCADE ENSEMBLE AVERAGES

A. Average of the Product of Two Velocities

As the hydrodynamic equation of Navier-Stokes or its Fourier transform is nonlinear to the second order, in view of the presence of the inertial term $\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}$, a nonlinear term of the form

$$\langle \underset{\sim}{v}^{\alpha+1}\, \underset{\sim}{v}^{\alpha+1} \rangle^{\alpha+1}$$

will appear as a Reynolds stress in the equation of evolution of $\underset{\sim}{u}^{\alpha}$. Therefore, we require that it be of the rank α, or equivalently $\langle \underset{\sim}{v}^{\alpha}\, \underset{\sim}{v}^{\alpha} \rangle^{\alpha}$ is of rank $\underset{\sim}{u}^{\alpha-1}$, as a condition of identification of like ranks. More specifically, we write

$$\langle \underset{\sim}{v}^{\alpha}\underset{\sim}{v}^{\alpha} \rangle^{\alpha} \equiv \langle \underset{\sim}{u}^{\alpha}\underset{\sim}{u}^{\alpha} \rangle^{\alpha} + \langle \underset{\sim}{u}^{\alpha}\underset{\sim}{u}^{\alpha+1} \rangle^{\alpha} + \cdots = \langle \underset{\sim}{u}^{\alpha}\underset{\sim}{u}^{\alpha} \rangle^{\alpha} \tag{16}$$

to be of rank $\underset{\sim}{u}^{\alpha-1}$. This requires that all the remaining products vanish upon averaging, as being of higher ranks:

$$\langle \underset{\sim}{u}^{\beta}\underset{\sim}{u}^{\gamma} \rangle^{\alpha} = 0 \tag{17}$$

because $\beta \geq \alpha$ and $\gamma > \alpha$. It is not difficult to verify the following

$$\begin{aligned} \langle \underset{\sim}{u}^{\beta}\underset{\sim}{u}^{\gamma} \rangle^{\alpha} &= \underset{\sim}{u}^{\gamma}\langle \underset{\sim}{u}^{\beta} \rangle^{\alpha} \\ &= 0 \end{aligned} \tag{18}$$

if $\beta \geq \alpha$, $\gamma < \alpha$

and

$$\langle \underset{\sim}{u}^{\beta} \underset{\sim}{u}^{\gamma} \rangle^{\alpha} = \underset{\sim}{u}^{\beta} \underset{\sim}{u}^{\gamma} \tag{19}$$

if $\beta < \alpha$ and $\gamma < \alpha$. The properties (16) - (19) enable one to select the non-vanishing correlations.

B. Average Related to Eddy Viscosity

As will be discussed in more detail later on, a turbulent Reynolds equation governing u_i^{α} will carry a Reynolds stress

$$-\langle u_i^{\alpha+1} u_j^{\alpha+1} \rangle^{\alpha+1}$$

On a phenomenological basis, this stress, representing the statistical effect of smaller scale fluctuations, results in a transfer of momentum from rank α to rank $\alpha+1$, proportional to the velocity gradient of the large scale motion $\underset{\sim}{u}^{\alpha}$, and to a transport property, called eddy viscosity $\underset{\approx}{\eta}^{\alpha+1}$ formed by small scales. Thus, we have

$$-\langle u_i^{\alpha+1} u_j^{\alpha+1} \rangle^{\alpha+1} = \text{const.}\ \eta_{js}^{\alpha+1} \frac{\partial u_i^{\alpha}}{\partial x_s} \tag{20}$$

From the energy consideration, this Reynolds stress will contribute to an eddy dissipation

$$\left\langle \left\langle u_i^{\alpha+1} u_j^{\alpha+1} \right\rangle^{\alpha+1} \frac{\partial u_i^\alpha}{\partial x_j} \right\rangle^\alpha = - \text{const.}\ \eta_{js}^{\alpha+1} \left\langle \frac{\partial u_i^\alpha}{\partial x_s} \frac{\partial u_i^\alpha}{\partial x_j} \right\rangle^\alpha \tag{21}$$

It is to be noted that an additional term, which could be obtained by interchanging i and j if symmetry is imposed upon (20), has not been written there, because it does not contribute to the above energy consideration in incompressible turbulence. As the left hand side of (21) has a rank value α-1, according to (16), we can presume that $\eta_{js}^{\alpha+1}$ is of lower value in rank than α-1, in order not to alter the rank value on the right-hand side of (21). In analogy with the derivation of transport coefficients in irreversible thermodynamics, the eddy viscosity $\eta_{js}^{\alpha+1}$ can be calculated from a time Lagrangian integration of the correlation between two velocities $\underset{\sim}{u}^{\alpha+1}$, on the basis of a Langevin equation of turbulence. As that velocity correlation is of rank value α, the smoothing procedure by that time integration is expected to lower its rank farther down, confirming the above rank evaluation of eddy viscosity, and giving

$$\begin{aligned} \left\langle \underset{\approx}{\eta}^\beta \nabla \underset{\sim}{u}^\alpha \right\rangle^\alpha &= \underset{\approx}{\eta}^\beta \left\langle \nabla \underset{\sim}{u}^\alpha \right\rangle^\alpha \\ &= 0 \end{aligned} \tag{22}$$

if β is of lower rank than α, including $\beta = \alpha+1$, $\alpha+2$. On the above thermodynamic basis, $\eta_{js}^{\alpha+1}$ is a tensor of second order and not higher. An analytical investigation of (20) will be given in Sec. VIII.

C. Average in Fourier Space

The stress tensor at a given point can be written in its Fourier transform

$$\left\langle u_i^\alpha(t,\underset{\sim}{x}) u_j^\alpha(t,\underset{\sim}{x})\right\rangle^\alpha = \iint_{-\infty}^{\infty} d\underset{\sim}{k}' d\underset{\sim}{k}'' \left\langle u_i^\alpha(t,\underset{\sim}{k}') u_j^\alpha(t,\underset{\sim}{k}'')\right\rangle^\alpha \exp\left[i(\underset{\sim}{k}'+\underset{\sim}{k}'')\cdot\underset{\sim}{x}\right] \quad (23)$$

In view of the homogeneity of turbulence, the foregoing expression should not depend on $\underset{\sim}{x}$. Therefore, upon taking a space average, we have

$$\left\langle u_i^\alpha(t,\underset{\sim}{k}') u_j^\alpha(t,\underset{\sim}{k}'')\right\rangle^\alpha = \chi^\alpha \left\langle u_i^\alpha(t,\underset{\sim}{k}') u_j^\alpha(t,-\underset{\sim}{k}')\right\rangle^\alpha \delta(\underset{\sim}{k}'+\underset{\sim}{k}'') \quad (24)$$

where

$$\chi^\alpha = (\pi/X^\alpha)^3$$

to reduce (23) to

$$\left\langle u_i^\alpha(t,\underset{\sim}{x}) u_j^\alpha(t,\underset{\sim}{x})\right\rangle^\alpha = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \chi^\alpha \left\langle u_i^\alpha(t,\underset{\sim}{k}') u_j^\alpha(t,-\underset{\sim}{k}')\right\rangle^\alpha$$

We shall write the stress tensor in $\underset{\sim}{x}$ space, as

$$E_{ij}^\alpha(t,\underset{\sim}{x}) = \frac{1}{2}\left\langle u_i^\alpha(t,\underset{\sim}{x}) u_j^\alpha(t,\underset{\sim}{x})\right\rangle^\alpha$$

and in $\underset{\sim}{k}$ space, as

$$E_{ij}^{\alpha}(t,\underset{\sim}{k}) = \frac{1}{2}\chi^{\alpha}\left\langle u_i^{\alpha}(t,\underset{\sim}{k})\,u_j^{\alpha}(t,-\underset{\sim}{k})\right\rangle^{\alpha} \tag{25}$$

in terms of the shear $E_{ij}^{\alpha}(t,\underset{\sim}{x})$ and its spectrum $E_{ij}^{\alpha}(t,\underset{\sim}{k})$ related by

$$E_{ij}^{\alpha}(t,\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}\; E_{ij}^{\alpha}(t,\underset{\sim}{k})$$

They contain those portions of the following functions

$$E_{ij}(t,\underset{\sim}{x}) = \frac{1}{2}\,\overline{u_i(t,\underset{\sim}{x})\,u_j(t,\underset{\sim}{x})}$$

and

$$E_{ij}(t,\underset{\sim}{k}) = \frac{1}{2}\chi\,\overline{u_i(t,\underset{\sim}{k})\,u_j(t,-\underset{\sim}{k})}$$

truncated between the limits appropriated to their rank. Hence, for an arbitrary function $f(\underset{\sim}{k})$, we have

$$\int_{-\infty}^{\infty} d\underset{\sim}{k}\; f(\underset{\sim}{k})\,E_{ij}^{\alpha}(t,\underset{\sim}{k}) = \int_{k^{\alpha-1}}^{k^{\alpha}} d\underset{\sim}{k}\; f(\underset{\sim}{k})\,E_{ij}(t,\underset{\sim}{k})$$

It is necessary to distinguish between "rank truncation" and "rank value." For example, the energy E_{ii}^{0}, being formed from velocity u^0, has its truncation between o and k^0, while it has a value -1 as a rank, in view of definitions (16); therefore, the average

$$E_{ij}^{0} = \frac{1}{2}\left\langle u_i^{0}\,u_j^{0}\right\rangle^{0} = \frac{1}{2}\,\overline{u_i^{0}\,u_j^{0}}$$

corresponds to a global average in a homogeneous space. In particular, E^{0}_{ii} will be called an energy spectrum in three dimensions. It is to be noted that the kinetic energy

$$E^{\circ}_{ii} = \frac{1}{2} \chi^{\circ} \left\langle u^{\circ}_{i}(\underset{\sim}{k}) u^{\circ}_{j}(-\underset{\sim}{k}) \right\rangle^{\circ}$$

has a rank value -1, according to the rank definition (16), and that the two ensemble averages, global and cascade $\langle\cdots\rangle^{0} = \overline{(\cdots)}$ are identical.

V. DYNAMICAL EQUATIONS OF CASCADE

A. Langevin Equation of Turbulence

Consider an incompressible fluid of permanent wind speed $\underset{\sim}{U}(\underset{\sim}{x})$ and velocity fluctuation $\underset{\sim}{u}(t,\underset{\sim}{x})$. While the total velocity $\tilde{\underset{\sim}{u}} \equiv \underset{\sim}{U}+\underset{\sim}{u}$ satisfies the Navier-Stokes equation of motion, the velocity fluctuation $\underset{\sim}{u}$ is governed by the following equations

$$\left[\frac{\partial}{\partial t} + (\underset{\sim}{U} + \underset{\sim}{u})\cdot\nabla\right]\underset{\sim}{u} = -\frac{1}{\rho}\nabla p - \underset{\sim}{u}\cdot\nabla U + \nu\nabla^{2}\underset{\sim}{u} + \overline{\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}} \tag{26}$$

$$\nabla\cdot\underset{\sim}{u} = 0 \tag{27}$$

Here, $p(t,\underset{\sim}{x})$ is the pressure fluctuation, ρ is the density, and ν is the kinematic viscosity.

In order to describe the mode coupling more conveniently, we make a Fourier transformation, reducing (26) and (27) to

$$\frac{\partial u_i(t,\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \Delta_{im}(\underset{\sim}{k})\, ik_s \, u_m(t,\underset{\sim}{k}')\, \tilde{u}_s(t,\underset{\sim}{k}-\underset{\sim}{k}')$$

$$= - \int_{-\infty}^{\infty} d\underset{\sim}{k}' \Delta_{im}(\underset{\sim}{k})\, ik_s \, U_m(\underset{\sim}{k}')\, u_s(t,\underset{\sim}{k}-\underset{\sim}{k}') - \nu k^2 u_i(t,\underset{\sim}{k}) \tag{28}$$

where

$$\Delta_{im}(\underset{\sim}{k}) = \delta_{im} - k_i k_m / k^2$$

Now, we can apply the rank average rules (4) and (15) to transform the Eq. (28) for $\underset{\sim}{u}$ into equations for $\underset{\sim}{V}^{\alpha+1}$ and $\underset{\sim}{V}^{\alpha}$, and subsequently into an equation for $\underset{\sim}{u}^{\alpha}$. Alternatively, we can transform (28) into an equation for $\underset{\sim}{v}^{\alpha}$ and subsequently into an equation for $\underset{\sim}{u}^{\alpha}$ with the aid of average rules (15). We list the equations of evolution of $\underset{\sim}{V}^{\alpha}$, $\underset{\sim}{v}^{\alpha}$, and $\underset{\sim}{u}^{\alpha}$ as follows:

$$\frac{\partial V_i^{\alpha}(t,\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \Delta_{im}(\underset{\sim}{k})\, ik_s \, V_m^{\alpha}(t,\underset{\sim}{k}')\, V_s^{\alpha}(t,\underset{\sim}{k}-\underset{\sim}{k}')$$

$$= - \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \Delta_{im}(\underset{\sim}{k})\, ik_s \, U_m^{\alpha}(\underset{\sim}{k}')\, V_s^{\alpha}(t,\underset{\sim}{k}-\underset{\sim}{k}')$$

$$- \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \Delta_{im}(\underset{\sim}{k})\, ik_s \left\langle v_m^{\alpha}(t,\underset{\sim}{k}')\, v_s^{\alpha}(t,\underset{\sim}{k}-\underset{\sim}{k}') \right\rangle^{\alpha}$$

$$D_t v_i^\alpha(t,\underset{\sim}{k}) \equiv \frac{\partial v_i^\alpha(t,\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \Delta_{im}(\underset{\sim}{k})\, i k_s\, v_m^\alpha(t,\underset{\sim}{k}')\, \tilde{u}_s(t,\underset{\sim}{k}-\underset{\sim}{k}')$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \Delta_{im}(\underset{\sim}{k})\, i k_s\, \tilde{V}_m(t,\underset{\sim}{k}')\, v_s^\alpha(t,\underset{\sim}{k}-\underset{\sim}{k}') - \nu k^2 v_i^\alpha(t,\underset{\sim}{k})$$

$$d_t u_i^\alpha(t,\underset{\sim}{k}) \equiv \frac{\partial u_i^\alpha(t,\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \Delta_{im}(\underset{\sim}{k})\, i k_s\, u_m^\alpha(t,\underset{\sim}{k}')\, \tilde{V}_s^{\alpha+1}(t,\underset{\sim}{k}-\underset{\sim}{k}')$$

$$= g_i^\alpha(t,\underset{\sim}{k}) - \nu k^2 u_i^\alpha(t,\underset{\sim}{k}) \tag{29}$$

where

$$g_i^\alpha(t,\underset{\sim}{k}) \equiv -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \Delta_{im}(\underset{\sim}{k})\, i k_s \Big\{ \tilde{V}_m^\alpha(t,\underset{\sim}{k}')\, u_s^\alpha(t,\underset{\sim}{k}-\underset{\sim}{k}') + \Big\langle u_m^{\alpha+1}(t,\underset{\sim}{k}')\, u_s^{\alpha+1}(t,\underset{\sim}{k}-\underset{\sim}{k}') \Big\rangle^{\alpha+1} \Big\}$$

and the term

$$\int_{-\infty}^{\infty} dk'\, \Delta_{im}(\underset{\sim}{k})\, i k_s \Big\langle v_m^\alpha(t,\underset{\sim}{k}')\, v_s^\alpha(t,\underset{\sim}{k}-\underset{\sim}{k}') \Big\rangle^\alpha$$

is omitted in (29), as being of lower rank.

Equation (29) describes the time evolution along the Lagrangian path of $\underset{\sim}{u}^\alpha$ which is endowed with a velocity $\tilde{\underset{\sim}{V}}^{\alpha+1} \equiv \tilde{\underset{\sim}{V}}^\alpha + \underset{\sim}{u}^\alpha$. The motion has a friction $-\nu k^2 u^\alpha$, and a driving force $\underset{\sim}{g}^\alpha$ which consists of a production due to the inhomogeneous mixing of $\underset{\sim}{u}^\alpha$

with its background, and an eddy degradation due to the Reynolds' stress $<u_m^{\alpha+1}(\underset{\sim}{k}')u_s^{\alpha+1}(\underset{\sim}{k}-\underset{\sim}{k}')>^{\alpha+1}$. In the operation of transforming (28) into (29), terms like

$$\left\langle u_m^{\beta}(\underset{\sim}{k}')\, u_s^{\gamma}(\underset{\sim}{k}-\underset{\sim}{k}')\right\rangle^{\alpha+1}$$

with $\beta<\alpha+1$ and $\gamma>\alpha+1$ or $\beta>\alpha+1$ and $\gamma<\alpha+1$ cannot intervene on account of the average rules (15).

Equation (29) can be considered an a Langevin equation of turbulent motion, when $\underset{\sim}{k}$ is treated as a parameter, and the variable t in d_t is a one-dimensional variable in the Lagrangian Fourier representation, instead of a variable of four dimensions in time and three wave numbers in the Eulerian representation. It can be formally integrated, giving

$$u_i^{\alpha}(t,\underset{\sim}{k}) = \int_0^t dt' \exp\left[-\nu k^2(t-t')\right] g_i^{\alpha}(t',\underset{\sim}{k}) + u_i^{\alpha}(0,\underset{\sim}{k}) \exp\left(-\nu k^2 t\right) \tag{30}$$

where $u_i^{\alpha}(o,\underset{\sim}{k})$ is the velocity at time t = o. We note that the upper limit t is a time of macroscopic scale of the order of the duration of correlation t_c, and belonging to the macroscopic component $\underset{\sim}{V}^{\alpha}$, so that any statistical quantity issued from the function $u_i^{\alpha}(t,\underset{\sim}{k})$ at time t, and would forget its initial value $u_i^{\alpha}(o,\underset{\sim}{k})$, thus simplifying (30) to:

$$u_i^{\alpha}(t,\underset{\sim}{k}) = \int_0^{\infty} d\tau \exp(-\nu k^2 \tau)\, g_i^{\alpha}(t-\tau,\underset{\sim}{k}) \tag{31}$$

B. Equation of Energy in Cascade Process

Equation (29) for $\alpha = o$ is an equation of momentum, which multiplied by $u_i^0(t,-\underset{\sim}{k})$, gives an equation of energy in wave number space:

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle u_i^o(\underset{\sim}{k})\, u_i^{(o)}(-\underset{\sim}{k})\right\rangle^o = -\nu k^2 \left\langle u_i^o(\underset{\sim}{k})\, u_i^{(o)}(-\underset{\sim}{k})\right\rangle^o$$

$$- i k_j \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, \frac{1}{2}\Big\{ \tilde{V}_i(\underset{\sim}{k}')\left\langle u_j^o(\underset{\sim}{k}-\underset{\sim}{k}')\, u_i^o(-\underset{\sim}{k})\right\rangle^o$$

$$+ \left\langle\left\langle u_i'(\underset{\sim}{k}')\, u_j'(\underset{\sim}{k}-\underset{\sim}{k}')\right\rangle' u_i^o(-\underset{\sim}{k})\right\rangle^o$$

$$+ (\underset{\sim}{k} \rightarrow -\underset{\sim}{k}) \Big\}$$

Upon further integrating with respect to $d\underset{\sim}{k}$, with the notation (25), we find the energy balance in the following non-equilibrium form

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\underset{\sim}{k}''\, \chi^o \left\langle u_i^o(\underset{\sim}{k}'')\, u_i^o(-\underset{\sim}{k}'')\right\rangle^o = -T^o + S^o - \Phi^o \tag{32}$$

with the transport functions

$$T^{\circ} = \iint_{-\infty}^{\infty} d\underset{\sim}{k}'' \, d\underset{\sim}{k}' \, i k_j'' \, \frac{1}{2} \left\{ \chi^{\circ} \left\langle \left\langle u_i'(\underset{\sim}{k}') u_j'(\underset{\sim}{k}'' - \underset{\sim}{k}') \right\rangle' u_i^{\circ}(-\underset{\sim}{k}'') \right\rangle^{\circ} + (\underset{\sim}{k}'' \to -\underset{\sim}{k}'') \right\} \tag{33}$$

$$S^{\circ} = -\iint_{-\infty}^{\infty} d\underset{\sim}{k}'' \, d\underset{\sim}{k}' \, i k_j'' \, \frac{1}{2} \left\{ \tilde{V}_i^{\circ}(\underset{\sim}{k}') \chi^{\circ} \left\langle u_j^{\circ}(\underset{\sim}{k}'' - \underset{\sim}{k}') u_i^{\circ}(-\underset{\sim}{k}'') \right\rangle^{\circ} + (\underset{\sim}{k}'' \to -\underset{\sim}{k}'') \right\} \tag{34}$$

$$\Phi^{\circ} = \nu \int_{-\infty}^{\infty} d\underset{\sim}{k}'' \, (k'')^2 \left\langle u_i^{\circ}(\underset{\sim}{k}'') u_i^{\circ}(-\underset{\sim}{k}'') \right\rangle^{\circ}$$

called transfer, production, and molecular dissipation functions, respectively. The notation $(\underset{\sim}{k}'' \to -\underset{\sim}{k}'')$ represents the complex conjugate part obtained upon replacing $\underset{\sim}{k}''$ by $-\underset{\sim}{k}''$.

Equation (32), with the transfer function (33), indeed forms a hierarchy, since the evolution of energy E_{ii}^{0} involves a triple correlation, and the evolution of the latter will call for a quadruple correlation, so that the sequence continues. In Sec. VI, we shall show how the quadruple correlation will be cut off, upon degeneration into a product of two double correlations: one to be expressed in terms of E_{ii}^{0}, and the other in terms of an eddy viscosity $\underset{\approx}{\eta}'$. The evolution of $\underset{\sim}{\eta}'$ generates its own hierarchy. The closure of the second hierarchy will be studied in Sec. VII.

V. TRANSPORT PROCESS: TRANSFER FUNCTION

A. Phenomenological Theory of Transfer and Production Functions

The transfer function determines the energy transfer across the spectrum. We shall first present a phenomenological theory which portrays the fundamental mechanism of transfer in a simple physical sense, and then in Sec. VIB, an analytic theory will follow, giving the mode coupling precisely, without arbitrary numerical constants.

As the phenomenological approach does not enter into the coupling between modes, it suffices to write the transfer function in the physical space, instead of Fourier space. Hence, we obtain an equation of energy of 0 - rank, identical to (32), with

$$T^{\circ} = - \left\langle \langle u_i' u_j' \rangle' \frac{\partial u_i^{\circ}}{\partial x_j} \right\rangle^{\circ} \tag{35}$$

$$S^{\circ} = - \langle u_i^{\circ} u_j^{\circ} \rangle^{\circ} \frac{\partial U_i}{\partial x_j}$$

$$\Phi^{\circ} = \nu \left\langle \left(\partial u_i^{\circ} / \partial x_j \right)^2 \right\rangle^{\circ} \tag{36}$$

where terms like

$$\frac{\partial}{\partial x_j} \langle (u_i^{\circ})^2 \rangle^{\circ}, \quad \frac{\partial^2}{\partial x_j^2} \langle (u_i^{\circ})^2 \rangle, \quad \frac{\partial}{\partial x_j} \langle (u_i^{\circ})^2 u_j^{\circ} \rangle^{\circ}, \quad \frac{\partial}{\partial x_j} \left\langle \langle u_i' u_j' \rangle' u_i^{\circ} \right\rangle^{\circ}$$

have been omitted since they vanish in a homogeneous turbulence.

Since the Reynolds stress $<u_i^\alpha u_j^\alpha>$ appears in both functions (35) and (36) with α = o and α = 1, respectively, we shall discuss it with a general α.

From the phenomenological point of view, a stress $<u_i^\alpha u_j^\alpha>^\alpha$ occurs when the background shear $\partial\tilde{V}_i^\alpha/\partial x_s$ transfers a momentum u_i^α,

$$u_i^\alpha = -\ell_s^\alpha \frac{\partial \tilde{V}_i^\alpha}{\partial x_s} \tag{37}$$

so that the stress is

$$\langle u_i^\alpha u_j^\alpha \rangle^\alpha = -\langle u_j^\alpha \ell_s^\alpha \rangle^\alpha \frac{\partial \tilde{V}_i^\alpha}{\partial x_s}$$

Here ℓ_s^α is a mixing length contributing to an eddy viscosity.

$$\langle u_j^\alpha \ell_s^\alpha \rangle^\alpha = c_\alpha \eta_{js}^\alpha \quad , \qquad c_\alpha = \text{const.}$$

Hence we can rewrite the Reynolds stress in the general form

$$\langle u_i^\alpha u_j^\alpha \rangle^\alpha = -c_\alpha \eta_{js}^\alpha \frac{\partial \tilde{V}_i^\alpha}{\partial x_s}$$

or, in particular

$$\langle u_i^o u_j^o \rangle^o = -c_o \eta_{js}^o \frac{\partial U_i}{\partial x_s} \tag{38}$$

$$\langle u_i' u_j' \rangle' = -c_1 \eta_{js}' \left(\frac{\partial u_i^o}{\partial x_s} + \frac{\partial U_i}{\partial x_s} \right) \tag{39}$$

The formula (38) agrees with the Boussinesq[18] formula for Reynolds stress.

Upon substituting (38) and (39) into (36) and (35), respectively, we find

$$
\begin{aligned}
S^{o} &= c_0\, \eta^{o}_{js} \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_s} \\
T^{o} &= c_1\, \eta'_{js} \left\langle \frac{\partial u_i^{o}}{\partial x_j} \frac{\partial u_i^{o}}{\partial x_s} \right\rangle^{o}
\end{aligned}
\tag{40}
$$

In isotropic form, T^{o} is

$$T^{o} = c_1\, \eta' \left\langle \left(\frac{\partial u_i^{o}}{\partial x_j} \right)^{2} \right\rangle^{o}$$

An additional term

$$c_1 \left\langle \eta'_{js} \frac{\partial u_i^{o}}{\partial x_j} \frac{\partial U_i}{\partial x_s} \right\rangle^{o}$$

from (38) will not contribute to T^{o}, since it vanishes on account of property (22).

The phenomenological method, based upon the assumption (37), puts the triple mode coupling in (40) in the form of the product of a pair of double correlations and thereby closes the hierarchy. The transfer function (40) represents the transfer of energy across a spectrum in the direction of high wave numbers, and indicates that the transfer in the opposite direction should be absent. However, the method is not able to determine the numerical constants c_0 and c_1. An analytic theory which follows in Sec. VIB, and is based upon the solution (31) of the Langevin equation, will calculate the Reynolds stresses and the triple mode coupling

$$\left\langle \left\langle u'_m(\underset{\sim}{k}')\, u'_s(\underset{\sim}{k}''-\underset{\sim}{k}') \right\rangle' u^0_i(-\underset{\sim}{k}'') \right\rangle^0$$

characteristic of the transfer function (33). The analytic theory will consolidate the result (40) of the phenomenological theory, and, in addition, will determine the numerical constants to be

$$c_0 = 1 \quad , \quad c_1 = 1/3 \tag{41}$$

In addition, it will determine the direction of transfer of energy across the spectrum. Readers who are less interested in such an elaboration of mathematical details could pass over Sec. VIB.

B. Analytic Theory of Transfer

1. General formulation of Reynolds stress and transfer function

We first calculate the Reynolds stress as appearing in (33)

$$\left\langle u'_i(t,\underset{\sim}{k})\, u'_j(t,\underset{\sim}{k}''-\underset{\sim}{k}) \right\rangle' \equiv \left\langle v'_i(t,\underset{\sim}{k})\, v'_j(t,\underset{\sim}{k}''-\underset{\sim}{k}) \right\rangle' \tag{42}$$

To this end, upon multiplying (31) by $u_j'(t,\underset{\sim}{k}''-\underset{\sim}{k})$, we find

$$\left\langle u_i'(t,\underset{\sim}{k})\, u_j'(t,\underset{\sim}{k}''-\underset{\sim}{k})\right\rangle' = P_{ij}^{o}(t,\underset{\sim}{k};\, t,\, \underset{\sim}{k}''-\underset{\sim}{k}) + D_{ij}^{o} \tag{42}$$

with

$$P_{ij}^{o}(t,\underset{\sim}{k};\, t,\, \underset{\sim}{k}''-\underset{\sim}{k}) \equiv -ik_s\, \Delta_{i\ell}(\underset{\sim}{k}) \int_{-\infty}^{\infty} d\underset{\sim}{k}'\ \tilde{V}_\ell(t,\underset{\sim}{k}) \cdot$$

$$\cdot \int_0^{t\to\infty} dt' \left\langle u_s'(t',\underset{\sim}{k}-\underset{\sim}{k}')\, u_j'(t,\underset{\sim}{k}''-\underset{\sim}{k})\right\rangle' \exp\left[-\nu k^2(t-t')\right] \tag{44}$$

$$D_{ij}^{o}(t,\underset{\sim}{k};\, t,\, \underset{\sim}{k}''-\underset{\sim}{k}) \equiv -ik_s\, \Delta_{i\ell}(\underset{\sim}{k}) \int_{-\infty}^{\infty} d\underset{\sim}{k}' \cdot$$

$$\cdot \int_0^{t\to\infty} dt' \left\langle \left\langle u_\ell''(t',\underset{\sim}{k}')\, u_s''(t',\underset{\sim}{k}-\underset{\sim}{k}')\right\rangle'' u_j'(t,\underset{\sim}{k}''-\underset{\sim}{k})\right\rangle' \exp\left[-\nu k^2(t-t')\right] \tag{45}$$

Here we have replaced $V_\ell'(t',k')$ by $V_\ell'(t,k')$ in view of its quasi-stationarity in regard to the motion of u' of higher rank.

P_{ij}^{o} and D_{ij}^{o} of (44) and (45) contribute to a supply T_P^{o} and a loss T_D^{o} in transfer function. We have from (33)

$$T^{o} = T_P^{o} + T_D^{o} \tag{46}$$

with

$$T_P^\circ = \iint_{-\infty}^{\infty} d\underset{\sim}{k}'' d\underset{\sim}{k}' i k_j'' \, \chi^\circ \frac{1}{2} \left\{ \left\langle P_{ij}^\circ (t, \underset{\sim}{k}'; t, \underset{\sim}{k}'' - \underset{\sim}{k}') \, u_i^\circ (-\underset{\sim}{k}'') \right\rangle^\circ + (k'' \to -k'') \right\} \tag{47}$$

$$T_D^\circ = \iint_{-\infty}^{\infty} d\underset{\sim}{k}'' d\underset{\sim}{k}' i k_j'' \, \chi^\circ \frac{1}{2} \left\{ \left\langle D_{ij}^\circ (t, \underset{\sim}{k}'; t, \underset{\sim}{k}'' - \underset{\sim}{k}') \, u_i^\circ (-k'') \right\rangle^\circ + (k'' \to -k'') \right\} \tag{48}$$

We shall calculate T_P^O and T_D^O separately in the following.

2. Supply (or transfer toward large wave numbers)

Let us introduce an eddy viscosity

$$\eta_{sj}' (\underset{\sim}{k}) = \int_0^\infty d\tau \, \chi' \left\langle u_s' (t-\tau, \underset{\sim}{k}) u_j' (t, -\underset{\sim}{k}) \right\rangle' \exp(-\nu k^2 \tau) \tag{49}$$

in the k space, corresponding to

$$\eta_{sj}' (\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k} \, \eta_{sj}' (\underset{\sim}{k}) \tag{50}$$

in the x space. The correlation in (49) is an even function in τ, i.e.,

$$\langle u'_s(t-\tau, \underset{\sim}{k})\, u'_j(t, -\underset{\sim}{k})\rangle' = \langle u'_s(t, \underset{\sim}{k})\, u'_j(t \pm \tau, -\underset{\sim}{k})\rangle'$$

By integrating (44) as indicated by (47) and applying the property of homogeneity (24), we can write

$$P^{\circ}_{ij}(t, \underset{\sim}{k}; t, \underset{\sim}{k}'' - \underset{\sim}{k}) = -i k_s \Delta_{i\ell}(\underset{\sim}{k})\, \tilde{V}'_{\ell}(t, \underset{\sim}{k}'')\, \eta'_{sj}(|\underset{\sim}{k} - \underset{\sim}{k}''|) \tag{51}$$

so that

$$T^{\circ}_{P} = \int_{-\infty}^{\infty} d\underset{\sim}{k}''\, \chi \langle u^{\circ}_{\ell}(\underset{\sim}{k}'')\, u^{\circ}_{i}(-\underset{\sim}{k}'')\rangle^{\circ} \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \Delta_{i\ell}(\underset{\sim}{k}')\, \frac{1}{2} \left\{ k''_j k'_s \eta'_{sj}(|\underset{\sim}{k}' - \underset{\sim}{k}''|) + (\underset{\sim}{k}'' \to -\underset{\sim}{k}'') \right\}$$

is reduced to an isotropic form

$$T^{\circ}_{P} = \frac{2}{3} \int_{-\infty}^{\infty} d\underset{\sim}{k}''\, \chi^{\circ} \langle \underset{\sim}{u}^{\circ}(\underset{\sim}{k}'') \cdot \underset{\sim}{u}^{\circ}(-\underset{\sim}{k}'')\rangle^{\circ} \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, \left| \underset{\sim}{k}' \cdot \underset{\sim}{k}''\, \eta'(|\underset{\sim}{k}' - \underset{\sim}{k}''|) \right| \tag{52}$$

recognizing that

$$\Delta_{ii} = 2$$

$$\langle u^{\circ}_{\ell}\, u^{\circ}_{i}\rangle^{\circ} = \frac{1}{3} \langle \underset{\sim}{u}^{\circ} \cdot \underset{\sim}{u}^{\circ}\rangle^{\circ} \delta_{\ell i}$$

$$\eta'_{sj} = \eta' \delta_{sj}$$

and that commutability in complex conjugates can be written as an absolute value

In view of the rapid decrease of $\eta'(|\underset{\sim}{k}'-\underset{\sim}{k}''|)$ with its argument as a result of the quasi-homogeneous η' having rank <-1 according to its property (22), we can approximate

$$\underset{\sim}{k}'\cdot\underset{\sim}{k}'' = (k'')^2\ \hat{\underset{\sim}{k}}'\cdot\hat{\underset{\sim}{k}}''$$

where $\hat{k}'$ and $\hat{k}''$ are unit vectors along the directions of $\underset{\sim}{k}'$ and $\underset{\sim}{k}''$, respectively, and reduce (52) to

$$T_p^{\circ} = \frac{2}{3}\int_{-\infty}^{\infty} d\underset{\sim}{k}''(k'')^2\chi^{\circ}\langle\underset{\sim}{u}^{\circ}(\underset{\sim}{k}'')\cdot\underset{\sim}{u}^{\circ}(-\underset{\sim}{k}'')\rangle^{\circ}\int_{-\infty}^{\infty} d\underset{\sim}{k}'\left|\hat{\underset{\sim}{k}}'\cdot\hat{\underset{\sim}{k}}''\right|\eta'(|\underset{\sim}{k}'-\underset{\sim}{k}''|) \tag{53}$$

In the above combination of double integration with respect to ranks 0 and 1, we can approximate

$$\eta'(|\underset{\sim}{k}'-\underset{\sim}{k}''|) \cong \eta'(k')$$

as $k'' \ll k'$. Thus, in spherical polar coordinates we can calculate the integral with respect to $d\underset{\sim}{k}'$ as

$$\begin{aligned}\int_{-\infty}^{\infty} d\underset{\sim}{k}'\ \hat{\underset{\sim}{k}}'\cdot\hat{\underset{\sim}{k}}''\ \eta'(|\underset{\sim}{k}'-\underset{\sim}{k}''|) &\cong \int_{-\infty}^{\infty} d\underset{\sim}{k}'\ \hat{\underset{\sim}{k}}'\cdot\hat{\underset{\sim}{k}}''\ \eta'(k') \\ &= \int_{k^{\circ}}^{k^{(1)}} d\underset{\sim}{k}'\ 2\pi(k')^2\int_{-1}^{+1} d\mu\ |\mu|\ \eta'(k') \\ &= \frac{1}{2}\eta'(\underset{\sim}{x})\end{aligned} \tag{54}$$

and write the integral with respect to dk" as

$$\int_{-\infty}^{\infty} d\underset{\sim}{k}'' (k'')^2 \chi^0 \langle \underset{\sim}{u}^0(\underset{\sim}{k}'') \cdot \underset{\sim}{u}^0(-\underset{\sim}{k}'') \rangle^0 = 2 \int_0^{k^0} dk'' (k'')^2 F(k'') \qquad (55)$$

where $\underset{\sim}{\mu} = \underset{\sim}{k}'' - \underset{\sim}{k}'$, F is the energy spectrum, k^0 and $k^{(1)}$ are the bounding wave number of η'. Upon substituting (54) and (55) we reduce (53) to

$$T_P^0 = c_1 \eta'(\underset{\sim}{x}/k^0) \, 2 \int_0^{k^0} dk'' (k'')^2 F(k'') \qquad (56)$$

with

$$c_1 = \frac{1}{3}$$

as mentioned in (41). The result (56) confirms the result (40) from the phenomenological theory. Equation (56) indicates that the transfer of energy across the spectrum takes the form of a cascade. A single cascade would suffice, if η' could be postulated. But the analytical determination of η' requires the knowledge of a relaxation frequency which will involve a chain of memories.

3. Loss (or transfer toward small wave numbers).

In order to show the hierarchy more explicityly, we shall study T_D^0 for an arbitrary rank. For the sake of abbreviation in writing, we shall omit the viscous effect and the explicit writing of any complex conjugate part. Introducing the notation

$$\phi^{\alpha}_{ms}(t',\underset{\sim}{k}';t,\underset{\sim}{k}''/t,\underset{\sim}{k}) = -\Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\,ik'_r\left\langle\left\langle u_\ell^{\alpha+1}(t',\underset{\sim}{k}')\,u_r^{\alpha+1}(t',\underset{\sim}{k}'')\right\rangle^{\alpha+1} u_s^{\alpha}(t,\underset{\sim}{k})\right\rangle^{\alpha} \tag{57}$$

we can generate (33) and (45) to be

$$T^{\alpha} = -\iint\limits_{-\infty}^{\infty} d\underset{\sim}{k}'\,d\underset{\sim}{k}''\,\chi^{\alpha}\phi^{\alpha}_{ii}(t,\underset{\sim}{k}';t,\underset{\sim}{k}''-\underset{\sim}{k}'/t,-\underset{\sim}{k}'') \tag{58}$$

$$D^{\alpha}_{ij}(t,\underset{\sim}{k},t,\underset{\sim}{k}''-\underset{\sim}{k}) = \int_0^{t\to\infty} dt'\int_{-\infty}^{\infty} d\underset{\sim}{k}'''\,\phi^{\alpha+1}_{ij}(t',\underset{\sim}{k}''';t',\underset{\sim}{k}-\underset{\sim}{k}'''/t,\underset{\sim}{k}''-\underset{\sim}{k}) \tag{59}$$

The evolut on of $u^{\alpha-1}$ n the Reynolds stress

$$\left\langle u_\ell^{\alpha+1}(t',\underset{\sim}{k}')\,u_r^{\alpha+1}(t',\underset{\sim}{k}'')\right\rangle^{\alpha+1}$$

similar to (43) is governed by (31) and involves $P^{\alpha}_{\ell r}$ and $D^{\alpha}_{\ell r}$; after substituting for $P^{\alpha}_{\ell r}$ and $D^{\alpha}_{\ell r}$ from expressions similar to (44) and (45) but of higher rank, we find the dynamical equation governing ϕ^{α}_{ms}

$$\begin{aligned}\phi^{\alpha}_{ms}(t',\underset{\sim}{k}';t',\underset{\sim}{k}''/t,\underset{\sim}{k})' = {} & \Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\Delta_{\ell q}(\underset{\sim}{k}')\,k'_r\,(k'_p+k''_p)\cdot \\ & \cdot\eta^{\alpha+1}_{pr}(t',\underset{\sim}{k}'')\left\langle u_q^{\alpha}(t',\underset{\sim}{k}'+\underset{\sim}{k}'')\,u_s^{\alpha}(t,\underset{\sim}{k})\right\rangle^{\alpha} \\ & + \int_0^{t\to\infty} dt''\int d\underset{\sim}{k}'''\,\Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\,ik'_r\left\langle \phi^{\alpha+1}_{\ell r}(t'',\underset{\sim}{k}''';t'',\underset{\sim}{k}'-\underset{\sim}{k}'''/t',\underset{\sim}{k}'')\,u_s^{\alpha}(t,\underset{\sim}{k})\right\rangle^{\alpha}\end{aligned} \tag{60}$$

Its reduction to ϕ^{0}_{ij} and substitution into (58) gives

$$T^{\alpha} = T_P^{\alpha} + T_D^{\alpha} \qquad , \qquad T^{0} = T_P^{0} + T_D^{0}$$

with

$$T_P^{\alpha} = \iint_{-\infty}^{\infty} d\underset{\sim}{k}''\, d\underset{\sim}{k}^{\alpha}\, \Delta_{\ell\lambda}(\underset{\sim}{k}^{\alpha})\, k_r^{\alpha} k_\rho^{0}\, \eta_{\rho r}^{\alpha+1}(|\underset{\sim}{k}^{\alpha} - \underset{\sim}{k}''|) \chi^{\alpha} \left\langle u_\lambda^{\alpha}(t, \underset{\sim}{k}'')\, u_\ell^{\alpha}(t, -\underset{\sim}{k}'') \right\rangle^{\alpha} \quad (61)$$

and

$$T_D^{\alpha} = \int_0^{t\to\infty} dt' \iiint_{-\infty}^{\infty} d\underset{\sim}{k}'\, d\underset{\sim}{k}^{\alpha}\, d\underset{\sim}{k}'''\, ik_r^{\alpha} \left\langle \phi_{\ell r}^{\alpha+1}(t', \underset{\sim}{k}'''; t, \underset{\sim}{k}^{\alpha} - \underset{\sim}{k}''' / t, \underset{\sim}{k}'' - \underset{\sim}{k}^{\alpha})\, u_\lambda^{\alpha}(t, -\underset{\sim}{k}'') \right\rangle^{\alpha} \quad (62)$$

Like T_P^{α} the counterpart T_D^{α} is also seen to involve a quadruple correlation

$$\left\langle \phi_{\ell r}^{\alpha+1}\, u_i^{\alpha} \right\rangle^{\alpha}$$

from the definition (57). While the quadruple correlation in T_P^{α} comes out in its decoupled form (61)

$$\eta_{\rho r}^{\alpha+1} \left\langle u_\lambda^{\alpha} u_i^{\alpha} \right\rangle^{\alpha}$$

thus immediately closing the hierarchy of T_P^{α}, we are faced with a new dynamical hierarchy (60) and (62) generated by T_D^{0}.

Noting that the function ϕ^{α} appears as an integrand in (60) and (62), it is expedient to introduce a new function

$$\psi_m^{\alpha+1}(\underset{\sim}{k}'/t,\underset{\sim}{k}'') \equiv \int_0^{t\to\infty} dt_1 \int_{-\infty}^{\infty} d\underset{\sim}{k}''' \Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\, ik_r'\, \phi_{\ell r}^{\alpha+1}(t_1 \underset{\sim}{k}''';\, t_1,\, \underset{\sim}{k}'-\underset{\sim}{k}'''/t,\, \underset{\sim}{k}'')$$

which enables the writing of (60) and (62) in a more compact form:

$$T_D^{\alpha} = \iint_{-\infty}^{\infty} d\underset{\sim}{k}\, d\underset{\sim}{k}^{\alpha} \chi^{\alpha} \left\langle \psi_i^{\alpha+1}(k^{\alpha}/t,\underset{\sim}{k}''-\underset{\sim}{k}^{\alpha})\, u_i^{\alpha}(t,-\underset{\sim}{k}'') \right\rangle^{\alpha} \tag{63}$$

and

$$\psi_m^{\alpha+1}(\underset{\sim}{k}'/t,\underset{\sim}{k}'') = \psi_m^{(D)\,\alpha+1}(\underset{\sim}{k}'/t,\underset{\sim}{k}'')$$
$$+ \int_0^{t\to\infty} dt' \int d\underset{\sim}{k}''' \Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\, ik_r' \left\langle \psi_\ell^{\alpha+2}(\underset{\sim}{k}'''/t,\underset{\sim}{k}'-\underset{\sim}{k}''')\, u_r^{\alpha+1}(t,\underset{\sim}{k}'') \right\rangle^{\alpha+1} \tag{64}$$

where

$$\psi_m^{(D)\,\alpha+1}(\underset{\sim}{k}'/t,\underset{\sim}{k}'') = \int_{-\infty}^{\infty} d\underset{\sim}{k}''' \Delta_{\ell\lambda}(\underset{\sim}{k}')\, \Delta_{\lambda\mu}(\underset{\sim}{k}''')\, \Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\, ik_\rho'''\, k_r'\, k_\sigma' \cdot$$
$$\cdot \int_0^{t\to\infty} dt' \left\langle \eta_{\sigma\rho}^{\alpha+2}(t',|\underset{\sim}{k}'-\underset{\sim}{k}'''|)\, u_\mu^{\alpha+1}(t',\underset{\sim}{k}')\, u_r^{\alpha+1}(t,\underset{\sim}{k}'') \right\rangle^{\alpha+1}$$

is a driving force of the integral equation (64).

When we consider $\eta_{\sigma\rho}^{\alpha+2}$ to be stationary in the gradient field of $u_\mu^{\alpha+1}$, according to the property (22) we can approximate

$$\eta_{\sigma\rho}^{\alpha+2}(t',\underset{\sim}{k}'-\underset{\sim}{k}''') \cong \eta_{\sigma\rho}^{\alpha+2}(t,\underset{\sim}{k}'-\underset{\sim}{k}''')$$

and simplify the driving force $\psi_m^{(D)\alpha+1}$ to the form

$$\psi_m^{(D)\alpha+1}(\underset{\sim}{k}'/t,\underset{\sim}{k}'') \cong \int_{-\infty}^{\infty} d\underset{\sim}{k}''' \Delta_{\ell\lambda}(\underset{\sim}{k}')\Delta_{\lambda\mu}(\underset{\sim}{k}''')\Delta_{m\ell}(\underset{\sim}{k}'+\underset{\sim}{k}'')\, ik_\rho''' k_\sigma' k_r'\, \delta(\underset{\sim}{k}'+\underset{\sim}{k}'') \cdot$$

$$\cdot\, \eta_{\sigma\rho}^{\alpha+2}(t,\underset{\sim}{k}'-\underset{\sim}{k}'')\, \eta_{\mu r}^{\alpha+1}(t,\underset{\sim}{k}')$$

When we substitute (64) into (63), we find the expression

$$T_D^{\alpha} = \iint_{-\infty}^{\infty} d\underset{\sim}{k}'' d\underset{\sim}{k}^{\alpha}\, \chi^{\alpha} \left\langle \psi_i^{(D)\alpha+1}(\underset{\sim}{k}^{\alpha}/t,\underset{\sim}{k}''-\underset{\sim}{k}^{\alpha})\, u_i^{\alpha}(t,-\underset{\sim}{k}'') \right\rangle^{\alpha}$$

$$+ \int_0^{t\to\infty} dt' \iint_{-\infty}^{\infty} d\underset{\sim}{k}'' d\underset{\sim}{k}^{\alpha} d\underset{\sim}{k}'''\, \Delta_{i\ell}(\underset{\sim}{k}'')\, ik_r'' \cdot$$

$$\cdot\, \chi^{\alpha} \left\langle \left\langle \psi_\ell^{\alpha+2}(\underset{\sim}{k}'''/t',\underset{\sim}{k}^{\alpha}-\underset{\sim}{k}''')\, u_r^{\alpha+1}(t,\underset{\sim}{k}''-\underset{\sim}{k}^{\alpha}) \right\rangle^{\alpha+1} u_i^{\alpha}(t,-\underset{\sim}{k}'') \right\rangle^{\alpha} \tag{65}$$

Repeated substitutions of $\psi_\ell^{\alpha+2}$, from (64) taken at higher ranks, generate correlations of the following types:

$$\eta_{\mu r}^{\alpha+1}(t,\underset{\sim}{k}^{\alpha}) \left\langle \eta_{\sigma\rho}^{\alpha+2}(t,\underset{\sim}{k}^{\alpha}-\underset{\sim}{k}''')\, u_i^{\alpha}(t,-\underset{\sim}{k}'') \right\rangle^{\alpha} \delta(\underset{\sim}{k}'') \tag{66}$$

$$\int_0^{t\to\infty} dt' \left\langle \left\langle \underset{\approx}{\eta}^{\alpha+3}(t',\underset{\sim}{k}_1)\, \underset{\sim}{u}^{\alpha+1}(t,\underset{\sim}{k}_2) \right\rangle^{\alpha+1} \underset{\approx}{\eta}^{\alpha+2}(t',\underset{\sim}{k}_3)\, \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k}_4) \right\rangle^{\alpha} \tag{67}$$

and similar high ranks, including products with a larger number of eddy viscosities of high ranks. They all vanish on account of the property of n^{β} stated in (22). In this respect, we note that the mechanism of repeated cascade involves couplings between neighbor ranks, as seen from the character of the dynamic equations (29) of cascade. The couplings in (66) and (67) govern non-neighbor ranks and do not have a physical basis in the cascade scheme. Thus, we can state that the total contribution (65) of T_D^o is negligible:

$$T_D^o \cong 0 \tag{68}$$

reducing (46) to

$$T^o = T_P^o$$

where T_P^o is given by (56).

We conclude that, through the dynamical equation (29) or (31), we have transformed the triple correlation in the transfer function T_P^o (47) into a product of two double correlations, as a degenerate form of a quadruple correlation, and thus closed the hierarchy for T_P^o. The hierarchy for T_D^o, which starts with fifth order correlations between non-neighbor ranks is found to be zero. The closed form of T_P^o involves an eddy viscosity, the dynamics of which generates a memory chain which is a new form of hierarchy to be discussed in Sec. VIII.

It is to be remarked that T_P^o is a transfer by degradation of energy of rank 0 into fluctuations of rank 1 forming an eddy viscosity, while T_D^o represents an energy flux in the opposite direction, i.e., from ranks 1 and 2 toward rank 0. The above derivations (56) and (68) demonstrate that the flux of energy in the cascade model occurs only toward the direction of increasing wave numbers. The phenomenological theory of Sec. VIA intuitively recognizes this fact by relying upon the Boussinesq hypothesis of transport (37).

An alternative analysis of the build up of the Reynolds stress (42) in the transfer theory is to use the equation for $D_t \underset{\sim}{v}^\alpha$ as a new Langevin equation in lieu of (29), and to calculate $\langle v_i' v_j' \rangle'$ of (42). Such a scheme automatically eliminates T_D^o, but bears the disadvantage of losing a mechanism of degradation of eddy viscosity by relaxations into smaller and smaller scales.

VII. TRANSPORT PROCESS: PRODUCTION FUNCTION

The production function S^o has been written in (34) as a coupling between modes, we shall transform it into an explicit expression of the spectral distribution. The same method used for the derivation of the transfer function as in Sec. VI will be followed.

We rewrite (34) in terms of the supply $P_{is}^{(-1)}$ defined by (44), and call this contribution S_P^o:

$$S_P^o = -\iint_{-\infty}^{\infty} d\underset{\sim}{k}'' d\underset{\sim}{k}' \, i k_j'' \, X^o \frac{1}{2} \left\{ \left\langle U_i(\underset{\sim}{k}') P_{ij}^{-1}(-\underset{\sim}{k}'', -\underset{\sim}{k}' + \underset{\sim}{k}'') \right\rangle^\alpha + (\underset{\sim}{k}'' \rightarrow -\underset{\sim}{k}'') \right\}$$

which, after substitution of (51) rewritten as

$$P_{ij}^{(-1)}(-\underset{\sim}{k}'', -\underset{\sim}{k}'+\underset{\sim}{k}'') = -ik_s' \ \Delta_{il}(-\underset{\sim}{k}'') \ \eta_{sj}^{\circ}(|\underset{\sim}{k}'-\underset{\sim}{k}''|) \ U_l(t, -\underset{\sim}{k}')$$

becomes

$$S_P^{\circ} = \int_{-\infty}^{\infty} d\underset{\sim}{k}'' \chi^{\circ} U_1(\underset{\sim}{k}') U_1(-\underset{\sim}{k}') \int_{-\infty}^{\infty} d\underset{\sim}{k}'' (k_3'')^2 \left(1 - \frac{(k_1'')^2}{(k'')^2}\right) \eta_{33}^{\circ}(|\underset{\sim}{k}''-\underset{\sim}{k}'|) \tag{69}$$

Here we have assumed that the mean wind has a velocity gradient of component $\partial U_1/\partial x_3$ only. When this gradient is strong, the wave number k_3 becomes negligible as compared to k_1. As mentioned earlier, $\eta_{33}^{0}(|\underset{\sim}{k}''-\underset{\sim}{k}'|)$ is a rapidly decreasing function, so that (69) can be reduced to

$$S_P^{\circ} = c_0 \eta_{33}^{\circ} \left(\frac{\partial U_1}{\partial x_3}\right)^2 , \quad c_0 = 1 \tag{70}$$

a result agreeing with the formula (40) found earlier from the phenomenological theory of Sec. VIA.

VIII. TRANSPORT PROCESS: EDDY VISCOSITY IN A WEAK WIND GRADIENT

A. Evolution of Correlation

The eddy viscosity, as defined by (49) and (50), can be found by a time integration of the Lagrangian correlation. To this end, we multiply (29) by

$$u_j^\alpha(t'',-\underset{\sim}{k})$$

yielding a gain $\underset{\sim}{Q}^\alpha$ and a loss $\underset{\sim}{L}^\alpha$ in the development of the correlation. We have

$$\chi^\alpha(d_t+\nu k^2)\left\langle u_i^\alpha(t,\underset{\sim}{k})\,u_j^\alpha(t'',-\underset{\sim}{k})\right\rangle^\alpha = \chi^\alpha\left\langle g_i^\alpha(t,\underset{\sim}{k})\,u_j^\alpha(t'',-\underset{\sim}{k})\right\rangle$$

$$= Q_{ij}^\alpha(t,\underset{\sim}{k};t'',-\underset{\sim}{k}) + L_{ij}^\alpha \tag{71}$$

with

$$Q_{ij}^\alpha(t,\underset{\sim}{k};t'',-\underset{\sim}{k}) = -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, ik_\beta'\,\Delta_{i\gamma}(\underset{\sim}{k})\,\tilde{V}_\gamma^\alpha(t,\underset{\sim}{k}')\,\chi^\alpha\left\langle u_\beta^\alpha(t,\underset{\sim}{k}-\underset{\sim}{k}')\,u_j^\alpha(t'',-\underset{\sim}{k})\right\rangle^\alpha$$

$$L_{ij}^\alpha(t,\underset{\sim}{k};t'',-\underset{\sim}{k}) = -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, ik_\beta'\,\Delta_{i\gamma}(\underset{\sim}{k})\,\chi^\alpha\left\langle\left\langle u_\gamma^{\alpha+1}(t,\underset{\sim}{k}')\,u_\beta^{\alpha+1}(t,\underset{\sim}{k}-\underset{\sim}{k}')\right\rangle^{\alpha+1} u_j^\alpha(t'',-\underset{\sim}{k})\right\rangle^\alpha \tag{72}$$

The contribution of Q_{ij}^α to $\underset{\sim}{\eta}^\alpha$ involves a background gradient $\nabla\underset{\sim}{V}^\alpha$, which may take both positive and negative values, and which consequently vanishes after an ensemble average in that coarse grain scale. Since $\underset{\sim}{\eta}^\alpha$ should not carry a rank denomination $\geq\alpha$-1, the contribution of Q_{ij}^α is not present

$$Q_{ij}^\alpha = 0$$

This means that the evolution of eddy viscosity η^{α} has a relaxation time characteristic of its approach to equilibrium, and that the time is contributed by smaller scales rather than larger ones, in agreement with the process of cascade toward smaller and smaller scales, demonstrated by (56) and (68). The eddy viscosity in a strong wind gradient follows a different process and will be studied in Sec. II.

For the calculation of (72), we use the same method as in Sec. VI, and write

$$L_{ij}^{\alpha}(t,\underset{\sim}{k};t'',-\underset{\sim}{k}) = -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\,\frac{1}{2}\Big\{ik'_{\beta}\Delta_{i\gamma}(\underset{\sim}{k})\,\chi^{\alpha}\Big\langle\Big[P_{\gamma\beta}^{\alpha}(t,\underset{\sim}{k}';t,\underset{\sim}{k}-\underset{\sim}{k}') + D_{\gamma\beta}^{\alpha}\Big]\cdot$$

$$\cdot\, u_j^{\alpha}(t'',-\underset{\sim}{k})\Big\rangle^{\alpha} + (\underset{\sim}{k}\to -\underset{\sim}{k})\Big\} \tag{73}$$

$$= -\chi^{\alpha}\Big\langle u_s^{\alpha}(t,\underset{\sim}{k})\,u_j^{\alpha}(t'',-\underset{\sim}{k})\Big\rangle^{\alpha}\int_{-\infty}^{\infty} d\underset{\sim}{k}'\,\frac{1}{2}\Big\{k'_{\beta}\,k_{\mu}\,\Delta_{i\gamma}(\underset{\sim}{k})\cdot$$

$$\cdot\,\Delta_{\gamma s}(\underset{\sim}{k}')\,\eta_{\mu\beta}^{\alpha+1}(|\underset{\sim}{k}'-\underset{\sim}{k}|) + (\underset{\sim}{k}\to -\underset{\sim}{k})\Big\} \tag{74}$$

The last term of (73), estimated by iteration, can be shown to give a negligible contribution, as in (68). Thus, we have succeeded in transforming the triple correlation in (72) into a quadruple correlation in (74).

By writing

$$\chi^{\alpha}\left\langle u_s^{\alpha}(t,\underset{\sim}{k})\, u_j^{\alpha}(t'',-\underset{\sim}{k})\right\rangle^{\alpha} = \frac{1}{3}\,\chi^{\alpha}\left\langle \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\alpha}(t'',-\underset{\sim}{k})\right\rangle^{\alpha}\delta_{sj}$$

using (51) and changing the variables $\underset{\sim}{k}' - \underset{\sim}{k} = \underset{\sim}{k}^{\alpha}$, we reduce (74) to an isotropic form

$$L_{ii}^{\alpha}(t,\underset{\sim}{k};t'',-\underset{\sim}{k}) = -\frac{1}{3}k^2\chi^{\alpha}\left\langle \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\alpha}(t'',-\underset{\sim}{k})\right\rangle^{\alpha}\int_{-\infty}^{\infty} d\underset{\sim}{k}^{\alpha}\,\eta^{\alpha+1}(\underset{\sim}{k}^{\alpha})\,\Gamma(\underset{\sim}{k},\underset{\sim}{k}^{\alpha}) \quad (75)$$

where, for the sake of abbreviation, we have written

$$\Gamma(\underset{\sim}{k},\underset{\sim}{k}^{\alpha}) = \frac{1}{2}\left[\left(1 + k_s k_s^{\alpha}/k^2\right)\Delta_{i\gamma}(\underset{\sim}{k})\Delta_{i\gamma}(\underset{\sim}{k}+\underset{\sim}{k}^{\alpha}) + \left(\underset{\sim}{k}\to-\underset{\sim}{k}\right)\right]$$

$$= 1 + \frac{\gamma^2(\gamma^2-3)\mu^2+\gamma^2+1}{(\gamma^2+1)^2 - 4\gamma^2\mu^2}$$

with

$$\gamma = \frac{k^{\alpha}}{k}\ , \quad \mu = \hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{k}}^{\alpha} = \cos\theta$$

The details of calculations are uninteresting and have been omitted.

Introduce

$$\bar{\Gamma}(\gamma) = \int_{-1}^{+1} d\mu\,\Gamma(\gamma,\mu) = \frac{7-\gamma^2}{2} + \frac{(\gamma^2-1)^2}{4\gamma}\ln\left(\frac{\gamma+1}{\gamma-1}\right)$$

which varies between

$$\bar{\Gamma}(\gamma = 1) = 3$$

and

$$\bar{\Gamma}(\gamma = \infty) = 2.5$$

can be considered as a slowly varying function, and therefore, permits the following approximation

$$\int_{-\infty}^{\infty} d\underset{\sim}{k}^{\alpha}\, \eta^{\alpha+1}(\underset{\sim}{k}^{\alpha})\, \Gamma(\gamma, \mu) \equiv \int_{k}^{\infty} d\underset{\sim}{k}^{\alpha}\, 2\pi (k^{\alpha})^2 \eta^{\alpha+1}(k^{\alpha})\, \bar{\Gamma}(\gamma)$$

$$\doteq \bar{\Gamma}(\gamma=1) \int_{k}^{\infty} d\underset{\sim}{k}^{\alpha}\, 2\pi (k^{\alpha})^2 \eta^{\alpha+1}(k^{\alpha})$$

$$= C_w\, \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k) \quad , \quad C_w = \frac{3}{2}$$

where

$$\tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k) = \int_{k}^{\infty} d\underset{\sim}{k}^{\alpha}\, \eta^{\alpha+1}(k^{\alpha}) \tag{76}$$

as analogous to (50). This reduces (75) to

$$L_{ii}^{\alpha}(t, \underset{\sim}{k}; t'', -\underset{\sim}{k}) = -\frac{1}{3} C_w k^2 \eta^{\alpha+1}(\underset{\sim}{x}/k)\, \chi^{\alpha} \langle \underset{\sim}{u}^{\alpha}(t, \underset{\sim}{k}) \cdot \underset{\sim}{u}^{\alpha}(t'', -\underset{\sim}{k}) \rangle^{\alpha}$$

Hence, we can rewrite (71) in the isotropic form as

$$(d_t + \nu k^2)\, \chi^{\alpha} \langle \underset{\sim}{u}^{\alpha}(t, \underset{\sim}{k}) \cdot \underset{\sim}{u}^{\alpha}(t'', -\underset{\sim}{k}) \rangle^{\alpha} = -\frac{1}{3} C_w k^2 \eta^{\alpha+1}(\underset{\sim}{x}/k)\, \chi^{\alpha} \langle \underset{\sim}{u}^{\alpha}(t, \underset{\sim}{k}) \cdot \underset{\sim}{u}^{\alpha}(t'', -\underset{\sim}{k}) \rangle^{\alpha}$$

giving the solution

$$\chi^{\alpha}\left\langle \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\alpha}(t-\tau,-\underset{\sim}{k})\right\rangle^{\alpha} e^{-\nu k^2\tau}$$

$$= \chi^{\alpha}\left\langle \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\alpha}(t,-\underset{\sim}{k})\right\rangle^{\alpha} \exp\left[-\left(2\nu+\tfrac{1}{3}C_W\tilde{\eta}^{\alpha+1}\right)k^2\tau\right] \qquad (77)$$

B. Relaxation and Memory

1. Memory chain of high ranks

The eddy viscosity is easily obtained by integrating (77) giving

$$\eta^{\alpha}(\underset{\sim}{x}) = \frac{1}{3}\int_{-\infty}^{\infty} d\underset{\sim}{k}' \int_{0}^{\infty} d\tau\, \chi^{\alpha}\left\langle \underset{\sim}{u}^{\alpha}(t,\underset{\sim}{k}')\cdot\underset{\sim}{u}^{\alpha}(t-\underset{\sim}{k})\right\rangle^{\alpha} \exp\left[-\left(2\nu+\tfrac{1}{3}C_W\tilde{\eta}^{\alpha+1}\right)(k')^2\tau\right]$$

$$= \frac{2}{3}\int_{-\infty}^{\infty} d\underset{\sim}{k}' \frac{E^{\alpha}(k')(k')^{-2}}{2\nu+\frac{1}{3}C_W\tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k')}$$

$$= C_2\int_{-\infty}^{\infty} d\underset{\sim}{k}' \frac{E^{\alpha}(k')(k')^{-2}}{2\nu'+\eta^{\alpha+1}(\underset{\sim}{x}/k')} \qquad (78)$$

with

$$C_2 = 2/C_W = 4/3$$

$$2\nu' = 6\nu/C_W = 4\nu$$

In accordance with the property (16), and the definition (76), we can write identically

$$\eta^{\alpha}(\underset{\sim}{x}) = \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1})$$

thus we can rewrite (78) in the form

$$\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1}) = C_2 \int_{k^{\alpha-1}}^{\infty} d\underset{\sim}{k}' \frac{(k')^{-2} F(k')}{2\nu' + \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k')} \tag{79}$$

2. Mechanism of cutoff of memory chain at high wave numbers

The sequence of memories is represented by the integral difference equation (79). It indicates that the approach to equilibrium of eddy viscosity is controlled by a relaxation process, consisting of a viscous relaxation frequency $2k'^2\nu'$ and an eddy relaxation frequency $k'^2\tilde{\eta}^{\alpha+1}(x/k')$, with $k' \geq k^{\alpha-1}$. In turn, $\tilde{\eta}^{\alpha+1}$ is governed by a higher order relaxation formula, similar to (79), and hence an infinite chain of memories continues.

On a simple scheme, we can assume that the relaxation frequency at a sufficiently high rank is viscous, thus approximating

$$\tilde{\eta}^{\alpha+1} \cong C_2 \int_{k^{\alpha}}^{\infty} dk'' \frac{(k'')^{-2} F(k'')}{2\nu'} \tag{80}$$

This helps to cut off the sequence (79), provided it be pushed to a sufficiently high rank. This is the scheme of an explicit viscous cutoff. In view of its slow convergence, we propose an alternate scheme, by means of a mechanism of dynamical cutoff, which relates $\tilde{\eta}^{\alpha+1}$ to a lower rank by

$$\tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) = \tilde{\eta}^{\alpha+1}\left[\underset{\sim}{x}/\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha})\right] \tag{81}$$

The scheme (81) will provide an implicit viscous cutoff, in lieu of (80) to close the hierarchy (79) to the form:

$$\tilde{\eta}'(\underset{\sim}{x}/k^{0}) = c_2 \int_{k^0}^{\infty} dk' \, \frac{(k')^{-2} F(k')}{\tilde{\eta}''(\underset{\sim}{x}/k')}$$

$$\tilde{\eta}''(\underset{\sim}{x}/k') = c_2 \int_{k'}^{\infty} dk'' \, \frac{(k'')^{-2} F(k'')}{\tilde{\eta}'''(\underset{\sim}{x}/k'')}$$

$$\vdots$$

$$\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1}) = c_2 \int_{k^{\alpha-1}}^{\infty} dk^{\alpha} \, \frac{(k^{\alpha})^{-2} F(k^{\alpha})}{\tilde{\eta}^{\alpha+1}\left[\underset{\sim}{x}/\tilde{\eta}(\underset{\sim}{x}/k^{\alpha})\right]} \tag{82}$$

3. Approximate formula of viscous cutoff

As a basis of providing a mechanism of degradation of $\tilde{\eta}^{\alpha+1}$ along the scheme (89), we use Eq. (32) which governs the energy balance, and formulas (56) and (68) which govern the energy transfer. In the isotropic form and in the absence of wind gradient, we have

$$(\nu + c_1 \tilde{\eta}^{\alpha+1}) J^{\alpha} = \frac{\partial}{\partial t} \int_{k^{\alpha-1}}^{k^{\alpha}} dk' \, F(k')$$

where

$$J^{\alpha} \equiv 2 \int_{k^{\alpha-1}}^{k^{\alpha}} dk' \, (k')^2 \, F(k')$$

A differentiation with respect to k^{α} gives

$$c_1 J^{\alpha} \frac{\partial \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha})}{\partial k^{\alpha}} + 2(k^{\alpha})^2 F(k^{\alpha}) \left[\nu + c_1 \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) \right] = \frac{\partial F(k^{\alpha})}{\partial t} \quad (83)$$

We choose $k^{\alpha-1} = 0$, and k near a cutoff wave number k_c so that we can approximate

$$J^{\alpha} \cong \varepsilon/\nu \quad , \quad 2(k^{\alpha})^2 F(k^{\alpha}) \cong 2 k_c^2 F(k_c)$$

where ε is the total rate of energy dissipation, and rewrite (83) in the form

$$\frac{c_1 \varepsilon}{\nu} \frac{\partial \tilde{\eta}^{\alpha+1}}{\partial k^{\alpha}} + 2 k_c^2 F(k_c) \left[\nu + c_1 \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) \right] = \frac{\partial F(k^{\alpha})}{\partial t}$$

or

$$\frac{\partial \tilde{\eta}^{\alpha+1}}{\partial k^{\alpha}} + k_c^{-1} \left[\tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k_{\alpha}) + \nu c_1^{-1} \right] = \frac{\nu}{c_1 \varepsilon} \frac{\partial F(k^{\alpha})}{\partial t} \quad (84)$$

with the frictional coefficient in (84) to be

$$k_c^{-1} = \frac{2\nu}{\varepsilon} k_c^2 F(k_c)$$

or

$$k_c = \left[\frac{2\nu}{\varepsilon} F(k_c) \right]^{-1/3} \tag{85}$$

If the cutoff is to follow a Heisenberg law of spectrum

$$F(k) = A_2 \left(\varepsilon/\nu^2\right)^2 k^{-7} \tag{86}$$

then we find, upon substituting (86) into (85),

$$k_c = A_c \left(\varepsilon/\nu^3\right)^{1/4}, \quad A_c = \left(2A_2\right)^{1/4} \tag{87}$$

Hence, the mechanism of cutoff is a viscous friction. It is to be noted that we are presently talking about the cutoff of the memory chain (79) into (82), by means of the degradation scheme (81) which is implicitly viscous. This cutoff is a relaxational cutoff, and is not related to the critical wave number k_ν dividing the inertia and viscous subranges, sometimes also inadequately called "viscous cutoff".

Equation (84) can be regarded as an equation of propagation $\tilde{\eta}^{\alpha'+1}$ across the spectrum, with a frictional coefficient k_c^{-1}, and with a driving force

$$\frac{\nu}{c_1 \varepsilon} \frac{\partial F(k^*)}{\partial t}$$

We would like to express the input in terms of $\tilde{\eta}^{\alpha}$. For this purpose, we write a similar equation of propagation for $\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1})$; at its own rank wave number $k^{\alpha-1}$; it is

$$\frac{\partial \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1})}{\partial k^{\alpha-1}} + k_c^{-1}\left[\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1}) + \nu/c_1\right] = \frac{\nu}{c_1 \varepsilon}\ \frac{\partial F(k^{\alpha-1})}{\partial t}$$

and at a wave number k^{α} higher than its own rank, it is

$$\frac{\partial \tilde{\eta}^{\alpha}(x/k^{\alpha})}{\partial k^{\alpha}} + k_c^{-1}\left[\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha}) + \nu/c_1\right] = \frac{\nu}{c_1 \varepsilon}\ \frac{\partial F(k^{\alpha})}{\partial t} \tag{88}$$

Since $\tilde{\eta}^{\alpha}$ is set at a higher wave number than that its rank would normally permit, it becomes negligible with respect to ν/c_1, reducing (88) to an approximate form

$$\frac{\partial \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha})}{\partial k^{\alpha}} + \frac{\nu}{c_1 k_c} = \frac{\nu}{c_1 \varepsilon}\ \frac{\partial F(k^{\alpha})}{\partial t}$$

This is to be subtracted from (84), giving an approximate equation of $\tilde{\eta}^{\alpha+1}$,

$$\frac{\partial \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha})}{\partial k^{\alpha}} + k_c^{-1}\ \tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) = \frac{\partial \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha})}{\partial k^{\alpha}} \tag{89}$$

with a frictional coefficient k_c^{-1}, and now with a driving force $\partial\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha})/\partial k^{\alpha}$ as an input.

In the process of resolving (89) we remark that $\tilde{\eta}^{\alpha}$, being of lower rank than $\tilde{\eta}^{\alpha+1}$, is quasi-stationary, so that $\partial\tilde{\eta}^{\alpha}/\partial k^{\alpha}$ behaves like a δ-function, simplifying the solution to

$$\tilde{\eta}^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) = \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha}) \exp\left(-k^{\alpha}/k_c\right) \tag{90}$$

suggesting an exponential cutoff as expected. Equation (90) is an approximate cutoff relation replacing (81).

4. Inviscid eddy viscosity

If $\nu = 0$, we find $k_c^{-1} = 0$ according to (85), and

$$\eta^{\alpha+1}(\underset{\sim}{x}/k^{\alpha}) = \tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha}) \tag{91}$$

according to (90). The closure of the hierarchy (82) at $\tilde{\eta}^{\alpha+1}$ is obtained, when we solve (82) and (91), which, after elimination of $\tilde{\eta}^{\alpha+1}$, can be rewritten in the form

$$\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1}) = c_2 \int_{k^{\alpha}}^{\infty} dk' \frac{(k')^{-2} F(k')}{\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k')}$$

The solution is

$$\tilde{\eta}^{\alpha}(\underset{\sim}{x}/k^{\alpha-1}) = \left[2c_2 \int_{k^{\alpha-1}}^{\infty} dk' (k')^{-2} F(k') \right]^{1/2} \tag{92}$$

As a first order closure, we take $\alpha = 1$, reducing (92), and therefore also the hierarchy (82), to

$$\tilde{\eta}^{1}(\underset{\sim}{x}/k^{0}) = \left[2c_2 \int_{k^{0}}^{\infty} dk' (k')^{-2} F(k') \right]^{1/2} \tag{93}$$

In applications studies in the following sections, we shall use diverse formulas of eddy viscosities (91), (92), or (93) depending on special interest.

IX. UNIVERSAL RANGE OF TURBULENCE

A. Fundamental System

With the aid and substitutions of (56), (68), and (70), we can write the energy equation (32) in terms of E_{ii} as follows

$$\frac{\partial}{\partial t}\int_0^k d\underset{\sim}{k}'\, E_{ii} = c_o\, \eta_{33}^{o}\left(\frac{\partial U_1}{\partial x_3}\right)^2 - 2\left(c_1 \tilde{\eta}'_{js} + \nu\delta_{js}\right)\int_0^k d\underset{\sim}{k}'\, k'_s k'_j\, E_{ii} \tag{94}$$

For infinite k, (94) reduces to

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{u_i^2} = c_o\, \eta_{33}\left(\frac{\partial U_1}{\partial x_3}\right)^2 - \varepsilon \tag{95}$$

where

$$\varepsilon = \nu\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}\ , \qquad \eta_{33} = \eta_{33}^{o} + \tilde{\eta}'_{33}$$

is the rate of dissipation, and the bar denotes a global average defined in (9).

Equation (94) describes the evolution and propagation of the kinetic energy E_{ii}^{o} in the portion of the spectrum between wave numbers from o to k^o. Since k^o is an independent variable, covering values between o to ∞, of an integral equation, such as (94), we have changed the independent variable k^o into k in the integral equation (94) for the sake of simplification in writing.

The integral equation (94) can be solved for the spectrum $E_{ii}(k)$ for any range of the spectrum for k between o and ∞, including non-universal and universal ranges. It is the purpose of most theories of turbulence to be concerned with the universal range. It holds for sufficiently large k, when the evolution (94) becomes independent of k. This turbulence is in quasi-equilibrium, as the total transport (including production, transfer, and dissipation) across the spectrum occurs at a constant rate independently of wave numbers. Under such a circumstance we can write, by equating (94) and (95):

$$c_0 \eta^0_{33} \left(\frac{\partial U_1}{\partial x_3} \right)^2 - 2 \left(c_1 \tilde{\eta}'_{js} + \nu \delta_{js} \right) \int_0^k d\underset{\sim}{k}' \, k'_j k'_s \, E_{ii}(k')$$

$$= c_1 \eta_{33} \left(\frac{\partial U_1}{\partial x_3} \right)^2 - \varepsilon$$

or, equivalently

$$(\eta_{33} - \eta^0_{33}) \left(\frac{\partial U_1}{\partial x_3} \right)^2 + 2 \left(c_1 \tilde{\eta}'_{js} + \nu \delta_{js} \right) \int_0^k d\underset{\sim}{k}' \, k'_j k'_s \, E_{ii}(k') = \varepsilon \qquad (96)$$

The spectral energy equation (96) together with the system (82) and (90) for viscous eddy viscosity, or the system (82) and (92) for inviscid eddy viscosity, forms the fundamental framework of study of turbulent spectra in various subranges. They will follow in the next sections.

B. Comparison Between Repeated Cascade and Single Cascade

It should be emphasized that the independent variable

k^o in (56) which is changed into k in (94), should not be confused or related to the critical wave number parameters, which are to be determined from external physical conditions: viscosity ν, rate of dissipation ε, wind shear, etc. We shall determine those parameters later on. It should also be emphasized that the fundamental system of equations (96), (79), (90), or (92) do not put any restriction on the above physical parameters, and even less on the domain of variations of the independent variables $k^o,\ldots k^\alpha\ldots$, except the domain normally understood for an universal spectrum.

The independent variables of repeated cascade $k^o\ldots k^\alpha\ldots$ had appeared in the two hierarchies (43) and (79) for stress and for eddy viscosity with memory. When the two hierarchies are closed and resolved in Sec. VI, VII, and VIII, there remains only one independent variable k^o, written as k in (96). It is not correct, based upon the appearance of (96) with one variable, to judge that the mechanism of (96) eventually involves a single cascade. Under the single cascade theory, it would not be possible to analytically derive the eddy viscosity $\tilde{\eta}'$, even if one managed to arrive at the form (96).

X. SPECTRUM OF ISOTROPIC AND HOMOGENEOUS TURBULENCE

A. General Formulation

When the wind shear is absent, and the turbulent motion is isotropic, we can write (94) in a reduced form

$$2\left(c_1\tilde{\eta}'+\nu\right)\int_0^k dk'(k')^2 F = \varepsilon \tag{97}$$

When we use the zero-order closure formula (93) for $\tilde{\eta}'$, we can transform F in terms of $\tilde{\eta}'$, and reduce (97) to

$$\tilde{\eta}'\left(c_1\tilde{\eta}'+\nu\right)^2 = \frac{1}{2}c_1 c_2 \varepsilon k^{-4} \tag{98}$$

The root $\tilde{\eta}'$, substituted into (93), yields the spectrum F, covering both the inertial subrange and the approach to the viscous subrange.

We shall study in Secs. XB and XC the spectra for the two subranges, separated by a critical wave number k_ν.

B. Inertia Subrange $k < k_\nu$

In this subrange the molecular viscosity is negligible, reducing to

$$\tilde{\eta}'^3 = \frac{c_2\varepsilon}{2c_1} k^{-4} \tag{99}$$

Upon substituting for $\tilde{\eta}'$, from (93) we can write (99) in the form

$$\left(2c_2\int_k^\infty dk'(k')^{-2}F\right)^{3/2} = \frac{c_2\varepsilon}{2c_1}k^{-4}$$

or

$$\int_k^\infty dk'(k')^{-2}F = \frac{1}{2c_2}\left(\frac{c_2\varepsilon}{2c_1}\right)^{2/3}k^{-8/3}$$

yielding, after differentiation,

$$F = F_K = A_1 \varepsilon^{2/3} k^{-5/3} \tag{100}$$

with

$$A_1 = \frac{2}{3}\left(2/c_1^2 c_2\right)^{1/3} \cong 1.586$$

This law (100) has been the focus[17] of recent analytical theories of turbulence, as a required prediction.

The solution (100) agrees with the power law of the Kolmogoroff[5] theory. The present theory, in addition, enables one to determine the numerical coefficient. The present theoretical value $A_1 = 1.58$ is in good agreement with the measurement in atmospheric turbulence reported by Wyngaard and Pao[24], and Wyngaard and Cote[25] as 1.6.

C. Law for the Approach to Viscous Subrange $k_\nu < k < k_c$

We shall determine k_ν later as the transition from the inertia to the viscous subrange. With $k > k_\nu$, we can approximate (98) by

$$\nu^2 \tilde{\eta}' = \frac{c_1 c_2}{2} k^{-4}$$

Use of formula (93) for η', and differentiation yield the solution F_H, in agreement with the Heisenberg[2] law

$$F = F_H = A_2 \left(\varepsilon/\nu^2\right)^2 k^{-7} \tag{101}$$

where

$$A_2 = C_1^2 C_2 = 0.148$$

D. Validity of the Heisenber Law

$k_c > k_\nu$

For the Heisenberg law to be valid, it is necessary that $k_c > k_\nu$. Therefore, it is important to determine those two critical wave numbers. For the transition wave number k_ν, we take the transition spectrum

$$F = A_2 (\varepsilon/\nu^2)^2 k^{-7} \left[1 + (k_\nu/k)^4\right]^{-4/3} \quad (102)$$

which satisfies the Heisenberg law (101) at large k. Since it should satisfy the Kolmogoroff law (100) at small k, we find

$$k_\nu = A_\nu (\varepsilon/\nu^3)^{1/4}, \quad A_\nu = (A_2/A_1)^{3/4} \cong 0.169 \quad (103)$$

On the other hand, from (87) we find

$$k_c = A_c (\varepsilon^3/\nu)^{1/4} \quad (104)$$

with

$$A_c = (2A_2)^{1/4} \cong 0.738$$

Hence by comparing (103) and (104) we find

$$\frac{k_c}{k_\nu} = \frac{A_c}{A_\nu} \cong 4.37 \tag{105}$$

Although dimensional theories could predict k_ν and k_c, to both be proportional to $(\varepsilon^3/\nu)^{1/4}$, but not their numerical coefficients A and A_c. Therefore, this created doubts as to whether the Heisenberg law of approach k^{-7} could ever be developed before it is cut off. Its validity on the basis of its divergence to high order moments was also doubtful. The present analysis of cutoff, with the ratio (105), establishes that there is a sufficient range of wave number for the Heisenberg law k^{-7} to be developed before its ultimate cutoff. In addition, the present theory reveals that the Heisenberg law is only valid as an approach to viscous subrange, and should be followed by a cutoff. Such a cutoff law will be studied in Sec. XE.

E. Cutoff of the Viscous Subrange

$k \sim k_c > k_\nu$

If $k_c > k$, so that the k^{-7} law of approach into the viscous subrange is permitted to develop, then the viscous cutoff of the memory (52) may follow it with a cutoff law.

We employ a different method than that used to derive (101).

Upon differentiating the energy balance (97) with respect to k, we can write:

$$c_1 J^{\circ} \frac{d\tilde{\eta}'}{dk} + (\nu + c_1 \tilde{\eta}') \frac{dJ_o}{dk} = 0$$

approximated by

$$c_1 \frac{d\tilde{\eta}'}{dk} = - \frac{\nu}{\varepsilon} 2k^2 F \tag{106}$$

since the tail of the spectrum occurs at large wave number permitting the approximation

$$J^{\circ} \cong J \cong \varepsilon/\nu \quad \text{and} \quad \nu + c_1 \tilde{\eta}' \cong \nu$$

We shall calculate the eddy viscosity from the memory chain (82), stopped at $\alpha = 2$, rewritten as follows:

$$\tilde{\eta}' = c_2 \int_k^{\infty} dk' \frac{F k'^{-2}}{\tilde{\eta}''}$$

$$\eta^{\circ} = c_2 \int_k^{\infty} dk'' \frac{F k''^{-2}}{\tilde{\eta}''}$$

$$\tilde{\eta}''' = \tilde{\eta}'' \exp(-k/k_c)$$

The system is resolved to give

$$\tilde{\eta}' = c_2 \int_k^{\infty} dk' \, F \, (k')^{-2} \left[2c_2 \int_k^{\infty} dk'' \, (k'')^{-2} F \exp(k''/k_c) \right]^{-1/2}$$

which, upon substitution, transforms (106) into the form

$$\left[2c_2 \int_k^\infty dk'' (k'')^{-2} F \exp(k''/k_c) \right]^{-1/2} = \frac{2\nu^2}{c_1 c_2 \varepsilon} k^4$$

Upon squaring and differentiating, we finally obtain

$$F = F_H \exp(-k/k_c) \tag{107}$$

The spectral cutoff law (107) consists of a Heisenberg spectrum F_H given by (101) accompanied by an exponential tail, which secures the convergence of moments constructed from the spectrum (107).

XI. TURBULENCE IN A STRONG WIND GRADIENT

A. Eddy Relaxation Frequency and Eddy Viscosity in a Strong Wind Gradient

The eddy relaxation frequency, as derived in Sec. X, has been based upon the Langevin eq. (29), when we approximate

$$|\nabla \underset{\sim}{U}| \ll |\nabla \underset{\sim}{u}^0|$$

In a strong wind gradient, the reverse is true, i.e.,

$$|\nabla \underset{\sim}{U}| \gg |\nabla \underset{\sim}{u}^0|$$

Under such a circumstance, the solution of that Langevin equation can be approximately as follows:

$$u_i'(t) = -\frac{\partial U_i}{\partial x_r}\int_0^t dt'\ u_r'(t')$$

and similarly

$$u_i'(t_1) = -\frac{\partial U_i}{\partial x_s}\int_0^{t_1} dt''\ u_s'(t'')$$

The product, averaged and integrated with Langrangian time T_1, is

$$\int_0^t dt_1 \langle u_i'(t) u_i'(t_1)\rangle' = \frac{\partial U_i}{\partial x_r}\frac{\partial U_i}{\partial x_s}\int_0^t dt_1 \int_0^t dt' \int_0^{t_1} dt'' \langle u_r'(t') u_s'(t'')\rangle'$$

$$\cong \frac{1}{3}\omega_s^2 \int_0^t dt_1 \int_0^t dt' \int_0^t dt'' \langle \underset{\sim}{u}'(t') \cdot \underset{\sim}{u}'(t'')\rangle' \qquad (108)$$

with the assumption of the isotropic property of small eddies of rank 1. Here,

$$\omega_s^2 = \left(\frac{\partial U_i}{\partial x_r}\right)^2$$

When we observe that the velocity correlations

$$\langle \underset{\sim}{u}'(t) \cdot \underset{\sim}{u}'(t_1)\rangle' = \langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(t-t_1)\rangle'$$

and

$$\langle \underset{\sim}{u}'(t')\cdot\underset{\sim}{u}'(t'')\rangle' = \langle \underset{\sim}{u}'(0)\cdot\underset{\sim}{u}'(t'-t'')\rangle'$$

are functions of the difference of the two arguments only, and choose the duration of correlation t_c as the upper limit of integrations in the place of t, we transform (108) into

$$\int_0^{t_c} d\tau \langle \underset{\sim}{u}'(0)\cdot\underset{\sim}{u}'(\tau)\rangle' = \frac{1}{3}\,\omega_s^2 \int_0^{t} dt' \int_0^{t_c} dt_1 \int_0^{t_1} dt'' \langle \underset{\sim}{u}'(0)\cdot\underset{\sim}{u}'(t'-t'')\rangle'$$

$$= \frac{1}{3}\,\omega_s^2 \int_0^{t_c} dt' \int_0^{t_c} dt'' \,(t_c - t'')\langle \underset{\sim}{u}'(0)\cdot\underset{\sim}{u}'(t'-t'')\rangle'$$

$$\cong \frac{1}{3}(\omega_s t_c)^2 \int_0^{t_c} ds \langle \underset{\sim}{u}'(0)\cdot\underset{\sim}{u}'(s)\rangle'$$

yielding the relaxation

$$\frac{1}{3}(\omega_s t_c)^2 = 1$$

which finds the eddy relaxation frequency to be

$$t_c^{-1} = \frac{1}{\sqrt{3}}\,\omega_s \tag{109}$$

By equating the expression for the relaxation frequency obtained in (109) and that as occurring in (82), we find

$$\frac{1}{\sqrt{3}}\,\omega_s = k^2 \tilde{\eta}'$$

or

$$\tilde{\eta}' = \frac{1}{\sqrt{3}}\, \omega_s\, k^{-2} \tag{110}$$

If we identify k^{-1} to be a mixing length, the formula (110) resembles the Prandtl formula for the eddy viscosity for flows in a boundary layer.

B. Energy Spectrum

The turbulent motion, in the production subrange of the spectrum, is primarily controlled by the wind gradient without viscous effect, so that we can write the equation of energy balance (96) in the following inviscid form:

$$-\eta^{\circ}_{33}\left(\frac{\partial U_1}{\partial x_3}\right)^2 + c_1 \tilde{\eta}' J^{\circ} = -\eta_{33}\left(\frac{\partial U_1}{\partial x_3}\right)^2 + \varepsilon \tag{111}$$

where the secondary effect of inertia is approximately represented in its isotropic form.

The eddy viscosity η^{o}_{33} may be considered proportional to η^{o}, so that the production function will be written approximately as

$$\eta^{\circ}_{33}\left(\frac{\partial U_1}{\partial x_3}\right)^2 \cong c_1 \eta^{\circ} \omega_s^2$$

where ω_s is the effective wind speed gradient, taking into account the above proportionality correction. Hence, the equation of energy balance (111) now simplifies to

$$-c_1\eta^0\omega_s^2 + c_1\tilde{\eta}'J^0 = -c_1\eta\omega_s^2 + \varepsilon$$

or to

$$c_1\tilde{\eta}'(\omega_s^2 + J^0) = \varepsilon \tag{112}$$

Upon applying the formula (110) for $\tilde{\eta}'$, we can rewrite Eq. (112) for the energy balance as follows

$$c_1(\omega_s^2 + J^0) = \sqrt{3}\,(\varepsilon/\omega_s)\,k^2$$

which, after differentiation, yields,

$$F = A_s\,(\varepsilon/\omega_s)\,k^{-1} \tag{113}$$

with

$$A_s = \sqrt{3}\,c_1^{-1} \equiv 1/\sqrt{3}$$

The power law k^{-1} in the production subrange (113) has been found earlier by Tchen[7,26] on a separate basis, and agrees with experimental observations[7,26-28].

The production subrange has a negligible dissipation spectrum and a negligible vorticity function; therefore, we have

$$\left| \frac{1}{2} c_1 J^0 \frac{d[(\tilde{\gamma}')^2]}{dk} \right| \ll 2c_1 k^2 F(\tilde{\gamma}') \qquad (114)$$

so that the end of the production subrange can be determined by (113) and (114) to be

$$k_s = a_s \left(\omega_s^3 / \varepsilon \right)^{1/2} \quad . \qquad (115)$$

with

$$a_s = \left(\frac{1}{4} c_1^2 c_2 A_s \right)^{1/2} \cong 0.145$$

XII. CONCLUSIONS

A. Double Hierarchies and Closure

In view of the nonlinearity to the second degree of the hydrodynamic equations, we expect that the velocity fluctuations in Fourier space, their double, triple, and quadruple correlations, etc. are coupled through their respective dynamic equations of evolution, forming the so-called hierarchy. One such hierarchy, as represented by (43) - (45), is for the development of energy and Reynolds stress, and the other is for the development of eddy viscosity (79) in the way of its approach to equilibrium.

The above double hierarchies are coupled.

The first hierarchy is closed at its quadruple correlation (56). By decomposing the velocity fluctuation into a series of ranks (1), forming a repeated cascade, the quadruple correlation becomes degenerate, as it appears in the form of the product of two double correlations of different ranks, see (56). The second hierarchy (79) describes the chain of memories governing the relaxation of eddy viscosity of different ranks. Thus, the eddy viscosity η^o of zeroth rank in (94) controls the transfer of energy from external wind shear into turbulent energy E^o_{ii} in an arbitrary portion of spectrum between o and $k = k^o$ (k, or k^o, is the independent variable of an integro-differential equation having values between o and ∞). The eddy viscosity $\tilde{\eta}'$ of rank 1 controls the transfer of energy away from E^o_{ii} into all higher wave numbers in (94). It turns out that the evolution, or approach to equilibrium, of an eddy viscosity $\tilde{\eta}^\alpha$ of rank α requires a relaxation frequency to be determined by an eddy viscosity $\tilde{\eta}^{\alpha+1}$ of higher rank, and so the chain (79) continues. The relation between $\tilde{\eta}^\alpha$ and $\tilde{\eta}^{\alpha+1}$ in the integral form (79) reproduces a memory chain with increasing ranks. It is true that the hierarchy of ranks involves many variables as k^α, but they enter as independent variables in multiple integral equations. This gives the advantage of leaving all wave numbers k^α, denominating the ranks, to be perfectly independent of any physical conditions or parameters. A scheme given by (81) for the cutoff of the memory chain is devised, on the basis of a dynamical equation (84), for the propagation eddy viscosity in the presence of mode transfer and molecular dissipation.

In summary, two fundamental assumptions are introduced: (a) A turbulent fluctuation can be decomposed into a series of ranks which exhibit relative macroscopic (coarse grain) and random behavior, and hence relative degrees of quasi-stationarity. This assumption permits the closure of the hierarchy on Reynolds stress which has its quadruple correlation degenerated. (b) The degradation of eddy viscosity follows a pattern of memory relaxation which is mainly due to eddy scrambling, but is cut off by a viscous dissipation. As a result, the hierarchy for the eddy viscosity becomes closed.

B. Homogeneous and Isotropic Turbulence

In the inertial subrange, the theory predicts the Kolmogoroff law (100) with a Kolmogoroff constant $A_1 = 1.58$ in good agreement with experiments.

As an approach to the viscous subrange, and with a compound eddy and molecular dissipation, but with an inviscid eddy viscosity appropriate to the approach region, the Heisenberg law k^{-7} (101) is recovered with a coefficient $A_2 = 0.148$. The inclusion of a relaxation cutoff modifies the Heisenberg k^{-7} law to that entailing an exponential cutoff (107), thus securing the convergence of the spectrum.

The condition of validity or occurrence of the above Heisenberg approach law, together with its entailing cutoff law, is also established. The cutoff wave number k_c is calculated in (105) to be larger than the transitic wave number k_ν, so that the Heisenberg law can have a chance of development.

C. Effect of Wind Gradient

A spectral law k^{-1} is found in (113), valid in the production subrange.

D. Critical Wave Numbers

The following critical wave numbers separate the production, inertial, and viscous subranges (115), (103), (104):

$$k_s = 0.145 \left(\omega_s^3/\varepsilon\right)^{1/2}$$

$$k_\nu = 0.169 \left(\varepsilon/\nu^3\right)^{1/4}$$

$$k_c = 0.738 \left(\varepsilon/\nu^3\right)^{1/4}$$

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