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THERMAL CALCULATION OF A FREEZING TUBE IN WATER-PERMEABLE GROUND--ETC(U)  
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B.V. Proskuryakov

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# THERMAL CALCULATION OF A FREEZING TUBE IN WATER-PERMEABLE GROUND

Doctor (Technical Science) B. V. Proskuryakov

## 1. Foreword

The problem of ground freezing has a theoretical and an experimental history. The problem of freezing homogeneous ground by a single tube or by a series of tubes has been fairly well clarified by both theory and experience. This cannot, however, be said about freezing ground in the presence of filtration water. With the exception of several attempts to solve this problem (see, for instance the article by docent Charnyy "Soviet Metropolitan", No. 4, 1940) which introduced a number of assumptions and roughly outlined the process, we simply find no rigorous solution to this problem, and are compelled to note that the literature lacks even data concerning experimental studies of this problem. However, in a number of cases practitioners encounter the necessity to freeze sections of ground which are sometimes highly water-permeable.

In this case it is impossible to evaluate by calculation neither the required power of the freezing devices, nor to select an economical interval between the tubes, nor to establish the freezing times, and the problem has to be solved during production by trial and error, causing superfluous expenditures to be made on freezing and losing time in experimenting during the process of carrying out the operations. It must, however, be noted here that there is practically no experience in freezing ground under these conditions. The goal of our study is to establish means of solving this problem.

## 2. Calculating the Freezing of Water-Permeable Ground by Means of a Single Tube (Steady-State Conditions)

Let us assume that a freezing tube of radius  $r$  has a surface temperature  $T$  which is kept constant throughout the freezing period.

This tube is surrounded by the filtration flow, the direction of which is perpendicular to the tube axis and the velocity of which when learning the tube is equal to  $v$ . The temperature of the ground and consequently of the filtration flow when approaching the tube is equal to  $T_{fi}$ .

The ground will continue to freeze until the quantity of heat emitted from the surface of the freezing column of ground to the tube is equal to the quantity of heat transmitted to this surface from the filtration flow. When these quantities of heat are equal, freezing will stop and a state of equilibrium will be established in which the power of the freezing mechanisms will be expended only to cool the filtration flow if the dimensions of the column of frozen ground are kept constant. We will examine in this way the problem under the conditions of established heat conditions specifically for the case where

$$Q_2 = Q_1,$$

where  $Q_2$  - is the quantity of heat extracted from the filtration flow at the boundary between the thawed and frozen ground;

$Q_1$  - is the quantity of heat taken from the surface of the frozen ground (zero isotherm) from the freezing tube.

In this case the problem is stated as follows:

"At a given velocity  $v$  of a filtration flow which has temperature  $T_{fi}$ , find the maximum radius  $R_0$  of a frozen column of ground if the radius of the freezing tube  $r_T$  and the temperature on its surface  $T$  are known."

In solving this problem, we are first obliged to state that the propagation of heat within the boundaries of the frozen portion of ground and within the limits of the thawed ground penetrated by the filtration flow obey different laws. In the first case heat propagates only due to the thermal conductivity of the ground, and in the second case heat is transferred from point to point by the filtration flow as well.

Thus, we are obliged to examine two problems: One concerns the propagation of heat in the frozen portion of ground and the second concerns the propagation of heat in the thawed ground in the presence of a filtration flow, or as we will refer to it for the sake of brevity, the internal problem and the external problem.

In solving the problem we will consider that the frozen column of ground is a circular cylinder which is eccentrically located relative to the freezing tube. This assumption, as experience shows, can be accepted without introducing any large errors.

#### A. The Internal Problem

Problem statement. We will designate the following:

1. The radius of the frozen column of ground or, what is the same thing, the radius of the zero isotherm  $R_0$ .
2. The outer radius of the freezing tube  $r_T$ .
3. The temperature on the surface of the freezing tube  $T$ .
4. Eccentricity  $e_0$  of the zero isotherm with radius  $R_0$  and of the isotherm with radius  $r_T$ .
5. The heat conductivity factor of the frozen ground  $\lambda_f$ .

Find: 1. The total amount of heat extracted from the zero isotherm.

2. The distribution of heat extracted from the zero isotherm.

In view of the fact that the heat in the frozen ground propagates only due to thermal conductivity, we may consider that in this case the Fourier equation is applicable:

$$\frac{dt}{d\tau} = a \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right), \quad (1)$$

where

$a$  - is the temperature conductivity factor;

$t$  - is the temperature at any point of the frozen ground;

$\tau$  - is time;

$x, y$  - are the coordinates of a point with temperatures  $t$ .

In view of the fact that we are examining established thermal conditions, at which the temperatures do not vary in time, we can write the Fourier equation as applied to our case in the following form:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0. \quad (2)$$

The solution to this equation as applied to the given problem conditions is a function which is known in the theory of a two-charge electrical field (a dipole field):

$$t = \frac{1}{C_1} \left\{ \ln \sqrt{x^2 + (y+a)^2} - \ln \sqrt{x^2 + (y-a)^2} + C_2 \right\}. \quad (3)$$

The position of the coordinate axes and the accepted designations are shown in Figure 1.

Differentiation readily demonstrates that function (3) satisfies thermal conductivity equation (2) and consequently is a partial integral of it.

Simple transformations convert function (3) into the following form:

$$x^2 + \left[ y - a \frac{e^{2C_1 t} - 2C_1 - 1}{e^{2C_1 t} + 2C_1 - 1} \right]^2 + 4a^2 \frac{e^{2C_1 t} - 2C_1}{[e^{2C_1 t} + 2C_1 - 1]^2} = 0. \quad (4)$$

The form of this equation makes it possible to state that the isothermic curves are a family of circles with radii:

$$r = 2a \frac{e^{C_1 t} - C_1}{e^{2C_1 t} + 2C_1 - 1} \quad (5)$$

and with centers located on axes  $y$ , while the ordinates of the circles' centers are equal to

$$y_c = a \frac{e^{2C_1 t} - 2C_2 + 1}{e^{2C_1 t} - 2C_2 - 1} \quad (6)$$

By solving the last two equations together, we find

$$a^2 = y_c^2 - r^2 \quad (7)$$

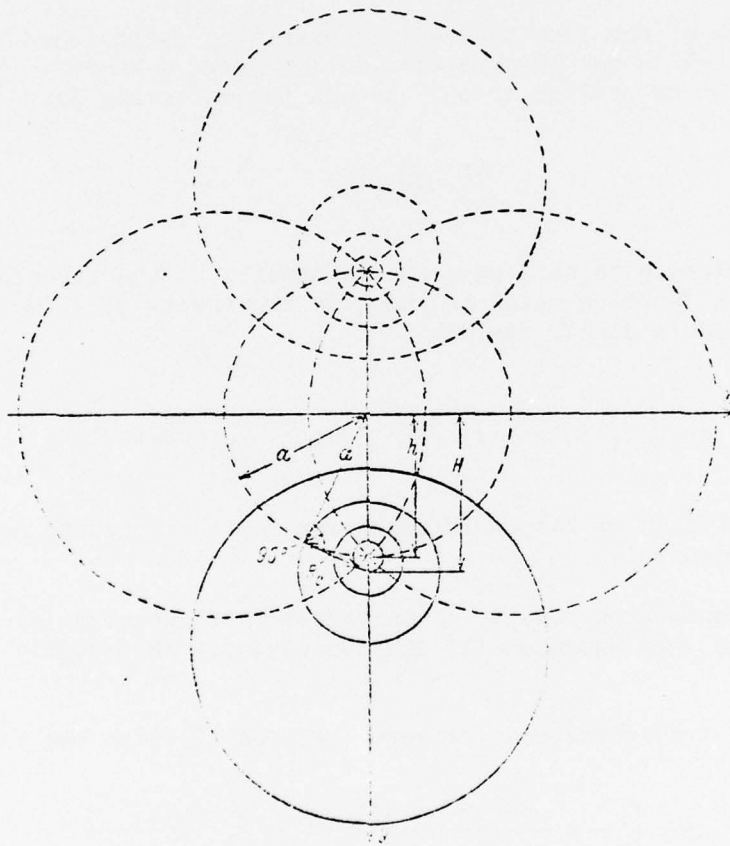


Figure 1.

We will determine the values of the integration constants from the following condition: when  $y_c = h$ ,  $x = 0$  we should have  $r = r_T$ , and  $t = T$ .

Then from (5) and (6) we have

$$C_1 T - C_2 = \ln \left( \frac{h}{r_T} + \sqrt{\frac{h^2}{r_T^2} - 1} \right). \quad (8)$$

From the second condition for the zero isotherm, we find: when  $y_c = H$  and  $x = 0$ , we should have  $r = R_0$  and  $t = 0$ ,

$$-C_2 = \ln \left( \frac{H}{R_0} + \sqrt{\frac{H^2}{R_0^2} - 1} \right). \quad (9)$$

By substituting the value of  $C_2$  from (9) into (8) we obtain an expression for the integration constant  $C_1$ :

$$C_1 = \frac{1}{T} \left[ \ln \left( \frac{h}{r_T} + \sqrt{\frac{h^2}{r_T^2} - 1} \right) - \ln \left( \frac{H}{R_0} + \sqrt{\frac{H^2}{R_0^2} - 1} \right) \right]. \quad (10)$$

In view of the fact that because of the nature of the problem we usually do not know the value of  $h$ ,  $a$  and  $H$ , it is necessary to replace these values with  $r_T$ ,  $R_0$  and  $e_0$ .

Using the conditions of the problem and Figure 1, we can write:

$$e_0 = H - h. \quad (11)$$

However, equality (7) makes it possible to establish the following functions:

$$\begin{aligned} H &= \sqrt{a^2 + R_0^2}, \\ h &= \sqrt{a^2 + r_T^2}. \end{aligned} \quad (12)$$

Combined solution of equations (12) and (11) yields:

$$e_0 = H - h = \sqrt{a^2 + R_0^2} - \sqrt{a^2 + r_T^2}; \quad (13)$$

from which we obtain the value of  $a$ :

$$a = \sqrt{\frac{(R_0^2 - r_T^2) - e_0^2}{4e_0}}. \quad (14)$$

From (12) and (14) we find the desired values:

$$h = \frac{(R_0^2 - r_T^2) - e_0^2}{2e_0}; \quad (15)$$

$$H = \sqrt{\frac{[(R_0^2 - r_T^2) - e_0^2]^2}{4e_0^2} + (R_0^2 - r_T^2)} = \frac{e_0^2 + (R_0^2 - r_T^2)}{2e_0}. \quad (16)$$

After obtaining the field of isotherm expressed by  $e_0$ ,  $R_0$  and  $r_T$ , we find the value of the heat flow directed from the zero isotherm to the surface of our freezing tube.

For this purpose we utilize equation (3) related to ordinate  $y = 0$ .

As is evident, in this case the isotherm is a straight line:

$$t_{y=0} = \frac{C_2}{C_1}. \quad (17)$$

The quantity of heat which passes through this isotherm and consequently through any other, in particular the zero isotherm (since the thermal conditions in our case have been established) can be expressed by the following equation:

$$Q = -\lambda \int_{-\infty}^{+\infty} \left. \frac{\partial t}{\partial y} \right|_{y=0} dx. \quad (18)$$

We find the general expression for the gradient  $\frac{\partial t}{\partial y}$  from equation (3):

$$C_1 \frac{\partial t}{\partial y} = \frac{y+a}{x^2 + (y+a)^2} = \frac{y-a}{x^2 + (y-a)^2}. \quad (19)$$

Hence, when  $y = 0$  the temperature gradient will be equal to:

$$\left. \frac{\partial t}{\partial y} \right|_{y=0} = \frac{2a}{C_1(x^2 + a^2)}. \quad (20)$$

By substituting the value  $\left. \frac{dt}{dy} \right|_{y=0}$  from equation (20) into equation (18) and by integrating the latter, we find:

$$Q = -\lambda_f \int_{-\infty}^{+\infty} \frac{2a}{C_1(x^2+a^2)} dx = -\frac{2\lambda_f \pi}{C_1}. \quad (21)$$

This equation can be rewritten as follows if  $C_1$  is expressed on the basis of (10):

$$Q = \frac{2\pi\lambda_f T}{\ln\left(\frac{h}{r_T} + \sqrt{\frac{h^2}{r_T^2} - 1}\right) - \ln\left(\frac{H}{R_0} + \sqrt{\frac{H^2}{R_0^2} - 1}\right)}. \quad (22)$$

The values, however,  $h$  and  $H$  can be replaced through (15) and (16), after which we find the final value for  $Q$ :

$$Q = \frac{2\pi\lambda_f T}{\ln\left[\frac{(R_0^2 - r_T^2) - e_0^2}{2e_0 r_T} + \sqrt{\frac{[(R_0^2 - r_T^2) - e_0^2]^2}{4e_0^2 r_T^2} - 1}\right] - \ln\left[\frac{e_0^2 + (R_0^2 - r_T^2)}{2e_0 R_0} + \sqrt{\frac{[e_0^2 + (R_0^2 - r_T^2)]^2}{4e_0^2 R_0^2} - 1}\right]}. \quad (23)$$

This solution is also the solution to the first portion of our set tasks: Finding the total quantity of heat extracted from the zero isotherm. However, expression (23) contains a value as yet unknown  $e_0$  which has to be determined. For this purpose we evaluate the distribution of the heat extracted from the zero isotherm. We can state that this distribution depends on the eccentricity  $e_0$  of the zero isotherm with regard to the isotherm of  $T$  (on the surface of the freezing tube), and the axes of symmetry is the  $y$  axis.

We will call the heat losses through half (a semi-circle) of the zero isotherm directed towards negative values of  $y$  (or, when it is the same thing in our case, towards the eccentricity)  $Q_e$ . We will attempt to find the relationship

$$\frac{Q_e}{Q} = E, \quad (24)$$

which will also characterize the non-uniformity of the distribution of heat flow from the zero isotherm.

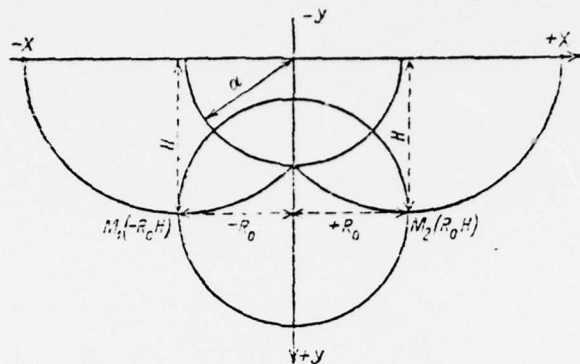


Figure 2.

Value  $Q_e$  is determined by integrating equation (21) within the limits from the flow line going from point  $M_1$  on the zero isotherm with coordinates  $x = -R_0$ ,  $y = H$  to a current line corresponding to point  $M_2$  on the zero isotherm with coordinates  $x = R_0$  and  $y = H$  (see Figure 2).

Designating these integration limits as  $-X$  and  $+X$ , we will express them through values already known to us. Returning to Figure 2, we see that

$$-X = -(R_0 + H); \quad (25)$$

$$+X = R_0 + H. \quad (26)$$

By integrating equation (21) within the interval  $-X$  to  $+X$ , we find:

$$Q_e = \frac{2\gamma f a}{C_1} \int_{-X}^{+X} \frac{dx}{x^2 + a^2} = \frac{2\gamma f}{C_1} \left( \arctan \frac{X}{a} - \arctan \frac{-X}{a} \right) = \frac{2\gamma f}{C_1} \arctan \frac{2X}{a \left( 1 - \frac{X^2}{a^2} \right)}. \quad (27)$$

By replacing  $X$  in this equation with its value according to equation (26), we obtain:

$$Q_e = \frac{2\gamma f}{C_1} \arctan \frac{2(H + R_0)}{a \left[ 1 - \frac{(H + R_0)^2}{a^2} \right]} = \frac{2\gamma f}{C_1} \arctan \frac{2a}{(H - R_0) - \frac{a^2}{(H - R_0)}}; \quad (28)$$

however, since  $a^2 = H^2 - R_0^2$ , then we can rewrite equation (28) in the following form:

$$Q_e = \frac{2\lambda_f r}{C_1} \arctan\left(-\frac{a}{R_0}\right) = \frac{2\lambda_f}{C_1} \arctan\left(-\sqrt{\frac{(R_0^2 - r_T^2 - e_0^2)^2 - r_T^2}{4R_0^2 e_0^2 - R_0^2}}\right). \quad (29)$$

By substituting into equation (24) the values we have found for  $Q_e$  and  $Q$ , we find:

$$E = \frac{1}{\pi} \arctan\left(-\sqrt{\frac{[R_0^2 - r_T^2 - e_0^2]^2 - r_0^2}{4R_0^2 e_0^2 - R_0^2}}\right). \quad (30)$$

This expression (30) is not by itself sufficient to find the value  $e_0$ . It is also necessary to examine the non-uniformity of the distribution of the heat flow passing through the outline of the zero isotherm based on the conditions of heat emission by the cylinder to the filtration flow. If in this process it is impossible to express the value  $E$  to known values, then the value of  $e_0$  will be determined from equation (30) and the problem will be completely solved.

Let us now examine heat exchange between the cylinder and the filtration flow moving around it.

#### B. External Problem

The solution of this problem amounts to determining the heat losses of a filtration flow moving around a cylindrical column of frozen ground.

As in solving the first portion of the problem, we will assume that the thermal conditions have been established, i.e., we will examine the boundary-equilibrium thermal state of the system.

In this case the differential equation of heat conductivity may be presented in the following form:<sup>1</sup>

$$a \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right) = v_x \frac{\partial t}{\partial x} + v_y \frac{\partial t}{\partial y}, \quad (31)$$

<sup>1</sup>See F. Frank and R. Mizes, *Differential and Interval Equations of Mathematical Physics*, ONTI, 1937, page 675.

where

$a$  - is the thermal conductivity coefficient of the water-permeable ground equal to  $\frac{\lambda_{wg}}{C_w \gamma_w}$  (where in turn  $\lambda_{wg}$  is the heat conductivity factor of ground saturated with water and  $C_w$  and  $\gamma_w$  are respectively the heat capacity and volumetric weight of water);

$v_x$  and  $v_y$  - are velocity projections of the filtration flow rate onto the coordinate axes.

The method of integrating equation (31) for the case where a cylinder of infinite length is surrounded by a flow has been proposed by Bussineskoe.<sup>1</sup>

We will replace the variables  $x$  and  $y$  by  $\xi$  and  $\eta$  with the aid of the following conversion formulas:

$$\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = v_x; \quad -\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = v_y, \quad (32)$$

where

$\eta$  - is the velocity potential which is constant along equipotential lines;

$\xi$  - is the "current function", which is constant along the current line.

The system of current line and equipotential lines is orthogonal and, as is known from hydrodynamics, satisfies the equation:

$$\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} = 0.$$

Examining the left-hand portion of equation (31) which is in parentheses and which utilizes the relationship between the variable  $\xi$  and  $\eta$  and the previous values of  $x$  and  $y$ , (32) makes it possible to write the following as the result of simple differentiation:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = (v_x^2 + v_y^2) \left( \frac{\partial^2 t}{\partial \xi^2} + \frac{\partial^2 t}{\partial \eta^2} \right). \quad (33)$$

---

<sup>1</sup>Analytical Theory of Heat, Paris, 1903.

With allowance for (32), however, the right-hand portion of equation (31) acquires the following form:

$$v_x \frac{\partial t}{\partial x} + v_y \frac{\partial t}{\partial y} = (v_x^2 + v_y^2) \frac{\partial t}{\partial r_1}. \quad (34)$$

These last two equations are rewritten with allowance for equation (31) in the following form:

$$a \left( \frac{\partial^2 t}{\partial \xi^2} + \frac{\partial^2 t}{\partial \eta^2} \right) = \frac{\partial t}{\partial r_1}. \quad (35)$$

If the axis of the cylinder passes through the origin of the coordinates, then the equations of the current lines and equipotentials will be:

$$\xi = \xi_0 - V \left( y - \frac{R_0^2 y}{x^2 + y^2} \right); \quad (36)$$

$$\eta = \eta_0 - V \left( x + \frac{R_0^2 x}{x^2 + y^2} \right), \quad (37)$$

where  $V$  - is the velocity of flow in infinity, where:

$$-V = v_{x \text{ inf}}, \text{ and } v_{y \text{ inf}} = 0. \quad (38)$$

We select the values  $\xi_0$  and  $\eta_0$  such that when  $\sqrt{x^2 + y^2} = R_0$   $\xi$  becomes 0; thus  $\xi_0 = 0$ , and when  $x = R_0$  and  $y = 0$  we can approximately assume  $\eta = 0$ , from which  $\eta_0 = 2R_0 V$ .

At these values of  $\xi_0$  and  $\eta_0$  equations (36) and (37) are rewritten in the following form:

$$\xi = -V \left( y - \frac{R_0^2 y}{x^2 + y^2} \right); \quad (39)$$

$$\eta = +V \left( 2R_0 - x - \frac{R_0^2 x}{x^2 + y^2} \right). \quad (40)$$

As applied to the conditions of the problem to be solved by us, it is possible to ignore the value  $\frac{\partial^2 t}{\partial \eta^2}$  [in equation (35)] in comparison with  $\frac{\partial^2 t}{\partial \xi^2}$ , assuming that:

$$\frac{\partial^2 t}{\partial \eta^2} \approx 0. \quad (41)$$

Then equation (35) acquired the form:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial \xi^2} \quad (42)$$

Equation (42) should be integrated under the following boundary conditions:

$$\begin{aligned} \text{On the body surface when } \xi = 0 \quad t = 0 \\ \text{when } \eta = 0 \quad t = T_{fi} \\ \text{and everywhere in infinity} \quad t = T_{fi} \end{aligned} \quad (43)$$

Whenever discussing the method used to integrate equation (42) under boundary conditions (43) and relying on the special guides of interest to us,<sup>1</sup> we will cite only the final expression for evaluating the quantity of heat which the flow surrounding our frozen column-cylinder loses (per one linear meter of cylinder length):

$$Q = \dots \sqrt{\frac{\lambda_{wg} \gamma_w C_w}{\pi}} (1 - \eta_1) T_{fi} \quad (44)$$

where

$\lambda_{wg}$  - is the heat conductivity factor of the ground saturated with water;

$\gamma_w$  - is the weight of a unit of water volume;

$C_w$  - is the heat capacity factor of the water;

$T_{fi}$  - is the temperature of the filtration flow on the approach to the cylinder (in infinity).

In this latter equation value  $\eta_1$  corresponds to  $x = R_0$  and  $y = 0$ , and therefore from (40) we obtain:

$$\eta_1 = 0 \quad (45)$$

When  $x = -R_0$  and  $y = 0$ , we find:

$$\eta_2 = 4R_0 V \quad (46)$$

<sup>1</sup>For instance, Frank and Mizes. Differential and Interval Equations of Mathematical Physics, ONTI, 1937, page 676.

By substituting the values of  $\eta_1$  and  $\eta_2$  obtained from (45) and (46) into equation (44), we find:

$$Q = 8T_{fi} \sqrt{\frac{\lambda_w g' w C_w R_0 V}{\pi}} \quad (47)$$

This equation is also the solution to our external problem.

For the purpose of further analysis, it is also necessary to determine the non-uniform distribution of heat losses over the cylinder surface. For this goal, as in solving the "internal" problem, we introduce the relationship:

$$\frac{Q_e}{Q} = E' \quad (48)$$

Here  $Q_e$  is used to indicate the heat losses from the flow to that half of the cylinder surface which is turned towards the filtration flow.

In order to determine the value  $Q_e$ , we examine equation (44) which in this case is rewritten as follows:

$$Q_e = 4 \sqrt{\frac{\lambda_w g' w C_w}{\pi}} (V \sqrt{\eta_c} - V \sqrt{\eta_1}) T_{fi} \quad (49)$$

where  $\eta_c$  is found from (40) when  $x = 0$  and  $y = R_0$ :

$$\eta_c = 2R_0 V \quad (50)$$

and  $\eta$  retains the value given to it in (45).

By substituting the values which we have found for  $\eta_1$  and  $\eta_c$  into equation (49), we obtain:

$$Q_e = 5,64 T_{fi} \sqrt{\frac{\lambda_w g' w C_w R_0 V}{\pi}} \quad (51)$$

Now, by solving (47), (48) and (51) together, we find the desire value of  $E'$ :

$$E' = 0,705 = \text{const.} \quad (52)$$

### C. Combined Solution to the "Internal" and "External" Problems

Having examined two portions of one problem, we can now turn to obtaining general computation functions.

In view of the fact that under the conditions of established thermal conditions the quantity of heat extracted from the filtration flow is equal to the quantity of heat extracted by the tube from a frozen column of ground, on the basis of equations (23) and (27) we can write:

$$\frac{2\pi l_f T}{\ln A_0 - \ln B_0} = 8 \sqrt{\frac{\rho_w g l_w C_w R_0 V}{\pi}} \cdot T_{fi}, \quad (53)$$

where

$$A_0 = \frac{(R_0^2 - r_f^2) - e_0^2}{2e_0 r_f} + \sqrt{\frac{[(R_0^2 - r_f^2) - e_0^2]^2}{4e_0^2 r_f^2} - 1}; \quad (54)$$

$$B_0 = \frac{(R_0^2 - r_f^2) + e_0^2}{2e_0 R_0} + \sqrt{\frac{[(R_0^2 - r_f^2) + e_0^2]^2}{4e_0^2 R_0^2} - 1}. \quad (55)$$

After very simple transformations, equation (53) can be reduced to the following form:

$$\frac{1}{4 \sqrt{\frac{\rho_w C_w}{l_w}} \sqrt{\frac{\rho_w g}{l_w}}} \cdot \frac{l_f}{V} \cdot \frac{2\pi T}{T_{fi}} = \sqrt{R_0 \ln \frac{A_0}{B_0}}. \quad (56)$$

In the terms of the essence of the problem, all of the values which belong to equations (54), (55) and (56) except for  $R_0$  and  $e_0$  should be given. Thus, equation (56) will contain two unknowns:  $R_0$  and  $e_0$ .

In order to determine  $e_0$  we can utilize equations (30) and (52) from which it follows that with established thermal conditions  $E = E'$ , or:

$$\frac{1}{\pi} \arctan \left( - \sqrt{\frac{R_0^2 - r_f^2 - e_0^2}{4R_0^2 e_0^2} - \frac{r_f^2}{R_0^2}} \right) = 0,705. \quad (57)$$

By transforming this equation and by solving it in terms of  $e_0$ , we obtain the quadratic equation:

$$e_0^2 + \sqrt{1,76R_0^2 + 4r_f^2} \cdot e_0 - (R_0^2 r - r_f^2) = 0. \quad (58)$$

From this equation it is easy to determine the desired  $e_0$ :

$$e_0 = 1,2R_0 - \sqrt{0,44R_0^2 + r_f^2} \quad (59)$$

Together with (56) this equation also serves as the basis for solving the problem about the boundary-equilibrium state of a frozen ground column in the field of a filtration flow.

### 3. Sample Problem Solution

Assume that we know:

- Temperature on the surface of freezing tube .....  $T = -20^{\circ}\text{C}$   
 Outer radius of freezing tube .....  $r_T = 0.10$  meters  
 Velocity of filtration flow .....  $V = 1$  m/hr  
 Temperature of filtration flow .....  $T_{fi} = +5^{\circ}\text{C}$   
 Thermal conductivity factor of frozen ground .....  $\lambda_f = 2.0 \frac{\text{kcal}}{\text{m}\cdot\text{hr}\cdot\text{deg}}$   
 Thermal conductivity factor of ground saturated with water  $\lambda_{wg} = 1 \frac{\text{kcal}}{\text{m}\cdot\text{hr}\cdot\text{deg}}$   
 Volumetric weight of water .....  $\gamma_w = 1000$  kg/m<sup>3</sup>  
 Heat capacity of water .....  $C_w = 1 \frac{\text{kcal}}{\text{kg}\cdot\text{deg}}$

Find  $R_0$ .

The computation equations are equations (54), (55), (56) and (59) which we will again rewrite here:

$$-\frac{\pi^{\frac{3}{2}}}{4\sqrt{\gamma_w C_w}} \cdot \frac{\lambda_f}{\sqrt{\lambda_{wg}}} \cdot \frac{T}{r_T \sqrt{V}} = \sqrt{R_0} \ln \frac{A_0}{B_0}, \quad (60)$$

or

$$A_0 = \frac{(R_0^2 - r_T^2) - e_0^2}{2e_0 r_T} + \sqrt{\frac{[(R_0^2 - r_T^2) - e_0^2]^2}{4e_0^2 r_T^2} - 1}; \quad (61)$$

$$B_0 = \frac{(R_0^2 - r_T^2) + e_0^2}{2e_0 R_0} + \sqrt{\frac{[(R_0^2 - r_T^2) - e_0^2]^2}{4e_0^2 R_0^2} - 1}; \quad (62)$$

$$e_0 = 1,2R_0 - \sqrt{0,44R_0^2 + r_T^2}. \quad (63)$$

By substituting our given values of the computation values into equation (60), we find:

$$\frac{3,14^{\frac{3}{2}}}{4\sqrt{1000 \cdot 1}} \cdot \frac{2,0}{\sqrt{1}} \cdot \frac{-20}{0,1\sqrt{1}} = \sqrt{R_0} \ln \frac{A_0}{B_0}$$

or

$$\sqrt{R_0} \ln \frac{A_0}{B_0} = 0,37 R_0^{\frac{1}{2}}. \quad (64)$$

Examining equation (63), we find:

$$e_0 = 1,2R_0 - \sqrt{0,44R_0^2 + 0,01} \text{ m.}$$

By substituting value  $e_0$  into equations (61) and (62), we obtain:

$$A_0 = \frac{(R_0^2 - 0,01) - (1,2R_0 - \sqrt{0,44R_0^2 + 0,01})^2}{0,24R_0 - 0,2\sqrt{0,44R_0^2 + 0,01}} + \quad (65)$$

$$+ \sqrt{\left[ \frac{(R_0^2 - 0,01) - (1,2R_0 - \sqrt{0,44R_0^2 + 0,01})^2}{0,24R_0 - 2\sqrt{0,44R_0^2 + 0,01}} \right]^2 - 1};$$

$$B_0 = \frac{(R_0^2 - 0,01) + (1,2R_0 - \sqrt{0,44R_0^2 + 0,01})^2}{2,4R_0^2 - 2R_0\sqrt{0,44R_0^2 + 0,01}} + \quad (66)$$

$$+ \sqrt{\left[ \frac{(R_0^2 - 0,01) + (1,2R_0 - \sqrt{0,44R_0^2 + 0,01})^2}{2,4R_0^2 - 2R_0\sqrt{0,44R_0^2 + 0,01}} \right]^2 - 1}.$$

In view of the fact that solving equation (64) in the presence of (65) and (66) relative to  $R_0$  is impossible, we will determine the desired  $R_0$  value by selection. Thus,

$$\text{when } R_0 = 0.2 \text{ meters: } \sqrt{R_0} \ln \frac{A_0}{B_0} = 0,225 \text{ m}^2;$$

$$\text{when } R_0 = 0.3 \text{ meters: } \sqrt{R_0} \ln \frac{A_0}{B_0} = 0,57 \text{ m}^2.$$

Thus, by comparing the latter solutions with equation (64), we see that the desired  $R_0$  lies:

$$0,2 < R_0 < 0,3.$$

Now assigning the value  $R_0 = 0.25$ , we find:

$$\sqrt{R_0} \ln \frac{A_0}{B_0} = 0,40 \text{ m}^2,$$

from which we may conclude that the desired solution will correspond to the value  $R_0 = 0.24$  meters.



- V - velocity of filtration flow;
- $\gamma$  - volumetric weight of water;
- $C_w$  - heat capacity of water;
- $\lambda_f$  - thermal conductivity factor of frozen ground;
- $\lambda_{wg}$  - thermal conductivity factor of thawed ground;
- $r_T$  - radius of freezing tube;
- $R_0$  - radius of frozen column of ground.

This example shows that the problem solutions are associated with fairly cumbersome calculations. Therefore, in order to make the calculations easier, we have drawn up the nomogram presented in Figure 3.

In order to plot the nomogram, we have introduced a new variable:

$$\psi = \frac{\lambda_f}{4 \sqrt{\gamma_w C_w}} \frac{1}{\sqrt{\lambda_{wg}} \sqrt{V}}. \quad (67)$$

The temperature on the surface of the freezing tube is plotted on scale I. On scale II is the relationship of the temperature on the freezing tube surface to the temperature of the filtration flow.

A unique line corresponds to each value of the filtration flow temperature in the field of the first quadrant.

The value  $\sqrt{R_0} \ln \frac{A_0}{B_0}$  is plotted on scale III. As follows from equation (60), with the aid of a system of straight lines in the second quadrant corresponding to various values of C, the multiplication operation is performed:

$$\psi \cdot \frac{T}{T_{fi}}.$$

If we examine the system of equations (60), (61), (62) and (63), then we see that:

$$\sqrt{R_0} \ln \frac{A_0}{B_0} = f(r_T, R_0). \quad (68)$$

Consequently, by assigning various values of  $r_T$  it is possible to absolutely determine the desired radius of the frozen ground column  $R_0$ . The solution to equation (68) is also carried out in the fourth quadrant with regard to  $R_0$ .

The above-cited solution through a numerical example of the problem is repeated in the nomogram, and the sequential passage through the quadrants of the nomogram is indicated by a dotted line with arrows showing the direction of the passage. When

$$\psi = \frac{3,14^{1,2} \cdot 2}{4 \sqrt{1000} \sqrt{1}} \cdot \frac{1}{\sqrt{1}} = 0,088$$

value  $R_0$  is equal to 0.25 m according to the nomogram, which corresponds fairly closely to the results obtained previously.