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Exponential Fourier Densities on $S^2/\pm 1$

and

Optimal Estimation for Axial Processes

by

James Ting-Ho Lo and Linda R. Eshleman

Mathematics Research Report No. 77-1

January, 1977

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An estimation error criterion, which is compatible with a Riemannian metric, is introduced. It is shown that the corresponding optimal axial estimates can be easily computed from the conditional probability distributions.

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I. INTRODUCTION

In this paper we consider the problem of estimating axes in three-dimensional space. An axis or axial vector is distinguished from a polar vector in that the former is invariant under inversion. Such axes occur in many diverse areas including the following: geophysical fluid dynamics to estimate the vorticity of a flow, paleomagnetism to estimate a magnetic field, crystallography to estimate the optic axis of a crystal, geology to estimate the direction of a normal to the axis of a fold in a layer of rock, and quantum mechanics to estimate the axis of rotation of a rigid body rotation.

Using densities of the form exp f where f is a linear combination of axially symmetric spherical harmonics, estimation problems which arise by examining various possible ways of obtaining a displacement of an axis will be solved in this paper. Although the state space under consideration is homeomorphic to a hemisphere of S^2, the results in [1] for estimation on S^2 cannot be applied for several important reasons: the displacements defined in that paper may result in a given point being displaced to a non-antipodal point in the opposite hemisphere, the densities discussed in [1] were not, in general, axially symmetric, and the error criterion used for S^2 is undesirable since it would result in a rejection of the antipode of the optimal estimate.

Using the various displacements and conditional densities obtained in this paper, detection for axial processes would be described by procedures similar to those used for S^2 and SO(3) which have been discussed in [1] and [2].

II. PRELIMINARIES AND AXIAL EXPONENTIAL FOURIER DENSITIES

The set of all axes centered at a fixed point can be identified in a one-to-one manner with the points on a hemisphere HS(p) of the unit sphere S^2 with pole at p. Hence the state space for the axial processes to be discussed can be considered to be the factor space obtained by partitioning

S^2 into equivalence classes such that two different points on S^2 belong to the same equivalence class if and only if they are antipodal points. This factor space, which is projective two-space, will be designated by the symbol $S^2/\pm 1$ throughout this paper.

Since the spherical coordinates (θ,ϕ), $(\pi-\theta,\phi+\pi)$, and $(\pi+\theta,\phi)$ will all represent the same axis, any probability density $p(\theta,\phi)$ on $S^2/\pm 1$ should have the property that it is antipodally symmetric on S^2, i.e.

$$p(\theta,\phi) = p(\pi-\theta,\phi+\pi) = p(\pi+\theta,\phi) \quad . \tag{1}$$

An example of such an antipodally symmetric density that has been previously studied is the Bingham distribution [3] which is a generalization of the Dimroth-Watson distribution of [4] and [5]. It can be expressed as a density containing spherical harmonics of the form

$$c \exp \sum_{m=-2}^{2} a_m Y_{2,m}(\theta,\phi) \quad ,$$

where c is a normalizing constant.

Clearly, given any density $p_1(\theta,\phi)$ defined on S^2, a corresponding density $p(\theta,\phi)$ can be obtained on $S^2/\pm 1$ regarded as the upper hemisphere

$$HS(n) = [0,\pi/2] \times [0,2\pi]$$

by defining

$$p(\theta,\phi) = p_1(\theta,\phi) + p_1(\pi-\theta,\phi+\pi) \quad , \quad (\theta,\phi) \in HS(n) \quad .$$

However, having seen in [1] that exponential Fourier densities were most useful in determining conditional densities that are needed for estimation, we shall instead consider those densities of the form

$$p(\theta,\phi) = \exp \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta,\phi)$$

for which (1) holds. For θ in the interval $[0,\pi]$ the spherical harmonics $Y_{\ell m}(\theta,\phi)$ are defined by

$$Y_{\ell m}(\theta,\phi) = (-1)^m \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)e^{im\phi} \quad . \tag{2}$$

where $P_\ell^m(x)$ is an associated Legendre function.

We extend this definition for θ outside this interval by defining $P_\ell^m(\cos\theta)$ in such a way that the values of the spherical harmonics remain independent of the choice of spherical coordinates. In particular, for $-\pi < \theta < 0$, (2) becomes

$$Y_{\ell m}(-\theta,\phi+\pi) = \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos(-\theta))e^{im\phi}$$

so that in order for the identity

$$Y_{\ell m}(-\theta,\phi+\pi) = Y_{\ell m}(\theta,\phi)$$

to hold we must define

$$P_\ell^m(\cos\theta) = (-1)^m P_\ell^m(\cos(-\theta)) \quad \text{for } -\pi < \theta < 0 \quad .$$

Similarly, we define

$$P_\ell^m(\cos\theta) = (-1)^m P_\ell^m(\cos(2\pi-\theta)) \quad \text{for } \pi < \theta < 2\pi \quad .$$

Suppose we now consider the point $(\theta+\pi,\phi)$ which is the antipodal point of (θ,ϕ) where $0 \leq \theta \leq \pi$. We have

$$Y_{\ell m}(\theta+\pi,\phi) = (-1)^m\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\,(\theta+\pi))e^{im\phi}$$

$$=\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\,(\pi-\theta))e^{im\phi}$$

$$=\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(-\cos\,\theta)e^{im\phi}$$

$$= (-1)^{m+\ell}\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\,\theta)e^{im\phi}$$

$$= (-1)^\ell\, Y_{\ell m}(\theta,\phi) \quad .$$

The fourth equality follows from the property of associated Legendre polynomials that

$$P_\ell^m(-\cos\,\theta) = (-1)^{m+\ell}\, P_\ell^m(\cos\,\theta) \qquad \text{for } \theta \in [0,\pi] \quad .$$

Since the spherical harmonics are independent of the choice of coordinates we can conclude that any spherical harmonic assumes the same value at antipodal points if and only if it is of even order and hence we have the following lemma which characterizes all EFD's for which antipodal points of S^2 have the same density:

<u>Lemma 1</u>: Let $p(\theta,\phi) = \exp f(\theta,\phi)$ where

$$f(\theta,\phi) = \sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\theta,\phi) \quad .$$

Relation (1) holds if and only if $a_{\ell m} = 0$ whenever ℓ is an odd integer.

Thus we define an <u>axial exponential Fourier density of order 2N</u> on $S^2/\pm 1$, denoted by AEFD(2N), to be a density of the form

$$p(\theta,\phi) = \exp \sum_{\ell=0}^{N}\sum_{m=-2\ell}^{2\ell} a_{2\ell,m}\, Y_{2\ell,m}(\theta,\phi) \quad . \tag{3}$$

In the special case where the density is isotropic so that $p(\theta,\phi)$ is independent of ϕ (3) is replaced by

$$p(\theta) = \exp \sum_{\ell=0}^{N} a_{2\ell}\, Y_{2\ell,0}(\theta,\phi) = \exp \sum_{\ell=0}^{N} a_{2\ell}\, P_{2\ell}(\cos\theta) \quad .$$

It should be noted that any twice continuously differentiable density can be approximated by an AEFD in the space of square integrable functions on $S^2/\pm 1$ as a consequence of Theorem 1 in [1].

In the determination of conditional densities for the several estimation models to be discussed, it will frequently be necessary to transform densities of the form (3) into expressions containing the trigonometric functions $\exp[i(m\theta+n\phi)]$. By Lemma 3 of [1] a spherical harmonic of even order can be expressed as a linear combination of trigonometric functions of the form

$$\exp \sum_{j=-N}^{N} \sum_{k=-2N}^{2N} b_{2j,k} \exp\,[i(2j\theta+k\phi)] \tag{4}$$

which we will refer to as an <u>axial exponential trigonometric function of order 2N</u>, denoted by AETF(2N) to differentiate it from an ETF(2N) of [1] where the coefficient of θ can be an odd integer. It should be observed that the use of the word "axial" is a misnomer since (1) does not hold for densities of the form (4).

Our choice of an error criterion for optimal estimation of a random axis is motivated by the desire that antipodal points be indistinguishable. If x and y are the column vectors of direction cosines of two points on S^2 and $\rho(x,y)$ denotes the principal angle between them, then we shall use the measure

$$1 - \cos 2\rho(x,y) = 2[1 - \cos^2 \rho(x,y)] = 2[1 - (x'y)^2]$$

which is similar to the measure used for estimation on SO(3) in [2]. Therefore, if x represents a random direction cosine vector, the axial estimation problem will be to find its estimate y which minimizes the criterion

$$J(x,y) = 2\,E[1 - (x'y)^2] = 2[1 - y'E(xx')y] \tag{5}$$

where the estimation is over an arbitrary hemisphere of S^2.

Since E(xx') is a positive definite 3×3 symmetric matrix, it is known that, just as in the SO(3) case, the quadratic form y'E(xx')y is maximized when y is the unit eigenvector associated with the largest eigenvalue of E(xx'). Hence

$$\min_y J(x,y) = 1 - \hat{x}'E(xx')\hat{x} = 1 - \lambda$$

where λ is the maximum eigenvalue of the matrix E(xx') and $\hat{x}$ is the unit eigenvector associated with λ.

Once $\hat{x}' = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3]$ is calculated, the spherical coordinates $(\hat{\theta},\hat{\phi})$ of the optimal estimate can be determined from the relations

$$\cos\hat{\theta} = \hat{x}_3 \qquad , \qquad 0 < \hat{\theta} < \pi$$

$$\cos\hat{\phi} = \frac{\hat{x}_1}{\sqrt{1 - \hat{x}_3^2}}$$

$$\sin\hat{\phi} = \frac{\hat{x}_2}{\sqrt{1 - \hat{x}_3^2}}$$

III. DISPLACEMENTS ON $S^2/\pm 1$

In order to use a hemisphere of S^2 as a representation for $S^2/\pm 1$ in the estimation of axial processes, it is necessary to use displacements $\circ$ which have the property that for any $x \in S^2/\pm 1$, the displaced points $x \circ y$ and $\bar{x} \circ y$ are either identical or antipodal points on S^2, $\bar{x}$ being the antipode of x.

While the displacements used for estimation on S^2 in [1] do not have this property, it will be seen that variations of these displacements which have this property can be defined which result in usable estimation models.

Let us consider the first displacement on the sphere S^2 with center O given in [1] of a point $s(\theta,\phi)$ by an ordered pair $v(\alpha,\beta)$. Assuming that (θ,ϕ) and (α,β) are both in $[0,\pi) \times [0,2\pi)$, let n denote the point (0,0,1) and let $\widehat{ns}$ be the smaller great circle arc between n and s. There is a unique semi-circle Γ with center at O having s as an endpoint such that the angle from $\widehat{ns}$ to Γ has measure β, using the right-hand screw rule in the direction Os. The point m which is the endpoint of the arc $\widehat{sm}$ on Γ of length α is shown in Figure 1 and its coordinates (λ,μ) are related to those of s and v by the equations

$$\begin{aligned} \cos\alpha &= \cos\lambda\cos\theta + \sin\lambda\sin\theta\cos(\mu-\phi) \\ \sin\alpha\cos\beta &= \cos\lambda\sin\theta + \sin\lambda\cos\theta\sin(\mu-\phi) \\ \sin\alpha\sin\beta &= \sin\lambda\sin(\phi-\mu) \end{aligned} \qquad (6)$$

By considering the special case where $\beta = 0$ it is seen that the spherical coordinates of m are $(\theta-\alpha,\phi)$; but the displacement of the signal's antipode $\bar{s}(\pi-\theta,\phi+\pi)$ is $(\pi-\theta-\alpha,\phi+\pi)$ which does not coincide with either m or $\bar{m}$ on S^2.

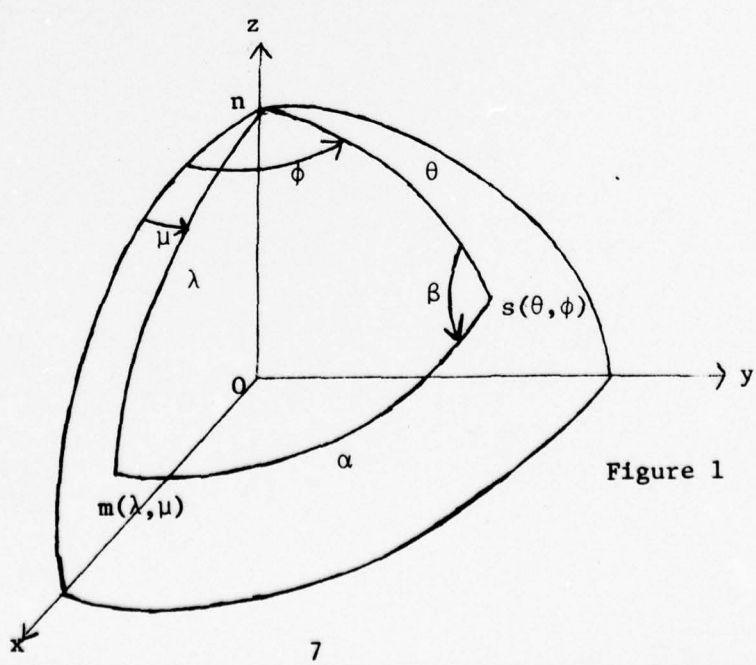

Figure 1

This difficulty can be resolved by making the restriction that a fixed hemisphere be used as the state space. For convenience we choose the upper hemisphere and require that its points have spherical coordinates (θ,ϕ) such that $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq 2\pi$. Throughout the remainder of this paper we will identify $S^2/\pm 1$ with this set

$$S^2/\pm 1 = [0,\pi/2] \times [0,2\pi) \quad . \tag{7}$$

If $s(\theta,\phi)$ and $v(\alpha,\beta)$ belong to $S^2/\pm 1$ let $\tilde{m}(\tilde{\lambda},\tilde{\mu})$ be the displaced point where $(\tilde{\lambda},\tilde{\mu})$ is in $[-\pi,\pi) \times [-2\pi,2\pi)$. Now we define $m(\lambda,\mu)$ as being a point of $S^2/\pm 1$ with coordinates

$$(\lambda,\mu) = \begin{cases} (\pi+\tilde{\lambda},\tilde{\mu} \bmod 2\pi) & \text{if } \tilde{\lambda} \leq -\pi/2 \\ (-\tilde{\lambda},(\tilde{\mu}+\pi) \bmod 2\pi) & \text{if } -\pi/2 < \tilde{\lambda} \leq 0 \\ (\tilde{\lambda},\tilde{\mu} \bmod 2\pi) & \text{if } 0 \leq \tilde{\lambda} \leq \pi/2 \\ (\pi-\tilde{\lambda},(\tilde{\mu}+\pi) \bmod 2\pi) & \text{if } \tilde{\lambda} > \pi/2 \end{cases}$$

and we indicate this relation of m to s and v by

$$m(\lambda,\mu) = s(\theta,\phi) \; \textcircled{a} \; v(\alpha,\beta) \quad .$$

Geometrically, it is seen that $\textcircled{a}$ is identical to the spherical displacement whenever m is in HS(n), while m is the antipode of $\tilde{m}$ otherwise. The requirement that the first spherical coordinate be between 0 and $\pi/2$ is made to avoid some unnecessary difficulties in the calculation of conditional densities to be discussed later.

A second displacement $\textcircled{b}$ arises by increasing each coordinate of the position of the signal. Let $s_0(\theta_0,\phi_0)$ be in $S^2/\pm 1$ and suppose $w(\xi,\zeta)$ is a pair of angles satisfying the condition

$$(\xi,\zeta) \in [-\pi/2,\pi/2) \times [0,2\pi) \quad .$$

Now set $\tilde{\theta}_1 = \theta_0 + \xi$ and $\tilde{\phi}_1 = \phi_0 + \zeta$. Let $s_1(\theta_1,\phi_1)$ be the point of $S^2/\pm 1$ with coordinates

$$(\theta_1,\phi_1) = \begin{cases} (-\tilde{\theta}_1, (\tilde{\phi}_1+\pi) \bmod 2\pi) & \text{if } -\pi/2 < \tilde{\theta}_1 \leq 0 \\ (\tilde{\theta}_1, \tilde{\phi}_1 \bmod 2\pi) & \text{if } 0 \leq \tilde{\theta}_1 \leq \pi/2 \\ (\pi-\tilde{\theta}_1, (\tilde{\phi}_1+\pi) \bmod 2\pi) & \text{if } \pi/2 < \tilde{\theta}_1 \leq \pi \end{cases} .$$

This relationship of s_1 to s_0 is denoted by

$$s_1(\theta_1,\phi_1) = s_0(\theta_0,\phi_0) \ \textcircled{b} \ w(\xi,\zeta)$$

If $0 \leq \xi \leq \pi/2$ then, since $0 \leq \theta_0 + \xi \leq \pi$, the two hypotheses

$$H_1: \ (\theta_1,\phi_1) = (\theta_0+\xi, (\phi_0+\zeta) \bmod 2\pi)$$
$$H_2: \ (\theta_1,\phi_1) = (\pi-\theta_0-\xi, (\phi_0+\zeta+\pi) \bmod 2\pi)$$

must be considered in the determination of s_0 if s_1 and w are known. If, instead, $-\pi/2 \leq \xi < 0$, these hypotheses are replaced by

$$H_1: \ (\theta_1,\phi_1) = (\theta_0+\xi, (\phi_0+\zeta) \bmod 2\pi)$$
$$H_2: \ (-\theta_0-\xi, (\phi_0+\zeta+\pi) \bmod 2\pi)$$

since $-\pi/2 < \theta_0 + \xi \leq \pi/2$.

It should be observed that (b) identifies s_1 with the antipode of $\tilde{s}_1$ when $\tilde{s}_1$ is not in HS(n) and is similar to the second displacement (2) on S^2 in [1] when $\tilde{s}_1$ is in HS(n).

The final displacement (c) to be used is obtained from a rotation of three-dimensional space. If s_0 is in $S^2/\pm 1$ and R is a rotation that rotates s_0 to the position $\tilde{s}_1$ on S^2, we define s_1 to be $\tilde{s}_1$ if $\tilde{s}_1$ is in HS(n) and to be the antipode of $\tilde{s}_1$ if $\tilde{s}_1$ is not in HS(n) and write

$$s_1 = s_0 \ \textcircled{d} \ R \quad .$$

It should be noted that since R will rotate the antipode $\bar{s}_0$ into $\bar{s}_1$, unlike the previous displacements, it is actually unnecessary to require that the fixed hemisphere (7) be used as the state space.

IV. ESTIMATION OF AN AXIS ON $S^2/\pm 1$

We now consider a signal s, represented by the point $s(\theta,\phi)$ on $S^2/\pm 1$, which we want to estimate, given an observation $m(\lambda,\mu)$ in $S^2/\pm 1$ and using the error criterion (5). The optimal estimate requires the calculation of the conditional density

$$p(s|m) = c\ p(m|s)\ p(s) \tag{8}$$

according to Bayes' rule. Because of the nature of the definitions of the various displacements in the previous section, m may be the antipode of the measurement of some displacement on S^2, and consequently care must be taken in determining the density $p(m|s)$.

Estimation Model E(a)

Let m be a measurement on $S^2/\pm 1$ of an axial signal s which can be described by the relation

$$m(\lambda,\mu) = s(\theta,\phi) \ \textcircled{a}\ v(\alpha,\beta)$$

where s and v have AEFD(2N)'s exp $f_s(\theta,\phi)$ and exp $f_v(\alpha,\beta)$, respectively, where

$$f_s(\theta,\phi) = \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} a_{2\ell,m}\, Y_{2\ell,m}(\theta,\phi)$$

and

$$f_v(\alpha,\beta) = \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} b_{2\ell,m}\, Y_{2\ell,m}(\alpha,\beta) \quad .$$

Consider the possible positions of $m(\lambda,\mu)$ on $S^2/\pm 1$. First, suppose $m(\lambda,\mu)$ is in the hemisphere HS(s) whose pole is at the point $s(\theta,\phi)$. Since $0 \le \alpha \le \pi/2$, it follows that we have

$$p(m|s) = \exp f_v(\alpha,\beta) = g_v(\lambda,\mu) \tag{9}$$

where $g_v(\lambda,\mu)$ is obtained by using the spherical trigonometric relations in (6).

On the other hand, if $m(\lambda,\mu)$ is not in HS(s), then it must be the antipode of a point $\tilde{m}(\tilde{\lambda},\tilde{\mu})$ not in HS(n) where

$$\tilde{\lambda} = \pi - \lambda \quad , \quad \tilde{\mu} = (\mu+\pi) \bmod 2\pi \quad .$$

But this means that there is a unique directional process $\tilde{v}(\tilde{\alpha},\tilde{\beta})$ such that

$$\tilde{m}(\tilde{\lambda},\tilde{\mu}) = s(\theta,\phi) \; \textcircled{a} \; \tilde{v}(\tilde{\alpha},\tilde{\beta})$$

where $0 \leq \tilde{\alpha} < \pi/2$ and hence

$$p(m|s) = p(\tilde{m}|s) = \exp f_{\tilde{v}}(\tilde{\alpha},\tilde{\beta}) = g_{\tilde{v}}(\tilde{\lambda},\tilde{\mu}) \quad . \tag{10}$$

If $g_v(\lambda,\mu) = g_{\tilde{v}}(\tilde{\lambda},\tilde{\mu})$ then $p(m|s)$ could be determined in the same manner as in model SE-1 of [1]. The following lemma gives the important result that any AEFD in the variables (α,β) has this property:

<u>Lemma 2:</u> If $Z_{\ell m}(\lambda,\mu)$ denotes the function obtained by substituting the relations (6) into the spherical harmonic $Y_{\ell m}(\alpha,\beta)$ then

$$Z_{\ell m}(\lambda,\mu) = (-1)^{\ell} \, Z_{\ell m}(\pi-\lambda,\mu+\pi) \quad .$$

Proof: By examining the relations (6) we see that the statement holds for the spherical harmonics of orders zero and one:

$$Z_{00}(\lambda,\mu) = 1$$

$$Z_{11}(\lambda,\mu) = -\frac{1}{\sqrt{2}} (\sin\alpha\cos\beta + i\sin\alpha\sin\beta)$$

$$Z_{10}(\lambda,\mu) = \cos\alpha$$

$$Z_{1,-1}(\lambda,\mu) = \frac{1}{\sqrt{2}} (\sin\alpha\cos\beta - i\sin\alpha\sin\beta) \quad .$$

Now assume the statement holds for all $\ell \leq L$. From the recursive relations for spherical harmonics we have

$$Z_{L+1,m}(\lambda,\mu) = [(L+m+1)(L-m+1)]^{-1/2}[(2L+1)Z_{10}(\lambda,\mu)Z_{Lm}(\lambda,\mu)$$
$$- \sqrt{(L-m)(L+m)}\; Z_{L-1,m}(\lambda,\mu)]$$

when $m = -L,-L+1,\ldots,L-1,L$ and

$$Z_{L+1,L+1}(\lambda,\mu) = \left[\frac{2L+1}{L+1}\right]^{1/2} Z_{11}(\lambda,\mu)\; Z_{LL}(\lambda,\mu) \quad .$$

If L is even, it is evident from the induction hypotheses that

$$Z_{L+1,m}(\pi-\lambda,\mu+\pi) = - Z_{L+1,m}(\lambda,\mu)$$

while for L odd we have

$$Z_{L+1,m}(\pi-\lambda,\mu+\pi) = Z_{L+1,m}(\lambda,\mu) \quad .$$

Hence the statement holds for all integers ℓ. □

This lemma assures us that (9) and (10) have the same value since $\exp f_v(\alpha,\beta)$ is an AEFD and hence without any loss of generality we can assume that $m(\lambda,\mu)$ is in HS(s) with α and β defined by (6).

By Lemma 1 of [1], $\exp f_v(\alpha,\beta)$ is an exponential trigonometric function of the form

$$\exp \sum_{m,n=-2N}^{2N} b_{mn} \exp[i(m\theta+n\phi)]$$

when transformed by (6) into a function of (θ,ϕ) so that

$$p(s|m) = c \exp\,[f_s(\theta,\phi)+f_v(\alpha,\beta)]$$

is also an exponential trigonometric function.

We remark that if $f_v(\alpha,\beta)$ has coefficients with the special property that

$$b_{2\ell,m} = (-1)^m \, b_{2\ell,-m}$$

then by Lemma 3 of [1], exp $f_v(\alpha,\beta)$ when transformed into a function of (θ,ϕ) is an AETF(2N) and hence $p(s|m)$ is an AETF(2N).

For the isotropic noise case where

$$f_v(\alpha,\beta) = \sum_{\ell=0}^{N} b_{2\ell} \, P_{2\ell}(\cos\alpha) = \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} b_{2\ell} \, \overline{Y_{2\ell,m}(\lambda,\mu)} \, Y_{2\ell,m}(\theta,\phi)$$

it is observed that $p(s|m)$ is itself an AEFD(2N).

A generalization of the noise density to include the situation where the density depends not only on the spherical coordinates (α,β) but also on the position of the signal $s(\theta,\phi)$ can be used. Suppose

$$f_v(\alpha,\beta) = \sum_{\ell=0}^{N_0} \sum_{m=-2\ell}^{2\ell} b_{2\ell,m}(\theta,\phi) \, Y_{2\ell,m}(\alpha,\beta)$$

where each $b_{2\ell,m}(\theta,\phi)$ is a smooth function of θ and ϕ on $S^2/\pm 1$ so that there is an integer $N(\ell,m)$ and coefficients $b(2\ell,m,2k,j)$ such that

$$\sum_{k=0}^{N(\ell,m)} \sum_{j=-2k}^{2k} b(2\ell,m,2k,j) \, Y_{2k,j}(\theta,\phi)$$

approximates $b_{2\ell,m}(\theta,\phi)$ sufficiently closely. Since $Y_{2\ell,m}(\alpha,\beta)$ and $Y_{2\ell,m}(\theta,\phi)$ can each be expressed as linear combinations of terms of the form $\exp[i(2j\theta+k\phi)]$, their product and hence $f_v(\alpha,\beta)$ are also such linear combinations. Therefore, the noise density is an exponential trigonometric density of order 2M where

$$M = N_0 + \max N(\ell,m) \quad .$$

If this generalized density has the additional property that

$$b_{2\ell,m}(\theta,\phi) = (-1)^m \, b_{2\ell,-m}(\theta,\phi) \quad ,$$

then

$$b_{2\ell,-m}(\theta,\phi)Y_{2\ell,-m}(\alpha,\beta) + b_{2\ell,m}(\theta,\phi)Y_{2\ell,m}(\alpha,\beta) =$$

$$= \sum_{k=0}^{N(\ell,m)} \sum_{j=-2k}^{2k} Y_{2k,j}(\theta,\phi)\{b(2\ell,m,2k,j)[Y_{2\ell,m}(\alpha,\beta)+(-1)^m Y_{2\ell,-m}(\alpha,\beta)]\}$$

By Lemma 3 of [1], we can thus conclude that $f_v(\alpha,\beta)$ is of the form

$$f_v(\alpha,\beta) = \sum_{\ell=0}^{N_0} \sum_{m=0}^{2\ell} \sum_{k=0}^{N(\ell,m)} \sum_{j=-2k}^{2k} C(2\ell,m,2k,j)\exp[i(2k\theta+j\phi)]Y_{2k,j}(\theta,\phi)$$

and hence that $\exp f_v(\alpha,\beta)$ is an AETF(2M) in (θ,ϕ).

V. ESTIMATION OF A TIME-VARYING AXIAL PROCESS ON $S^2/\pm 1$

Using displacements (b) and (c) to define a sequence of signals in conjunction with measurements obtained by using the displacement (a) as well as the important case of additive white noise, we now describe several time-varying estimation models for axial processes.

A. Estimation Model E(b,a)

Suppose signal and measurement processes are described by

$$s_{k+1}(\theta_{k+1},\phi_{k+1}) = s_k(\theta_k,\phi_k) \;\; (b) \;\; w_k(\xi_k,\zeta_k)$$

$$m_k(\lambda_k,\mu_k) = s_k(\theta_k,\phi_k) \;\; (a) \;\; v_k(\alpha_k,\beta_k)$$

where $-\pi/2 \le \xi_k < \pi/2$, $0 \le \zeta_k < 2\pi$, and the densities of s_0 and v_k are AEFD(2N)'s.

Let us assume that $0 \le \xi_0 < \pi/2$ so that

$$p(s_1) = p(s_1|H_1)P(H_1) + p(s_1|H_2)P(H_2) \quad ,$$

where

$$P(H_1) = \int_0^{2\pi} \int_0^{\pi/2-\xi_0} p(s_0)\, ds_0$$

and

$$P(H_2) = \int_0^{2\pi} \int_{\pi/2-\xi_0}^{\pi/2} p(s_0)\, ds_0 \quad .$$

Let

$$g_1(s,m) = \begin{cases} 1 & \text{if } \theta \geq \xi \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2(s,m) = \begin{cases} 1 & \text{if } \theta \geq \pi/2 - \xi \\ 0 & \text{otherwise} \end{cases}$$

so that if $p(s_0) = p(s_0|m^0) = \exp f_1(\theta_0,\phi_0)$ then

$$p(s_1|m^0) = g_1(s_1,m_0)\exp f_1(\theta_1-\xi_0,\phi_1-\zeta_0)\int_0^{2\pi} \int_0^{\pi/2-\xi_0} p(s_0)\, ds_0$$

$$+ g_2(s_1,m_0)\exp f_1(\pi-\theta_1-\xi_0,\phi_1-\zeta_0-\pi)\int_0^{2\pi}\int_{\pi/2-\xi_0}^{\pi/2} p(s_0)\, ds_0 \quad .$$

Now if f_1 is the exponent of an AEFD so that it can be written in the form

$$f_1(\theta,\phi) = \sum_{m=-N}^{N} \sum_{n=-2N}^{2N} a_{mn}^0 \exp[i(2m\theta+n\phi)] \quad ,$$

then

$$p(s_1|m^0) = R_{01} \exp\left[\sum_{m=-N}^{N} \sum_{n=-2N}^{2N} a_{mn}^0 \exp[-i(2m\xi_0-n\zeta_0)]\exp[i(2m\theta_1+n\phi_1)]\right]$$

$$+ R_{02} \exp\left[\sum_{m=-N}^{N} \sum_{n=-2N}^{2N} \{(-1)^n a_{-m,n}^0 \exp[i(2m\xi_0-n\zeta_0)] \exp[i(2m\theta_1+n\phi_1)]\}\right]$$

where

$$R_{01} = g_1(s_1,m_0) \int_0^{2\pi} \int_0^{\pi/2-\xi_0} p(s_0)\, ds_0$$

and

$$R_{02} = g_2(s_1, m_0) \int_0^{2\pi} \int_{\pi/2-\xi_0}^{\pi/2} p(s_0)\, ds_0 \quad .$$

Since $p(m_1|s_1)$ is a density of the form

$$\exp\left[\sum_{m=-2N}^{2N} \sum_{n=-2N}^{2N} b_{mn}^1 \exp[i(m\theta_1+n\phi_1)]\right]$$

the conditional density $p(s_1|m^1)$ will be the sum of two exponential functions:

$$p(s_1|m^1) = c_1 \left\{ R_{01} \exp\left[\sum_{m,n=-2N}^{2N} a_{mn}^{11} \exp[i(m\theta_1+n\phi_1)]\right] + R_{02} \exp\left[\sum_{m,n=-2N}^{2N} a_{mn}^{12} \exp[i(m\theta_1+n\phi_1)]\right]\right\},$$

with

$$a_{mn}^{11} = \begin{cases} a_{mn}^0 \exp[-i(m\xi_0+n\zeta_0)] + b_{mn}^1 & \text{if } m \text{ is even} \\ b_{mn}^1 & \text{if } m \text{ is odd} \end{cases},$$

$$a_{mn}^{12} = \begin{cases} (-1)^n a_{-m,n}^0 \exp[i(m\xi_0-n\zeta_0)] + b_{mn}^1 & \text{if } m \text{ is even} \\ b_{mn}^1 & \text{if } m \text{ is odd} \end{cases}.$$

If instead $-\pi/2 \leq \xi_0 < 0$, then

$$P(H_1) = \int_0^{2\pi} \int_{-\xi_0}^{\pi/2} p(s_0)\, ds_0 \,, \quad P(H_2) = \int_0^{2\pi} \int_0^{-\xi_0} p(s_0)\, ds_0$$

so that if we let

$$g_3(s,m) = \begin{cases} 1 & \text{if } 0 < \pi/2 + \xi \\ 0 & \text{otherwise} \end{cases}$$

$$g_4(s,m) = \begin{cases} 1 & \text{if } 0 \leq -\xi \\ 0 & \text{otherwise} \end{cases}$$

then

$$p(s_1|m^0) = g_3(s_1,m_0) \exp f_1(\theta_1-\xi_0,\phi_1-\zeta_0)\cdot\int_0^{2\pi}\int_{-\xi_0}^{\pi/2} p(s_0)\, ds_0$$

$$+ g_4(s_1,m_0) \exp f_1(-\theta_1-\xi_0,\phi_1-\zeta_0)\cdot\int_0^{2\pi}\int_{-\xi_0}^{\pi/2} p(s_0)\, ds_0$$

so that

$$p(s_1|m^1) = c_1\left\{R_{01} \exp\left[\sum_{m,n=-2N}^{2N} a_{mn}^{11} \exp[i(m\theta_1+n\phi_1)]\right]\right.$$

$$\left. + R_{02} \exp\left[\sum_{m,n=-2N}^{2N} a_{mn}^{12} \exp[i(m\theta_1+n\phi_1)]\right]\right\},$$

where

$$a_{mn}^{11} = \begin{cases} a_{mn}^{0} \exp[-i(m\xi_0+n\zeta_0)] + b_{mn}^{1} & \text{if } m \text{ is even} \\ b_{mn}^{1} & \text{if } m \text{ is odd} \end{cases},$$

$$a_{mn}^{12} = \begin{cases} a_{-m,n}^{0} \exp[i(m\xi_0-n\zeta_0)] + b_{mn}^{1} & \text{if } m \text{ is even} \\ b_{mn}^{1} & \text{if } m \text{ is odd} \end{cases},$$

where $R_{01} = g_3(s_1,m_0)P(H_1)$, and $R_{02} = g_4(s_1,m_0)P(H_2)$. The computation of

$$p(s_2|m^2) = p(s_2|m^1,H_1)P(H_1|m^1) + p(s_2|m^1,H_2)P(H_2|m^1)$$

will not be included here although it proceeds in a similar manner. We observe that for any k, $p(s_k|m^k)$ will be a linear combination of 2^k terms of the form

$$\exp\left[\sum_{m,n=-2N}^{2N} a_{mn}^{kr} \exp[i(m\theta_k+n\phi_k)]\right]$$

and it will be necessary to determine twice as many parameters as for $p(s_{k-1}|m^{k-1})$.

B. Estimation Model E(c,a) Using Isotropic Noise

Suppose the signal process $\{s_k\}$ is described by

$$s_{k+1} = s_k \textcircled{c} R_k$$

where $\{R_k\}$ is a sequence of rotations in three-dimensional space. From the definition of $\textcircled{c}$ we have, using direction cosine vectors for s_0 and s_1 and the direction cosine matrix representation for R_0

$$s_0 = R_0^{-1} s_1 \qquad \text{or} \qquad s_0 = R_0^{-1} \bar{s}_1$$

where $\bar{s}_1$ is the antipode of s_1 having spherical coordinates $(\theta_1+\pi,\phi_1)$. By [6, p.150]

$$Y_{2\ell,m}(\theta_0,\phi_0) = \sum_{n=-2\ell}^{2\ell} D^{2\ell}(R_0)_{nm} Y_{2\ell,n}(\theta_1,\phi_1)$$

for either case, so that if

$$p(s_0) = \exp \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} a_{2\ell,m}^{0} Y_{2\ell,m}(\theta_0,\phi_0) \quad ,$$

then

$$p(s_1) = \exp \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} \left[\sum_{n=-2\ell}^{2\ell} a_{2\ell,m}^{0} D^{2\ell}(R_0)_{mn}\right] Y_{2\ell,m}(\theta_1,\phi_1) \quad .$$

Now if there is a measurement process such that

$$m_k(\lambda_k,\mu_k) = s_k(\theta_k,\phi_k) \quad \text{ⓐ} \quad v_k(\alpha_k,\beta_k)$$

where v_k has an <u>isotropic</u> noise density of the form

$$p(v_k) = \exp \sum_{\ell=0}^{N} b_{2\ell}^{k} P_{2\ell}(\cos \alpha_k) \quad ,$$

then

$$p(m_k|s_k) = \exp \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} b_{2\ell}^{k} \overline{Y_{2\ell,m}(\lambda_k,\mu_k)}\, Y_{2\ell,m}(\theta_k,\phi_k) \quad .$$

Therefore,

$$p(s_k|m^k) = \exp \sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} a_{2\ell,m}^{k} Y_{2\ell,m}(\lambda_k,\mu_k)$$

where

$$a_{2\ell,m}^{k} = \sum_{n=-2\ell}^{2\ell} a_{2\ell,n}^{k-1} D^{2\ell}(R_{k-1})_{mn} + b_{2\ell}^{k} \overline{Y_{2\ell,m}(\lambda_k,\mu_k)} \quad .$$

C. <u>Estimation Model E(b,+) with Additive White Noise</u>

Let $\{s_k\}$ be a sequence of signals such that

$$s_{k+1}(\theta_{k+1},\phi_{k+1}) = s_k(\theta_k,\phi_k) \quad \text{ⓑ} \quad w_k(\xi_k,\zeta_k)$$

and suppose $\{m_k\}$ is a sequence of vectors such that

$$m_k = h(s_k) + v_k \quad ,$$

where $\{v_k\}$ is a sequence of independent Gaussian vectors with mean zero and covariance matrix $R_k = E[v_k v_k']$ and h is a vector-valued function on $S^2/\pm 1$. The completeness of the spherical harmonics of even order on $S^2/\pm 1$ allows us to consider h(s) to be approximately of the form

$$\sum_{\ell=0}^{N} \sum_{m=-2\ell}^{2\ell} h_{2\ell,m} Y_{2\ell,m}(\theta,\phi) \quad .$$

Now

$$p(v_k) = (2\pi)^{-p/2}(\det R_k)^{-1/2}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij}\, v_k^i\, v_k^j\right]$$

where R_k has matrix elements R_k^{ij} and vector v_k has entries v_k^i. We therefore have as the conditional density

$$p(m_k|s_k) = (2\pi)^{-p/2}(\det R_k)^{-1/2}\exp\Bigg[c_k + \sum_{\ell=0}^{N}\sum_{m=-2\ell}^{2\ell} c_k(\ell,m)Y_{2\ell,m}(\theta_k,\phi_k)$$

$$+ \sum_{\ell,\ell'=0}^{N}\sum_{m=-2\ell}^{2\ell}\sum_{m'=-2\ell'}^{2\ell'} c_k(\ell,\ell',m,m')Y_{2\ell,m}(\theta_k,\phi_k)\; Y_{2\ell',m'}(\theta_k,\phi_k)\Bigg] \qquad (11)$$

with

$$c_k = -\frac{1}{2}\sum_{i,j=1}^{p} m_k^i\, m_k^j\, R_k^{ij}$$

$$c_k(\ell,m) = \frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij}\,[m_k^i\, h_{2\ell,m}^j + m_k^j\, h_{2\ell,m}^i]$$

$$c_k(\ell,\ell',m,m') = -\frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij}\, h_{2\ell,m}^i\, h_{2\ell',m'}^j \quad .$$

(11) is an AETF(4N) since each spherical harmonic of even order is an AETF and hence so is the product of two such spherical harmonics.

If s_0 has an AEFD(2N), then using (11) the conditional density, obtained in the same manner as in model E(b,a), will be a finite linear combination of exponential trigonometric functions of order 4N.

D. <u>Estimation Model E(c,+)</u>

As our final model consider the signal and measurement processes given by

$$s_{k+1}(\theta_{k+1},\phi_{k+1}) = s_k(\theta_k,\phi_k) \;\textcircled{c}\; R_k$$

$$m_k = h(s_k) + v_k$$

where h and v_k are defined as for model E(b,+).

If $p(m_k|s_k)$ given by (11) can be written as an AEFD(4N) of the form

$$p(m_k|s_k) = \exp \sum_{\ell=0}^{2N} \sum_{m=-2\ell}^{2\ell} b^k_{2\ell,m} Y_{2\ell,m}(\theta_k,\phi_k) \tag{12}$$

and if

$$p(s_0) = \exp \sum_{\ell=0}^{2N} \sum_{m=-2\ell}^{2\ell} a^0_{2\ell,m} Y_{2\ell,m}(\theta_0,\phi_0) \quad ,$$

then

$$p(s_k|m^k) = \exp \sum_{\ell=0}^{2N} \sum_{m=-2\ell}^{2\ell} a^k_{2\ell,m} Y_{2\ell,m}(\theta_k,\phi_k)$$

where

$$a^k_{2\ell,m} = \sum_{n=-2\ell}^{2\ell} a^{k-1}_{2\ell,n} D^{2\ell}(R_{k-1})_{mn} + b^k_{2\ell} \overline{Y_{2\ell,m}(\lambda_k,\mu_k)} \quad .$$

The apparent difficulty in requiring that (12) holds is resolved by the following lemma:

<u>Lemma 3</u>: The product of two spherical harmonics of even orders is a sum of spherical harmonics of even orders.

Proof: The product of two spherical harmonics of orders 2ℓ and $2\ell'$ where $\ell' \geq \ell$ is specified by [6,p.165]:

$$Y_{2\ell,m}(\theta,\phi)\, Y_{2\ell',m'}(\theta,\phi) = \sum_{L=2\ell'-2\ell}^{2\ell'+2\ell} a_{L,m+m'} Y_{L,m+m'}(\theta,\phi)$$

where

$$a_{L,m+m'} = (-1)^{m+m'}(2L+1)\begin{pmatrix} 2\ell & 2\ell' & L \\ m & m' & -m-m' \end{pmatrix}\begin{pmatrix} 2\ell & 2\ell' & L \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

Now the proof follows as an immediate consequence of Lemma 1.

VI. CONCLUSIONS

Several models for the estimation of discrete-time axial processes have been proposed and have been found to be solvable by introducing axial exponential Fourier densities which employ spherical harmonics of even orders. When these densities are expressed as trigonometric functions a closure property for the conditional densities is seen to exist that permits their computation by updating a finite number of parameters.

The three displacements defined on $S^2/\pm 1$ which were used to obtain the estimation models, together with the properties of spherical harmonics of even order yielded conditional densities which resembled the conditional densities arising in estimation models for directional processes on S^2 that appear in [1]. It should be kept in mind, however, that S^2 and $S^2/\pm 1$ have inherent geometrical differences as is evidenced by the need to use different error criteria on the spaces involved.

The displacements defined for $S^2/\pm 1$, with the only exception being (c), required that $S^2/\pm 1$ be represented by a fixed hemisphere of S^2. The question arises as to whether or not it is possible to define other displacements for which this requirement can be removed.

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