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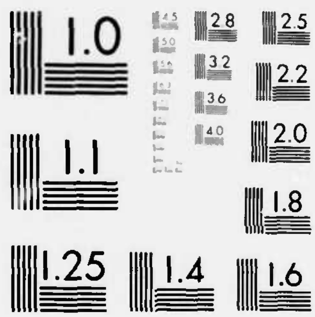

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ESTIMATION OF OPTIMAL DEPOT STOCK IN TWO-ECHELON INVENTORY SYSTEMS FOR RECOVERABLE ITEMS

by

John A. Muckstadt

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ABSTRACT

In 1968 Sherbrooke formulated the well known METRIC model for a two-echelon inventory system consisting of a set of bases and a supporting depot. The items stocked in the system are called recoverable items, that is, they are subject to repair when they fail. Sherbrooke and others have proposed a variety of algorithms for determining optimal base and depot stock levels. A substantial portion of the computational requirement associated with each of the algorithms is related to the search for the optimal depot stock level. The purpose of this paper is to describe an easily implementable method for estimating the optimal depot stock level. The computational experience reported in the paper indicates the proposed method provides an excellent estimate of the optimal depot stock level particularly for high demand items. Furthermore, the proposed method significantly reduces the computational requirements for any known algorithm for solving Sherbrooke's problem.

1. INTRODUCTION

In 1968 Sherbrooke[2] presented a mathematical model of a two-echelon inventory system consisting of a set of bases, at which primary demands occur, and a supporting depot which resupplies the bases when necessary. The items stocked in this system are called recoverable items, that is, items subject to repair when they fail. Both the bases and depot serve two functions; they are both inventory stocking points as well as locations at which maintenance takes place. Each time a demand for an item is levied on a base supply organization, a corresponding requirement exists to perform maintenance on a failed item. A base may lack adequately skilled technicians or equipment needed to accomplish the repair. Only in these instances will a failed item be sent to the depot for repair; otherwise, the base maintenance organization will repair the item. The repairing location is responsible for resupplying the base supply organization at which the original demand was placed. All resupply is done on a one-for-one basis.

Sherbrooke describes both the problem and its formulation and analysis in detail in reference 2. In addition, Sherbrooke proposed a marginal analysis algorithm for solving the problem. Another approach has been suggested by Fox and Landi[1] for determining optimal base and depot stock levels. A substantial portion of the computational requirement associated with both of these algorithms is related to the search for the optimal depot stock level. As will be discussed, this search is particularly time consuming for items having a high expected number of items in transit to the depot and in depot repair.

The purpose of this paper is to develop a method for estimating the optimal depot stock level which is both accurate and easy to implement. Computational experience reported in this paper indicates the proposed method provides a very good estimate of the optimal depot stock level. Furthermore, the proposed method significantly reduces the overall computational requirement for the algorithms developed by Fox and Landi[1] and Sherbrooke[2].

2. BACKGROUND

The optimization problem posed in reference 2 by Sherbrooke for finding optimal stock levels for the depot-base system described in Section 1 is

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

$$\text{subject to} \quad \sum_{i=1}^{m} c_i \sum_{j=0}^{n} s_{ij} \leq C, \tag{1}$$

$$s_{ij} = 0,1,\ldots,$$

where n is the number of bases,

m is the number of items,

s_{ij} is the stock level at location j for item i,

s_{i0} is the depot stock level for item i,

λ_{ij} is the expected daily demand rate for item i at base j,

c_i is the unit cost for item i,

C is the budget constraint,

$T_{ij}(s_{i0})$ is the average resupply time for base j for item i given the depot stock level for item i is s_{i0},

and $p(x|\lambda_{ij}T_{ij}(s_{i0})) = e^{-\lambda_{ij}T_{ij}(s_{i0})}(\lambda_{ij}T_{ij}(s_{i0}))^x/x!$, where $p(x|\cdot)$ represents the probability of having x units in the resupply system at any point in time.

Furthermore, Sherbrooke shows that $T_{ij}(s_{i0})$ can be expressed as

$$T_{ij}(s_{i0}) = r_{ij}A_{ij} + (1-r_{ij})(B_{ij} + \delta(s_{i0})\cdot D_i), \qquad (2)$$

where A_{ij} is the average base repair time for item i at base j,

r_{ij} is the proportion of demands requiring base repair for item i at base j,

B_{ij} is the average order-and-ship time at base j for item i,

D_i is the average depot repair cycle time for item i,

$\delta(s_{i0})\cdot D_i \overset{\Delta}{=} \frac{1}{\lambda_i} \sum_{x>s_{i0}} (x-s_{i0})p(x|\lambda_i D_i)$, the expected depot delay per demand for item i,

and $\lambda_i \overset{\Delta}{=} \sum_{j=1}^{n} (1-r_{ij})\lambda_{ij}$, the expected daily depot demand rate for item i.

Subsequently Fox and Landi[1] proposed a Lagrangian formulation of Problem 1 and suggested a relatively simple method for solving the Lagrangian problem. This approach requires only a small fraction of the time consumed by Sherbrooke's method to obtain the optimal solution. Their Lagrangian formulation of the problem is

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{x>s_{ij}} (x-s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0})) + \theta \sum_{j=0}^{n} \sum_{i=1}^{m} c_i s_{ij} \qquad (3)$$

subject to $s_{ij} = 0,1,\ldots,$

where θ is a Lagrangian multiplier. Fox and Landi propose to solve Problem 1 by solving Problem 3 for a fixed finite set of multiplier values $\theta_0,\ldots,\theta_M$. In their approach, the "optimal" value of θ is the θ_K, $K\epsilon\{0,\ldots,M\}$, whose total investment cost is closest to C. We will now see how the solution to Problem 3 is found for a given value of θ.

Observe that Problem 3 is separable by item. Consequently for each θ, its solution can be obtained by solving m independent item problems. To simplify the notation we drop the item designator in what follows. Then for each item for a given θ we solve a problem of the form

$$\min \sum_{j=1}^{n} \sum_{x>s_j} (x-s_j)p(x|\lambda_j T_j(s_0)) + \theta \sum_{j=0}^{n} cs_j$$

subject to $s_j = 0,1,\ldots,$ where $j = 0,\ldots,n$. This problem can be partitioned as follows:

$$\min_{s_0=0,1,\ldots} \{\theta cs_0 + \sum_{j=1}^{n} \min_{s_j}\{\sum_{x>s_j} (x-s_j)p(x|\lambda_j T_j(s_0)) + \theta cs_j : s_j=0,1,\ldots,\ s_0 \text{ fixed}\}\}. \tag{4}$$

Problem 4 is not necessarily convex in s_0; however, the inside minimization problems are convex for a fixed value of s_0. We may restate Problem 4 as follows:

$$\min_{s_0=0,1,\ldots} Z(s_0;\theta)$$

where

$$Z(s_0;\theta) \overset{\Delta}{=} \theta cs_0 + \sum_{j=1}^{n} \min_{s_j}\{\sum_{x>s_j} (x-s_j)p(x|\lambda_j T_j(s_0)) + \theta cs_j : s_j = 0,1,\ldots,\ s_0 \text{ fixed}\}. \tag{5}$$

To determine $Z(s_0;\theta)$, we must solve the n base problems of the form

$$\min_{s_j = 0,1,\ldots} \sum_{x>s_j} (x-s_j)p(x|\lambda_j T_j(s_0)) + \theta cs_j. \tag{6}$$

It is easy to see that the optimal s_j for Problem 6 is the smallest nonnegative integer for which

$$\sum_{x>s_j} p(x|\lambda_j T_j(s_0)) \leq \theta c. \tag{7}$$

To find the optimal solution to Problem 5 we must implicitly examine $Z(s_0;\theta)$ for $s_0 = 0,1,\ldots$. In practice, however, the number of values for s_0 that need to be examined explicitly is small. This is due to the asymptotic behavior of $\delta(s_0)$. Specifically, $\delta(s_0)$ approaches 0 quite rapidly as s_0 increases beyond λD. (Experience gained by the author in applying the Fox-Landi technique to real Air Force problems suggests that no more than 15 values for s_0 need to be explicitly examined for any item.) Observe that the amount of time required to solve Problem 1 using the Fox-Landi algorithm is roughly proportional to the number of depot stock levels for which $Z(s_0;\theta)$ is explicitly evaluated. Computation time required by Sherbrooke's algorithm to obtain the solution to Problem 1 is also proportional to the number of depot stock levels examined. Consequently, if the number of depot stock levels that are examined explicitly can be reduced, then the total time required to find the optimal solution to Problem 1 can also be reduced.

In the next section we develop a method that provides an excellent estimate of the optimal depot stock level. This method significantly reduces the computation time required to find an optimal solution to Problem 1 using either the Fox-Landi or the Sherbrooke algorithm.

3. THE ESTIMATION PROCEDURE

In the preceding section we indicated how the optimal base stock level, call it s_j^*, can be determined for a given value of s_0 (and θ). In particular, we showed that s_j^* is optimal if it is the smallest nonnegative integer for which

$$\sum_{x>s_j} p(x|\lambda_j T_j(s_0)) \leq \theta c.$$

We begin this section by developing a different but equivalent way of characterizing s_j^*. We then show how to find an estimate of the optimal depot stock level by solving a relaxed version of Problem 4 in which both the nonnegativity and integrality requirements for s_0 are dropped.

Define the backorder function for base j for a given depot stock level s_0 as

$$B_j(s_j;s_0) \stackrel{\Delta}{=} \sum_{x>s_j} (x-s_j)p(x|\lambda_j T_j(s_0)), \text{ for } s_j \geq 0 \text{ and integer,}$$

and the convex piecewise linear completion of B_j, call it $\hat{B}_j$, as follows:

$$\hat{B}_j(t;s_0) \stackrel{\Delta}{=} \begin{cases} B_j(t;s_0), \text{ if } t \text{ is a nonnegative integer.} \\ [B_j(s_j;s_0) - B_j(s_j-1;s_0)](t-(s_j-1)) + B_j(s_j-1;s_0),\ s_j-1<t<s_j; \\ \quad \text{where } s_j \text{ is a nonnegative integer, and } B_j(-1;s_0) \stackrel{\Delta}{=} \infty. \end{cases}$$

Let $\Delta\hat{B}_j(s_j;s_0) \stackrel{\Delta}{=} \hat{B}_j(s_j;s_0) - \hat{B}_j(s_j-1;s_0)$ when s_j is a nonnegative integer, and

$$D(s_j;s_0) \stackrel{\Delta}{=} \{v:\ \Delta\hat{B}_j(s_j;s_0) < v \leq \Delta\hat{B}_j(s_j+1;s_0)\}.$$

Observe that $D_j(s_j;s_0)\cup\{\Delta\hat{B}_j(s_j;s_0)\}$ is the set of subgradients of $\hat{B}_j$ at s_j. Then an alternative way of verifying that s_j^* is an optimal base stock level is to show that $-\theta c \in D(s_j^*;s_0)$. This alternative characterization of s_j^* will be subsequently used in the development of the estimation procedure.

Next let $F(s_1,s_2,\ldots,s_n;\ s_0) \stackrel{\Delta}{=} \sum_{j=1}^{n} (B_j(s_j;s_0) + \theta c s_j)$. By dropping both the integrality and nonnegativity restrictions on s_0, we obtain the following relaxation of Problem 4:

$$\min_{s_0} \{\theta c s_0 + \min_{s_j=0,1,\ldots} \{F(s_1,\ldots,s_n;s_0)\colon\ s_0\ \text{fixed}\}\}. \tag{8}$$

If s_0 is the optimal solution to Problem 4, then

$$\frac{\partial F}{\partial s_0} + \theta c = 0. \tag{9}$$

But
$$\frac{\partial F}{\partial s_0} = \sum_{j=1}^{n} \frac{\partial B_j}{\partial T_j} \frac{\partial T_j}{\partial s_0}.$$

Furthermore, by writing $B_j(s_j;s_0)$ as $\sum_{K=1}^{\infty} Kp(K+s_j|\lambda_j T_j(s_0))$, we see that

$$\frac{\partial B_j}{\partial T_j} = \sum_{K=1}^{\infty} \lambda_j K e^{-\lambda_j T_j(s_0)} \frac{(\lambda_j T_j(s_0))^{K+s_j-1}}{(K+s_j-1)!} - \sum_{K=1}^{\infty} K\lambda_j e^{-\lambda_j T_j(s_0)} \frac{(\lambda_j T_j(s_0))^{K+s_j}}{(K+s_j)!}$$

$$= -\lambda_j \Delta\hat{B}(s_j;s_0).$$

Let $B_0(s_0) \triangleq \sum_{x>s_0} (x-s_0)p(x|\lambda D)$. Data gathered during the conduct of the study showed that $B_0(s_0)$ can be closely approximated by an exponential function. This result should not be entirely surprising. The only probabilities entering the backorder calculation are the so-called "tail" probabilities. In the upper tail, the Poisson distribution behaves almost like a geometric distribution, that is, each succeeding probability is approximately a constant proportion of the preceding value. This is particularly true for large values of s_0. Consequently, when s_0 is large the expected number of backorders existing at any point in time at the depot is approximately a geometric function of s_0. Thus an exponential function provides a continuous approximation to the relationship between depot backorders and depot stock. We therefore will approximate $B_0(s_0)$ by the exponential function of the form

$a_0 e^{-b_0 s_0}$, where a_0 and b_0 are positive real numbers.

Then using this approximation we rewrite $T_j(s_0)$ as

$$T_j(s_0) = r_j A_j + (1-r_j)(B_j + \frac{1}{\lambda} a_0 e^{-b_0 s_0})$$

and observe that

$$\frac{\partial T_j}{\partial s_0} = -\frac{(1-r_j)}{\lambda} a_0 b_0 e^{-b_0 s_0} .$$

Upon combining these observations we see that

$$\frac{\partial F}{\partial s_0} \cong \sum_{j=1}^{n} \lambda_j \, \Delta \hat{B}(s_j;s_0) \frac{(1-r_j)}{\lambda} a_0 b_0 e^{-b_0 s_0} .$$

Recall that $-\theta c \in D(s_j^*;s_0)$. Consequently $-\theta c$ approximates the marginal reduction in backorders at base j when the stock level at that base is s_j^*. After making this substitution and representing this further approximation of $\frac{\partial F}{\partial s_0}$ by $\frac{\hat{\partial F}}{\partial s_0}$, we see that

$$\frac{\hat{\partial F}}{\partial s_0} = -\sum_{j=1}^{n} (1-r_j)\lambda_j \frac{1}{\lambda} \theta c a_0 b_0 e^{-b_0 s_0} = -\theta c a_0 b_0 e^{-b_0 s_0} .$$

Substituting this approximation into Equation 9 we obtain the following estimate of the optimal depot stock level:

$$\hat{s}_0 = -\frac{1}{b_0} \ln\{\frac{1}{a_0 b_0}\} . \qquad (10)$$

Recall the value of $\hat{s}_0$ was derived based on an exponential approximation of $B_0(s_0)$. As the average number of units in the depot repair cycle increases, that is, as λD increases, the quality of this exponential approximation improves in the region in which the optimal depot stock level should be located.

Consequently, the approximation should be most accurate in these cases. But the items for which the search for the optimal depot stock level is most time consuming for both the Sherbrooke and Fox-Landi algorithms are the items having a large number of units in depot repair. Therefore the proposed approximation method will be most appropriate for those items requiring the greatest amount of computation.

The approach we have described for estimating the optimal depot stock level has been coded and tested using a sample of 40 items found on the Air Force's new F-15 aircraft. The test consisted of two sets of runs. In the first set, monthly flying was divided among 3 bases; in the second set, the same monthly flying program was divided among 5 bases. The total budget distributed among the 40 items ranged from \$34 million to \$65 million in the first set of runs, and from \$34 million to \$88 million in the second set. Table 1 contains the data indicating both the optimal and estimated depot stock level for each item in both runs.

The data show that there usually is no single optimal depot stock level for an item. Rather the optimal value depends on the amount of total item system stock available for distribution among the depot and bases. The data displayed in Table 1 also indicate the estimate of optimal depot stock is quite close to the optimal value in all cases. Furthermore, the increase in expected system backorders using the estimated depot stock levels rather than the optimal levels is generally small. For most items the increase is substantially less than .1 backorders.

The results of the tests indicate that it is possible to estimate closely the optimal depot stock level using Equation 10. Additionally, incorporating this method for estimating the optimal depot stock into the Fox-Landi and Sherbrooke algorithms will reduce the search required to find the optimal depot stock level and will therefore reduce the computational time needed to solve Problem 1 using these algorithms. In the tests, the computation time for high demand items ($\lambda D \geq 20$) was reduced by over 90 per cent. Consequently, this approximation method is recommended for use particularly in situations where expected depot demand is large.

Item	Optimal Depot Stock Levels Case I (3 Bases)	Optimal Depot Stock Levels Case II (5 Bases)	Estimated Optimal Depot Stock Levels
1	4-7	5-9	6
2	1,2	1-3	1
3	6	6,7	6
4	0-2	2,3	1
5	10,11	8-12	10
6	18-21	18-21,25	19
7	1,2	1,2	1
8	2	3,4	2
9	5,6	6,7	6
10	1	1,2	1
11	4,5	4-6	5
12	1	1	0
13	0-2	0,1	0
14	1-3	1-3	2
15	2-4	3,4	3
16	8,9	8,9	8
17	1,2	1,2	1
18	3,4	3-5	3
19	12-14	13-14	12
20	9-12	10-13	10
21	21-27	22-28	23
22	4,5	4-6	5
23	1	1-3	1
24	1,2	2,3	2
25	5-7	6,7	6
26	16	16	16
27	3	3,4	3
28	40-42	41-43	40
29	8-10	9,10	9
30	1	2	9
31	1,2	1,2	1
32	8,9	8,9	8
33	4,5	5,6	5
34	9-11	9,10	10
35	6,7	7,8	7
36	1-3	2	2
37	1,2	1,2	1
38	7,8	7-9	7
39	2,3	3,4	3
40	41-43	42-44	41

TABLE 1

Comparison of Optimal and Estimated Depot Stock Levels

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2. Sherbrooke, Craig C., "METRIC: A Multi-Echelon Technique for Recoverable Item Control," Operations Research, Vol. 16, 1968, pp. 122-141.

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In 1968 Sherbrooke formulated the well known METRIC model for a two-echelon inventory system consisting of a set of bases and a supporting depot. The items stocked in the system are called recoverable items, that is, they are subject to repair when they fail. Sherbrooke and others have proposed a variety of algorithms for determining optimal base and depot stock levels. A substantial portion of the computational requirement associated with each of the algorithms

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