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1976 R P GILBERT, G C HSIAO

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20. ABSTRACT cont'd

→ Taylor (Laurent) expansion in powers of the time has as its first term the classical fundamental solution of VEKUA. Integral representations are obtained for the initial boundary value problems.

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CONSTRUCTIVE FUNCTION THEORETIC METHODS FOR
HIGHER ORDER PSEUDOPARABOLIC EQUATIONS*

R.P. Gilbert and G.C. Hsiao

0. INTRODUCTION

In this work we will develop a constructive method for solving pseudoparabolic equations of order $2n$ in the plane. More precisely, we investigate equations of the form

$$(0.1) \quad \mathcal{L}[u] := M[u_t] + L[u] ,$$

where M and L are the respective elliptic operators

$$(0.2) \quad M[u] := \Delta^n u + \sum_{k=1}^n M_k (\Delta^{n-k} u) ,$$

$$M_k[\phi] := \sum_{p+q \leq k} \sum_{p,q=0} a_k^{pq}(x,y) \frac{\partial^{p+q} \phi}{\partial x^p \partial y^q} ,$$

and

$$(0.3) \quad L[u] := \Delta^m u + \sum_{k=1}^m L_k (\Delta^{m-k} u) , \quad m < n ,$$

$$L_k[\psi] := \sum_{p+q \leq k} \sum_{p,q=0} b_k^{pq}(x,y) \frac{\partial^{p+q} \psi}{\partial x^p \partial y^q} .$$

The coefficients of M_k , L_k are taken, furthermore, to be analytic functions of x and y for $(x,y) \in D \subset C^1$.

Recently, the integral operator methods of BERGMAN [2] and VEKUA [22], which have been very successful for developing representations for solutions of elliptic equations in the plane, have been extended by COLTON to treat the cases of parabolic [9] and second order pseudoparabolic equations [10], [11] in the plane. BROWN, GILBERT and HSIAO [8] and BROWN and GILBERT [7] developed analogous techniques for fourth-order pseudoparabolic equations using

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respectively the methods of VEKUA and BERGMAN to treat elliptic operators. BROWN [6], on the other hand, completed the study of fourth-order, analytic, parabolic equations in two space variables.

Investigations concerning integral operators which generate solutions to parabolic and pseudoparabolic equations in three and four space dimensions have been made by RUNDELL [12], RUNDELL and STECHER [13], and by BHATNAGAR and GILBERT [3], [4], [5].

Pseudoparabolic equations arise in a variety of physical problems, such as the velocity of a non-steady flow of a viscous fluid [21], the theory of seepage of homogeneous fluids through fissured rock [1], hydrostatic excess pressure during the consolidation of clay [19], and the stability of liquid filled shells [18], [23], [24]. GILBERT and ROACH are presently investigating the last mentioned problem as an application of some of the ideas presented in this current work.

I. THE FUNDAMENTAL SOLUTION

If the coefficients of M_k as analytic functions of x, y have an analytic extension to $(z, z^*) \in D \times D^*$, $D^* := \{z: \bar{z} \in D\}$, where $z = x + iy$, $z^* = x - iy$, then $M[u]$ has a representation

$$(1.1) \quad M[u] := \sum_{k,j=0}^n A_{kj}(z, z^*) \frac{\partial^{k+j} u}{\partial z^k \partial z^{*j}}, \quad U(z, z^*) := u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right),$$

with $A_{nn} \equiv 1$. We assume also that the operator L has a complex form

$$(1.2) \quad L[U] := \sum_{k,j=0}^{n-1} B_{kj}(z, z^*) \frac{\partial^{k+j} U}{\partial z^k \partial z^{*j}}.$$

The adjoint operator to L is given by $L^* = M$,

$$(1.3) \quad M[U] := M^*[U_t] - L^*[U],$$

with

$$(1.4) \quad M^*[U] := \sum_{k,j=0}^n (-1)^{k+j} \frac{\partial^{k+j}}{\partial z^k \partial z^{*j}} (A_{kj} U),$$

$$(1.5) \quad L^*[U] := \sum_{k,j=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j}}{\partial z^k \partial z^{*j}} (B_{kj} U).$$

A function S of the form

$$(1.6) \quad S(x, y, t; \xi, \eta, \tau) := A(x, y, t; \xi, \eta, \tau) \ln \frac{1}{r} + B(x, y, t; \xi, \eta, \tau)$$

where $r = [(x-\xi)^2 + (y-\eta)^2]^{\frac{1}{2}}$ will be called a fundamental solution of (1.1) if it satisfies the following conditions

(c-1) As a function of (x, y, t) , S is a solution of the adjoint equation $\mathcal{M}[S] = 0$ and is an analytic function of its argument except at $r = 0$, where $\Delta^{n-1}S$ has a logarithmic singularity.

(c-2) At the parameter point $x = \xi$, $y = \eta$ we have

$$\frac{\partial^{p+q} A_t}{\partial x^p \partial y^q} = 0 \quad \text{for } p+q \leq 2n-3, \quad \text{and} \quad \Delta^{n-1} A_t = 4^{n-1}.$$

(c-3) A and B are analytic functions of (x, y, t) at $r = 0$ and vanish at $t = \tau$.

Remark: The above implies that A may be written as

$r^{2n-2} \hat{A}(x, y, t; \xi, \eta, \tau)$ with $\hat{A}$ regular at the parameter point. The above notation is computationally easy to work with.

We intend to show that it is possible to develop the coefficients A and B as analytic functions with the expansions

$$(1.7) \quad A(z, z^*, t; \zeta, \zeta^*, \tau) = \sum_{j=1}^{\infty} A_j(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!},$$

$$(1.8) \quad B(z, z^*, t; \zeta, \zeta^*, \tau) = \sum_{j=1}^{\infty} B_j(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!}.$$

Furthermore, we shall identify $A_1(z, z^*; \zeta, \zeta^*)$ as the Riemann function corresponding to the operator $\mathcal{M}$. The other coefficients will be seen to satisfy homogeneous Goursat conditions on $z = \zeta$ and $z^* = \zeta^*$.

Inserting (1.6) into $\mathcal{M}[U] := \mathcal{M}^*[U_t] - \mathcal{L}^*[U] = 0$ we obtain, after some manipulation of terms

$$(1.9) \quad \mathcal{M}[S] \equiv \mathcal{M}[A] \ln \frac{1}{r} + \mathcal{M}[B] + I_n + I_n^* = 0$$

with

$$\begin{aligned}
 I_n := & \left\{ \frac{1}{2} (n-1)! \sum_{j=0}^n (-1)^j \frac{\partial^j}{\partial z^{*j}} (A_{nj} A_t) \right\} (z-\zeta)^{-n} \\
 & + \frac{1}{2} \sum_{p=1}^{n-1} \left\{ \binom{n}{n-p} \sum_{j=0}^n (-1)^{n+j} \frac{\partial^{n+j-p}}{\partial z^{n-p} \partial z^{*j}} (A_{nj} A_t) \right. \\
 (1.10) \quad & + \sum_{k=p}^{n-1} \binom{k}{k-p} \left[(-1)^{k+n} \frac{\partial^{k+n-p}}{\partial z^{k-p} \partial z^{*n}} (A_{kn} A_t) \right. \\
 & \left. \left. + \sum_{j=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_t - B_{kj} A) \right] \right\} \frac{(-1)^p (p-1)!}{(z-\zeta)^p}
 \end{aligned}$$

and

$$\begin{aligned}
 I_n^* := & \left\{ \frac{1}{2} (n-1)! \sum_{k=0}^n (-1)^k \frac{\partial^k}{\partial z^k} (A_{kn} A_t) \right\} (z^*-\zeta^*)^{-n} \\
 & + \frac{1}{2} \sum_{q=1}^{n-1} \left\{ \binom{n}{n-q} \sum_{k=0}^n (-1)^{k+n} \frac{\partial^{k+n-q}}{\partial z^k \partial z^{*n-q}} (A_{kn} A_t) \right. \\
 (1.11) \quad & + \sum_{j=q}^{n-1} \binom{j}{j-q} \left[(-1)^{n+j} \frac{\partial^{n+j-q}}{\partial z^n \partial z^{*j-q}} (A_{nj} A_t) \right. \\
 & \left. \left. + \sum_{k=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (A_{kj} A_t - B_{kj} A) \right] \right\} \frac{(-1)^q (q-1)!}{(z^*-\zeta^*)^q} .
 \end{aligned}$$

Because of the multivaluedness of the logarithmic singularity it is necessary to set $\mathcal{N}[A] = 0$. To cancel the poles at $z = \zeta$ and $z^* = \zeta^*$ we ask that the coefficients of $(z-\zeta)^p$, $(z^*-\zeta^*)^p$, $(p=1, \dots, n)$ vanish. These latter conditions provide us with so-called Goursat data for the A_j , B_j coefficients in the representations (1.9). Setting (1.7) into (1.10) and (1.11) we obtain the following conditions which $A_1(z, z^*; \zeta, \zeta^*)$ must satisfy

$$\begin{aligned}
 (1.12) \quad & \sum_{k=p}^n \binom{k}{p} \sum_{j=0}^n (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} \left[A_{kj}(\zeta, z^*) A_1(\zeta, z^*; \zeta, \zeta^*) \right] = 0 \\
 & (p=1, 2, \dots, n), \quad z = \zeta,
 \end{aligned}$$

and

$$(1.13) \quad \sum_{j=q}^n \binom{j}{q} \sum_{k=0}^n (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} \left[A_{kj}(z, \zeta^*) A_1(z, \zeta^*; \zeta, \zeta^*) \right] = 0$$

$$(q=1, 2, \dots, n), \quad z^* = \zeta^* .$$

We recall that condition (c-2) implies that

$$(1.14) \quad \frac{\partial^{p+q}}{\partial z^p \partial z^{*q}} A_j(\zeta, \zeta^*; \zeta, \zeta^*) = 0, \quad \text{for } j=1, 2, 3, \dots,$$

$$(p, q=0, 1, 2, \dots, 2n-3, \text{ with } p+q \leq 2n-3) .$$

In order to show that A_1 is actually the Riemann function, it is sufficient for us to show that the above conditions are equivalent to the characteristic conditions which uniquely determine it. We recall from VEKUA [22], Chapter V, the following conditions imposed on the Riemann function $R(z, z^*; \zeta, \zeta^*)$ and for convenience we label these conditions using his equation numbers :

$$(37.28) \quad M^*[R] = 0$$

$$(37.29) \quad \left. \frac{\partial^k R}{\partial z^k} (z, z^*; \zeta, \zeta^*) \right|_{z=\zeta} = 0, \quad \left. \frac{\partial^k R}{\partial \zeta^k} (z, z^*; \zeta, \zeta^*) \right|_{z^*=\zeta^*} = 0,$$

$$(k=0, 1, 2, \dots, n-2)$$

$$(37.30) \quad \left. \frac{\partial^{n-1} R}{\partial z^{n-1}} (z, z^*; \zeta, \zeta^*) \right|_{z=\zeta} = X(z^*, \zeta^*, \zeta),$$

$$\left. \frac{\partial^{n-1} R}{\partial z^{*n-1}} (z, z^*; \zeta, \zeta^*) \right|_{z^*=\zeta^*} = X^*(z, \zeta, \zeta^*) .$$

Here X , and X^* are solutions respectively of the ordinary differential equations

$$(37.31) \quad \sum_{k=0}^n (-1)^k \frac{\partial^k}{\partial z^{*k}} \left[A_{nk}(\zeta, z^*) X \right] = 0, \quad \text{and}$$

$$\sum_{k=0}^n (-1)^k \frac{\partial^k}{\partial z^k} \left[A_{kn}(z, \zeta^*) X^* \right] = 0 .$$

Furthermore, X and X^* are seen to satisfy the initial conditions

$$(37.32) \quad \left. \frac{\partial^k X}{\partial z^{*k}} (z^*, \zeta^*, \zeta) \right|_{z^*=\zeta^*} = 0, \quad \left. \frac{\partial^{n-1} X}{\partial z^{*n-1}} (z^*, \zeta^*, \zeta) \right|_{z^*=\zeta^*} = 1,$$

$$(k=0, 1, \dots, n-2)$$

$$\left. \frac{\partial^k X^*}{\partial z^k} (z, \zeta, \zeta^*) \right|_{z=\zeta} = 0, \quad \left. \frac{\partial^{n-1} X^*}{\partial z^{n-1}} (z, \zeta, \zeta^*) \right|_{z=\zeta} = 1.$$

$$(k=0, 1, \dots, n-2)$$

We note that $A_1(z, z^*; \zeta, \zeta^*)$ automatically satisfies (37.28), (37.29), (37.32) by virtue of our conditions (c-1) and (c-2). This suggests that we check to see if the remaining conditions (37.30), (37.31) are compatible with ours. Using (37.29) in combination with condition (1.12) we obtain for $p > 1$

$$(1.15) \quad \sum_{j=0}^n (-1)^{n+j} \frac{\partial^{n+j-p}}{\partial z^{n-p} \partial z^{*j}} \left[A_{nj}(z, z^*) A_1(z, z^*; \zeta, \zeta^*) \right] \Big|_{z=\zeta} = 0.$$

In the case where $p = 1$, this becomes

$$(1.16) \quad \sum_{j=0}^n (-1)^{n+j} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \frac{\partial^j}{\partial z^{*j}} \left[\frac{\partial^{n-1-\ell}}{\partial z^{n-1-\ell}} A_{nj}(z, z^*) \frac{\partial^\ell A_1}{\partial z^\ell}(z, z^*; \zeta, \zeta^*) \right] \Big|_{z=\zeta}$$

$$= \sum_{j=0}^n (-1)^{n+j} \frac{\partial^j}{\partial z^{*j}} \left[A_{nj}(\zeta, z^*) \frac{\partial^{n-1} A_1}{\partial z^{n-1}}(z, z^*; \zeta, \zeta^*) \right] \Big|_{z=\zeta} = 0.$$

Identifying temporarily A_1 with R and identifying the $(n-1)^{st}$ derivative with respect to z as X as given in (37.30), (1.16) becomes the first of equations (37.31). Repeating this analysis with (1.13) and identifying X^* as the $(n-1)^{st}$ derivative with respect to z^* , (37.30) yields likewise the second equation of (37.31). This exhausts our conditions and leaves us free to impose the additional initial conditions on X and X^* prescribed by (37.32). We conclude that we are permitted to identify $A_1 \equiv R$.

Lemma: The first coefficient $A_1(z, z^*; \zeta, \zeta^*)$ for A in the representation (1.7) may be taken as the Riemann function associated

with $M[u] = 0$. The coefficients $A_p(z, z^*; \zeta, \zeta^*)$, $p \geq 2$, are determined recursively by the nonhomogeneous equations

$$(1.17) \quad M^*[A_p] = L^*[A_{p-1}] ,$$

and the homogeneous Goursat data ,

$$(1.18) \quad \begin{aligned} \frac{\partial^\ell}{\partial z^\ell} A_p(z, z^*; \zeta, \zeta^*) \Big|_{z=\zeta} &= 0 , \quad \ell=0,1,\dots,n-1 , \\ \frac{\partial^\ell}{\partial z^{*\ell}} A_p(z, z^*; \zeta, \zeta^*) \Big|_{z^*=\zeta^*} &= 0 , \quad \ell=0,1,\dots,n-1 . \end{aligned}$$

Proof: Since $S = A \log \frac{1}{r} + B$ satisfies the adjoint equation, it follows that $M[A] = 0$. This in turn implies $M^*[A_1] = 0$, and $M^*[A_{j+1}] = L^*[A_j]$, ($j=0,1,\dots$). A moment's reflection concerning our condition (c-1) indicates that it can be satisfied using the above conditions (1.18), when A_1 is taken to be the Riemann function for $M[U] = 0$.

II. DETERMINATION OF THE COEFFICIENTS $A_p(z, z^*; \zeta, \zeta^*)$

The series representations (1.9) for the coefficients A and B of the singular solution

$S(z, z^*; t; \zeta, \zeta^*, \tau) := A(z, z^*; t; \zeta, \zeta^*, \tau) \ln \frac{1}{r} + B(z, z^*; t; \zeta, \zeta^*, \tau)$ suggest that we try to determine A_p and B_p successively. To this end we develop a Green's formula based on the formal identity

$$(2.1) \quad v M[U] - U M^*[v] \equiv \frac{\partial P}{\partial z}(z, z^*) + \frac{\partial P^*}{\partial z^*}(z, z^*) ,$$

where M and M^* are given by (1.4) and (1.7 and

$$(2.2) \quad P(z, z^*) := \sum_{j=0}^n \sum_{k=1}^{k-1} (-1)^p \frac{\partial^p}{\partial z^p} (v A_{kj}) \frac{\partial^{k+j-p-1} U}{\partial z^{k-p-1} \partial z^{*j}} ,$$

and

$$P^*(z, z^*) := \sum_{k=0}^n \sum_{q=0}^{j-1} (-1)^{k+q} \frac{\partial^{q+k}}{\partial z^{*q} \partial z^k} (v A_{kj}) \frac{\partial^{j-q-1} U}{\partial z^{*j-q-1}} .$$

If $\mathcal{M}_s$ is meant as the $\mathcal{M}$ operator with s, s^* replacing z, z^* , then, setting $U(s, s^*) := A_1(z, z^*; s, s^*)$ and noting that A_1 is also a solution of $\mathcal{M}_s[U] = 0$, the identity (2.1) yields

$$(2.4) \quad -U(s, s^*) \mathcal{M}_s^*[V] \equiv \frac{\partial P}{\partial s}(s, s^*) + \frac{\partial P^*}{\partial s^*}(s, s^*) .$$

Lemma 2: The coefficients $A_{p+1}(z, z^*; \zeta, \zeta^*)$ may be formally determined by the recursive scheme

$$(2.5) \quad A_{p+1}(z, z^*; \zeta, \zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds A_1(z, z^*; s, s^*) \mathcal{L}^*[A_p(s, s^*; \zeta, \zeta^*)] ,$$

$p=1, 2, \dots$

Proof: Integrating (2.4) yields

$$- \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds U(s, s^*) \mathcal{M}_s^*[V] = \int_{\zeta^*}^z P(z, s^*) ds^* + \int_{\zeta}^z P^*(s, z^*) ds - Q(z, z^*; \zeta, \zeta^*) ,$$

with

$$Q(z, z^*; \zeta, \zeta^*) = \int_{\zeta^*}^{z^*} P(\zeta, s^*) ds^* - \int_{\zeta}^z P^*(s, \zeta^*) ds .$$

Using the conditions (37.29) and recalling $U := A_1$, we can simplify the integral of P above as

$$(2.6) \quad \int_{\zeta^*}^{z^*} P(z, s^*) ds^* = \sum_{j=0}^n \int_{\zeta^*}^{z^*} V(z, s^*) A_{nj}(z, s^*) \frac{\partial^{n-1+j} U(z, s^*)}{\partial z^{n-1} \partial s^{*j}} ds^* .$$

According to VEKUA [22] we may identify the function

$$(37.10) \quad g(s^*; z, z^*) := \frac{\partial^{n-1}}{\partial z^{n-1}} A_1(z, z^*; z, s^*) ,$$

which, furthermore, satisfies an ordinary differential equation

$$(37.9) \quad \frac{d^n g}{d\zeta^n} + \sum_{m=0}^{n-1} A_{nm}(z, \zeta) \frac{d^m g}{d\zeta^m} = 0 .$$

Using (37.9), the right-hand side of (2.6) is seen to vanish identically.

We next turn our attention to the P^* -integral, and note that (37.29) implies that $\frac{\partial^{j-q-1} U}{\partial z^{*j-q-1}}(s, z^*) = 0$, except for $q = 0, j = n$.

Hence

$$(2.7) \quad \int_{\zeta}^z P^*(s, z^*) ds = \sum_{k=0}^n (-1)^k \int_{\zeta}^z \left\{ \frac{\partial^k}{\partial s^k} \left[V(s, z^*) A_{kn}(s, z^*) \right] \cdot \frac{\partial^{n-1} U(s, z^*)}{\partial z^{*n-1}} \right\} ds.$$

The associated functions

$$(37.10) \quad g^*(s; z, z^*) := \frac{\partial^{n-1}}{\partial s^{*n-1}} A_1(z, z^*; s, s^*) \Big|_{s^*=z^*}$$

are known [24] to satisfy the ordinary differential equation

$$(37.9) \quad \frac{d^n g^*}{ds^n} + \sum_{m=0}^{n-1} A_{mn}(s, z^*) \frac{d^m g^*}{ds^m} = 0,$$

and the initial conditions

$$(37.11) \quad \begin{aligned} \frac{\partial^{\ell} g^*}{\partial s^{\ell}}(s; z, z^*) \Big|_{s=z} &= 0, \quad (\ell=0, 1, \dots, n-2) \\ \frac{\partial^{n-1} g^*}{\partial s^{n-1}}(s; z, z^*) \Big|_{s=z} &= 1. \end{aligned}$$

Consequently, after some regrouping, it may be seen that

$$(2.8) \quad \begin{aligned} \int_{\zeta}^z P^*(s, z^*) ds &= \sum_{k=1}^n \int_{\zeta}^z \frac{\partial}{\partial s} \left\{ \sum_{\ell=0}^{k-1} (-1)^{\ell+k} \frac{\partial^{k-1-\ell}}{\partial s^{k-1-\ell}} \left[V(s, z^*) A_{kn}(s, z^*) \right] \cdot \frac{\partial^{\ell} g^*(s; z, z^*)}{\partial s^{\ell}} \right\} ds \\ &= (-1)^{2n-1} V(z, z^*) A_{nn}(z, z^*) \frac{\partial^{n-1}}{\partial z^{n-1}} g^*(z; z, z^*) + Q^*(\zeta; z, z^*) \\ &= -V(z, z^*) + Q^*(\zeta; z, z^*), \end{aligned}$$

where

$$(2.9) \quad \begin{aligned} Q^*(\zeta; z, z^*) &:= - \sum_{k=1}^n \sum_{\ell=0}^{k-1} (-1)^{\ell+k} \frac{\partial^{k-1-\ell}}{\partial s^{k-1-\ell}} \left[V(\zeta, z^*) A_{kn}(\zeta, z^*) \right] \\ &\quad \cdot \frac{\partial^{\ell} g^*(\zeta; z, z^*)}{\partial \zeta^{\ell}}. \end{aligned}$$

Putting together the above terms we note that the following representation for $V(z, z^*)$ has been obtained

$$(2.10) \quad V(z, z^*) = Q^*(\zeta; z, z^*) - Q(z, z^*; \zeta, \zeta^*) + \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds A_1(z, z^*; s, s^*) M^*[V(s, s^*)] .$$

Setting $V(z, z^*) := A_{p+1}(z, z^*; \zeta, \zeta^*)$ and recalling the initial conditions (1.18), we conclude $Q^*(\zeta; z, z^*) \equiv 0$. The expression for $Q(z, z^*; \zeta, \zeta^*)$ may also be seen to vanish identically by virtue of the identities

$$\int_{\zeta^*}^{z^*} P(\zeta, s^*) ds^* = \sum_{j=0}^n \sum_{k=1}^{k-1} (-1)^j \int_{\zeta^*}^{z^*} ds^* \frac{\partial^j}{\partial s^{*j}} \left[A_{p+1}(\zeta, s^*; \zeta, \zeta^*) A_{kj}(\zeta, s^*) \right] \cdot \frac{\partial^{k+j-l-1}}{\partial z^{k-l-1} \partial s^{*j}} U(\zeta, s^*) \equiv 0 ,$$

and

$$\int_{\zeta}^z P^*(s, \zeta^*) ds = \sum_{k=0}^n \sum_{j=1}^{j-1} (-1)^{k+q} \int_{\zeta}^z \frac{\partial^{q+k}}{\partial z^{*q} \partial s^k} \left(A_{p+1}(s, \zeta^*; \zeta, \zeta^*) \cdot A_{kj}(s, \zeta^*) \right) \frac{\partial^{j-q-1}}{\partial z^{*j-q-1}} U(s, \zeta^*) ds \equiv 0 .$$

Our Lemma is proved at this point by recognizing that

$$M^*[A_{p+1}] = L^*(A_p) .$$

Lemma 3: The coefficients $A_{p+1}(z, z^*; \zeta, \zeta^*)$ may be computed recursively by

$$(2.11) \quad A_{p+1}(z, z^*; \zeta, \zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds A_p(z, z^*; s, s^*) L^*[A_1(s, s^*; \zeta, \zeta^*)] ,$$

$p=1, 2, \dots$. Furthermore, the series representation for

$A(z, z^*; t; \zeta, \zeta^*, \tau)$ converges uniformly in the domain $(G \times G^* \times T)^2$ where G is the domain of regularity of the coefficients in the x - y coordinates and T is a disk in the complex t -plane.

Proof: Recalling the form of L and L^* , we note that we may write

$$\begin{aligned}
& A_1(z, z^*; s, s^*) \mathbb{L}^*[A_p(s, s^*; \zeta, \zeta^*)] - A_p(s, s^*; \zeta, \zeta^*) \mathbb{L}[A_1(z, z^*; s, s^*)] \\
&= - \sum_{j=0}^{n-1} \frac{\partial}{\partial s} \left\{ \sum_{k=1}^{k-1} (-1)^k \frac{\partial^k}{\partial s^k} \left[A_p(s, s^*; \zeta, \zeta^*) B_{kj}(s, s^*) \right] \frac{\partial^{k+j-\ell-1}}{\partial s^{k-\ell-1} \partial s^{*j}} A_1(z, z^*; s, s^*) \right\} \\
&- \sum_{k=0}^{n-1} (-1)^k \frac{\partial}{\partial s^*} \left\{ \sum_{m=0}^{j-1} (-1)^m \frac{\partial^{m+k}}{\partial s^{*m} \partial s^k} \left[A_p(s, s^*; \zeta, \zeta^*) B_{kj}(s, s^*) \right] \frac{\partial^{j-m-1}}{\partial s^{*j-m-1}} A_1(z, z^*; s, s^*) \right\}
\end{aligned}$$

Integration gives the following identity

$$\begin{aligned}
& \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds A_1(z, z^*; s, s^*) \mathbb{L}^*[A_p(s, s^*; \zeta, \zeta^*)] \\
&= \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds A_p(s, s^*; \zeta, \zeta^*) \mathbb{L}[A_1(z, z^*; s, s^*)] \\
&\quad + H + H^*,
\end{aligned}$$

where

$$\begin{aligned}
(2.12) \quad H &:= - \sum_{j=0}^{n-1} \sum_{k=1}^{k-1} (-1)^k \int_{\zeta^*}^{z^*} ds^* \left\{ \frac{\partial^k}{\partial s^k} \left[A_p(s, s^*; \zeta, \zeta^*) B_{kj}(s, s^*) \right] \frac{\partial^{k+j-\ell-1}}{\partial s^{k-\ell-1} \partial s^{*j}} \right. \\
&\quad \left. \cdot A(z, z^*; s, s^*) \right\} \Bigg|_{s=\zeta}^{s=z}
\end{aligned}$$

and

$$\begin{aligned}
(2.13) \quad H^* &:= \sum_{k=0}^{n-1} (-1)^k \sum_{m=0}^{j-1} (-1)^m \int_{\zeta}^z ds \left\{ \frac{\partial^{m+k}}{\partial s^{*m} \partial s^k} \left[A_p(s, s^*; \zeta, \zeta^*) B_{kj}(s, s^*) \right] \right. \\
&\quad \left. \cdot \frac{\partial^{j-m-1}}{\partial s^{*j-m-1}} A_1(z, z^*; s, s^*) \right\} \Bigg|_{s^*=\zeta^*}^{s^*=z^*}.
\end{aligned}$$

Using the conditions (1.18) for A_{p+1} and (37.29) for A_1 , the terms H and H^* are seen to identically vanish. This establishes the recursive definition (2.11). To show that the series for A converges we note that according to VEKUA [22], p. 186 $A_1(z, z^*; \zeta, \zeta^*)$ is dominated by

$$\frac{|z-\zeta|^{n-1} |z^*-\zeta^*|^{n-1}}{[(n-1)!]^2} \quad \text{in} \quad [G \times G^*]^2.$$

Consequently, using (2.11) it follows by induction that

$$(2.14) \quad |A_p(z, z^*; \zeta, \zeta^*)| \leq C \frac{|z-\zeta|^{n+p-2} |z^*-\zeta^*|^{n+p-2}}{[(n+p-2)!]^2} \|L[A_1]\|^{p-1}$$

where

$$\|L[A_1]\| := \sup_{(G \times G^*)^2} |L[A_1]|$$

and

$$C := \sup_{(G \times G^*)^2} |A_1|.$$

From (2.14) it is clear that the series for A converges in the stated domain.

III. DETERMINATION OF $B(z, z^*, t; \zeta, \zeta^*, \tau)$

From (1.10) we obtain as the partial differential equation for B ,

$$(3.1) \quad \mathcal{M}[B] = -I_n - I_n^*,$$

where I_n and I_n^* are given by (1.11) and (1.12) respectively.

Putting the series expansion (1.8) for B into the left-hand side of (3.1) yields then a recursive scheme for the coefficients B_j , namely

$$(3.2) \quad M^*[B_1] = f_0(z, z^*; \zeta, \zeta^*) := -\frac{1}{2} \sum_{p=1}^n \frac{(-1)^p (p-1)!}{(z-\zeta)^p} \left\{ \sum_{k=p}^n \binom{k}{p} \cdot \sum_{j=0}^n (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_1) \right\} \\ - \frac{1}{2} \sum_{q=1}^n \frac{(-1)^q (q-1)!}{(z^*-\zeta^*)^q} \left\{ \sum_{j=q}^n \binom{j}{q} \sum_{k=0}^n (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (A_{kj} A_1) \right\},$$

$$(3.3) \quad M^*[B_{\ell+1}] = f_{\ell}(z, z^*; \zeta, \zeta^*) := L^*[B_{\ell}] - \frac{1}{2} \frac{(n-1)!}{(z-\zeta)^n} \sum_{j=0}^n (-1)^j \frac{\partial^j}{\partial z^{*j}} (A_{nj} A_{\ell+1}) \\ - \frac{1}{2} \sum_{p=1}^{n-1} \frac{(-1)^p (p-1)!}{(z-\zeta)^p} \left\{ \sum_{k=p}^n \binom{k}{p} \sum_{j=0}^n (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_{\ell+1}) \right. \\ \left. - \sum_{k=p}^{n-1} \binom{k}{p} \sum_{j=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (B_{kj} A_{\ell}) \right\} +$$

$$\begin{aligned}
& - \frac{1}{2} \frac{(n-1)!}{(z^* - \zeta^*)^n} \sum_{k=0}^n (-1)^k \frac{\partial^k}{\partial z^k} (A_{kn} A_{\ell+1}) \\
& - \frac{1}{2} \sum_{q=1}^{n-1} \frac{(-1)^q (q-1)!}{(z^* - \zeta^*)^q} \left\{ \sum_{j=q}^n \binom{j}{q} \sum_{k=0}^n (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (A_{kj} A_{\ell+1}) \right. \\
& \quad \left. - \sum_{j=q}^{n-1} \binom{j}{q} \sum_{k=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (B_{kj} A_{\ell}) \right\}, \\
& \ell=1, 2, \dots
\end{aligned}$$

If we specify that the B_{ℓ} satisfy the homogeneous Goursat data

$$(3.4) \quad \begin{aligned}
& \frac{\partial^k B_{\ell}(z, z^*; \zeta, \zeta^*)}{\partial z^{*k}} \Big|_{z^* = \zeta^*} = 0, \\
& \frac{\partial^k B_{\ell}(z, z^*; \zeta, \zeta^*)}{\partial z^k} \Big|_{z = \zeta} = 0, \quad \begin{array}{l} k=0, 1, 2, \dots, n-1 \\ \ell=1, 2, \dots \end{array}
\end{aligned}$$

then

$$B_{\ell+1}(z, z^*; \zeta, \zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^z ds f_{\ell}(s, s^*; \zeta, \zeta^*) A_1(z, z^*; s, s^*), \quad \ell=0, 1, \dots$$

The majoration for the $B_{\ell+1}$, while technically involved, proceeds by the usual methods [10], [11], [8].

We summarize the discussion of Sections II and III in the following theorem.

Theorem 1: Assume that the coefficients A_{jk} , B_{jk} are analytic functions of two complex variables z, z^* in the bicylinder $D \times D^*$. Then $A(z, z^*; t; \zeta, \zeta^*, \tau)$ and $B(z, z^*; t; \zeta, \zeta^*, \tau)$ are analytic functions of their six independent variables for all (complex) t, τ , and $z, \zeta \in D$, $z^*, \zeta^* \in D^*$. Moreover, both A and B can be represented by a uniformly convergent series expansion in the form of (1.7) and (1.8) respectively.

IV. BOUNDARY VALUE PROBLEMS

We now proceed to use the fundamental solution (1.8) to develop representations for boundary value problems. The solutions to boundary value problems we will seek in the class

$$(4.1) \quad \mathcal{Y} := \{u(x, y, t) : u \in C^{2n-2}(\bar{D} \times T); u_t \in C^{2n}(D \times T) \cap C^{2n-1}(\bar{D} \times T)\},$$

where $T := \{t : 0 \leq t < t_0\}$ and where t_0 is a fixed constant.

We begin by considering the identity

$$(4.2) \quad v_t \mathcal{L}[u] - u_t \mathcal{L}[v] \equiv \{v_t M[u_t] - u_t M^*[v_t]\} \\ - \{v L[u_t] - u_t L^*[v]\} + \frac{\partial}{\partial t} \{v L[u]\}.$$

If we replace v by the fundamental solution

$$S(x, y, t; \xi, \eta, \tau) = A \log \frac{1}{r} + B,$$

then (4.2) leads to an integral representation for solutions which satisfy the homogeneous initial data $u(x, y, 0) = 0$ in D , namely

$$(4.3) \quad u(\xi, \eta, \tau) = \frac{1}{2\pi[(n-1)!]^2} \int_0^\tau dt \int_{\partial D} H[u(x, y, t), S(x, y, t; \xi, \eta, \tau)]$$

Here H is a bilinear form in $\frac{\partial^{k+j} u_t}{\partial x^k \partial y^j}, \frac{\partial^{k+j} S_t}{\partial x^k \partial y^j}, (k+j \leq 2n-1)$ as

well as $\frac{\partial^{k+j} u}{\partial x^k \partial y^j}, \frac{\partial^{k+j} S}{\partial x^k \partial y^j}, (k+j \leq 2n-3)$. The introduction of a

special, Green-type, fundamental solution, namely one which satisfies on ∂D

$$(4.4) \quad S_t = \frac{\partial S_t}{\partial v} = \dots = \frac{\partial^{n-1} S_t}{\partial v^{n-1}} = 0 \quad (v = \text{inward normal})$$

permits us to reduce the bilinear form $H[,]$ to the case where only the boundary data of the first kind

$$(4.5) \quad u_t^+ = f_0, \quad \frac{\partial u_t^+}{\partial v} = f_1, \quad \dots, \quad \frac{\partial^{n-1} u_t^+}{\partial v^{n-1}} = f_n \quad \text{on } \partial D$$

appears.

This representation is more easily computed using the complex notation. To this end we recall the elementary identities,

$$(4.6) \quad \frac{d}{ds} \left(\frac{\partial^{k+m} u}{\partial z^k \partial z^{*m}} \right)^+ \equiv \left(\frac{\partial^{k+m+1} u}{\partial z^{k+1} \partial z^{*m}} \right)^+ \frac{dz}{ds} + \left(\frac{\partial^{k+m+1} u}{\partial z^k \partial z^{*m+1}} \right)^+ \frac{dz^*}{ds},$$

$$\left(\frac{d^k u}{dv^k} \right)^+ = i^k \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{\partial^k u}{\partial z^{k-\ell} \partial z^{*\ell}} \right)^+ \left(\frac{dz}{ds} \right)^{k-2\ell},$$

which hold on ∂D . The first two brackets of (4.2) we compute directly as

$$(4.7) \quad \{V_t M[U_t] - U_t M^*[V_t]\} - \{V L[U_t] - U_t L^*[V]\}$$

$$= \frac{\partial}{\partial z} \tilde{P} + \frac{\partial}{\partial z^*} \tilde{P}^*,$$

where

$$(4.8) \quad \tilde{P} := \sum_{k=1}^n \sum_{\ell=0}^{k-1} (-1)^\ell \frac{\partial^\ell}{\partial z^\ell} (V_t A_{kj}) \frac{\partial^{k+j-\ell-1} U_t}{\partial z^{k-\ell-1} \partial z^{*j}}$$

$$- \sum_{k=1}^{n-1} \sum_{\ell=0}^{k-1} (-1)^\ell \frac{\partial^\ell}{\partial z^\ell} (V B_{kj}) \frac{\partial^{k+j-\ell-1} U_t}{\partial z^{k-\ell-1} \partial z^{*j}},$$

$$(4.9) \quad \tilde{P}^* := \sum_{k=0}^n (-1)^k \sum_{m=0}^{j-1} (-1)^m \frac{\partial^{m+k}}{\partial z^{*m} \partial z^k} (V_t A_{kj}) \frac{\partial^{j-m-1} U_t}{\partial z^m}$$

$$- \sum_{k=0}^{n-1} (-1)^k \sum_{m=0}^{j-1} (-1)^m \frac{\partial^{m+k}}{\partial z^{*m} \partial z^k} (V B_{kj}) \frac{\partial^{j-m-1} U_t}{\partial z^m}.$$

Then (4.3) takes on the form

$$(4.10) \quad U(z, z^*, \tau) = \int_0^\tau dt \int_{\partial D} N_O(U, S) ds,$$

where

$$(4.11) \quad N_O(u, v) := i \left(\tilde{P} \frac{dz^*}{ds} - \tilde{P}^* \frac{dz}{ds} \right).$$

This in turn may be expressed in terms of tangential and normal derivatives of the data using (4.6). When S is a Green's function, then (4.10) is directly evaluated in terms of (4.5).

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