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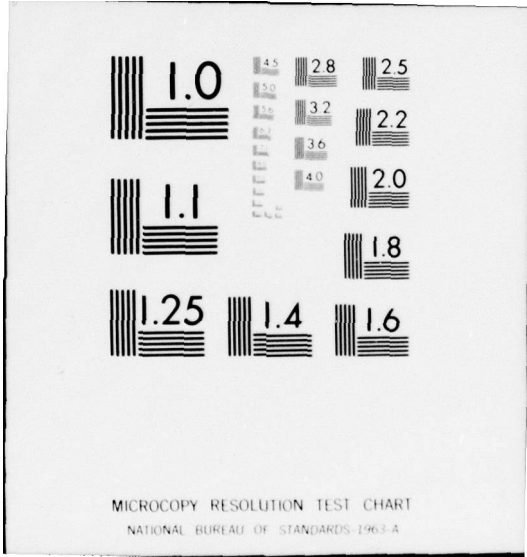
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THE CALCULATION OF e^{At} WITH SOME APPLICATIONS

Elmo J. Stewart

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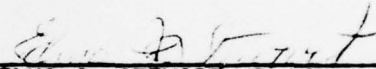
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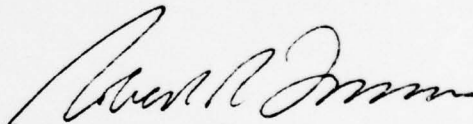
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Given A an $n \times n$ matrix with real or complex elements, then e^{At} is calculated as the unique solution of an initial value problem. In the process of obtaining this solution n -unknown matrices become involved and must be computed. Characterizing properties of these matrices to be computed become known: such properties as pairwise-orthogonal, idempotent and nilpotent. Finally some applications of the above calculations are given in the field of solutions to systems of differential equations.		

THE CALCULATION OF e^{At} WITH SOME APPLICATIONS

1. Introduction

Throughout this paper A will be an $n \times n$ matrix with real or complex elements, having $f(\lambda)$ as its characteristic function and eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ (not necessarily distinct). In this paper we calculate the exponential matrix e^{At} , specify some properties of certain matrices that must be determined in order to describe e^{At} and finally indicate some applications of this calculation.

2. An Initial Value Problem and e^{At}

In [3] e^{At} is obtained as the unique solution to the following initial value problem. With $f(\lambda)$ as specified above and $D \equiv \frac{d}{dt}$ we wish to obtain the solution to:

$f(D)G(t) = 0$, $G(t)$ an $n \times n$ matrix with elements functions of t and such that $G(0) = I$, $G'(0) = A$, \dots , $G^{(n-1)}(0) = A^{n-1}$.

If e^{At} is defined by the equation

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!},$$

then

$$e^{At} \Big|_{t=0} = I, \quad D(e^{At}) \Big|_{t=0} = A, \quad \dots \quad D^{n-1}(e^{At}) \Big|_{t=0} = A^{n-1},$$

and $f(D)e^{At} = f(A)e^{At} = 0$ by the Cayley-Hamilton Theorem. Therefore, e^{At} is the unique solution to this initial value problem. Suppose $\alpha_1, \alpha_2, \dots, \alpha_s$ are the distinct eigenvalues of A with multiplicities $\mu_1, \mu_2, \dots, \mu_s$, we may write

$$(1) \quad e^{At} = \sum_{k=1}^s (C_{k1} + C_{k2}t + \dots + C_{k\mu_k} t^{\mu_k-1}) e^{\alpha_k t},$$

where the C_{kj} are $n \times n$ matrices. From the initial conditions we have:

$$I = \sum_{k=1}^s C_{k1}$$

$$A = \sum_{k=1}^s (\alpha_k C_{k1} + C_{k2})$$

(2)

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$A^{n-1} = \sum_{k=1}^s (\alpha_k^{n-1} C_{k1} + (n-1)\alpha_k^{n-2} C_{k2} + \dots + \frac{(n-1)! \alpha_k^{n-\mu_k}}{(n-\mu_k)! (\mu_k-1)!} C_{k\mu_k}),$$

from which the $n C_{kj}$ can be determined. We note from the system (2) that a solution for a C_{kj} will be a linear combination of the left side of (2) and therefore, any C_{kj} will be a polynomial in A of degree at most $(n-1)$. Being polynomials in A , the C_{kj} commute.

3. Properties of the C_{k1} when all roots of $f(\lambda)$ are distinct.

We will not use the system (2) to completely solve for the $n C_{kj}$, rather will we obtain another representation for e^{At} satisfying the initial value problem and then equate coefficients of like terms " $t^k e^{\alpha_k t}$ ". Before proceeding to this representation let us consider the simple case in which all roots of $f(\lambda)$ are distinct. This case will provide insights on how to handle the more general case of multiple roots.

If all roots of $f(\lambda)$ are distinct, then (1) becomes:

$$(1)' \quad e^{At} = \sum_{k=1}^n C_{k1} e^{\alpha_k t}.$$

The inverse of (1)' is

$$(1)'' \quad e^{-At} = \sum_{k=1}^n C_{k1} e^{-\alpha_k t}.$$

Multiplying (1)' and (1)" and using the commutative property for the C_{k1} we obtain

$$(3) \quad I - \sum_{k=1}^n C_{k1}^2 = \sum_{k=1}^n \sum_{\substack{j=1 \\ k \neq j}}^n C_{k1} C_{j1} e^{(\alpha_k - \alpha_j)t}.$$

Equation (3) is true for all t , however, the left side of (3) is independent of t which suggests that $C_{k1} C_{j1} = 0$ for $k \neq j$, i.e., the C_{k1} are pairwise orthogonal. (This will be shown below.)

Applying the initial conditions to (1)' we obtain:

$$(2)' \quad \begin{aligned} I &= C_{11} + C_{21} + \dots + C_{n1} \\ A &= \alpha_1 C_{11} + \alpha_2 C_{21} + \dots + \alpha_n C_{n1} \\ &\vdots \\ A^{n-1} &= \alpha_1^{n-1} C_{11} + \alpha_2^{n-1} C_{21} + \dots + \alpha_n^{n-1} C_{n1}. \end{aligned}$$

Since the α_k are distinct, the coefficient matrix for the system (2)' is a Vandermonde matrix and is non-singular. Therefore, the system (2)' has unique solutions for the C_{k1} . If we solve the system (2)' for C_{k1} , then the coefficient of A^{n-1} in this solution is:

$$(4) \quad (-1)^{n+k} \left| \begin{array}{cccc|ccc} 1 & \dots & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_{k-1} & \alpha_{k+1} & \dots & \alpha_n & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^{n-2} & \dots & \alpha_{k-1}^{n-2} & \alpha_{k+1}^{n-2} & \dots & \alpha_n^{n-2} & \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{array} \right|$$

Both numerator and denominator of this coefficient are non-zero Vandermonde determinants and, therefore, C_{k1} ($k = 1, 2, \dots, n$) is a polynomial

in A of degree precisely $n-1$. Moreover, the coefficient of A^{n-1} in this solution for C_{k1} is

$$1 / \prod_{\substack{j=1 \\ j \neq k}}^n (\alpha_k - \alpha_j),$$

which agrees with the expansion of (4).

Next, suppose we wish to eliminate C_{k1} from the second through the n^{th} equation in (2)', which yields $n-1$ equations in the $n-1$ unknown matrices $C_{11}, C_{21}, \dots, C_{k-1,1}, C_{k+1,1}, \dots, C_{n1}$. We do this in order: subtract α_k times the first equation from the second, α_k^2 times the first from the third, to finally α_k^{n-1} times the first from the n^{th} . Deleting the first of this new system of equations we obtain $n-1$ equations with the unknown C_{k1} missing. The left members of this new system will be $A - \alpha_k I, A^2 - \alpha_k^2 I, \dots, A^{n-1} - \alpha_k^{n-1} I$, each of which has a factor $A - \alpha_k I$, as does any linear combination of these left members. Therefore, each of the solutions for $C_{11}, C_{21}, \dots, C_{k-1,1}, C_{k+1,1}, \dots, C_{n1}$ will have a factor $A - \alpha_k I$. From this we conclude further that:

$$(5) \quad C_{j1} = a_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (A - \alpha_\ell I), \quad j = 1, 2, \dots, n$$

for some scalar a_j . We note from (5) that C_{j1} is precisely a polynomial of degree $(n-1)$ in A with leading coefficient a_j which must be

$$(6) \quad a_j = 1 / \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\alpha_j - \alpha_\ell) \quad (\text{from (4)}) .$$

As an example using (5) and (6), let A be any 3×3 matrix with distinct eigenvalues $\alpha_1, \alpha_2, \alpha_3$, then;

$$e^{At} = \frac{(A - \alpha_2 I)(A - \alpha_3 I)e^{\alpha_1 t}}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_3 I)e^{\alpha_2 t}}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_2 I)e^{\alpha_3 t}}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}.$$

3.1 Orthogonality and Idempotency of the C_{k1} .

A further conclusion to be made at this point occurs when, using (5), we multiply C_{j1} and C_{i1} ($i \neq j$); this yields

$$\begin{aligned} C_{j1}C_{i1} &= a_j a_i \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (A - \alpha_\ell I) \prod_{\substack{\ell=1 \\ \ell \neq i}}^n (A - \alpha_\ell I) \\ &= a_j a_i f(A) \prod_{\substack{\ell=1 \\ \ell \neq i, \ell \neq j}}^n (A - \alpha_\ell I) = 0 \end{aligned}$$

by the Cayley-Hamilton Theorem. Therefore, as conjectured earlier, the C_{k1} are pairwise orthogonal when the eigenvalues of A are distinct.

Using the first of the initial conditions in (2)' and this orthogonality we have $C_{j1} = C_{j1} \sum_{k=1}^n C_{k1} = C_{j1}^2$, $j = 1, 2, \dots, n$, i.e., each C_{j1} is idempotent.

We summarize section (3) in the Theorem I. Given A and $f(\lambda)$ with distinct eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ we have

$$a) \quad e^{At} = \sum_{k=1}^n C_{k1} e^{\alpha_k t}$$

$$b) \quad C_{k1} = \prod_{\substack{\ell=1 \\ \ell \neq k}}^n \frac{(A - \alpha_\ell I)}{(\alpha_k - \alpha_\ell)} \quad k = 1, 2, \dots, n$$

$$c) \quad C_{i1}C_{j1} = \begin{cases} C_{i1} & \text{if } i = j \text{ (Idempotent)} \\ 0 & \text{if } i \neq j \text{ (Orthogonal)} \end{cases}$$

d) All C_{k1} are polynomials of degree $n - 1$ in A and they commute.

4. Properties of the C_{kj} when $f(\lambda)$ has multiple roots.

4.1 Minimal Polynomial for a given matrix.

First consider the example:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for which } f(\lambda) = (\lambda - 1)^4.$$

According to (1)

$$e^{At} = (C_{11} + C_{12}t + C_{13}t^2 + C_{14}t^3)e^t.$$

Using the initial conditions we obtain:

$$C_{11} = I, \quad C_{12} = (A - I), \quad C_{13} = \frac{1}{2!} (A - I)^2, \quad C_{14} = \frac{1}{3!} (A - I)^3.$$

However:

$$C_{13} = (A - I)^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = 0,$$

and, therefore, $C_{13} = C_{14} = 0$. For the given matrix $(\lambda - 1)^4 = 0$ is the characteristic equation and $(A - I)^4 = 0$. But it is also true for this matrix that $(A - I)^3 = (A - I)^2 = 0$ and $A - I \neq 0$. In this case A not only satisfies its characteristic equation, it also satisfies the equations $(\lambda - 1)^3 = 0$ and $(\lambda - 1)^2 = 0$. For a general square matrix A, the lowest degree monic (leading coefficient equal to 1) polynomial that A satisfies is called the minimal polynomial for A. In the example above $(\lambda - 1)^2$ is the minimal polynomial for the given A and for this matrix $(D - 1)^2 e^{At} = 0$. Our solution for e^{At} should then have been written

$$e^{At} = (C_{11} + C_{12}t)e^t.$$

In general, if $\psi(\lambda)$ is the minimal polynomial for a given matrix A and the degree of $\psi(\lambda)$ is $m(\leq n)$, then e^{At} satisfies the initial value problem:

$$\psi(D)e^{At} = 0$$

and

$$D^k(e^{At})_{t=0} = A^k \quad k = 0, 1, 2, \dots, m-1.$$

4.2 A Redefining of the C_{kj}

Let the distinct eigenvalues of A be $\alpha_1, \alpha_2, \dots, \alpha_s$ with $\psi(\lambda) = \prod_{k=1}^s (\lambda - \alpha_k)^{\mu_k}$ the minimal polynomial for A (written in factored form), $\mu_k \geq 1$, and $\sum_{k=1}^s \mu_k = m (\leq n)$ the degree of $\psi(\lambda)$. Following the theory presented in [2] we define $X_k(\lambda) = \psi(\lambda)/(\lambda - \alpha_k)^{\mu_k}$; then $X_k(\lambda)$ and $(\lambda - \alpha_k)^{\mu_k}$ are relatively prime. Therefore, there exist polynomials $p_k(\lambda), q_k(\lambda)$ (degree of $p_k(\lambda) < \mu_k$) such that:

$$(7) \quad p_k(\lambda)X_k(\lambda) + q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} \equiv 1, \quad k = 1, 2, \dots, s.$$

Then define:

$$(8) \quad \begin{aligned} E_k(\lambda) &= p_k(\lambda)X_k(\lambda) \quad \text{and} \\ E_k(A) &= E_k = p_k(A)X_k(A) \quad k = 1, 2, \dots, s. \end{aligned}$$

We note from (8) that for $k \neq \ell$ $E_k(\lambda)E_\ell(\lambda)$ is a polynomial multiple of $\psi(\lambda)$ and, therefore, $E_k \cdot E_\ell = 0$ for $k \neq \ell$ (i.e. the E_k are pairwise orthogonal).

From (7) we form the product:

$$\prod_{k=1}^s q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} = \psi(\lambda) \prod_{k=1}^s q_k(\lambda) = \prod_{k=1}^s (1 - E_k(\lambda)).$$

Replacing λ by A in this last equation we obtain

$$\psi(A) \prod_{k=1}^s q_k(A) = 0 = \prod_{k=1}^s (I - E_k) = I - \sum_{k=1}^s E_k,$$

which follows from the definition of $\psi(\lambda)$ and the orthogonality of the E_k . From this we have

$$(9) \quad I = \sum_{k=1}^s E_k.$$

Multiplying through (9) by E_ℓ , using the orthogonality, we have

$$E_\ell = E_\ell^2 \quad \ell = 1, 2, \dots, s$$

i.e., the E_ℓ are idempotent.

Next, define:

$$(10) \quad \begin{aligned} N_k(\lambda) &= (\lambda - \alpha_k)E(\lambda) && \text{and} \\ N_k(A) &= N_k = (A - \alpha_k I)E_k && \text{(for } \mu_k > 1). \end{aligned}$$

If $\mu_k = 1$, then α_k is a simple root of $\psi(\lambda)$ and for such roots

$$N_k(A) = \psi(A) = 0. \text{ We note from (10) that } N_k^{\mu_k} = (A - \alpha_k I)^{\mu_k} E_k = \psi(A) = 0$$

and $N_k^{\mu_k-1} \neq 0$. The N_k are said to be nilpotent of index μ_k .

Additional conclusions from definitions (8) and (10) are:

(a) All E_k, N_k are polynomials in A and, therefore, commute.

(b) $E_k N_k = N_k, E_k N_j = N_k N_j = 0$ ($k \neq j$).

(c) We have the identity:

$$(11) \quad A = \sum_{k=1}^s E_k (\alpha_k I + N_k)$$

which can be seen as follows:

$$\begin{aligned} \sum_{k=1}^s E_k (\alpha_k I + N_k) &= \sum_{k=1}^s (\alpha_k E_k + A E_k - \alpha_k E_k) \\ &= \sum_{k=1}^s A E_k = A \sum_{k=1}^s E_k = A I = A. \end{aligned}$$

If we replace A in e^{At} by the identity (11), we have:

$$(12) \quad \begin{aligned} e^{At} &= e^{(\sum_{k=1}^s E_k (\alpha_k I + N_k))t} = e^{\sum_{k=1}^s E_k \alpha_k t} \cdot e^{\sum_{k=1}^s N_k t} \\ &= \sum_{k=1}^s e^{\alpha_k t} \left\{ E_k + N_k t + \frac{(N_k t)^2}{2!} + \dots + \frac{(N_k t)^{\mu_k-1}}{(\mu_k-1)!} \right\}, \end{aligned}$$

and this must be identical with (1) i.e.

$$= \sum_{k=1}^s e^{\alpha_k t} \{ C_{k1} + C_{k2} t + \dots + C_{k\mu_k} t^{\mu_k-1} \}$$

Rewriting (12) using the definition (10) we have:

$$(13) \quad e^{At} = \sum_{k=1}^s E_k e^{\alpha_k t} \left\{ I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1}}{(\mu_k - 1)!} t^{\mu_k - 1} \right\}.$$

In (13) we observe that we have only the E_k ($k = 1, 2, \dots, s$) to determine. These can be calculated from the definition (8) or from (2) calculating only the $C_{k1} = E_k$ ($k = 1, 2, \dots, s$).

4.3 Summary of section 4 and examples.

We summarize this section in:

Theorem 2. Given A with minimal polynomial $\psi(\lambda) = \prod_{k=1}^s (\lambda - \alpha_k)^{\mu_k}$, $m \leq n$,

we have:

$$e^{At} = \sum_{k=1}^s E_k e^{\alpha_k t} \left\{ I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1}}{(\mu_k - 1)!} t^{\mu_k - 1} \right\},$$

in which the E_k satisfy the following:

$$(a) \quad I = \sum_{k=1}^s E_k$$

$$(b) \quad E_k E_j = \begin{cases} 0 & \text{if } i \neq j \\ E_k & \text{if } k = j \end{cases}$$

$$(c) \quad (A - \alpha_k I)E_k = N_k \text{ are nilpotent of index } \mu_k$$

$$(d) \quad \text{all } E_k \text{ and } N_k \text{ are polynomials in } A.$$

Up to this point we have determined e^{At} for any $A(3 \times 3)$ with distinct eigenvalues. Now let us complete this calculation for any 3×3 matrix A . To this end we have the following cases and calculations:

(i) All eigenvalues of A are equal to α , and $\psi(\lambda) = f(\lambda) = (\lambda - \alpha)^3$.

In this case:

$$e^{At} = e^{\alpha t} \left(E_1 + N_1 t + \frac{N_1^2 t^2}{2!} \right) \text{ in which } E_1 = I \text{ and } N_1 = (A - \alpha I).$$

(ii) Again all eigenvalues equal α , but $\psi(\lambda) = (\lambda - \alpha)^2$. In this case $E_1 = I$ and $e^{At} = e^{\alpha t}(I + (A - \alpha I)t)$.

(iii) $\psi(\lambda) = (\lambda - \alpha)$. In this case $E_1 = I$ and $e^{At} = e^{\alpha t}I$ a scalar matrix.

(iv) $\psi(\lambda) = f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$, $\alpha_1 \neq \alpha_2$.

In this case:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2.$$

We can solve for E_1 and E_2 ($N_1 = (A - \alpha_1 I)E_1$) using the initial conditions (2) or definitions (7) and (8). In view of (7) we have (replacing λ by A):

$$E_1 + q_1(A)(A - \alpha_1 I)^2 = I.$$

However, we know that $E_1 + E_2 = I$ and, therefore, $E_2 = q_1(A)(A - \alpha_1 I)^2$ which means we obtain both E_1 and E_2 simultaneously by using (7) and (8). Accordingly using (7) and (8): $(a\lambda + b)(\lambda - \alpha_2) + c(\lambda - \alpha_1)^2 \equiv 1$ which must hold for all λ and, therefore, we have:

$$a = \frac{-1}{(\alpha_2 - \alpha_1)^2}, \quad b = \frac{-(\alpha_2 - 2\alpha_1)}{(\alpha_2 - \alpha_1)^2}, \quad c = \frac{1}{(\alpha_2 - \alpha_1)^2}.$$

Therefore:

$$E_1 = \frac{-1}{(\alpha_2 - \alpha_1)^2} (A - \alpha_2 I)(A + (\alpha_2 - 2\alpha_1)I)$$

$$E_2 = \frac{1}{(\alpha_2 - \alpha_1)^2} (A - \alpha_1 I)^2 \quad \text{and}$$

$$N_1 = \frac{-(A - \alpha_1 I)}{(\alpha_2 - \alpha_1)^2} (A - \alpha_2 I)(A - 2\alpha_1 I + \alpha_2 I).$$

Accordingly:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2.$$

(v) The last case is that in which $f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$ as in (iv) but $\psi(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)$.

In this case α_1 and α_2 are simple roots of $\psi(\lambda)$ and therefore,

$$e^{At} = \frac{e^{\alpha_1 t}}{\alpha_1 - \alpha_2} (A - \alpha_2 I) + \frac{e^{\alpha_2 t}}{\alpha_2 - \alpha_1} (A - \alpha_1 I)$$

by virtue of the results in section 3.

In view of the example in section 3 and the 5 cases above we have obtained e^{At} for any 3 x 3 matrix.

It is of interest to note that for any $A(n \times n)$ in which either (a) A has n equal eigenvalues or the opposite extreme (b) A has n distinct eigenvalues we have that e^{At} can be written immediately as:

$$(a) e^{At} = e^{\alpha t} \sum_{k=0}^{n-1} \frac{(A - \alpha I)^k t^k}{k!}.$$

If in this case $\psi(\lambda) = (\lambda - \alpha)^m$, $m < n$, then the summation would extend only to $m - 1$ since $(A - \alpha I)^m = 0$.

$$(b) e^{At} = \sum_{k=1}^n e^{\alpha_k t} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{(A - \alpha_j I)}{(\alpha_k - \alpha_j)}.$$

5. Other representations for e^{At} .

In [4] and [6] e^{At} is obtained by use of the Lagrange-Sylvester interpolation polynomial. By using the eigenvalues of A as the interpolation points we obtain the form of e^{At} in equation (12). In [5] and [7] representations of e^{At} are obtained in one case in powers of A and in others in powers of $A - \alpha_i I$ (α_i - eigenvalues of A). In any case, if all these representations were given in powers of the same $(A - \beta_i I)$ then they would, of course, all be the same.

Another representation for e^{At} , which has applications to solutions of first order linear systems of simultaneous differential equations

with constant coefficients, is obtained as follows. Suppose we have given:

$$(14) \quad x'(t) = Ax(t)$$

A an $n \times n$ matrix with constant elements, $x(t)$ an $n \times 1$ vector function of t . Suppose we have found a fundamental set of solutions for (14) namely $x_1(t), x_2(t), \dots, x_n(t)$. Then define:

$$(15) \quad X(t) = (x_1(t), x_2(t), \dots, x_n(t)),$$

which is an $n \times n$ matrix whose columns are the elements of the fundamental set. $X(0)$ is nonsingular and we define

$$(16) \quad G(t) = X(t)(X(0))^{-1};$$

then

$$G(t) = e^{At}.$$

This equation follows from differentiating (16), which yields

$$G'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))(X(0))^{-1}.$$

By (14) $G'(t)$ can be written:

$$\begin{aligned} G'(t) &= (Ax_1(t), Ax_2(t), \dots, Ax_n(t))(X(0))^{-1} \\ &= AX(t)(X(0))^{-1} = AG(t). \end{aligned}$$

From this last equation it follows that:

$$G^{(k)}(t) = A^k G(t) \quad k=0, 1, \dots,$$

which in turn yields:

$$G^{(k)}(0) = A^k \quad k=0, 1, \dots.$$

Moreover,

$$f(D)G(t) = f(A)G(t) = 0$$

by the Cayley-Hamilton theorem.

Therefore, $G(t)$ satisfies the initial value problem which is also satisfied by e^{At} . By uniqueness of such solutions we conclude:

$$G(t) = X(t)(X(0))^{-1} = e^{At}.$$

6. Some applications of e^{At} .

From the last remarks in section 5, and since $(X(0))^{-1}$ is nonsingular, $X(t)(X(0))^{-1} = e^{At}$ has columns that are linear combinations of the columns of $X(t)$ and form another fundamental set for the differential equation (14):

$$x'(t) = Ax(t),$$

Let us use this fact to solve the system

$$x'(t) = \begin{pmatrix} 4 & -5 & 3 \\ 2 & -3 & 2 \\ -1 & 1 & 0 \end{pmatrix} x(t),$$

The given matrix has characteristic polynomial $f(\lambda) = \psi(\lambda) = (\lambda - 1)^2(\lambda + 1)$.

Using case (iv) in section 4 with $\alpha_1 = 1$ and $\alpha_2 = -1$ we have:

$$\begin{aligned} e^{At} &= e^t \left\{ \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} t \right\} + e^{-t} \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t(2+t) - e^{-t} & -e^t(2+t) + 2e^{-t} & e^t(1+t) - e^{-t} \\ e^t - e^{-t} & -e^t + 2e^{-t} & e^t - e^{-t} \\ -te^t & te^t & e^t(1-t) \end{pmatrix} \end{aligned}$$

The columns of the latter matrix constitute a fundamental set for the differential equation and its general solution is:

$$x(t) = e^{At}x(0),$$

where $x(0)$ forms the initial conditions for $x(t)$ (given at $t = 0$).

As a matter of fact, having obtained e^{At} for all 3×3 matrices A we have therefore obtained fundamental sets for all systems of differential equations $x'(t) = Ax(t)$ with A a 3×3 matrix of constants and $x(t)$ a 3×1 vector function of t .

Knowing e^{At} and e^{-At} we define

$$\cosh At = 1/2(e^{At} + e^{-At}) \quad \text{and}$$

$$\sinh At = 1/2(e^{At} - e^{-At}).$$

Equally well we know e^{iAt} and e^{-iAt} ($i = \sqrt{-1}$) and define:

$$\cos At = 1/2(e^{iAt} + e^{-iAt})$$

$$\sin At = \frac{1}{2i}(e^{iAt} - e^{-iAt}).$$

As an example, we indicate the expansion of $\cosh At$:

$$\cosh At = \sum_{k=1}^s \cosh \alpha_k t \left(E_k + \frac{N_k^2 t^2}{2!} + \dots \right) + \sum_{k=1}^s \sinh \alpha_k t \left(N_k t + \frac{N_k^3 t^3}{3!} + \dots \right)$$

with each of these two sums terminating with the term containing either t^{μ_k-2} or t^{μ_k-1} , depending upon whether μ_k is even or odd.

In [1] T. M. Apostol considers the system of differential equations

$$Y''(t) = AY(t)$$

and writes the solution in terms of two matrix functions

$$C(t) = \sum_{k=0}^{\infty} \frac{t^{2k} A^k}{(2k)!}, \quad S(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1} A^k}{(2k+1)!}.$$

$C(t)$ is precisely $\cosh \sqrt{A} t$ and $S(t)$ is $(\sqrt{A})^{-1} \sinh \sqrt{A} t$ provided that \sqrt{A} is defined and nonsingular. Clearly one would define \sqrt{A} to be that matrix B such that $B^2 = A$. It turns out that B is not unique (as one would suspect), in fact, there may be as many as 2^n matrices B such that $B^2 = A$. However, these B -matrices may be calculated as follows. If A is similar to a diagonal matrix so that

$$A = T^{-1} \text{Diag} \{ \alpha_1, \alpha_2, \dots, \alpha_n \} T,$$

then

$$\sqrt{A} = T^{-1} \text{Diag} \{ \pm \alpha_1^{1/2}, \pm \alpha_2^{1/2}, \dots, \pm \alpha_n^{1/2} \} T.$$

If A is not similar to a diagonal matrix and is nonsingular, then \sqrt{A} can be obtained as follows: From (11)

$$A = \sum_{k=1}^s E_k (\alpha_k I + N_k) = \sum_{k=1}^s E_k \alpha_k \left(I + \frac{N_k}{\alpha_k} \right);$$

then

$$\sqrt{A} = \sum_{k=1}^s \alpha_k^{1/2} E_k \left(I + \frac{N_k}{\alpha_k} \right)^{1/2}$$

If $\left(I + \frac{N_k}{\alpha_k} \right)^{1/2}$ is expanded by the binomial theorem then this expansion would terminate with the term $N_k^{\mu_k - 1}$ since N_k is nilpotent of index μ_k .

Knowing how to compute \sqrt{A} in some cases we then have for these cases the solutions to

$$(a) \quad Y''(t) + AY(t) = 0$$

given by

$$Y(t) = (\cos \sqrt{A} t) Y_1 + (\sin \sqrt{A} t) Y_2;$$

or

$$(b) \quad Y''(t) - AY(t) = 0$$

given by

$$Y(t) = (\cosh \sqrt{A} t) Y_1 + (\sinh \sqrt{A} t) Y_2.$$

In (a) and (b)

$$Y(0) = Y_1 \text{ and } Y'(0) = \sqrt{A} Y_2.$$

As a last application we will consider: let A be a 3×3 matrix with characteristic function $f(\lambda) = \psi(\lambda) = (\lambda - \alpha_1)^2 (\lambda - \alpha_2)$ (case (iv) in section 4), and suppose we are given the nonhomogeneous system

$$(17) \quad x'(t) = Ax(t) + a_1 e^{\alpha_1 t},$$

in which $x(t)$ is a 3×1 vector function of t to be determined and a_1 is a 3×1 constant vector. Multiplying through (17) by e^{-At} yields

$$e^{-At}x'(t) - Ae^{-At}x(t) = D(e^{-At}x(t)) = e^{-At}a_1e^{\alpha_1 t}.$$

Integrating this last equation we obtain

$$(18) \quad x(t) = e^{At} \int e^{-A\tau} a_1 e^{\alpha_1 \tau} d\tau + e^{At} a_2,$$

in which a_2 is a 3×1 vector whose elements are arbitrary constants.

From case (iv)

$$(19) \quad e^{At} = e^{\alpha_1 t} (E_1 + N_1 t) + e^{\alpha_2 t} E_2.$$

Substituting (19) into (18), using the orthogonality, idempotency and nilpotency of E_1 , E_2 and N_1 and integrating we obtain

$$x(t) = e^{\alpha_1 t} \left\{ \left(E_1 + \frac{N_1 t}{2} \right) t + \frac{E_2}{\alpha_1 - \alpha_2} \right\} a_1 + \left\{ e^{\alpha_1 t} (E_1 + N_1 t) + e^{\alpha_2 t} E_2 \right\} a_2$$

as the complete solution to (17).

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