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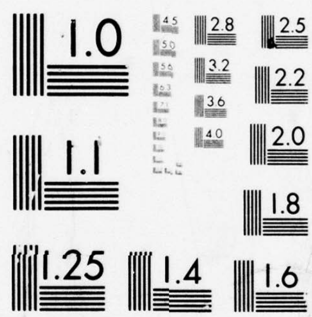

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AFFDL-TR-76-130

ESTIMATION OF ENGINE REMOVAL TIMES AND PREDICTION OF REPLACEMENT REQUIREMENTS

TEXAS A&M UNIVERSITY
DEPARTMENT OF INDUSTRIAL ENGINEERING
COLLEGE STATION, TEXAS 77843

DECEMBER 1976

TECHNICAL REPORT AFFDL-TR-76-130
FINAL REPORT FOR PERIOD 1 DECEMBER 1975 - 30 SEPTEMBER 1976

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

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This report has been reviewed by the Information Office (IO) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

MAX DUGGINS, LtCol, USAF
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The Air Force actuarial method uses engine operating time till removal for many reports, but it requires a large amount of data for accuracy, and there is less data on more engine types. Fortunately there are methods to use all the information now collected on engine ages. This report recommends several degrees of modification to the actuarial methods for:

1. estimation of engine lives,
2. updating the data set used to estimate lives, and
3. prediction of replacement requirements,

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20. ABSTRACT (continued)

and it outlines the additional research leading to integrated engine management based on full use of engine performance data and Air Force mission requirements. Computer programs have been written to accomplish the modifications.

The report contains a review of the actuarial method for estimation of engine lives, a suggestion to reduce the variance of the estimate by using variable age intervals, a description of alternative estimators that use all data on engine removal ages, a comparison of estimators, a sequential two sample test for obtaining representative data set of engine lives, a review of the actuarial method and a simulation program for predicting replacement requirements, a next event simulation program for predicting engine requirements, and suggestions for more comprehensive models of engine performance and replacement. The first appendix contains the derivation of the maximum likelihood estimator for engine lives from a multiple risk model with a progressively censored sample. The second appendix describes variance reduction by antithetic variates for a next event replacement simulation. The third appendix describes the next event type simulation of operation of a fleet of aircraft with a single type of engine.

PREFACE

This final report was prepared by the Texas A&M University, Department of Industrial Engineering, College Station, Texas, for the Analysis and Optimization Branch, Structures Division, Air Force Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson AFB, Ohio under contract F33615-76-C-3042. This research was conducted under Project 7071, "Research in Applied Mathematics" and Task 707102, "Mathematical Statistics". LtCol Max Duggins (AFFDL/FBR) was project monitor on this contract with technical assistance provided by Dr. H. Leon Harter (AFFDL/FBR).

The principal investigator on this research was Dr. Laurence L. George of the Texas A&M University. The work was performed primarily to support the Air Force Logistics Command, DCS Material Management, Directorate of Propulsion Systems (AFLC/MMP) at Wright-Patterson AFB, Ohio.

SUMMARY

The need that led to this research project was to obtain more information about engines from existing data without inordinate increase in computation. The primary focus was on the two fundamental parts of engine management, the estimation of engine removal times and the use of these estimates to predict replacement requirements. The secondary focus was on procedures for estimating spare engine requirements, improving simulation, and updating estimators.

The actuarial method has served long and well to provide fundamental information for engine management decisions, and technological development has progressed to the point where improvement in the accuracy of actuarial estimates can be achieved without a major increase in computation. The statisticians dream of more information from the same data can be achieved because all the information now available is not used. For the sake of computational convenience, information collected on engine removal times is summarized in counts of the numbers of engine exposures and removals in actuarial age intervals. Theoretical developments now allow the engine removal times themselves to be used for estimation of removal times and for prediction of replacement requirements without inordinate computation.

Many recommendations for improvement of the actuarial method and for improvement of engine management follow from the developments that will be described in this report. Some are recommendations for development of promising improvements and some are changes that can be made now permanently or perhaps temporarily until further improvement is possible. The nature and immediacy of each recommendation will be indicated.

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REVALUATION OF THE ACTUARIAL METHOD

1. Summary of Revaluation

There are several potential improvements that could be made in estimation of engine removal times and failure rates. Some can be done now, some require further study, testing and programming, and some involve eventual replacement of the actuarial failure rate estimator with an alternative estimator.

The first recommendation is not to smooth the crude failure rates and not to throw out data where data is sparse. The latter suggestion raises the question of scarce data and the confidence in very low failure rate estimates. This question can be resolved by recommendation two, an actuarial type estimator that uses "flexible" age intervals that contain either equal numbers of exposures or removals in each age interval. The actuarial estimator with flexible intervals was not used for comparison with the currently used crude actuarial estimators because still better estimators were available.

The actuarial estimator of failure rates summarizes the data on engine removal times, operating times, and ages of operating engines into counts of the numbers of exposures and removals in each actuarial age interval. This summary of data reduces the amount of information in the sample and consequently the accuracy of any estimator. An estimator that uses the actual engine removal times should be better than an estimator that uses only counts of the numbers of removals during age intervals. An estimator that also incorporates the ages of engines still operating should be better yet. And an estimator that incorporates the operating time during a period should give the sharpest picture of engine operation during that period. An estimator that uses all three types of information has not been derived yet, but that is the next step for research.

The "product limit" estimator of Kaplan and Meier [4] uses the removal times and the ages of engines still in operation to give a distribution free estimate of the cumulative distribution function of engine removal times. It can be adapted to also give a cumulative failure rate function estimator which in turn can be used to obtain interval failure rates comparable to actuarial failure rates for any desired interval widths.

For one additional assumption on the underlying nature of engine removals, a maximum likelihood estimator of the cumulative distribution function is derived in Appendix A that is simpler than the product limit estimator and can also be used in engine diagnosis. The assumption is that inspection removals and usage removals are statistically independent, a testable hypothesis.

All three estimators, here called the actuarial estimator, the product limit estimator, and the maximum likelihood estimator will be compared on the basis of how closely they estimate the theoretical distribution of simulated engine removal time data and on the basis of how well they predict replacement requirements. On theoretical grounds, the actuarial estimator is the least accurate because all three estimators are in fact maximum likelihood, Barlow and Proschan [5], but for different sample information and for different engine removal time models. The actuarial estimator uses summary counts of exposures and removals. The actuarial and product limit estimators make no assumption about the engine removal time mechanism. The product limit and maximum likelihood estimators use the removal time data and the ages of surviving engines, but the maximum likelihood estimator is derived from a removal time model that assumes statistical independence between inspection and usage removal times. Whenever a random process is known to have more statistical structure, more information can be inferred from data about the process, so it is believed that the maximum likelihood estimator is the most efficient estimator. Analytical proof of this has not been completed.

It is only fair to point out that the existing actuarial method has provision for counting only flying experience during a limited period of time. The product limit and maximum likelihood estimators use removal time data and ages on engines which may have begun operation at any time in the past. Comparison of all three estimators was made on an equal basis, all engines new at the start of the data gathering process although all may not have failed by the end of the data collection period. In this comparison, the product limit and maximum likelihood estimators were noticeably better even for a very small sample. However, the actuarial estimator will probably yield a better answer to questions of comparing the experience of one calendar quarter to another. Research will be continued to obtain product limit and maximum likelihood estimators freely comparable to the actuarial estimator.

An updating program has been written that compares current data on engine removal times and ages of engines that continue to operate with past data of the same type. It provides a measure of how likely it is that the current and all past data subsets came from populations of engine removal data that are the same. These measures will provide the Aerospace Engine Life Committee with guidance in selection of data to be included in the Official Failure Rate. The program uses a Wilcoxon type rank test statistic first proposed by Gehan [6]. This program is proposed as an alternative to the currently used "statistical test" for determining whether the actuarial failure rate during an age interval has changed. The program is subject to the same qualification mentioned earlier, that the actuarial test compares only experience obtained during the current calendar quarter while the updating program compares a removal time data set that may include engines which had begun operation prior to the current quarter. It remains to be seen whether this distinction is important. Meanwhile research has been started on a modified Gehan type updating program that compares current quarter experience with past experience. Even so, the Wilcoxon and Gehan type rank test is a temporary alternative proposed for use only until an updating program based on the product limit or maximum likelihood estimators can be developed and until either the question of use of current quarter experience can be resolved or until the product limit and maximum likelihood estimators can be modified to incorporate engine operation experience from a fixed calendar interval.

2. Review of the Actuarial Estimator of Failure Rates

The actuarial estimator of the failure rate during an engine operating age interval is basically the number of engines with operating age at removal recorded during that interval divided by the number of engines that operated until the beginning of that age interval. Let Δt be the width of an age interval, d_i the number of engines removed during the interval with ages between $(i-1)\Delta t$ and $i\Delta t$, and n_i the number of engines that operated at least until age $(i-1)\Delta t$, $i=1,2...,k$, then the actuarial failure rate estimator is

$$\hat{R}_i = d_i/n_i \qquad i = 1,2,...,k. \quad (2.1)$$

These simple estimators must be modified to account for data on engines that have been operated but not yet failed at the time data was collected otherwise the actuarial failure rate estimators will be biased high. This additional information referred to here as survivors' ages is used to modify the denominator n_i by counting the total numbers of engines, survivors and removed, entering each age interval and counting a fractional exposure in the age interval if a survivor's age is part of the way through an age interval. E.g. if an engines' age at time of data collection was 750 hours and the actuarial interval was from 720 to 760, three quarters of an exposure would be credited to the age interval and number of exposures n_{18} in that interval.

A further modification is used if it is desired to estimate failure rates based only on flying experience during a particular calendar period. Fractional exposures are counted for engines whose age at the beginning of the period was part way through an age interval, and a full exposure is counted for engines removed during an interval even though it may have begun the calendar period with operating age in the same actuarial age interval within which it was removed. The resulting ratio of deaths to exposures is the "crude actuarial failure rate" estimator of T.O. 00-25-128 [7] and T.O. 00-25-217 [8].

The crude actuarial failure rates are then smoothed with a weighted moving average up to age intervals where there are insufficient exposures and/or removals. Thereafter, actuarial failure rates are extrapolated by linear regression to the maximum operating time.

The crude failure rates (2.1) have several desirable large sample statistical properties -

1. They are maximum likelihood estimators for the sample information $\{(d_i, n_i), i = 1,2,\ldots,k\}$.
2. They are asymptotically jointly normally distributed with mean zero and, surprisingly, with zero covariance as the sample size grows under the condition that the probability of an engine being removed at age zero is zero.

The first assertion follows from a related assertion in Barlow and Proschan [5] which has been known for many years, and the second assertion follows from theorem 1.2 in Barlow and Proschan [5] which is proved for an estimator that is $\hat{R}_i/\Delta t$.

The actuarial estimator was developed for use in the life insurance industry for predicting human failure rates. A typical human failure rate estimate is shown in figure 1.

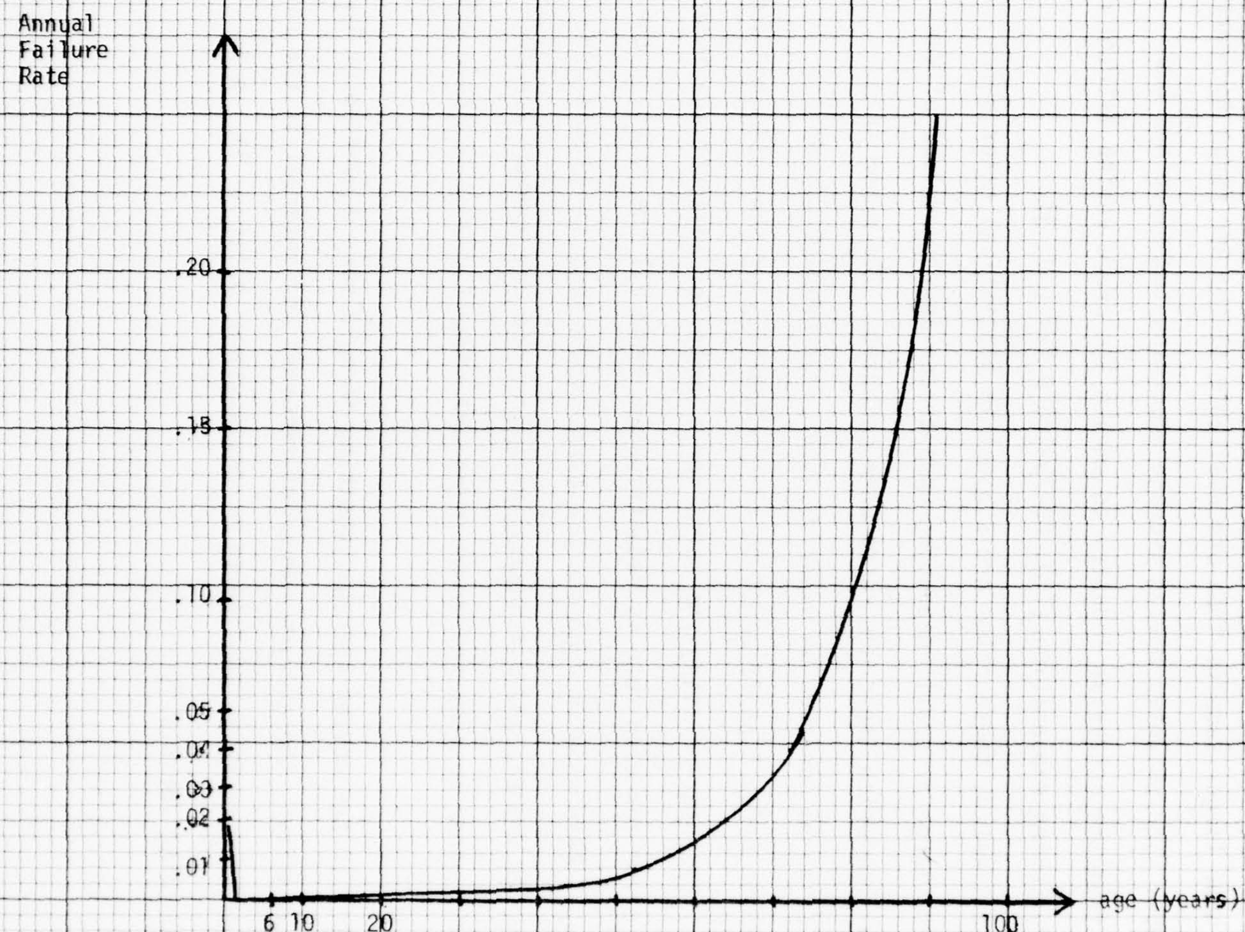

Figure 1. Annual Failure Rates for the Total Population of the United States in 1959-1961.

It is inherently a smooth function except in the first few years because there is nothing in the mechanism causing deaths that is age specific such as components that wear out at specific ages. Consequently, actuaries dependent on sampling to construct their failure rates feel justified in smoothing any irregularities that appear in their failure rate estimators. And they have larger samples than are available to the Air Force for each engine type. The weighted moving average smoothing technique has been borrowed from the actuarial industry to smooth the crude engine failure rates for the following explanation (T.O. 00-25-128 p 4-1 [7]).

> "As estimated, these rates are subject to statistical error which may be large or small, depending upon chance and the volume of the data from which the crude rates were computed. *The error is as likely to be positive as negative, i.e., the probability of the estimate being high is equal to the probability of the estimate being low.* By inter-relating the rates for a number of adjacent age intervals the error for each rate is redistributed among all the rates in the group. *When this is done the positive errors tend to balance with negative errors, and all unusually high errors are spread over all of the rates so that the crude rates are brought closer to their true rates."*

The weights in the moving average smoothing formula vary depending on the "failure density", the average number of removals per interval in the range where smoothing is to be applied (T.O. 00-25-217 p. 7-11[8]). A 7-point smoothing formula is

$$\bar{R}_i = (-.0587)\,(\hat{R}_{i-3} + \hat{R}_{i+3}) +$$
$$+\ .0587\,(\hat{R}_{i-2} + \hat{R}_{i+2}) +$$
$$+\ .2937\,(\hat{R}_{i-1} + \hat{R}_{i+1}) +$$
$$+\ .4126\ \hat{R}_i \qquad (2.2)$$

<u>It is recommended that smoothing be stopped immediately</u> for the following reasons:

1. The underlying mechanism of engine removals contains factors that create major changes in failure rates from one interval to the next, factors such as the high probability of removal at inspection and the possibility that certain engine components may fail within an age interval. Smoothing suppresses the information contained in inspection removal times and useful diagnostic information of high usage failure rates possibly attributable to particular engine components.
2. Smoothing is appropriate primarily to time series such as stock market prices or human ages which advance on or close to a one for one relation with calendar time. Engine ages do not advance in direct proportion to calendar time and so smoothing an interval failure rate over neighboring intervals is not appropriate.
3. The explanation (italics) that errors are as likely to be high as low is not known by the author to be true except asymptotically.
4. The explanation (italics) that positive errors tend to cancel negative errors is fallacious. Due to reason 1, it is <u>not</u> known whether a high crude failure rate is in error to the high side. The summarization of removal times into counts of removals

per age interval may mean that a high failure rate is in error on the low side of the true value!

5. The negative weighting factors in the smoothing formula (2.2) are appropriate only when there is a detected cyclic error in the failure rates, i.e., it is known that the crude failure rates 3 age intervals ago and 3 intervals ahead (for the 7-point smoothing formula) are in error in the opposite direction to the central age interval's failure rate. This is almost assuredly not true because the asymptotic covariance of failure rates is zero (Barlow and Proschan [5]) and because there is no cyclic mechanism in the engine failure process with such periodic behavior. Furthermore, rather sophisticated means are required to detect cyclic behavior and do not call for a changing period for the negative weight as occurs when changing from the 7-point to another smoothing formula.

Despite this condemnation of the smoothing procedure, the crude failure rates are reasonable and convenient estimators, and are easier to compute than smooth failure rates. Objection number 1 regarding the underlying mechanism of engine failure has led to the proposal of an engine removal time model, section 4, for which the maximum likelihood estimator of the usage removal time cumulative distribution function and inspection removal probabilities have been derived, Appendix A. That estimator is based on the actual removal times and survivor's ages. Its performance is compared to the crude actuarial failure rate estimator in section 5, and it is found to be superior. Nevertheless, the crude failure rate estimator may be improved by changing to flexible age intervals as described in the next section.

3. Variable Actuarial Age Intervals

Examination of the variance of the percent survivor function or cumulative distribution function, resulted in the conclusion that its variance could be minimized by having the same number of exposures in each age interval! The variance of any estimate is a measure of the precision of that estimate, and, if the estimate is unbiased, then the estimate with lowest variance is also the most accurate.

The conclusion about minimizing variance as a function of the number of exposures was arrived at indirectly by examining the variance of an *actuarial type* estimate of the percent survivor function or tail cumulative distribution function. First, some notation must be defined, and the conditions for the derivation must be outlined. The definitions and formulas are taken from Gross and Clark [3] Chapter 2.

$\{t_i\}$ is the sequence of age interval boundaries $i = 0, 1, \ldots, k+1$ where t_{k+1} may be ∞,

$\{d_i\}$ is the sequence of numbers of removals in the i^{th} interval (t_{i-1}, t_i),

$\{n_i\}$ is the sequence of numbers of engine exposures during the i^{th} interval, and

$\{\hat{P}(t_i)\}$ is the sequence of estimated percent surviving to time t_i estimated as the cumulative proportion in the sample data

$$\hat{P}(t_0) = 1.0,\ \hat{P}(t_1) = d_0/n_0 \cdot \hat{P}(t_0),\ \text{etc.}\ \hat{P}(t_i) = d_{i-1}/n_{i-1} \cdot \hat{P}(t_{i-1})$$

(The true unknown value of percent surviving is denoted $P(t_i)$).

It should be noted that the authors Gross and Clark assume that partial engine exposures are uniformly distributed over each interval so that the number of exposures is (approximately) equal to the number of engines that enter an age interval minus half the number of engines that have partial exposures during the interval (other than engines removed during the interval that are counted as a full exposures). Consequently, conclusions regarding the variance of estimators as a function of the number of exposures per interval are indicative rather than absolutely accurate conclusions. In fact, the variance formula to be used in obtaining the conclusion is itself an approximation.

The variance of $\hat{P}(t_i)$, the estimate of the percent surviving to t_i, is approximately (Gross and Clark p. 41 [3])

$$\text{Var } \hat{P}(t_i) \cong [P(t_i)]^2 \sum_{j=1}^{i-1} d_j/(n_j(n_j-d_j)) \qquad (1)$$

as a function of both the sequences $\{n_j\}$ and $\{d_j\}$. The numbers of removals d_j in each interval will be assumed equal to 1.0 for convenience in showing that the variance can be minimized by choosing intervals to contain equal numbers of exposures n_j. (It is recognized that these are possibly incompatible events unlikely to occur except by accident, but the conclusion is used only as an indication). Then the problem is to minimize the variance of $\hat{P}(t_i)$ by choice of the numbers of exposures to be included in each age interval (by adjusting the age interval widths).

$$\min_{n_1, n_2, \ldots, n_{i-1}} = \text{Var}[\hat{P}(t_i)] = \min \ [P(t_i)]^2 \sum_{j=1}^{i-1} (n_j(n_{j-1}))^{-1}$$

where the $\{n_j\}$ are constrained by the requirement that

$$\sum_{j=1}^{i-1} n_j = N(t_i)$$

where $N(t_i)$ is the number of exposures up to time t_i.

Constrained minimization can be done by the LaGrange multiplier method by replacing the variance minimization by the LaGrangian. ($[\hat{P}(t_i)]^2$ has been omitted from the minimization because it is constant).

$$\min_{n_1, n_2 \ldots, n_{i-1}, \lambda} \sum_{j=1}^{i-1} (n_j(n_{j-1}))^{-1} + \lambda \left(\sum_{j=1}^{i-1} n_j - N(t_i) \right) = \min L$$

where λ is the "LaGrange multiplier". Calculus is used to find the minimum,

$$\begin{cases} \dfrac{\delta L}{\delta n_j} = -(2n_j - 1)/(n_j(n_{j-1}))^2 + \lambda = 0, \; j=1, 2, \ldots, i-1 \\ \dfrac{\delta L}{\delta \lambda} = \sum_{j=1}^{i-1} n_j - N(t_i) = 0 \end{cases}$$

Since all the derivatives $\delta L/\delta n_j$ yield the same equation subject to the constraint

$$\sum_{j=1}^{i-1} n_j = N(t_i),$$

the optimal allocation of exposures to intervals is to choose equal numbers n_j in each interval (as closely as possible because $N(t_i)$ may not be evenly divisible by i-1).

The conclusion is that to reduce the variance of the estimate of a particular percent surviving, choose age intervals to contain the same number of exposures. This indicates that the age intervals should be short where there are many exposures and long where there are few. Since many exposures usually result in more removals, the conclusion above also implies that age intervals should be short where there are many removals and long where there are few. In fact, if the installation and operation of engines makes the number of exposures sequence $\{n_j\}$ mathematically independent of the sequence of removals $\{d_j\}$ then the minimization of Var $\hat{P}(t_i)$ with respect to the sequence $\{d_j\}$ yields the same conclusion, that the age intervals should be chosen to have an equal number of removals in each interval to minimize Var $\hat{P}(t_i)$.

The "product limit" estimator of Kaplan and Meier [4] carries the concept of equal numbers of removals per interval to its ultimate extension, one removal per interval. Further investigation of the optimal interval widths to minimize variance was not done because this ultimate extension was available and because the data on engine removal times and ages was already at hand. However, the calculus method used in optimization above may be used to choose interval widths for actuarial estimators if it is desired to continue to use actuarial estimators. Further work is required however since a computer program has not been written for actuarial estimators with equal numbers of removals or exposures per age interval and since the simplifying assumptions made in the derivation may not be true in practice. Additional calculation is needed to obtain optimal age intervals when removals and exposures are mathematically dependent.

4. Alternative Estimators

Examination of crude failure rates for many types of engines shows high failure rates in the age intervals surrounding inspection times. This is because the elaborate testing done at inspection is much more likely to disclose an impending cause for removal than during usage. On the other hand, there is reason to believe that normal maintenance carried out at inspections serves only to permit an engine to achieve its normal operating age at eventual removal for repair or rebuilding. Inspection maintenance seems no more likely to rejuvenate an engine than to damage it further. These observations led to proposal of a statistical model of engine removal times that assumes independence of inspection and usage removal times (George [9]). This model assumes a cumulative distribution function $F_1(t)$ for usage removal times and independent propabilities p_i, i = 1, 2, ..., k, for removal at the i^{th} inspection. The overall cumulative distribution function of engine removal time is the probability that removal time occurs prior to or at age t

$$F(t) = 1- (1-F_1(t)) \prod_{i=1}^{i(t)} (1-p_i) \qquad (4.1)$$

where i(t) is the index of the last inspection prior to or at age t. This engine removal time model assumes that inspections occur at fixed times and that inspection removal times and usage removal times are statistically independent. The latter is a testable hypothesis and a likelihood ratio test to do so has been prepared but not programmed or tested. The former assumption is an idealization from reality that may be an adequate representation since engine removal codes distinguish inspection removals. If not, a similar model could be constructed to represent inspection times which vary around a fixed value.

The maximum likelihood estimator of $F_1(t)$ and the p_i has been derived in Appendix A. It is based on a sample consisting of engine usage and inspection removal times and survivors' ages. When combined according to formula (4.1) to obtain an estimate of engine removal time cumulative distribution function regardless of cause, the product limit estimator of Kaplan and Meier [4] is obtained,

$$\hat{F}(t) = 1-\prod_{r}(N-r)/(N-r+1) \qquad (4.2)$$

where N is the total sample size and the index r ranges over the indexes of ordered data from smallest to largest including in the product above only those indexes for removal times prior to or at time t. An example of the computation of this estimator is in Appendix A.

It is very easy to convert the product limit estimator to an estimator of the cumulative failure rate function

$$R(t) = -\log(1-F(t)). \qquad (4.3)$$

The estimator is obtained from the logarithm of (4.2)

$$\hat{R}(t) = \sum_{r} (N-r)/(N-r+1)$$

The relation between actuarial failure rates and the cumulative failure rate function R(t) is

$$R(i\Delta t) = \sum_{j=1}^{i} R_i$$

where R_i is the true interval failure rate or the true probability of engine removal between ages $(i-1)\Delta t$ and $i\Delta t$ given survival to age $(i-1)\Delta t$. Consequently comparison of estimators can be made on the basis of interval failure rates or cumulative failure rates, or even on cumulative distribution functions since the actuarial failure rates can be converted to a cumulative distribution function by

$$F(t) = 1\text{-exp}\ (-\sum_{j=1}^{i(t)-1} R_j - R_{i(t)}\ (t-(i-1)\Delta t)\ )$$

where i(t)-1 is the index of the actuarial age interval containing age t. This comparison is done in Section 5.

Furthermore, it is possible to use each estimator, actuarial, product limit and maximum likelihood, to generate engine lives and construct simulated operation of planes for comparison of how well the estimators predict engine replacement requirements. This comparison is made in Part II, Section 2. However, it is a well known fact that functions of maximum likelihood estimators are also maximum likelihood, and since all three estimators are maximum likelihood estimators for their sample information and statistical models of removal times, any estimators obtained from them of replacement requirements are maximum likelihood estimators. Theoretical comparison of the three estimators can be done by computing their information matrices Wilks [10] Chapter 12.

A computer program has been prepared to calculate the maximum likelihood estimator of $F_1(t)$ and the p_i i = 1, 2, ..., k and the product limit estimators of engine removal time, cumulative distribution function, and failure rate function. (The flow chart is shown in Figure 2 and the procedure is demonstrated in the next section).

It is necessary to point out again that the product limit estimators and the maximum likelihood estimators are based on sample data containing engine removal times and survivors' ages, not the information on when the engines began operating during the period of data collection. Further derivation is necessary to obtain maximum likelihood estimators based on sample removal times and ages and including operating time experience in a fixed calendar period. It is also worth questioning whether it would be better not to limit experience to a fixed calendar period for some purposes. It seems that limitation of data to a particular calendar period is necessary only when deciding whether the experience during that period differs from earlier experience. If the decision is that there is no difference, then the product limit and maximum likelihood estimators may be used to obtain interval failure rates or any of the other reports computed from actuarial failure rates.

One last remark about the product limit and maximum likelihood estimators computation should be made. They both require some sorting of data on removal times and survivors' ages. Modern computers can sort huge amounts of data in little time, but the maximum likelihood estimator is by its nature quicker to calculate since removal times and survivors ages may be sorted separately. The product limit estimator must first order all observations. The actuarial estimator requires only that numbers of exposures and deaths per age interval be counted. However, the flexible actuarial estimator proposed in Section 3 will require sorting.

5. Comparison of the Estimators

Estimation of simulated engine removal ages has been done here to show what the estimators do but in a much simpler example than real life. The objective is to simulate removal age data, compute all three estimators of the removal age cumulative distribution function and failure rates, and compare these with the theoretical cumulative distribution function used to simulate the removal age data.

The input parameters are twenty engines with starting age zero and max time, two hundred hours. There will be one inspection at one hundred hours and the probability of an engine is removed at this inspection is one-fifth. The probability of removal at the maximum time of 200 hours is one to insure that all surviving engines are removed at max time.

The removal age data was generated from three formulas. The usage removal age random variable is generated from an exponential distribution with a mean removal age of two hundred hours. The survivor's age is a random variable Y generated using a uniform distribution from age zero to four hundred. Four hundred hours was used to assure that there would not be too many survivors up to the maximum operating time of two hundred hours. The inspection removal time random variable Z is generated for one hundred and two hundred hours by the probabilities given earlier. The engine removal age or survivors' age is the minimum of these three variables.

The resulting random variable, min (X,Y,Z), has the theoretical cumulative distribution function

$$1 - e^{-t/200} \cdot (1-t/400) \prod_{i=1}^{i(t)} (1-p_i) \qquad (5.1)$$

where

$$i(t) = \begin{cases} 0 & t < 100 \text{ hours prior to inspection} \\ 1 & 100 \le t < 200 \text{ hours} \\ 2 & t \ge 200 \text{ hours.} \end{cases}$$

(If $i(t) = 0$, the product is 1.0

The objective is to estimate the engine removal age min (X,Z) which is the minimum of usage removal age or inspection removal time (including max time as the second inspection) in the presence of the nuisance variable Y, survivor's age.

The theoretical cumulative distribution function of min (X,Z) the engine removal age is

$$F(t) = 1-e^{-t/200} \prod_{i=1}^{i(t)} (1-p_i) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/200} & 0 \le t < 100 \\ 1 - .8e^{-t/200} & 100 \le t < 200 \\ 1 & t > 200 \end{cases} \qquad (5.2)$$

which is plotted in Figure 3. The maximum likelihood estimator also gives separate estimates of the cumulative distribution function of usage removal age and the inspection removal probabilities for use in engine diagnosis and prediction of replacement requirements due to inspection removals.

The construction of Table 1 is as follows: Column 1 is the index of engines available, for this example that is twenty engines. These could be the engine serial numbers for real data. Columns 2, 4, and 6 are random numbers taken from Mihram [2]. A different random number is generated for each engine's age at usage removal, survivor's age, and inspection removal times. Column 3 uses the inverse transformation method (Fishman [1]) for generating exponentially distribution random variables.

TABLE 1

SIMULATED ENGINE REMOVAL AGE DATA

Column Number	1	2	3	4	5 hours	6	7	8
Column Number	Engine Number	Random [a]Numbers	X=-LOG(RN)*200	RN's	Y=RN*400	RN's	Z	MIN(X,Y,Z)[b]
	1	.71264	68	.362	145	.295	200	68
	2	.84665	32	.469	188	.555	200	32
	3	.24364	282	.015	6	.341	200	6
	4	.96418	7	.902	361	.307	200	7
	5	.21437	308	.541	216	.634	200	200
	6	.95026	10	.957	383	.088	100	10
	7	.02731	720	.066	26	.519	100	26
	8	.24453	282	.631	252	.474	200	200
	9	.15006	379	.812	325	.933	200	200
	10	.35218	209	.581	232	.077	100	100
	11	.44093	164	.497	199	.513	200	164
	12	.54629	121	.25034	100.1	.788	200	100
	13	.66392	82	.729	292	.907	200	82
	14	.04394	625	.326	130	.586	200	130
	15	.28697	250	.444	178	.386	200	178
	16	.95000	10	.263	105	.451	200	10
	17	.15711	370	.246	98	.175	100	98
	18	.56211	115	.102	41	.708	200	41
	19	.14635	384	.568	227	.477	200	200
	20	.63087	92	.713	285	.489	200	92

a. Mihram [2] Random Number Tables p. 499.

b. Survivor's ages are circled.

Column 5 uses a uniform distribution and rescales the random number to the interval 0-400 hours. Column 7 generates inspection or max time removals. If the random number is less than 0.2, Z is an inspection time of one hundred hours, otherwise Z is the max time of two hundred hours. Column 8 is the minimum time of columns 3, 5 or 7. These ages constitute the data of a simulation of twenty engines operating to a possible max time of two hundred hours.

The removal ages and survivors ages from Column 8 are used to compute the actuarial failure rates in Table 2. In this table, there are five actuarial age intervals assumed to have width forty hours, and all maximum time removals are collected as a sixth interval of zero width. Column 1 is the interval index. Column 2 gives the age span for each of the intervals. Column 3 is the number of removals during each interval. These are obtained from the removal times in Table 1., Col. 8. For instance, engines number 2, 4, 6, and 16 are removed at times 33, 7, 10 and 10 respectively so there are four removals in interval one. Column 4 is the number of survivors' ages contained in each interval. These are found in the same way as the removals times in column 3. Column 5 gives the number of full exposures for each interval. This is the number of engines that survived to an age greater than the end of the interval plus the number removed during the interval. The number of partial exposures is the proportion of an age interval during which engines recorded as survivors' ages operated. For instance, during interval one, engine 3 operated 6 hours or .15 of the interval and engine 7 operated .65 of the interval for a total of .8 partial exposures in interval one.

TABLE 2

COMPUTATION OF ACTUARIAL FAILURE RATES

Column Number	1	2	3	4	5		6	7	8
Column Name	Interval	Span	#Removals	#Survivor	Full #Exposures		Partial Exposures	#Exposures	Failure Rate
	1	0-40	4	2	18	+	.8	18.8	.213
	2	40-80	1	1	13	+	.025	13.025	.077
	3	80-120	4	1	11	+	.45	11.45	.349
	4	120-160	0	1	6	+	.25	6.25	0
	5	160-200	1	1	5	+	.45	5.45	.32
	max time	200	4	—	4			4	1
			14	6					

20 engines total

TABLE 3

COMPARISON OF ACTUARIAL ESTIMATOR AND THEORETICAL DISTRIBUTION

Column Number	1	2	3	4	5		6
Column Name	Interval	R_i Failure Rate	cdf. end of Interval	Survivor Prob. [1-F(t)]	Theoretical c.d.f. of Removal Times		Theoretical Interval Failure Rate
	1 (40)	.213	.192	.808	.181	-	.2
	2 (80)	.077	.252	.748	.330	-	.2
	3 (120)	.349	.472	.528	.561	-	.483
	4 (160)	0	.472	.528	.641	-	.2
	5 (200^-)	.184	.561	.439	.706	-	.2
	max time						
	MT (200)	1	1	0	1.0		1.0

Column 7 is the total number of exposures which is the sum of Columns 5 and 6. Column 8 is the failure rate for each interval which is obtained by dividing the number of removals (Column 3) by the number of exposures (Column 7).

In Table 3, there is a comparison of the cumulative distribution function and failure rates for the actuarial estimator and theoretical distribution (5.2). Column 1 gives the number of the interval. Column 2 gives the actuarial failure rate, R_i, which is column 8 of Table 2. Column 3 is the actuarial cumulative distribution function, $F(t_i)$, where t_i is the end of the interval. This function is obtained from

$$F_A (t_i) \quad 1 - \exp [- \sum_{j=1}^{i} R_j]$$

where R_j is the failure rate in column 2. The survivor probabilities in column 4 are the complements of column 3, $1-F(t_i)$. Column 5 gives the theoretical cumulative distribution function of removal times by the equation (5.2) where t is evaluated at the end of each interval. To find the theoretical interval failure rates, column 5 is used and the theoretical interval failure rate function $R(t_i)$ is found solving iteratively

$$F(t_i) = 1 - \exp [- \sum_{j=1}^{i} R(t_j)] \quad i = 1,2,\ldots,6$$

for $R(t_i)$ in each interval where t_i = 40, 80, 120, 180, 200^- and 200.

Figure 2 shows the comparison of the actuarial and theoretical interval failure rates as a graphical representation. Figure 3 shows the comparison of the actuarial and theoretical cumulative distribution functions. It should be recognized that deviation of the actuarial estimator from the theoretical value is due the very small sample size of 14 observed removal times and the fact that the data was simulated from the theoretical distribution function (5.1).

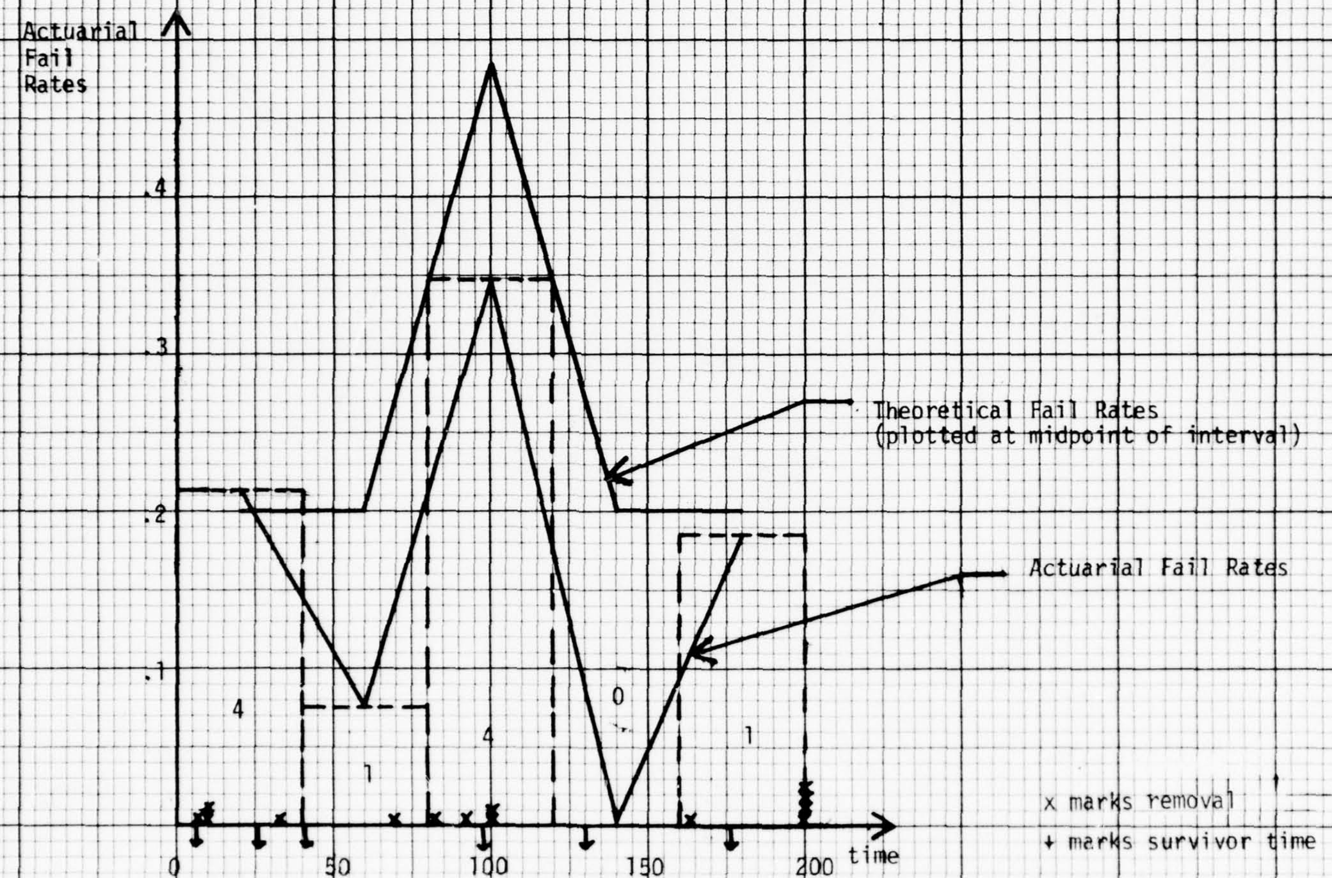

Figure 2. Actuarial Failure Rates and Theoretical Interval Failure Rates

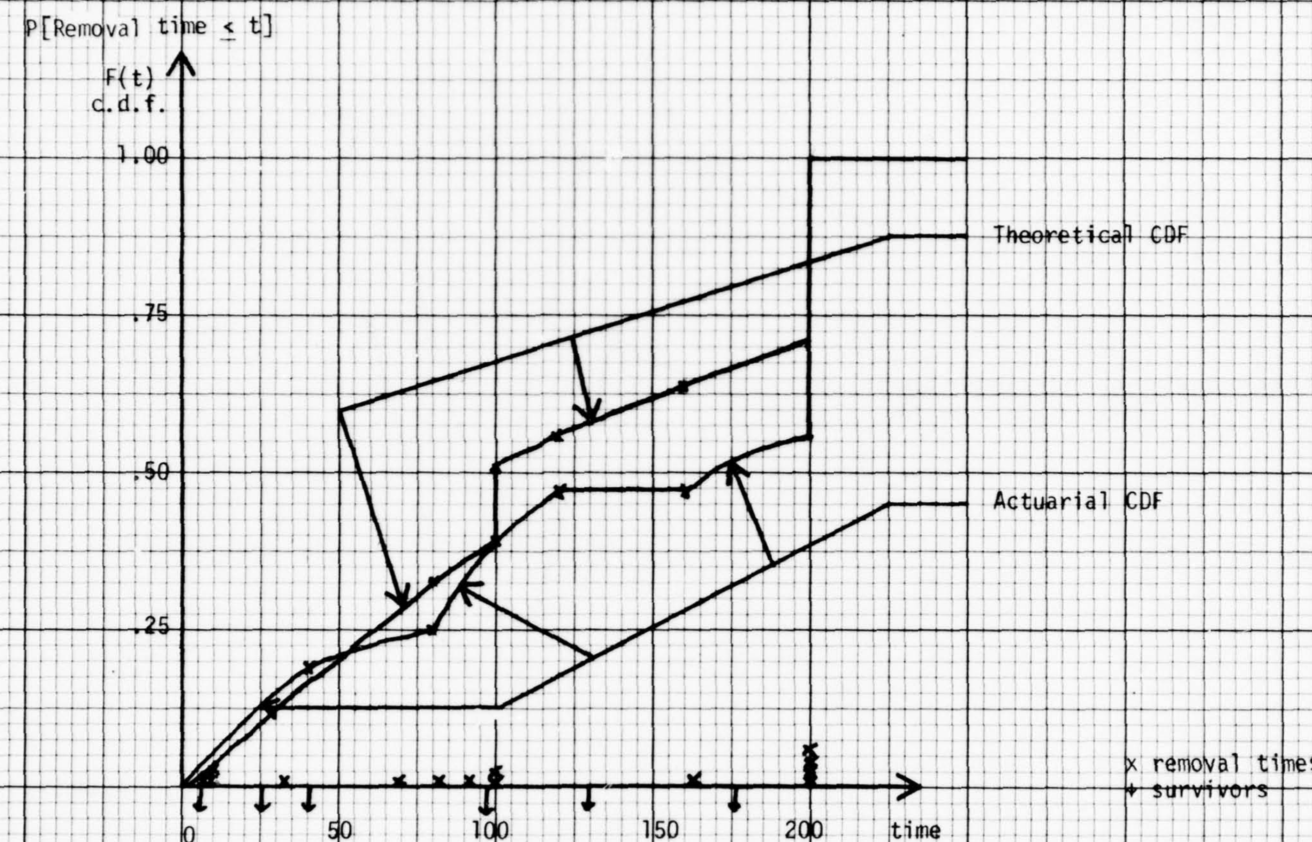

Figure 3. Removal Time Distribution from Actuarial Failure Rates.

The product limit estimator of the cumulative distribution function and failure rates is calculated from the data in Table 1 as follows: Columns 1 and 3 are the same as columns 1 and 8 in Table 1 except that the removal times are reordered along with their indexes. Column 2 is the removal code useful here and in calculation of the maximum likelihood estimator. Column 4 is the product limit estimate of the percent surviving or tail cumulative distribution function at the time or age values in column 3. The formula as given in Kaplan and Meier [4] is

$$\prod_r (N-r)/(N-r+1)$$

where N is the sample size 20 and the index r ranges from 1 to 20 taking on only the values for inspection and usage removals in column 3. Column 5 is the complement of column 4, the product limit estimator of the cumulative distribution function (5.2). Column 6 is the value of the failure rate function assuming that it is constant during the interval. It is the log of (N-r)/(N-r+1) for intervals between engine removal age r and the next subsequent engine removal. Column 7 is the cumulative failure rate function at the age in column 3, which is also the natural logarithm of Column 4.

The product limit estimator and the theoretical cumulative distribution functions are plotted in figure 4 and their corresponding cumulative failure rates are in figure 5. For comparison of the product limit estimator with the actuarial estimator of failure rates, the product limit estimator of failure rates during each 40 hour age interval must be calculated from the cumulative failure rate estimator. The comparison of product limit estimator with the theoretical interval failure rates is shown in figure 6.

TABLE 4

PRODUCT LIMIT CUMULATIVE FAILURE RATE FUNCTION

Column Number	1	2	3	4	5	6	7	
Column Name	Index	Code	Age	$\prod_{r}(N-r)/(N-r+1)$	$F_{PL}(t)$	r_i*	R(t)	
	1	Survivor	6	-		.0		
	2	Usage	7	.947	.053	.054	.054	
	3	Usage	10	.895	.105	.057	.111	
	4	Usage	10	.842	.158	.061	.172	
	5	Survivor	26	-		.061	-	
	6	Usage	32	.786	.214	.069	.241	
	7	Survivor	41	-		.064	-	
	8	Usage	68	.726	.274	.080	.321	
	9	Usage	82	.665	.335	.087	.408	
	10	Usage	92	.605	.395	.095	.503	
	11	Survivor	98	-		.095	-	
	12	Inspection	100	.537	.453	.118	.621	
	13	Inspection	100	.470	.530	.134	.755	.82
	14	Survivor	130	-		.134	-	
	15	Usage	164	.392	.608	.182	.927	.93
	16	Survivor	178	-		.182	-	
	17	Max time	200	.294	.706	.288	1.225	1.73
	18	Max time	200			.405	1.630	
	19	Max time	200			.693	2.32	
	20	Max time	200	0	1.0	∞	∞	

*Failure rate function is assumed constant between observations of actual removals.

In comparison with actuarial failure rates in Figure 2, the product limit estimator is closer than the actuarial estimator to the theoretical interval failure rate except in intervals 1 and 5. expected that this superiority will increase as sample size grows because the product limit estimator uses the actual values of removal times and survivors' ages instead of lumping the time values into interval counts exposures and removals.

It is very easy to calculate interval failure rates for any width of age intervals from the cumulative failure rate function in column 8 or any cumulative failure rate function R(t), $t \geq 0$. If Δt is the desired width of age interval, then the interval failure rate here denoted as R_{pLi} is

$$R_{pLi} = R(i\Delta t) - R((i-i)\Delta t).$$

For example, in order to obtain the product limit estimator of the failure rate in interval 2, 40 to 80 hours, use linear interpolation between usage removals at 33 and 68 hours to obtain R(40) = .25 and between usage removals at 68 and 82 hours to obtain R(80) = .38. (These values were read from figure 5). The interval failure rate R_{PL2} is the difference, .13.

The maximum likelihood estimators for the usage removal time cumulative distribution function and the probability of removal at inspection have also been computed according to formulas in Appendix A. (The maximum likelihood estimator of the removal time cumulative distribution function regardless of whether it is an inspection or usage removal is the product limit estimator.) The estimate of the probability of removal at the first inspection is

$$\hat{p}_i = \frac{\text{number removed at inspection}}{\text{number surviving to inspection}} = \frac{2}{9} = .222$$

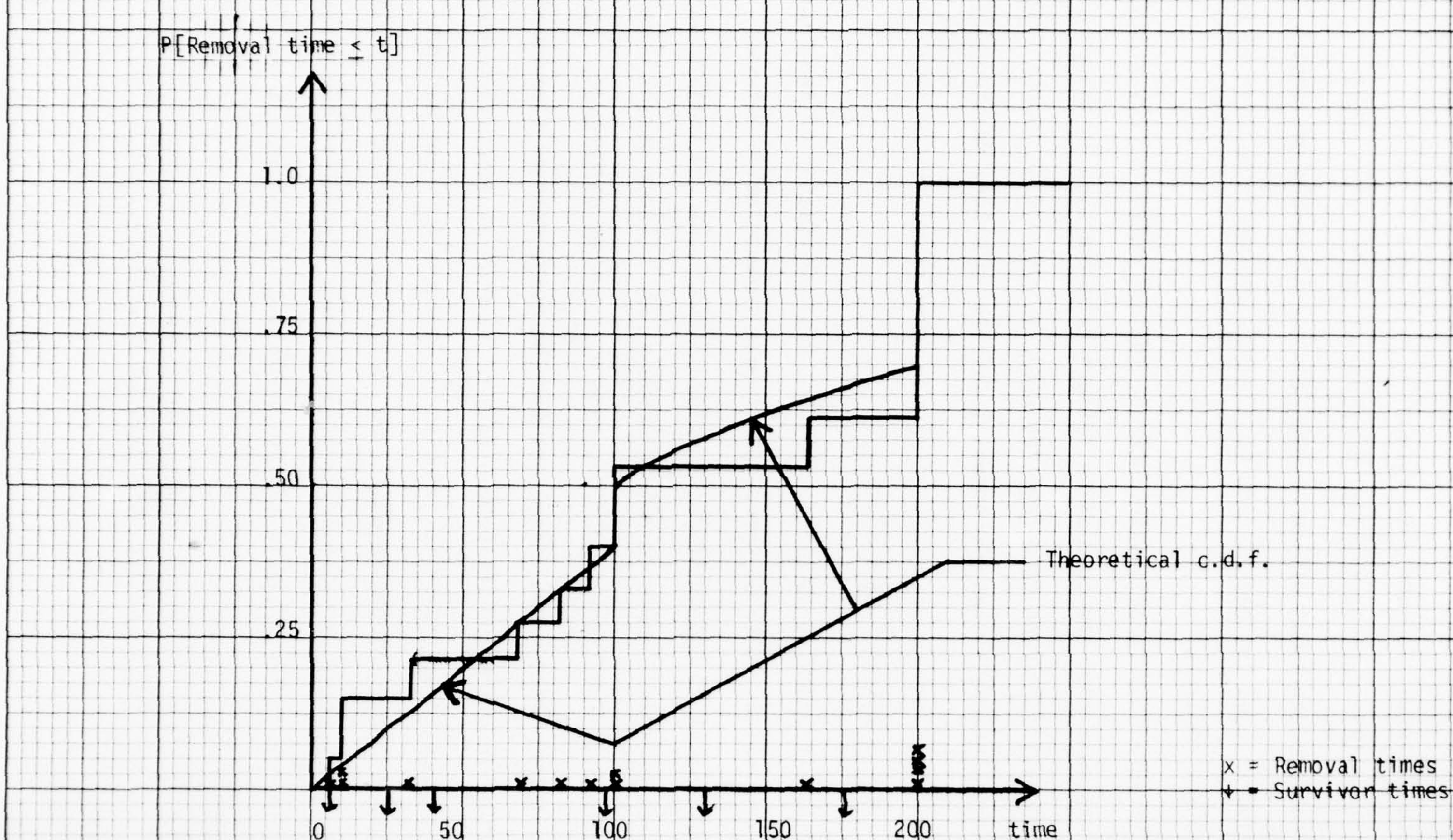

Figure 4. Product Limit c.d.f.

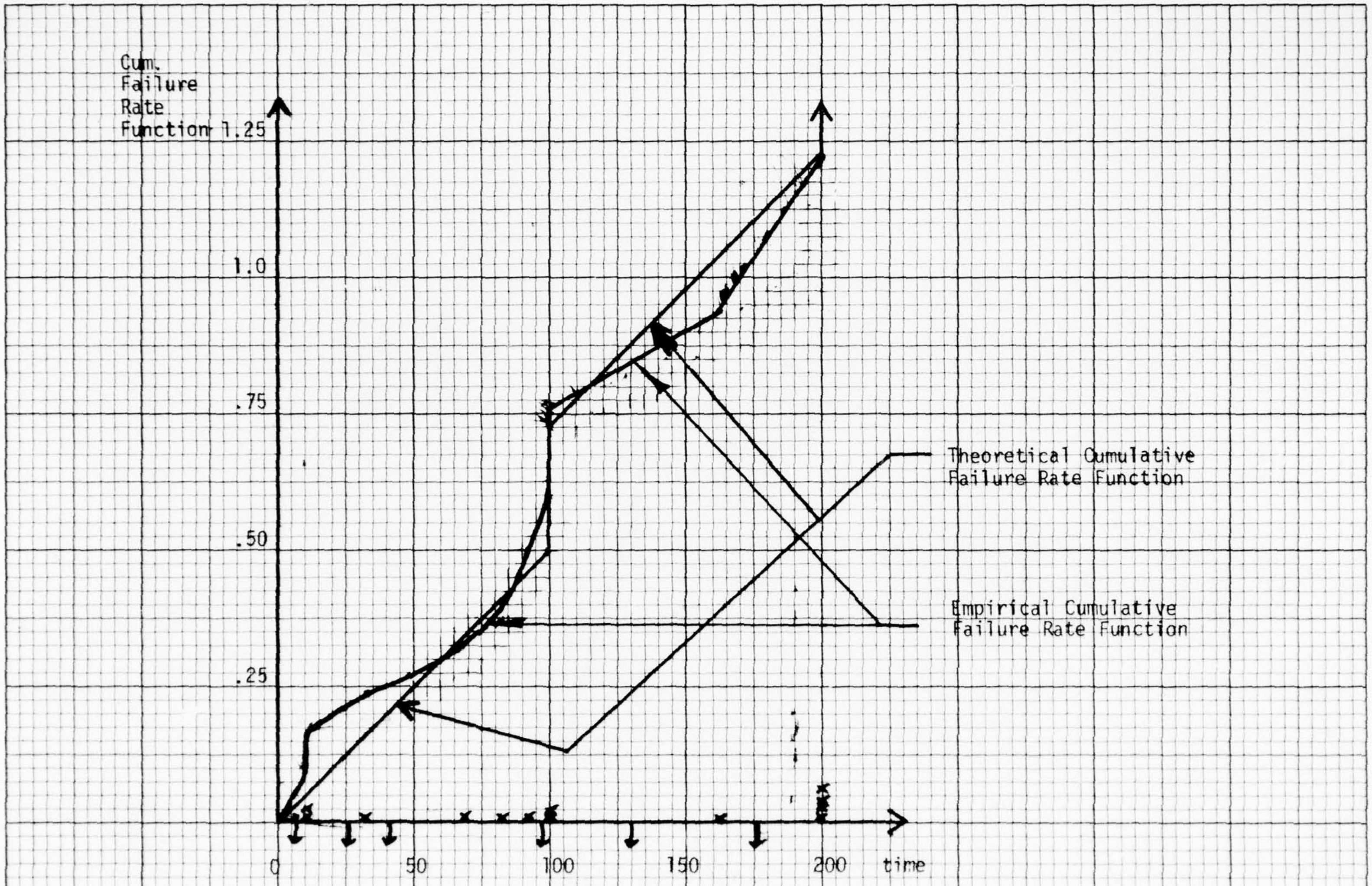

Figure 5. Empirical Cumulative Failure Rate Function.

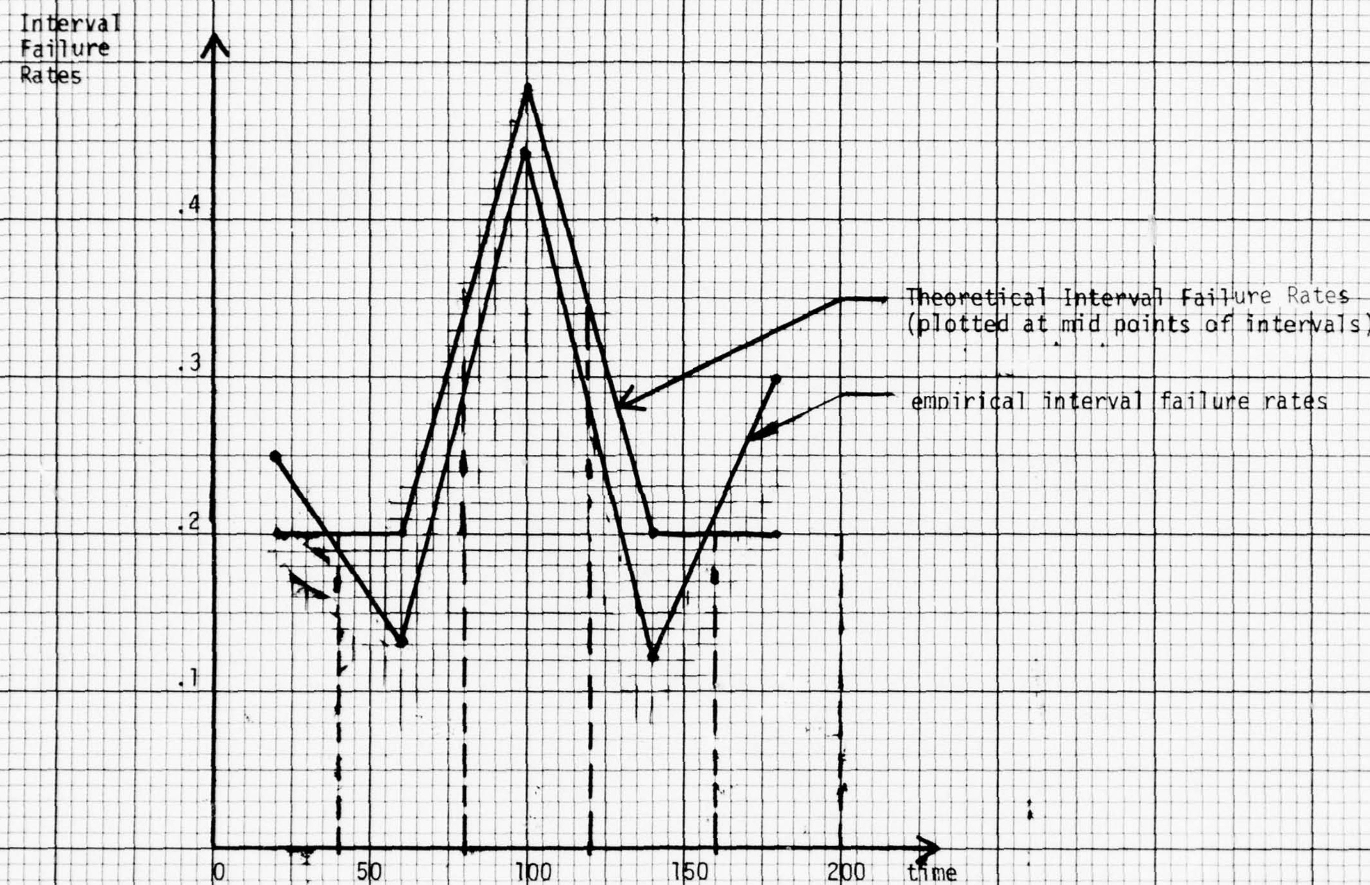

Figure 6. Empirical Interval Failure Rates.

(the theoretical value was 0.2) and the estimate of the usage removal time cumulative distribution function is equivalent to Figure 4 without the jump at 100 hours. This is plotted against the theoretical formula used to generate usage removal times in Figure 7. The jump at 200 hours represents the probability of surviving usage removal to max time.

A computer program for comparison of estimators on several bases has been written. In the remainder of this section, estimators are compared on the basis of their cumulative distribution function. In Part II, Section 8, estimators are compared on the basis of how well they predict replacement requirements. The flow chart for computation of the maximum likelihood estimators is shown in Figure 8. In Figure 9 the actuarial and product limit estimators are compared with the theoretical distribution with three inspections at 50, 100 and 150 hours. No survivors' ages were generated (no censoring), but the rest of the program generated data and computed estimators as was done earlier in this section. Figures 10 and 11 are the maximum likelihood estimators of the usage removal time cumulative distribution function which was calculated from data generated from an exponential usage removal time with mean 133.5 and inspection at 100 hours. Figure 11 shows the estimator in the presence of some survivors' ages. Figure 12 shows the cumulative failure rate function from figure 10 plotted along with the theoretical failure rate function. The slight low side bias in all three figures is due to the fact that the simulated data contained fewer usage removals than expected, <u>not</u> because of systematic bias in the estimators. A best fit linear regression to the cumulative failure rate estimator plotted in Figure 12 gave a slope of .0070 and an intercept of -.0064 with an R^2 value of .9966 indicating the data almost

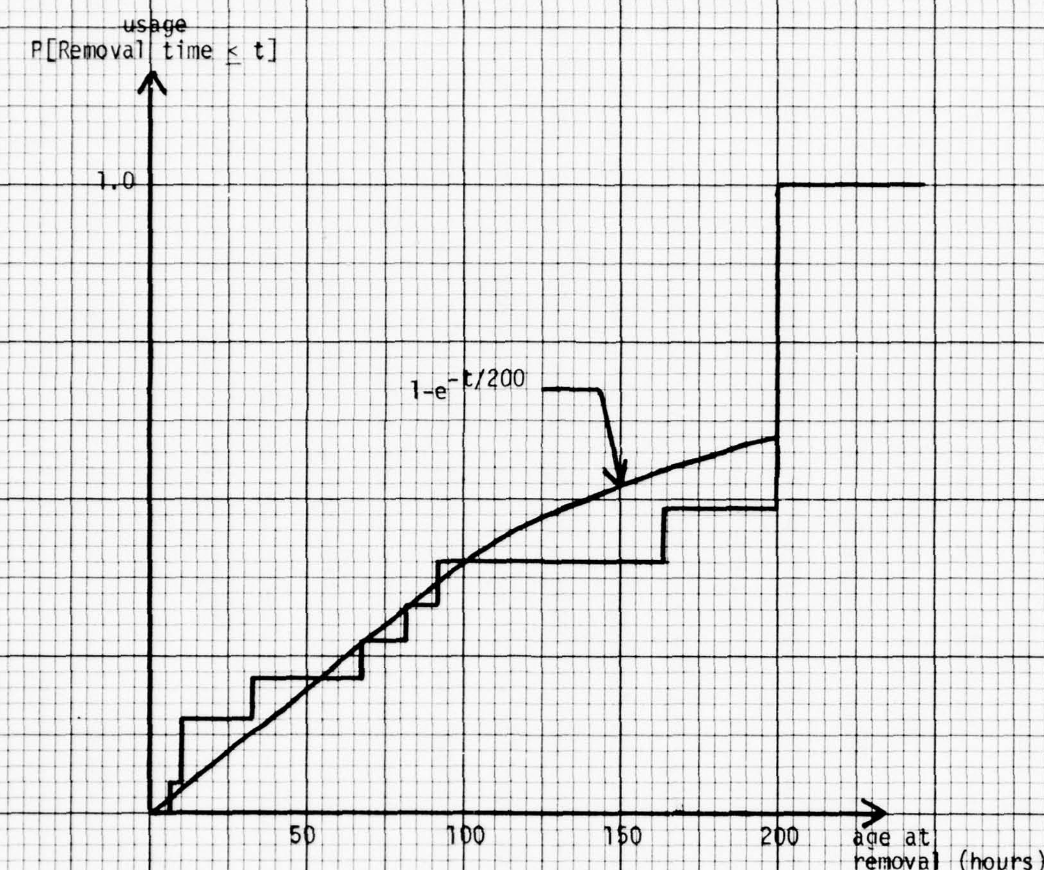

Figure 7. Maximum Likelihood Estimate of Usage Removal Time c.d.f.

certainly come from a linear model for a cumulative failure rate function which it did since the cumulative failure rate function for exponential usage removal ages is linear.

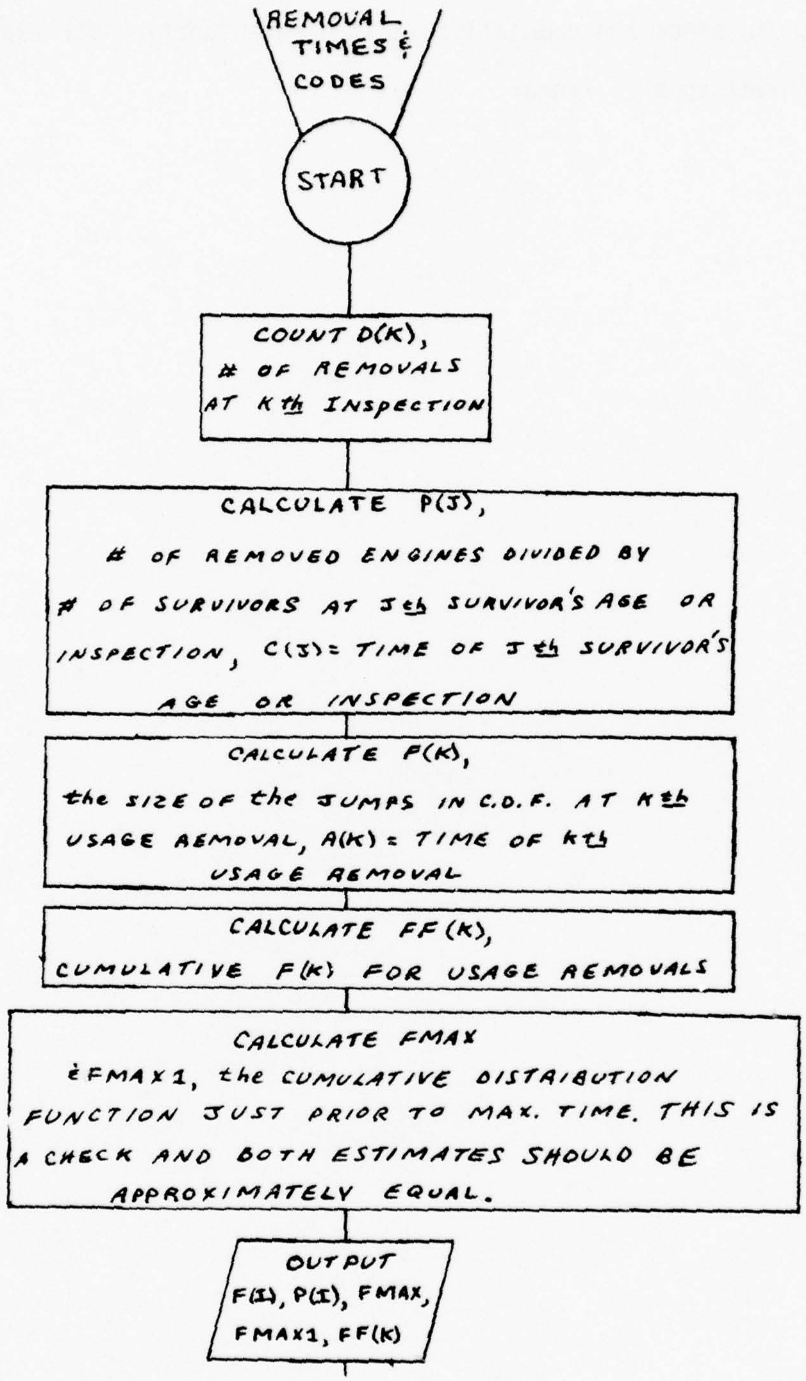

Figure 8, Maximum Likelihood Estimator

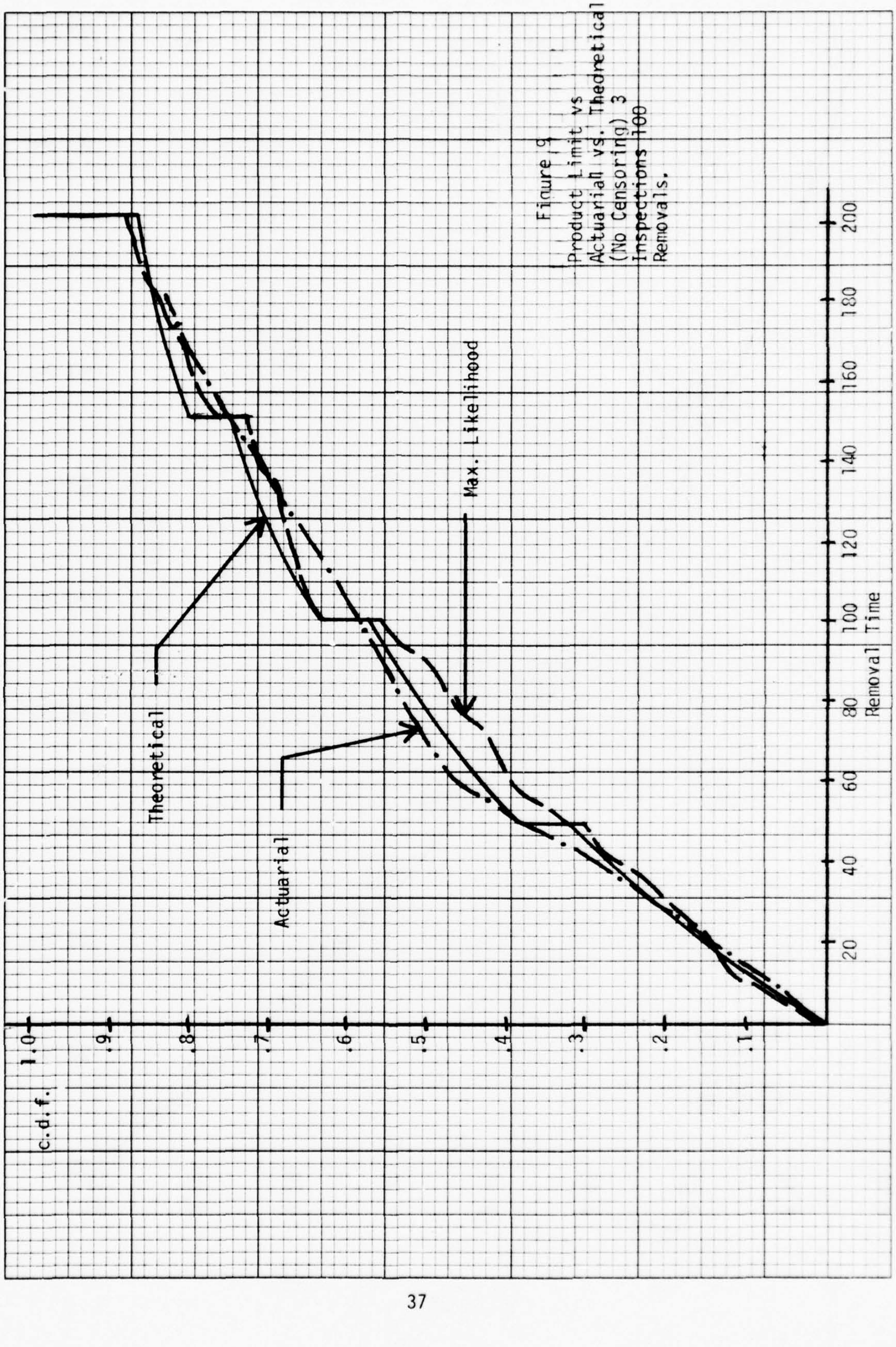
Figure 9
Product Limit vs
Actuarial vs. Theoretical
(No Censoring) 3
Inspections 100
Removals.
c.d.f.
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
20
40
60
80
100
120
140
160
180
200
Removal Time
Theoretical
Actuarial
Max. Likelihood

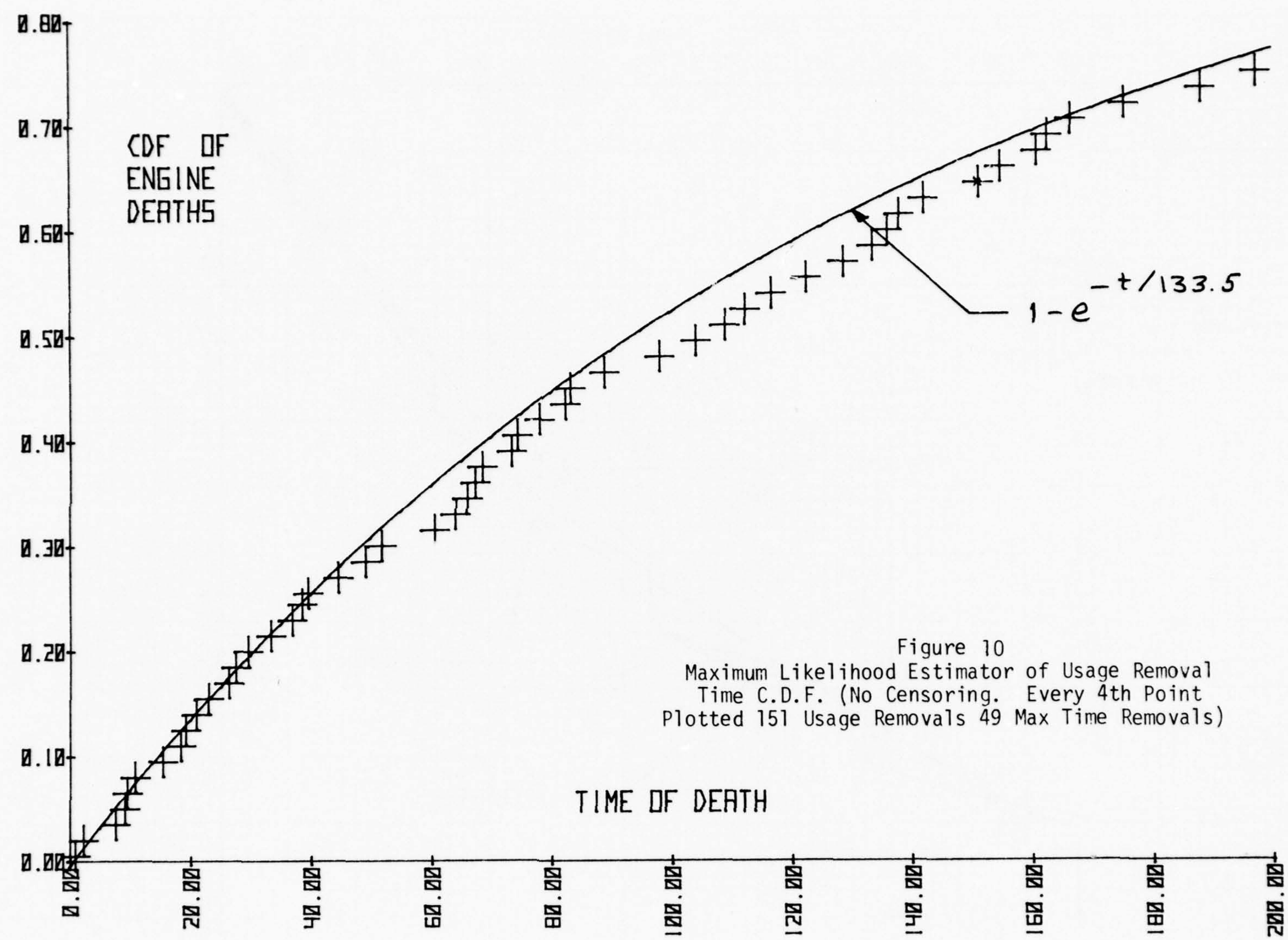

Figure 10
Maximum Likelihood Estimator of Usage Removal Time C.D.F. (No Censoring. Every 4th Point Plotted 151 Usage Removals 49 Max Time Removals)

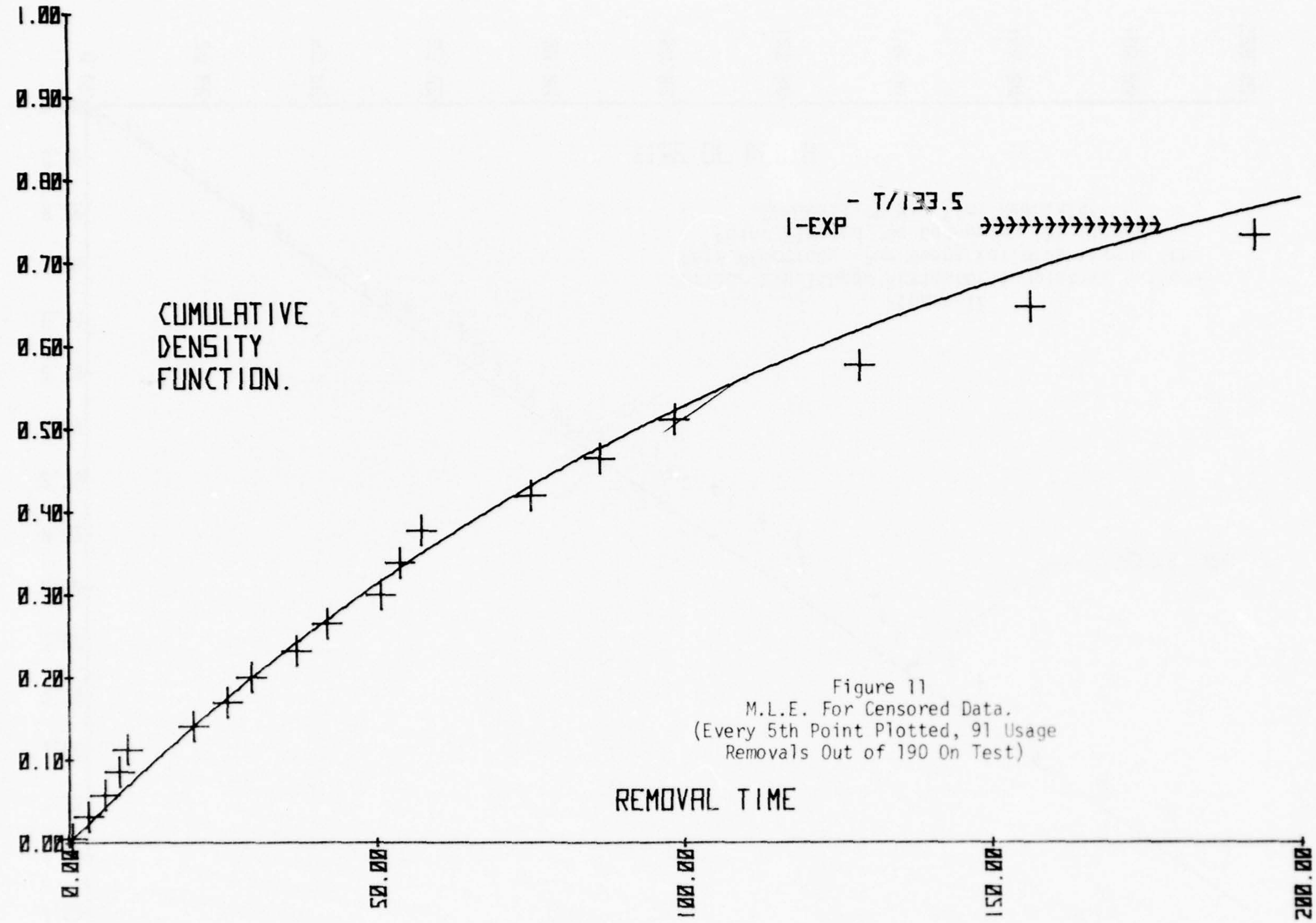

Figure 11
M.L.E. For Censored Data.
(Every 5th Point Plotted, 91 Usage
Removals Out of 190 On Test)

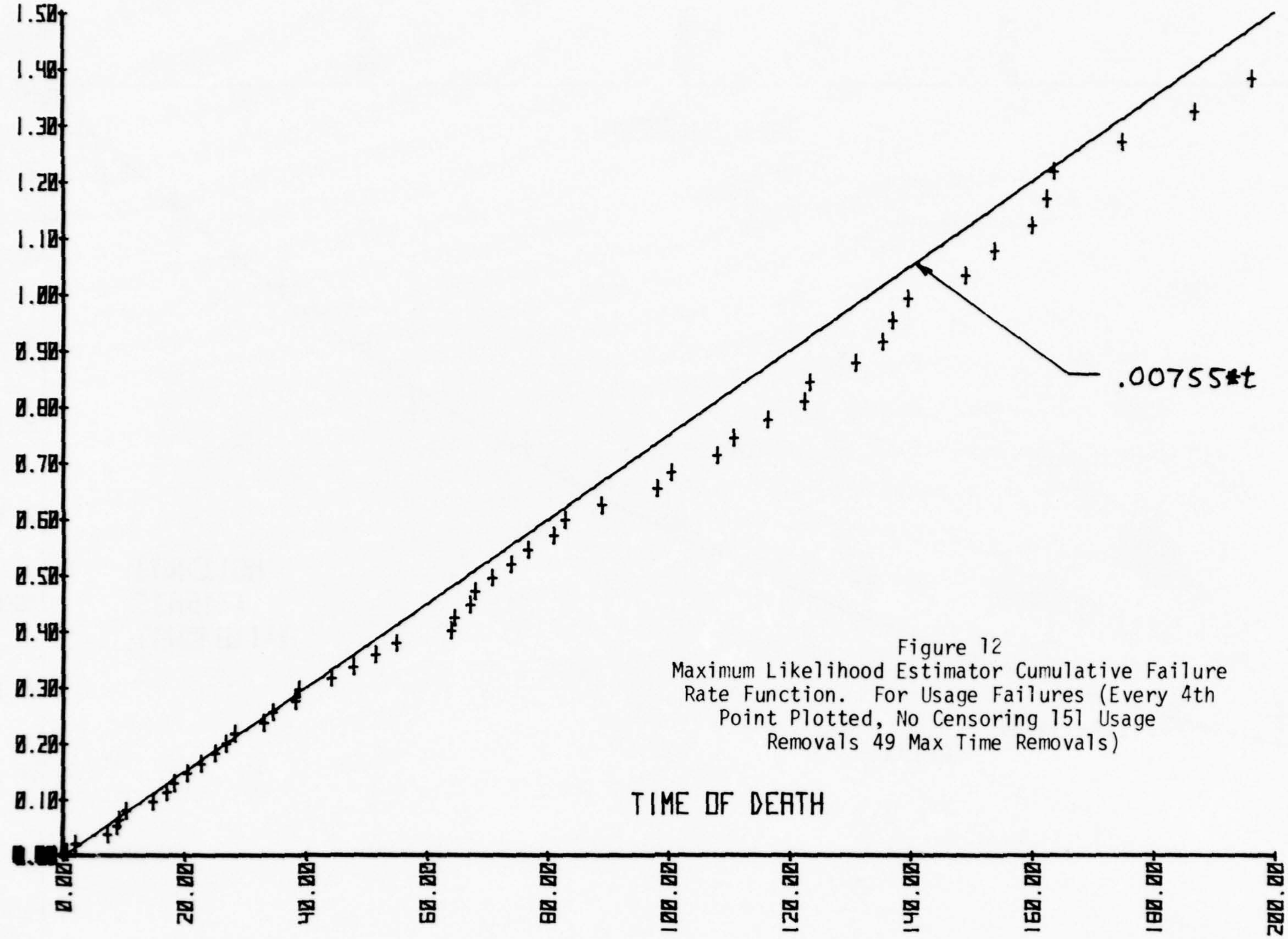

Figure 12
Maximum Likelihood Estimator Cumulative Failure Rate Function. For Usage Failures (Every 4th Point Plotted, No Censoring 151 Usage Removals 49 Max Time Removals)

6. Updating the Data Set for Computing Official Failure Rates

There are several tests used to infer whether engine performance is changing. (All of these are found in AFLCM 66-17 [11]). The "Control Factor" test uses the normed difference between actual and expected failures experienced in an actuarial age interval. If the magnitude of the control factor is too large, the official failure rate used to estimate expected failures is suspect (or perhaps usage by a base or squadron differs significantly from the official failure rate). This is method A section 9 of AFLCM 66-17. The "statistical test" (page 4-10) is computed to compare crude and official failure rates (or crude and smoothed failure rates) on an interval by interval basis. If the interval failure rate is significantly different from the official failure rate, it may be changed proportional to the ratio of the official actuarial life expectancy and the actual life expectancy. The third test, "Method D" of section 9, monitors flying hours per failure over several calendar time periods with a control chart.

Only in method D is inference based on the experience of an entire population of one type of engines. The other two tests base inference on test statistics calculated for each actuarial age interval, and consequently the conclusion for adjacent age intervals may be contradictory. These two tests may be very sensitive for engine diagnosis to changes in engine component performance, but they may not be the appropriate way to determine whether replacement requirements will change.

The general questions that seem to be addressed in the statistical tests are whether the estimates of engine performance have changed and whether engines seem to be failing at a particularly high rate during some age interval. There are two purposes here, engine diagnosis and replacement requirements prediction. Engine diagnosis can be assisted by determining whether there is a significant number of failures near some age. If so then there may be an engine component whose operating life is significantly less than the engine life itself and the component should be replaced prior to its failure. The two tests, "control factor" and "statistical test" have their discriminatory power reduced by the smoothing technique mentioned in section 2 of this report. Smoothing tends to smooth out peaks in failure rates that may be caused by failure of some engine component. Those two statistical tests will have to be revaluated if smoothing is stopped and more sensitive engine diagnosis is possible using the estimators proposed in section 4 .

The remainder of this section is to determine whether the official failure rates are adequate for prediction of replacement requirements. Future engine requirements depend on the entire set of failure rates throughout the operating time from new to max time, so comparison of the official failure rates to current failure rates should be made on an aggregate basis. It is advocated that the comparison should be made of the original data instead of the failure rates computed from the data for the same reason in section 1 , that the actuarial failure rates are based on a summary of the data. The engine life data itself should be used to answer questions about engine lives and consequent replacement requirements.

There are several ways to statistically formulate the question of whether the official failure rates still indicate engine performance.

1. Does the current calendar period of engine experience resemble the experience used to compute official failure rates? Here experience

refers to recorded engine removal times, and exposures during the calendar periods being compared.

2. Do the recorded ages at removal during the current calendar period and the ages of engines still operating at the end of this period come from the same population of engine lives as the corresponding data used to calculate the official failure rates?

These questions differ because an engine removed during the current calendar period may have acquired a significant proportion of its operation in prior calendar periods. A statistical procedure for answering question 2 is provided in this section, and a method for modifying this procedure to answer question 1 will be described. The statistical procedure computes a measure of how likely it is that the current calendar period's data could have come from the same population as past data. This measure is computed for <u>every</u> past calendar period for which data has been collected taking continguous periods ending with the beginning of the current period. The Aerospace Engine Life Committee may then use these measures and their knowledge of the actual operating conditions during the current period to decide whether to update the data set to include current data <u>and</u> whether to eliminate the oldest data.

The test uses a Wilcoxon type rank test suggested by Gehan [6]. The Gehan test statistic was used without the later modification by Efron [19]. A flowchart of this test is in Figure 13. The performance of this test appears to be as good as any two sample test with the possible exception of the Kolmogorov-Smirnov type test. (Several two sample tests were compared in Lee, Desu and Gehan [17] and Gehan and Thomas [16].) The reason for the exception of the Kolmogorov-Smirnov type test is that it has not yet been adapted to incorporate censoring such as experienced when sample information includes the ages of surviving engines. This can be done by extension of the truncated Kolmogorov-Smirnov two sample test in Barr and Davidson [20] and Koziol and Byar [21].

The Wilcoxon type rank test used in the updating program adds a plus or minus one to the test statistic W if it can be concluded that a engine age at removal from the current period is or will be larger than an engine age at removal in the past. The proposed modification to this test to answer question 1, whether the current period's experience is the same as the past, is to weight the plus or minus one by the proportion of an engine's exposure that is in the current period. This modification has not been evaluated.

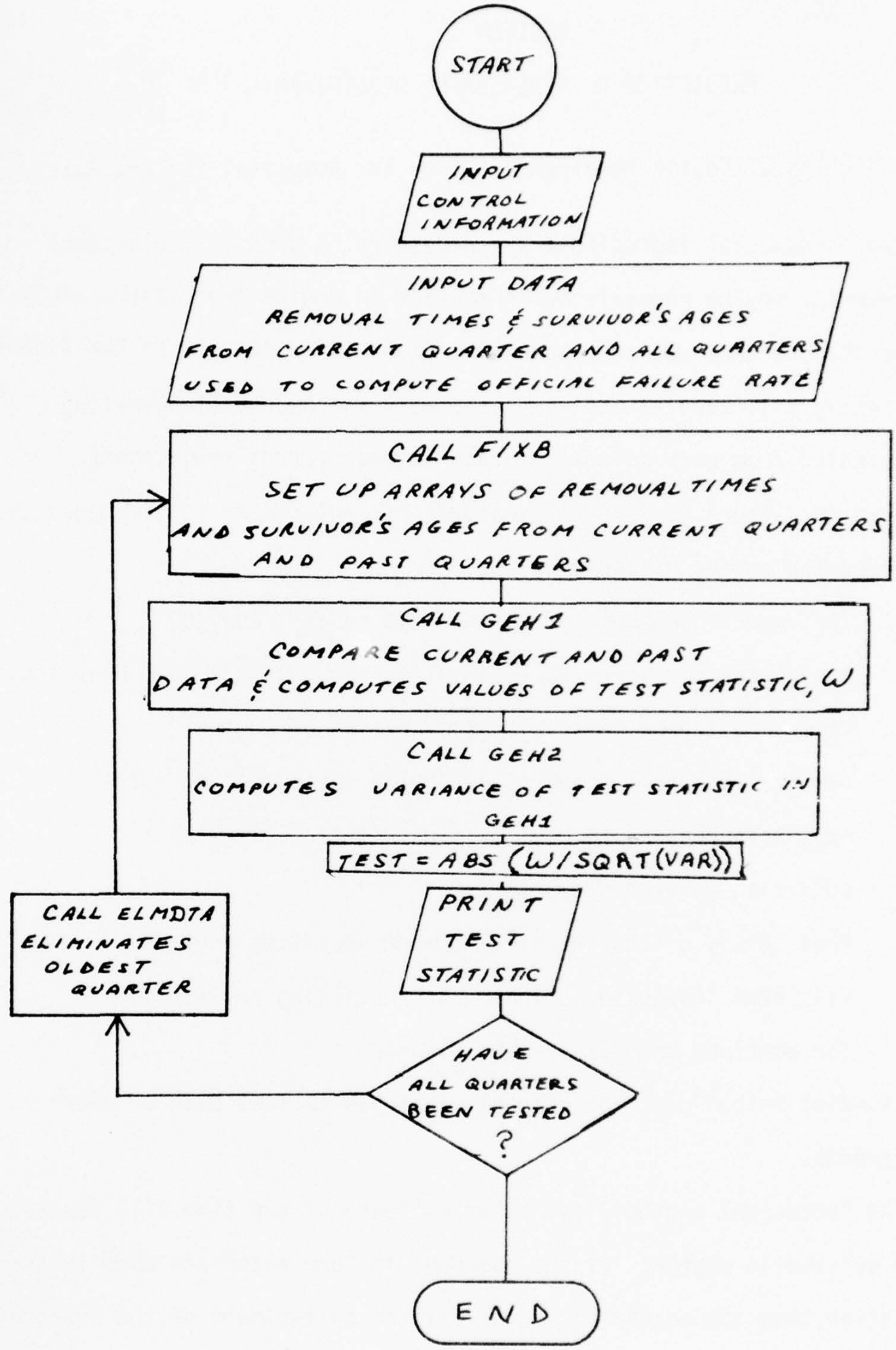

Figure 13
Update Main Program

SECTION II

PREDICTION OF REPLACEMENT REQUIREMENTS

7. Prediction of Engine Requirements from the Actuarial Failure Rates

Two fundamental ingredients are necessary to predict replacement requirements, engine removals which require an engine from spares stock to replace the one that was removed; some probability measure of the amount of operating till removal must be mixed with the amount of operating time demanded from each aircraft to obtain measures of replacement requirements. There are several measures of replacement requirements that are needed for engine management:

1. The expected numbers of engine repairs and rebuilds are needed for spare parts provisioning and work load planning.
2. Upper confidence intervals or quantiles of engine requirements are needed to determine the number of spare engines necessary to meet mission requirements with a sufficiently high probability.
3. Predictions of the actual time when operating engines will need inspection, repair, or rebuilding are necessary for workload and maintenance planning.

The actuarial method contains elements designed to meet each of these requirements.

The "actuarial engine life" is an estimate of the time till removal of a new or rebuilt engine. If the overhaul failure rates are used in the computation, then the actuarial engine life is an estimate of the expected time till overhaul. If base maintenance removal rates are used, then the actuarial engine life is an estimate of the time till first removal for base repair starting with a new or rebuilt engine. The expected time till removal could be converted into an estimate of demand requirements except that not every engine is new at the beginning of the period for which replacement predictions are needed. The actuarial method includes a procedure for "operating" on paper the entire inventory through a calendar period to determine replacement requirements from the input information on inventory, ages, flying hour program and actuarial failure rates. This procedure is described in T.O. 00-25-128 Chapter IX [7], T.O. 00-25-217 Chapter IX [8] and AFLC Manual 66-17 Chapter 6 [11]. It is equivalent to computing the conditional expected operating time till replacement given engine age at the beginning of the calendar period under consideration. This expected residual operating time is in turn used to estimate replacement requirements to meet the flying hour program by assuming the flying hours are equally allocated among all aircraft and each engine contributes its expected residual life to meet the program.

This procedure for predicting expected replacement requirements is based on two fallacies. For example, if every engine is new and the expected operating time till replacement is 300 hours, then it is natural and convenient to conlcude that to fly 1000 hours will require two replacements or three engines total. In general this procedure is as follows: let ET denote the expected time till replacement, t the flying hours required from a single engine plane, and N(t) the number of engines required to meet the flying hour requirement of t hours. An obvious estimate of N(t) is

$$N(t) = t/ET.$$

Unfortunately this estimate is unbiased only in the very special case of a single engine plane with new engines and only when the engine lives are independent identically distributed exponential random variables (Cox Chapter 2 [12]). This is fallacy #1. It is faintly possible that the procedure for estimating replacement requirements is asymptotically unbiased for a large fleet of simultaneously operating aircraft. However, one condition for this to be true is that all engines have identically distributed operating times until replacement (Cox Chapter 6 [12]). This assumption is not true for engines that have accumulated some operating hours at the beginning of the calendar period under consideration unless all engines have exponentially distributed times till replacement. Some indication of the error and direction of the bias in the estimate of N(t) is possible from analytical results. This is discussed further in section 9, part II. Meanwhile, the only unbiased estimator of the expected engine requirements <u>must be</u> obtained by simulation.

A procedure for computing the number of spare engines needed to meet flying hour requirements with 90% probability is given in AFM 400-1 Chapter 8 [13]. The procedure is based on the approximation alluded to in the previous paragraph. The approximation is that replacement requirements will have a Poisson distribution with a parameter which is the inverse of the mean time between replacements. The 90th percentile of the Poisson distribution is used to determine replacement requirements. Again, this approximation is not valid because engine ages at replacement are not exponential, but it may be possible to strengthen the approximation or change to an alternative normal approximation. This is considered in Section 3, Part II. Meanwhile, simulation remains the only unbiased procedure for predicting confidence limits on spare engine requirements.

The second fallacy is to use the age of an engine at the beginning of the current calendar quarter to compute the residual operating time to removal as if the engine had survived to the age recorded at the beginning of the quarter. Some engines used during a quarter had previously failed, and their ages at the beginning of the quarter may be their age at failure rather than the survival time. The question of whether this fallacy is biasing predictions of engine operating times requires further testing. Meanwhile, we will use the engine age at the beginning of a quarter as a survivor's age rather than a failure time to keep results comparable with Air Force results.

Prediction of the third measure needed for engine management, the calendar times of replacement of particular engines, has not been done by the actuarial method although the ingredients are available to do that. It is possible to do this by simulation.

L.A. Coco and Dale Plank of AFLC have written computer programs to simulate engine requirements based on the flying hour program, the actuarial failure rates, and the ages of available engines. A simplified flow chart of this program is shown in Figure 14. It is essentially a randomized version of the actuarial method for estimating replacement requirements. Instead of estimating the proportion of removals for engines in each actuarial age interval as the actuarial failure rate, the program simulates whether each engine survives each actuarial age interval required of it to achieve the flying hour program. The simulation is done by generating a random decimal between zero and 1.0. If the decimal is larger than the failure rate, the engine survives the age interval and contributes that amount of time to the flying hour program. Since failure rates are small, many random decimals are needed and the program execution time is so long that only one run through the flying hour program is possible. This procedure yields an unbiased estimate of replacement requirements but no indication of its accuracy and no way to predict confidence limits on spare engine requirements. Repeated runs through the program would generate many values of replacement requirements for each calendar period in the flying hour program from which an indication of the accuracy of estimated replacements and of the confidence limits could be obtained. However, the alternative simulation approach in the next section is more efficient and is not dependent on the type of estimator used to simulate engine age at replacement.

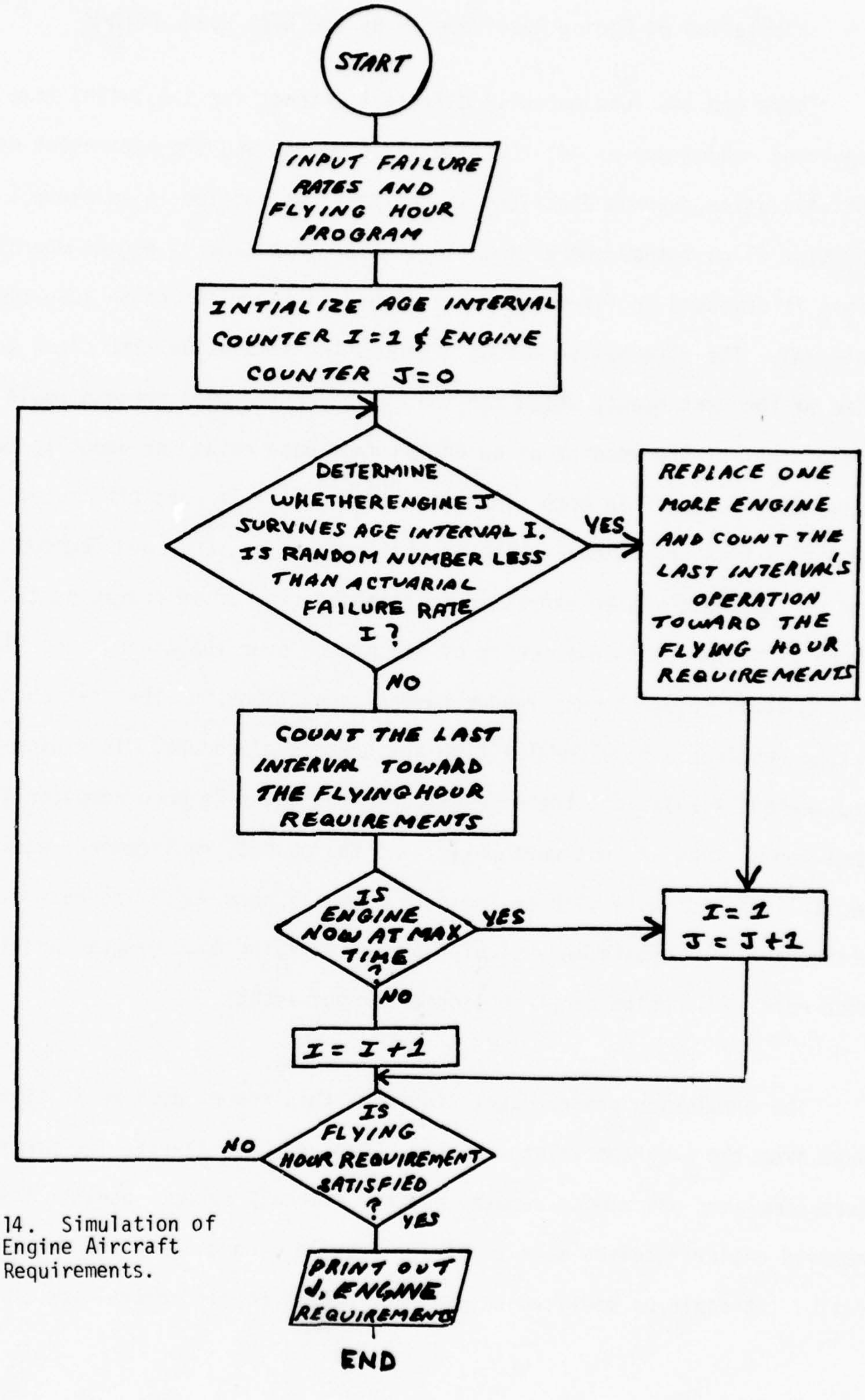

Figure 14. Simulation of Single Engine Aircraft Engine Requirements.

8. Simulation of Engine Requirements by the Next Event Method

There are two fundamentally different methods for simulating time dependent random processes, the time slice method and the next event method. The simulation program described in the previous section is an example of the time slice method where time (in that program time is engine operating time) is advanced in fixed increments equal to the width of an actuarial age interval. The alternative method advances the simulation time clock to the time of the next event, which for the engine replacement process would be a replacement or the receipt of an engine from base repair or depot to be returned to stock. In both simulation methods, it is possible to synchronize operating time and calendar time so that returns to stock and demands for replacements may be expressed in calendar time or in operating time. There is no inherent superiority of one method over the other. The time slice method requires many random number generations, but the next event method requires a more complex time advancement procedure. In engine replacement simulation, the time slice method takes so much computer time that repeat runs are not worthwhile, and the output, replacement requirements, is an estimate with no indication of its accuracy. The next event method quickly repeats many simulations of a flying hour program providing much more information about replacement requirements.

The simulation program submitted with this report differs in several ways from the programs written by L.A. Coco and Dale Plank. The program here simulates one random removal age and randomly selects whether the removed engine receives base repair or rebuild according to the return ratio. It could be modified to generate both a repair removal age and a rebuild removal age and direct the engine to base repair or depot depending whether the repair removal age or rebuild removal age was smaller. On the other hand, a replacement model will be described in Section 9, Part II which generates combined removals and splits removed engines for repair and rebuild according to an age dependent probability of rebuild. This model may be appropriate since older engines are more likely to be rebuilt. The age dependent rebuild probability could be incorporated into the computer program. It is not clear at this time which option is preferable.

Another difference of the program submitted with this report is that no attempt has been made to accurately represent the repair and rebuild time. Arbitrary calendar time intervals were chosen after which an engine removed for repair or rebuild rejoins the spare engine inventory. Its age is reset to zero if it was rebuilt. These repair and rebuild times could be randomized if desired, but AFLC already has a computer program, the JEMS (Jet Engine Management Simulator), which thoroughly represents every aspect of repair and rebuilding. However, it simulates calendar time between removals as exponential random variables. The simulation program submitted with this report may be adapted to run as a FORTRAN subroutine of the JEMS program or it may be worthwhile to rewrite both programs in a language more suitable for representation of the replacement and repair processes with their different time scales, operating time and calendar time. GASP IV is a candidate language (Pritsker [15]) since it is a FORTRAN based language which can handle events occurring in continuous time such as removals in operating time and events occurring in discrete time such as repairs or rebuilds which need not be monitored as accurately as is possible with the next event simulation method.

One objective of the actuarial method is to obtain an estimate of replacement requirements. Obtaining a good estimate of the actuarial failure rates is only part of the job. The replacement age estimate must then be transformed into an estimate of replacement requirements. The only theoretical property true of replacement requirement estimates obtained from actuarial failure rates, the product limit estimator, and the maximum likelihood estimator is that they are all maximum likelihood estimators for their respective sample information and replacement age models. Further comparison must be done by comparing their information matrices or by empirical means such as simulating a replacement process and comparing the results where the engine life data is generated from the estimators of replacement age being compared. That will be done now, to illustrate the next event method of simulation and to compare the estimates of replacement requirements obtained from estimates of the removal age failure rates and cumulative distribution function.

In order to illustrate the next event method and to compare it with the time slice method, 20 repeat runs of a 300 hour flying program for one single engine aircraft will be simulated by hand. The number of engines required to achieve the required flying hours will be collected for each run from which the mean, variance, percentiles, and the cumulative distribution function will be estimated.

In Table 5 engine ages at removal are simulated from each of the estimators including the "actuarial cumulative distribution function" obtained from the actuarial failure rates. This function was graphed in Figure 3, Section 5, Part I. Table 6 shows the calculation of engine requirements based on simulated operating times from Table 5. Figure 15 illustrates the results simulated from the theoretical and product limit cumulative distribution functions. Table 7 summarizes the frequency data from Table 6 and graphs for comparison are in Figures 16,17, and 18. The product limit estimates were as close or closer to the theoretical values of all measures as any estimates. The actuarial failure rate estimates didn't stand a chance because the engine lives generated from actuarial failure rates used many random numbers not used in the other simulations. Because of this, the actuarial failure rates were transformed into an actuarial cumulative distribution function from which engine lives were generated from the same random numbers used to generate engine lives from the product limit and theoretical distributions. (The maximum likelihood estimator mentioned in part I and derived in Appendix A is equivalent to the product limit estimator for simulation of replacement requirements.)

In Table 5, Column 1 is the index number of the engines needed to operate twenty planes over a period of three hundred flying hours each. Only ten engines are shown. Approximately 65 engines were required. Column 2 are the random numbers taken from Mihram [2]. The first random number is used to generate the engine lives for columns 3, 4, and 5. All the random numbers listed were used for column 6, the actuarial failure rate engine life. Column 3 is the engines life generated from the cumulative distribution function obtained from the actuarial failure rates. That cumulative distribution function is plotted in Figure 3 . Column 4 is the engine life generated from the product limit cumulative distribution function in Figure 4 . Column 5 is the engine life generated from the theoretical cumulative distribution function in Figure 3 . The procedure for columns 3, 4, and 5 is to take the random number in column 2, find it on the vertical axis of the graphs in Figures 3 and 4, move across until it meets the function, look down to the time axis, and record the removal time. Column 6 is the actuarial failure rate removal time. These times were generated from Table 2, the actuarial interval failure rates. The first random number listed in column 2 is compared with the first interval failure rate in Table 2, Column 8. If the random number is less than the interval failure rate, the actuarial failure rate life becomes the mid-point of the actuarial age interval. If the random number is greater than the interval failure rate, take the next random number in column 2 and compare it to the second interval failure rate in Table 2. Continue in this way until the engine life time for engine number one is found, then repeat this procedure for the next engine. For maximum time, the failure rate is infinity, so if the engine flies this long it will have flown two hundred hours.

The number of engines required for twenty aircraft varies slightly for each method. Each of the engine lives generated has been assumed to be for a new engine, so each time an engine is removed a new engine replaces it. Table 6, Column 1 gives the number of the aircraft that was used. Columns 2, 4, 6, and 8 are the engines lives used by each plane to fly three hundred hours for the actuarial cumulative distribution function, product limit, theoretical cumulative distribution function, and actuarial failure rates respectively. Columns 3, 5, 7, and 9 are the numbers of engines required for each plane for the actuarial cumulative distribution function, product limit, theoretical cumulative distribution function, and actuarial failure rates respectively. The engine lives for each plane to fly three hundred hours are taken from Table 5. The engine lives were taken in chronological order from Table 5 until the total flying time was greater than or equal to three hundred hours. For the actuarial cumulative distribution function, the first aircraft took six engines to reach three hundred flying hours. The first six removal times from Table 6, Column 2 are 33, 90, 47, 33, 16, and 200 which totals 419 hours. The excess time was not carried over to the next aircraft, so the engine lives for the second aircraft are 200 and 200. In Column 3 the number of engines required is the number of engine lives needed to reach or exceed three hundred flying hours, in this case, six engines are needed for the first aircraft.

Figure 15 is a chart of the product limit engine lives and the theoretical engine lives. It shows when the engine replacements occurred over the time period of three hundred hours. Frequently data is tabulated in Table 7. Column 1 is the number of engines that are required for an aircraft. Columns 2, 3, 4, and 5 are the number of aircraft that required the number of engines specified in Column 1. Figures 16, 17, and 18 are the cumulative distribution functions taken from the frequency table for each of the three ways engine requirements were generated. They are compared to the theoretical engine requirements cumulative distribution function.

A computer program has been written for comparing estimators on the basis of how well they simulate engine requirements for one single engine aircraft. The program does not yet simulate engine requirements from the theoretical distribution of engine lives but this will be incorporated soon and full scale testing could be done for larger samples than used in this section.

TABLE 5

RANDOM ENGINE LIVES FOR REPLACEMENT SIMULATION

Column Number	1	2	3	4	5	6
Column Name	Engine Number	Random Number	Actuarial c.d.f. Engine Life	Product Limit Life	Theoretical Life	Actuarial Failure Rate Life
	1	.153	3.3	10	38	20
	2	.337,.079 .165	90	92	85	100
	3	.203	47	33	52	20
	4	.162	33	33	41	20
	5	.078	16	10	21	20
	6	.876,.58, .09	200	200	200	100
	7	.778,.30,.46, .35,.10	200	200	200	180
	8	.909,.79,.45, .69,.18	200	200	200	180
	9	.931,.63,.53, .10,.43	200	200	200	200
	10	.326,.92,.56, .63,.07	89	82	82	180

TABLE 6

REPLACEMENT REQUIREMENTS FOR 300 FLYING HOURS

Column Number	1	2	3	4	5	6
Column Name	Aircraft Number	Engine Times from Actuarial c.d.f.	Engine Requirements	Engine Times from Product Limit	Engine Requirements	Engine Times for Theoretical Lives
	1	33+90+47+33+16+200	6	10+92+33+33+10+200	6	38+85+52+41+21+200
	2	200+200	2	200+200	2	200+200
	3	200+89+200	3	200+82+200	3	200+82+200
	4	200+200	2	200+200	2	164+200
	5	56+32+63	4	200+68+10+68	4	200+57+34+68

	7	8	9
Aircraft Number	Engine Requirements	Engine Times from Actuarial Failure Rates	Engine Requirements
1	6	200+100+20+20+20+100+180	7
2	2	180+200	2
3	3	180+100+180	3
4	2	200+100	2
5	4	200+100	2

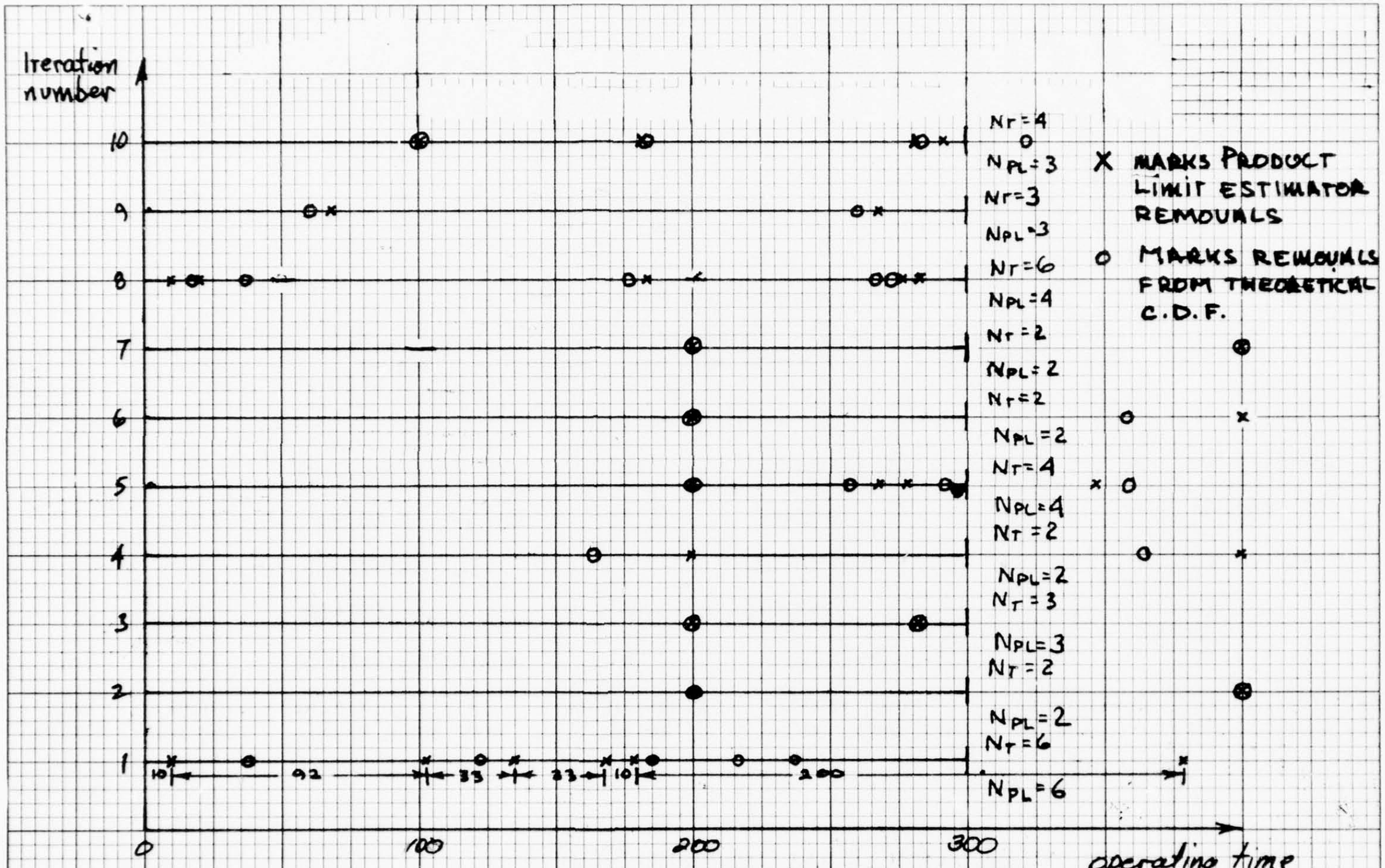

Figure 15. Diagram of Chart of Simulation: Product Limit and Theoretical C.D.F. Used for Generating Engine Lives.

TABLE 7

FREQUENCY DATA ON ENGINE REQUIREMENTS

Column Number	1	2	3	4	5
Column Name	Number of Engines Required	Actuarial c.d.f. Frequency	Product Limit Frequency	Theoretical c.d.f. Frequency	Actuarial Failure Rates Frequency
	2	8 (.4)	7 (.35)	7 (.35)	9 (.45)
	3	6 (.3)	7 (.35)	7 (.35)	9 (.45)
	4	4 (.2)	1 (.05)	3 (.15)	1 (.05)
	5	0	2 (.1)	0	0
	6	2 (.1)	3 (.15)	2 (.1)	0
	7	0	0	1 (.05)	1 (.05)
		20	20	20	20
90th Percentile		6	6	7	7
80th Percentile		6	6	6	4
70th Percentile		4	6	6	3
Mean		3.10	3.35	3.30	2.75
Variance		1.57	2.13	2.22	.93
Standard Deviation		1.25	1.46	1.49	.97

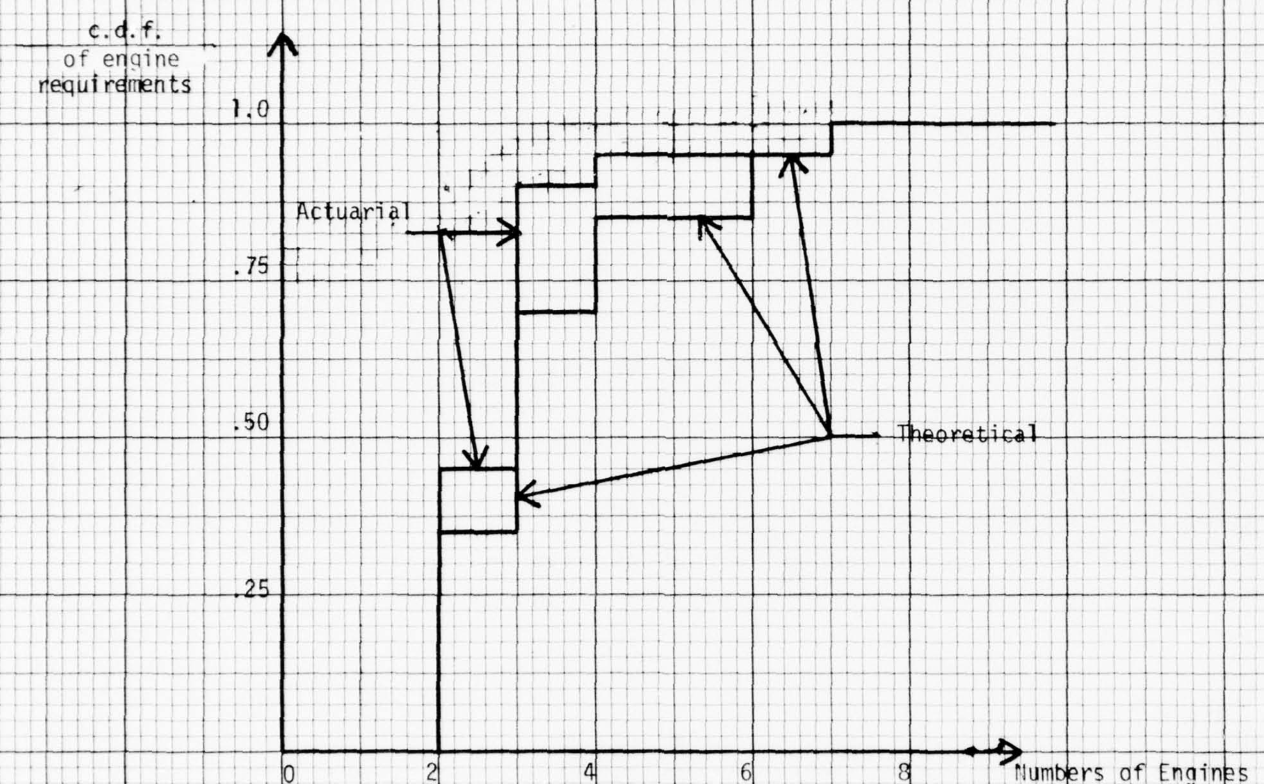

Figure 16. Actuarial Failure Rates Used to Simulate Replacement Requirements.

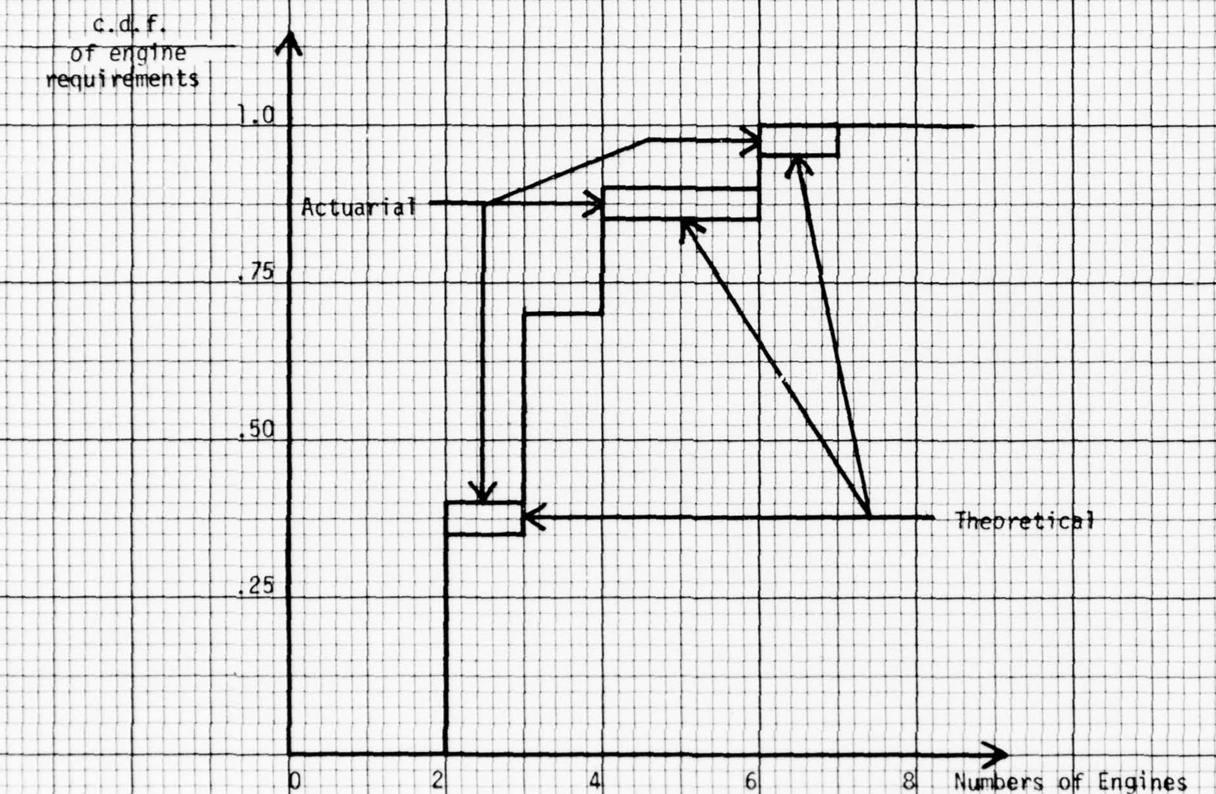

Figure 17. Actuarial c.d.f. Used to Simulate Replacement Requirements.
(20 Iterations 300 Flying Hours)

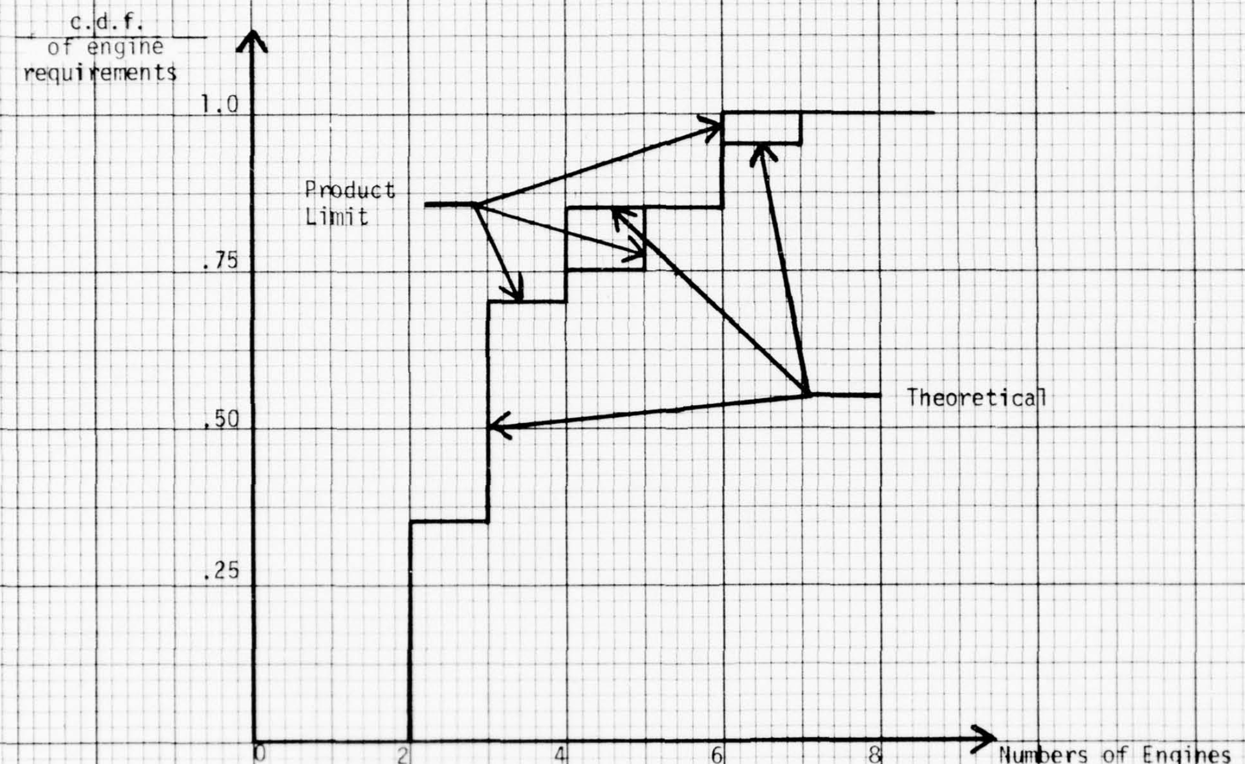

Figure 18. Product Limit Estimator Used to Simulate Replacement Requirements.

9. The Actuarial Poisson Approximation to Spare Engine Requirements and Other Analytical Approximations

It is not sufficient for engine management to predict only the expected number of engine replacements that require repair or rebuilding. The Air Force must plan spares requirements to give a comfortably high probability of meeting mission requirements, usually 80% or 90%, i.e., the number of spares must be sufficient to meet replacement requirements for the flying hour program with probability .8 or .9. In order to determine these numbers, the 80th and 90th precentiles of the replacement requirements' distribution must be estimated. The actuarial method contains a procedure to do this. The simulation program can be easily revised to produce these estimates. Other analytical approximations may be possible, perhaps combined with simulation. Each of these approaches will be described in this section and the foundations for the Poisson approximation will be reviewed to show a potential improvement in its accuracy.

The actuarial method for estimating safety stock levels in AFM 400-1 Chapter 8 [13] is based on the assumption that demand for spare engines has a Poisson distribution with rate parameter or expected demand per unit calendar time equal to the rate that engines require repair or rebuilding. (If there are fewer spares than will eventually be in repair or are being rebuilt, then some aircraft will not have engines.) The Safety Level Table Figure 8-1 of AFM 400-1 is a table of the 90th percentile of a Poisson distribution for different values of its parameter. The Poisson assumption is unlikely to be true exactly, but the conditions for use of a Poisson approximation may be near realization. Those conditions are (Cox [12] Chapter 6):

1. There are many simultaneously operating or installed engines, and

2. The distribution of the residual calendar times from engine installation to removal is the same for all engines and those times are independent.

Condition 2 may be violated for multiple engine planes and it may also be violated because some engines are not new at installation.

The conditions above should be tested if the Poisson approximation for spare engine requirements is to be used. If the conditions are realized, then the Po on approximation should be modified to take into account the initial failure rate after installation. Unless the calendar time between engine installation and removal is exponentially distributed, the initial failure rate can be used to improve the Poisson approximation. (If the initial failure rate is higher than average, more spares are required.)

Meanwhile, the next event simulation of replacement requirements can give all information about the distribution of replacements including percentiles. For instance the 90th percentile of simulated replacement requirements are obtainable from Figures 16 , 17, and 18 , and they are shown in Table 7. They are obtained by reading the figures across from the vertical axis at probability value 0.90 and reading down to the horizontal axis to obtain the value of engine requirements that was not exceeded in 90% of the simulation runs. The accuracy of this estimate could be estimated from many repeated simulation runs.

Another approximation to the number of replacement requirements is possible by an adaptation of the central limit theorem. It is known that the number of replacements with new, independent, identical components has approximately a normal distribution after a long time, regardless of the life distribution of the components (Cox Chapter 3 [12]). The central limit theorem is remarkably robust and will tolerate some variation in the residual operating time distribution of the components, (Lindberg Conditions, Gnedenko, Chapter 8 [23]). This tolerance is probably sufficient so that the normal approximation to replacement requirements is still adequate for engines even though some engines are not new at installation. This long time approximation could be combined with a simulation to determine replacement requirements for the near future. Some additional research is required here to develop the normal approximation.

The temptation is always present to assume a simple model of engine operation for convenience in predicting replacement requirements. It would also be desirable to incorporate additional information about an engine's history into a model of engine operation to obtain more accurate prediction of removal times and replacement requirements. Several approaches are possible that should be explored.

1) Postulate that the failure rate function of the removal times (regardless of whether the removal is for repair or rebuild) is of the form (Cox [14])

$$r(t) = r_0(t)e^{-Z\beta}$$

where Z is a row vector of concomittant variables with some relation to engine age at removal and β is a column vector of constants. Likely concomittant variables are the number of prior repairs and the age at last installation.

2) Postulate that the process causing combined removals interacts with an age dependent probability that determines whether the engine is repaired or rebuilt. (Older engines are more likely to be rebuilt than nearly new engines when removed.)

3) Postulate that the replacement requirements process is a Markov process in operating time and that there is a known random transformation from operating time to calendar time which may depend on the Markov process. The number of replacements required is also a Markov renewal process under rather general conditions.

Each of these three postulated models has properties that make them useful for different aspects of engine management. The first model is convenient to estimate and can be used to determine trends in the performance of engines (Tarone [18]). The second model is convenient for determining operating time between engine overhauls. The third model converts replacements occurring in operating time to replacements in calendar time as a function of the flying hour program and the rate at which each aircraft contributes to the program.

SECTION 3

RECOMMENDATIONS

Recommendation 1: Do not smooth the "crude" actuarial failure rates.

The smoothing procedure is a vestige of the life insurance industry where there was no reason to assume human failure rates were not smooth. The converse is true for engine failure rates due to component failure and the high probability of removal at inspection. To smooth the failure rates is to suppress the information most useful for engine diagnosis and for prediction of replacement requirements. Recommendation 1 can be adopted immediately and will only reduce computation.

The actuarial method employs a linear extrapolation of earlier failure rates where data grows sparse and the actual data is not used. This data can and should be used in calculation of the engine removal time distribution and failure rates. Several recommendations to follow will use all data. Even if none of them is adopted, it is still possible to get enough observations in each actuarial age interval to estimate meaningful failure rates by making some of the intervals larger. The ultimate extension of variable actuarial age interval estimators is the estimator with exactly one removal per interval. Because this estimator, called the "product limit" estimator, was already available, no further study of actuarial type estimators was done. Unfortunately this estimator does not take into account the age of engines at the beginning of some calendar period of operation, information which now may be useful for some purposes. This leads to two recommendations for further development.

Recommendation 2: Develop actuarial estimators with flexible age intervals and modify the product limit estimator to take into account engine ages at the beginning of some period or their ages at the latest installation if they were installed after the beginning of the period.

Recommendation 3: Determine whether it is worthwhile to use the additional information on ages for all actuarial purposes.

The estimator with one removal per age interval uses all engine operating time information, whether a removal time or the age of a still operating engine, to obtain the cumulative failure rates or the cumulative distribution function of engine removal times. This type of estimator also can provide every information product now computed from the actuarial failure rates, and the resulting products will usually be more accurate because the actual removal time information is fully used. The product limit estimator requires more computation than the actuarial estimator, but the required computation is well within the capability of current computers.

The second recommendation requires applied statistical research and the third requires some testing. If the second recommendation is successful, there is no reason to pursue the third. There is every reason to expect success on the second recommendation because there are possible estimators that exist or could easily be developed to use all the information now available on engine ages and removal times. As part of this research contract, the maximum likelihood estimator for an engine removal time model that distinguishes usage, inspection and max time removals was derived for a sample of removal times and survivors ages. The derivation can be modified to incorporate engine ages at the beginning of some calendar period or at last installation. That is recommendation 4.

Recommendation 4: Derive the maximum likelihood estimator for the removal time model with statistically independent usage and inspection removal times. Test whether this model is appropriate for engine removals.

If the model is appropriate, an estimator will be available that is expected to be simpler to use than the product limit estimator, and it also gives an estimate of usage removal times, useful for engine diagnostics and for curve fitting attempts to identify a parametric model of engine usage removal times.

One question that should be addressed by the actuarial statistical test is - does the official failure rate reflect the performance of the current population of engines? The answer is provided by comparing the data set from which the official failure rates were computed with the data set containing current engine removal data. The updating program submitted with this report provides a statistically valid measure that the two data sets really came from the same population. It is designed to compare all past data sets with current data on removal times and survivors' ages.

Recommendation 5: Adopt the engine removal data updating program submitted with this report to replace the current statistical test for changing the official failure rates age interval by age interval.

This updating program does not use ages of engines at the beginning of any period or at installation. A modification has been proposed that will do this but it requires further development.

Recommendation 6: Develop the modified statistical two sample Wilcoxon rank test to incorporate engine removal times, survivors' ages at the end of the operating periods, and ages at the beginning of the period or at installation whichever is later.

The actuarial procedure for computing confidence limits for spare engine requirements recognizes that demands do not always equal expected demand, so additional spares are required to keep the probability of running out to an acceptable level. However the procedure is based on the assumption that demands have a Poisson distribution. This is true only under very special conditions or when the number of operating engines is very large and all replacements are new. The Poisson assumption should be revaluated with two possibilities in mind. First, the Poisson approximation may be strengthened by a second order term involving the initial failure rate. Second, the normal approximation may be adapted to provide a better estimate of engine requirements after the engines have been in use for some time. These observations lead to recommendation 7.

Recommendation 7: Revaluate the spare engine requirements computation now based on the assumption of a Poisson demand distribution.

Other data related to engine removal times is available and should be used to obtain more accurate estimates of engine removal times and consequently of replacement requirements; data such as sorties, engine cycles, prior repairs, X-ray and chemical analyses. Development and testing of more comprehensive statistical estimators is necessary for significant improvement beyond the expected improvement from recommendations 1-6.

Recommendation 8: Support the research necessary to develop more comprehensive estimators of engine removal times.

Presently, simulation is the only statistically valid way to estimate engine requirements. The simulation program now used to predict engine requirements simulates each engine through each actuarial age interval. The result is a time consuming program which gives only an estimate of expected replacement requirements but no confidence limits and no indication of the accuracy of the requirements estimate. The next event method can simulate fleet operation many times and obtain information on confidence limits, accuracy of estimates, and even estimates of when engine removals will occur. The method can be used whether actuarial failure rates used or the product limit estimator of recommendation 2 is used.

Recommendation 9: Change the simulation program for replacement requirements from an interval by interval simulation to a next event type simulation.

The program prepared for this report contains the ingredients for accurate simulation of engine removals but does not make any pretense of accurately representing the repair and rebuilding of engines. This has already been done in the JEMS program developed at AFLC.

Recommendation 10: Combine the next event type simulation of the engine removals and the JEMS program simulating repair and rebuilding. This will have to be done in close cooperation with AFLC because of their thorough knowledge of the repair process for engines.

The resulting program is likely to be the largest program that can be handled in a reasonable amount of computer time. There is still a need for more accurate and more convenient methods to predict replacement requirements. A normal approximation conditioned on the ages of the present stock of engines should be developed for estimating engine requirements.

Recommendation 11: Support the research and development necessary for a simulation of short term engine requirements combined with a long term estimate based on a normal approximation.

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APPENDIX A

THE MAXIMUM LIKELIHOOD ESTIMATOR FOR A

MULTIPLE RISK MODEL WITH A CENSORED SAMPLE

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I

INTRODUCTION

The ultimate objective is to derive the maximum likelihood estimator of the cumulative distribution function (c.d.f.) of engine removal times from a progressively censored sample. The c.d.f. is assumed to be of the form

$$F(t) = 1-\overline{F}_1(t) \left\{ \prod_{i=1}^{i(t)} (1-p_i) \right\} \qquad (1)$$

where $\overline{F}_1(t)$ is the tail c.d.f. of another non-singular c.d.f. possibly truncated and representing usage removals, and the p_i are the unconditional probabilities of removal at the i^{th} inspection time (fixed) where $i(t)$ is the index of last inspection prior to or at time t. Throughout this derivation time is measured in flying hours, not calendar time. Progressive censoring occurs because estimation of the c.d.f. takes place at fixed calendar times, and in addition to data on the flying hours at removal of engines that have been replaced, the current accumulated flying hours of installed engines that have not yet been removed is also known.

The plan of this derivation is in three stages. First, for the sake of review, the empirical distribution function for an ordered random sample of engine removal times $T_1 \leq T_2 \leq \ldots \leq T_N$

$$\hat{F}_n(t) = \begin{cases} 0 & t < T_1 \\ i/N & T_i \leq t < T_{i+1} \\ 1 & t \geq T_N \end{cases} \qquad (2)$$

will be derived by the method of maximum likelihood. Then the maximum likelihood estimators of $F_1(t)$ and the p_j, j = 1, 2,..., k, will be derived for an uncensored sample of engine removal times. Substitution of those estimators into the c.d.f. (1) yields the empirical distribution function as an estimate of the model (1) but with additional information about usage removals and inspection removals. Last, the maximum likelihood estimators of $F_1(t)$ and the p_j are derived from a progressively censored sample. Substitution into (1) yields the "product limit" estimator of Kaplan and Meier [1].

SECTION II

MAXIMUM LIKELIHOOD DERIVATION OF THE EMPIRICAL DISTRIBUTION FUNCTION

Given an ordered random sample from an arbitrary c.d.f., it is frequently assumed that the mass of the estimator will be placed only at observed data in the sample. Then the likelihood function of the sample is the product of masses f_i, $i=1,2,\ldots,N$, at the data values $t_1 \leq t_2 \leq \cdots \leq t_N$,

$$L(t_1,\ldots,t_N;\ F(t)) = \prod_{i=1}^{N} f_i$$

and the masses are subject to the constraints of non-negativity and

$$\sum_{i=1}^{N} f_i = 1$$

so that the estimator of F(t) is a c.d.f. The values of f_i that maximize the likelihood function are all equal to 1/N. This can be verified by the Lagrange multiplier method or by substitution of the constraint

$$f_N = 1 - \sum_{i=1}^{N-1} f_i$$

into the likelihood function and setting derivatives of the log likelihood function with respect to f_i equal to zero,

$$\frac{\delta \log L}{\delta f_i} = \frac{\delta}{\delta f_i} \left\{ \sum_{j=1}^{N-1} \log f_j + \log\left(1 - \sum_{j=1}^{N-1} f_j\right) \right\}$$

or

$$\frac{\delta \log L}{\delta f_i} = \frac{1}{f_i} - \frac{1}{1-\sum_{j=1}^{N-1} f_j} = 0$$

which gives

$$f_i = 1 - \sum_{j=1}^{N-1} f_j$$

for all $i=1,2,\ldots,N-1$. Therefore, all such f_i are equal to $1/N$. This gives the familiar step function that jumps only at observed engine removal times.

For example, suppose there are five observations at times 10, 15, 25, 50 and 100. The empirical c.d.f. is shown in Figure 1.

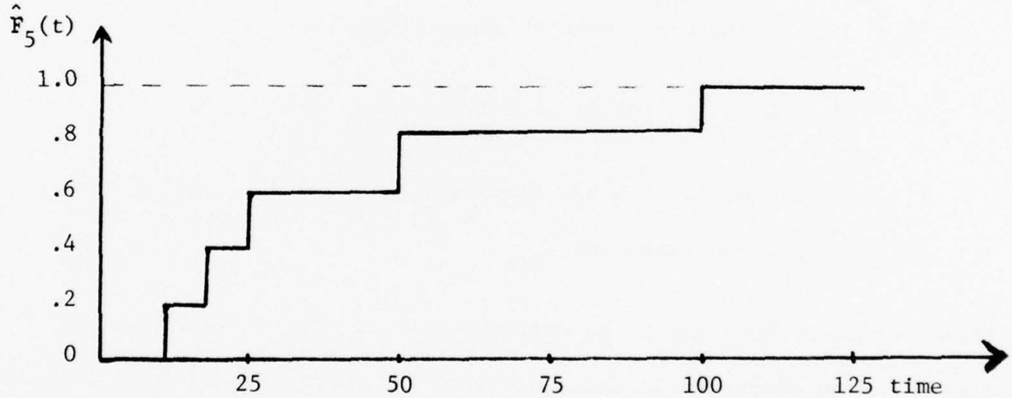

Figure 1. The Empirical c.d.f.

The properties of the empirical c.d.f. as an estimator are well known; Gnedenko 2 shows the strong consistency of the estimator.

SECTION III

MAXIMUM LIKELIHOOD DERIVATION OF THE FULL SAMPLE ESTIMATOR

The engine removal times $T_1,\ldots,T_N$ in the sample are assumed to be independent and identically distributed according to F(t), (1). In addition, the engine removal code specifies whether the removal was during usage, at inspection, or at maximum time. Define the following sequences:

$\{n_j\}$, the number of removals at the jth inspection and t_{max}, $j=1,2,\ldots,k+1$; ($t_{max} = t_{k+1}$)

$\{m_j\}$, the number of usage removals between the (j-1)st and the jth inspection;

$\{M_j\}$, the cumulative number of usage removals,

$$M_j = \sum_{i=1}^{j} m_i, \quad j=1,2,\ldots,k+1;$$

$\{t_i\}$, the sequence of usage removal times, $i=1,2,\ldots,M_{k+1}$; and

$\{t_j'\}$, inspection times and t_{max}.

These sequences turn out to be sufficient statistics. The sample size N can be expressed in terms of usage removals, inspection removals and maximum times removals as

$$N = M_{k+1} + \sum_{j=1}^{k+1} n_j$$

The likelihood function for the sample information is

$$\prod_{i=1}^{M_{k+1}} \left[f_1(t_i) \prod_{j=1}^{j(t_i)} (1-p_j) \right] \prod_{j=1}^{k} \left[p_j \prod_{i=1}^{j-1} (1-p_i) \quad (1-F_1(t'_j)) \right]^{n_j}$$

$$\left[\left\{ 1-F_1(t^-_{max}) \prod_{j=1}^{k} (1-p_j) \right\} \right]^{n_{k+1}} C$$

with $f_1(t_i)$ denoting the density of $F_1(t)$, discrete or continuous, at t_i, $t^-_{max} = t_{max} - \varepsilon$ for small $\varepsilon > 0$, and the notation $j(t_i)$ is used to indicate the index of the last inspection prior to the usage removal at t_i. The first term of the likelihood function is the probability of the usage removals, the second term is the probability of inspection removals, and the last is the probability of all the survivals to t_{max}. C is a combinatorial constant.

In order to make the likelihood function positive, all of the $f_1(t_i)$ should be positive, and to maximize it, $1-F_1(t'_j)$ and $1-F_1(t_{max})$ should be as large as possible consistent with the constraints on the distribution function $F(t)$; $F(0^-)=0$, $F(\infty)=1$, and $F(t)$ non-decreasing in t. As in the maximum likelihood derivation of the empirical distribution function, all the mass of $F_1(\cdot)$ should be placed at the observations $\{t_i\}$ and t_{max}.

Then

$$1-F_1(t'_j) = 1 - \sum_{i=1}^{M_j} f_1(t_i)$$

and

$$1-F_1(t^-_{max}) = 1 - \sum_{i=1}^{M_{k+1}} f_1(t_i)$$

However, not all of the jumps in $\hat{F}_1(\cdot)$ will be of the same size as in the empirical distribution function. The derivative of the natural logarithm of the likelihood function with respect to $f_1(t_i)$ gives

$$\delta \log L/\delta f_1(t_i) = 1/f_1(t_i) - f_1(t_i)/\{1 - \sum_{i=1}^{M_j} f_1(t_i)\}$$

for all $i=M_{j-1}+1,\ldots,M_j$, $j=1,2,\ldots,k+1$. This indicates that the maximum likelihood estimators of the jumps will be the same in intervals before the first inspection, between inspections, and after the last inspection. These jump sizes will be denoted by $f_1,f_2,\ldots,f_{k+1}$. The likelihood function now simplifies to

$$\prod_{i=1}^{k+1}\left[f_i \prod_{j=1}^{i-1}(1-p_j)\right]^{M_i - M_{i-1}} \prod_{j=1}^{k+1}\left[p_j \prod_{i=1}^{j-1}(1-p_i)\left\{1 - \sum_{i=1}^{j} f_i(M_i-M_{i-1})\right\}\right]^{n_j} C$$

where $p_{k+1} = 1$. The log likelihood function is

$$\sum_{i=1}^{k+1}(M_i-M_{i-1})\{\ln f_i + \sum_{j=1}^{i-1}\ln(1-p_j)\} +$$

$$+ \sum_{j=1}^{k+1} n_j\{\ln p_j + \sum_{i=1}^{j-1}\ln(1-p_i) + \ln(1-\sum_{i=1}^{j}(M_i-M_{i-1})f_i)\} + \ln C$$

Take the derivative of the log likelihood function with respect to p_j and set it equal to zero, $j=1,2,\ldots,k$,

$$\frac{\delta \log L}{\delta p_j} = \sum_{i=j+1}^{k+1} (M_i - M_{i-1}) \left(\frac{-1}{1-p_j}\right) + \frac{n_j}{p_j} - \sum_{i=j+1}^{k+1} \frac{n_i}{1-p_j} = 0$$

Remove the $1-p_j$ terms from the summations to obtain

$$\frac{n_j}{p_j} = \frac{1}{1-p_j} \sum_{i=j+1}^{k+1} (M_i - M_{i-1} + n_i)$$

or

$$n_j = p_j \left\{ M_{k+1} - M_j + \sum_{i=j}^{k+1} n_i \right\}$$

This gives the maximum likelihood estimator

$$\hat{p}_j = n_j / \left\{ M_{k+1} - M_j + \sum_{i=j}^{k+1} n_i \right\}$$

which may be rewritten as

$$\hat{p}_j = n_j / \left\{ N - M_j - \sum_{i=1}^{j-1} n_i \right\} \quad j=1,2,\ldots,k$$

where the denominator is the number of survivors to the jth inspection. Thus $\hat{p}_j$ is simply the proportion of survivors removed at the jth inspection.

Take the derivative of the log likelihood function with respect to f_i and set it equal to zero, $i=1,2,\ldots,k+1$,

$$\frac{\delta \ln L}{\delta f_i} = (M_i - M_{i-1}) \frac{1}{f_i} - \sum_{h=i}^{k+1} \frac{n_h (M_i - M_{i-1})}{1 - \sum_{j=1}^{h} (M_j - M_{j-1}) f_j} = 0$$

Divide out $(M_i - M_{i-1})$ to obtain the general relation for the maximum likelihood estimators of f_i

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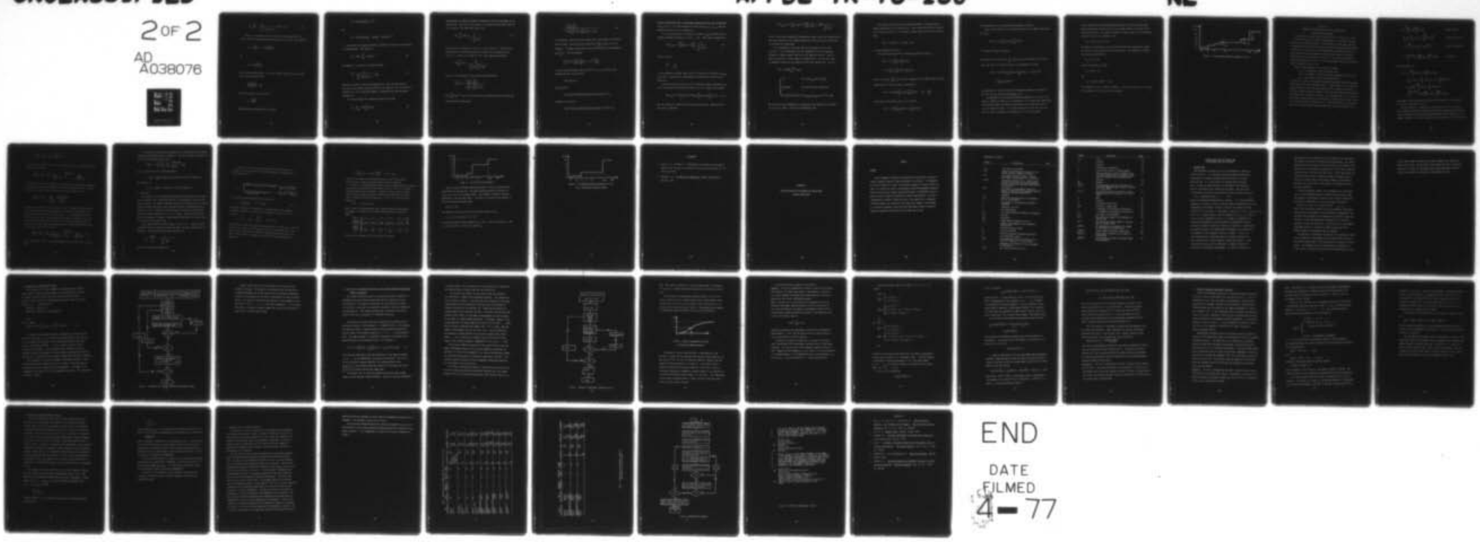

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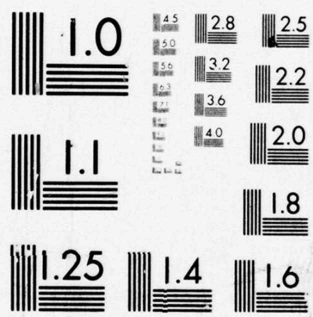

MICROCOPY RESOLUTION TEST CHART
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$$\frac{1}{f_i} = \sum_{h=i}^{k+1} \frac{n_h}{1 - \sum_{j=1}^{h} (M_j - M_{j-1}) f_j} \qquad i=1,2,\ldots,k+1 \tag{3}$$

Suppose for the moment that there is only one inspection time, k=1. Then to obtain the maximum likelihood estimators of f_1 and f_2, the equations

$$\frac{1}{f_1} = \frac{n_1}{1-m_1 f_1} + \frac{n_2}{1-m_1 f_1 - m_2 f_2}$$

and

$$\frac{1}{f_2} = \frac{n_2}{1-m_1 f_1 - m_2 f_2}$$

must be solved simultaneously. The first equation contains $1/f_2$ as the last term, and rearrangement gives

$$\frac{1-m_1 f_1 - n_1 f_1}{f_1(1-m_1 f_1)} = \frac{1}{f_2}$$

The second equation can be solved for

$$f_2 = \frac{1-f_1 m_1}{n_2+m_2}$$

Combining these two expressions for f_2 gives

$$\hat{f}_1 = 1/(n_1+n_2+m_1+m_2) = 1/N$$

and

$$\hat{f}_2 = (1/N)(n_1+n_2+m_2) \;/\; (n_2+m_2) = (1/N)(1-\hat{p}_1)^{-1}$$

Now return to the general problem of solving the system of k+1 equations (3) simultaneously. The answer is

$$\hat{f}_i = (1/N) \{ \prod_{j=1}^{i-1} (1-\hat{p}_j)\}^{-1} \tag{4}$$

The equation (3) yields a recursion relation

$$\frac{1}{f_i} = \frac{n_i}{1-\sum_{j=1}^{i} (M_j-M_{j-1}) f_j} + \frac{1}{f_{i+1}} \tag{5}$$

which may be used to verify the estimators (4). This has been done for k=2 and is very tedious, and the result (4) for larger k will be verified by induction on i for an arbitrary number of inspections $k \geq i$ from recursion relation (5).

If it can be shown that recursion relation (5) yields

$$\frac{1}{f_i} - \frac{1}{f_{i+1}} = N\hat{p}_i \prod_{j=1}^{i-1} (1-\hat{p}_j), \tag{6}$$

the induction is complete because the difference above is equivalent to the solution (4). The rest of this proof is to verify that the center ratio of (5) is equal to the right hand side of (6),

$$N\hat{p}_i \prod_{j=1}^{i-1} (1-\hat{p}_j) = \frac{n_i}{1-\sum_{j=1}^{i} m_j f_j} . \tag{7}$$

Substitute the solution (4), j=1,2,...,i, for t_j into (7). This is fair because under the induction method, the result (4) is assumed true for j=1,2,...,i, and is to be shown for i+1. This substitution yields

$$N\hat{p}_i \prod_{j=1}^{i-1} (1-\hat{p}_j) = \frac{n_i N}{N-\sum_{j=1}^{i} [m_j / \prod_{k=1}^{j-1} (1-\hat{p}_k)]} ,$$

and use of the definition of $\hat{p}_i$ allows some cancellation,

$$\prod_{j=1}^{i-1} (1-\hat{p}_j) = \frac{N-\sum_{j=1}^{i} m_j - \sum_{j=1}^{i-1} n_j}{N-\sum_{j=1}^{i} [m_j / \prod_{k=1}^{j-1} (1-\hat{p}_k)]} .$$

Now $[\prod_{k=1}^{i-1} (1-\hat{p}_k)]^{-1}$ will be factored from the denominator and will cancel the left hand side leaving only

$$1=\frac{N-\sum_{j=1}^{i} m_j-\sum_{j=1}^{i-1} n_j}{(N-m_1)\prod_{k=1}^{i-1}(1-\hat{p}_k)-m_2\prod_{k=2}^{i-1}(1-\hat{p}_k)-\ldots-m_{i-1}} \tag{8}$$

to be verified. This will be done term by term. Each product in the denominator includes 1 which will have coefficients $N-\sum_{j=1}^{i} m_j$ exactly as in the numerator. It remains to show that the rest of the terms in the denominator give $\sum_{j=1}^{i-1} n_j$. Use the expansion

$$\prod_{k=1}^{i-1}(1-\hat{p}_k)=1-\sum_{k=1}^{i-1}\hat{p}_i+\sum_{j\neq k}^{i-1}\hat{p}_i\hat{p}_j-\ldots(-1)^{i-1}\prod_{k=1}^{i-1}\hat{p}_k$$

to obtain the coefficients of $\hat{p}_k$ in terms of N, m_j, n_j, and $\hat{p}_j$ for $j\leq k$. Starting with k=1, the $\hat{p}_1$ term is

$$(N-m_1)(-\hat{p}_1)=-n_1.$$

The $\hat{p}_2$ term is

$$(N-m_1)(-\hat{p}_1)+(N-m_1)\hat{p}_1\hat{p}_2-m_2(-\hat{p}_2)=(N-m_1-m_2-n_1)(-\hat{p}_2)=-n_2.$$

Similarly the $\hat{p}_3$ term is

$$(N-m_1)[-\hat{p}_3+\hat{p}_1\hat{p}_3+\hat{p}_2\hat{p}_3-\hat{p}_1\hat{p}_2\hat{p}_3]-m_2[-\hat{p}_3+\hat{p}_2\hat{p}_3]-m_3[-\hat{p}_3]=-n_3,$$

and it is hoped that this is sufficient demonstration that the coefficients of $\hat{p}_k$, $k=1,2,\ldots,i-1$ when collected as above yield $n_1, n_2, \ldots, n_{i-1}$ with the negative sign just as in the numerator.

It is of some interest to estimate the jump at t_{max} resulting from the maximum likelihood estimator $\hat{f}_i$, $i=1,2,\ldots,k+1$. That jump is estimated by

$$\hat{F}_1(t^-_{max}) = 1-\sum_{i=1}^{k+1} m_i\hat{f}_i = 1-\frac{1}{N}\sum_{i=1}^{k+1}\frac{m_i}{\prod_{j=1}^{i-1}(1-\hat{p}_j)} \quad (9)$$

For $k=1$, this is

$$\frac{n_2}{N} \cdot \frac{1}{1-\hat{p}_1}$$

a very reasonable estimator since n_2/N is the fraction of survivors to t_{max} and $(1-\hat{p}_1)^{-1}$ represents the conditioning on surviving the first and only inspection.

A similar result may be proved for an arbitrary number inspections k by use of the method that verified the form of $\hat{f}_i$ (4). Equation (9) becomes

$$\hat{F}(t^-_{max}) = N^{-1}\left[\prod_{i=1}^{k}(1-\hat{p}_i)\right]^{-1}\left[(N-m_1)\prod_{i=1}^{k}(1-\hat{p}_i)-m_2\prod_{i=2}^{k}(1-\hat{p}_i)-\ldots-m_{k+1}\right]$$

when the product is factored from every term in the sum. Relation (8) is then used to show that

$$\hat{\hat{F}}(t^-_{max}) = N^{-1}[\prod_{i=1}^{k}(1-\hat{p}_i)]^{-1}[N-\sum_{i=1}^{k+1} m_i - \sum_{i=1}^{k} n_i] = \frac{n_{k+1}}{N}\frac{1}{\prod_{i=1}^{k}(1-\hat{p}_i)},$$

and the conditional probability interpretation again yields the interpretation that this estimator is the sample proportion of removals n_{k+1}/N conditioned on surviving all inspections.

It is reassuring to find that when the estimators for $F_1(t)$ and $p_j, j=1,2,\ldots,k$ are substituted into the assumed cdf (1), the resulting estimator of engine removal time cdf is the empirical cdf (2). This is easily verified for a small number of inspections k. For k=0, the result is immediate because the cdf jumps by 1/N at each removal time. For k=1,

$$\hat{F}(t) = 1-\hat{\hat{F}}_1(t)\prod_{i=1}^{i(t)}(1-\hat{p}_i) =$$

$$= \begin{cases} i/N & \text{for } t_i \le t < t_{i+1} \text{ and } t < t_1', i \le m_1 \\ (m_1+n_1)/N & \text{for } t=t_1' \text{ the first inspection} \\ 1-(1-m_1\hat{f}_1-i\hat{f}_2)(1-\hat{p}_1) & \text{for } t_{m_1+i} \le t < t_{m_1+i+1} \text{ and } t>t_1',\ i \le m_2, \end{cases}$$

The third term above simplifies to $(n_1+m_1+i)/N$, the proportion of removals at or prior to time t, so $\hat{F}(t)$ is the empirical cdf.

This result may be proved for an arbitrary number of inspections k by another application of the relation (8). First verify that $\hat{F}(t)$ is h/N prior to any inspection where h is the number of usage removals that have occurred, $h \leq m_1$;

$$\hat{F}(t) = 1-[1-h\hat{f}_1] = 1-(1-h/N) = h/N$$

by simple substitution for $\hat{f}_1$.

Then verify the result for an arbitrary inspection time t_j';

$$\hat{F}(t) = 1-[1-\sum_{i=1}^{j} m_i\hat{f}_i] \prod_{i=1}^{j} (1-\hat{p}_i)$$

$$= 1-[1-\sum_{i=1}^{j} [m_i/\prod_{k=1}^{i-1} (1-\hat{p}_k)]] \prod_{i=1}^{j} (1-\hat{p}_i).$$

Factor the product $\prod_{k=1}^{j-1} (1-\hat{p}_k)$ from the denominator of the $\hat{F}(t)$ term and cancel common terms of the two products, resulting in

$$\hat{F}(t) = 1-[(N-m_1)\prod_{k=1}^{j-1} (1-\hat{p}_k)-m_2\prod_{k=2}^{j-1} (1-\hat{p}_k)-\ldots-m_j] \cdot \frac{1-\hat{p}_j}{N}$$

Use relation (8) and the value of $\hat{p}_j$ to obtain

$$\hat{F}(t) = 1-\frac{1}{N}[N-\sum_{i=1}^{j} (m_i+n_i)] = \sum_{i=1}^{j} (m_i+n_i)/N$$

(The denominator of $1-\hat{p}_j$ cancelled the numerator of $\hat{\hat{F}}_1(t)$.)

Last, verify that $\hat{F}(t)$ is the empirical cdf for a t between inspections, $t'_j<t<t'_{j+1}$,

$$\hat{F}(t) = 1-[1-\sum_{i=1}^{j} m_i\hat{f}_i-h\hat{f}_{j+1}]\prod_{i=1}^{j}(1-\hat{p}_i),$$

for $h\leq m_{j+1}$ and $t_{m_j+h}\leq t<t_{m_j+h+1}$.

Substitute for $\hat{f}_i$, factor out $\prod_{k=1}^{j}(1-\hat{p}_k)$ from the denominator of the $\hat{\hat{F}}_1(t)$ term, and cancel it with the product in the numerator to obtain

$$\hat{F}(t) = 1-\frac{1}{N}\,[(N-m_1)\prod_{k=1}^{j}(1-\hat{p}_k)-m_2\prod_{k=2}^{j}(1-\hat{p}_k)-\ldots-m_j(1-\hat{p}_j)-h]$$

$$= [\sum_{i=1}^{j}(m_i+n_i)+h]/N$$

the empirical cdf from the relation (8) slightly modified to account for the fact that the product index ranges up to j.

The appearance of the maximum likelihood estimator of (1) from a full sample is similar to Figure 1 of the empirical c.d.f. except that jump sizes increase where there are multiple observations at the same time such as occur at inspection times. It is more interesting to note that the estimator of $F_1(t)$ is similar to the empirical c.d.f., but the jumps vary in height from one interinspection interval to the next because fewer engines survive the inspection and go on to usage removal in the next interinspection interval. For example, suppose the usage removal times distributed according to $F_1(t)$ were observed to be

$$\{t_i\} = \{10, 15, 25, 40, 80\}$$

and suppose an inspection was held at 50 hours when two engines were removed and three more were removed at max time, 100 hours. The estimator of inspection removal probability is

$$\hat{p}_1 = 2/6 = .333$$

and for the estimate of $F_1(t)$

$$\hat{f}_1 = 1/10 = .10$$

and

$$\hat{f}_2 = (1/10)(1-.333)^{-1} = .150$$

The estimate of $F_1(t)$ is shown in Figure 2. The mass at max time is included in the estimate of $F_1(t)$ but that is optional.

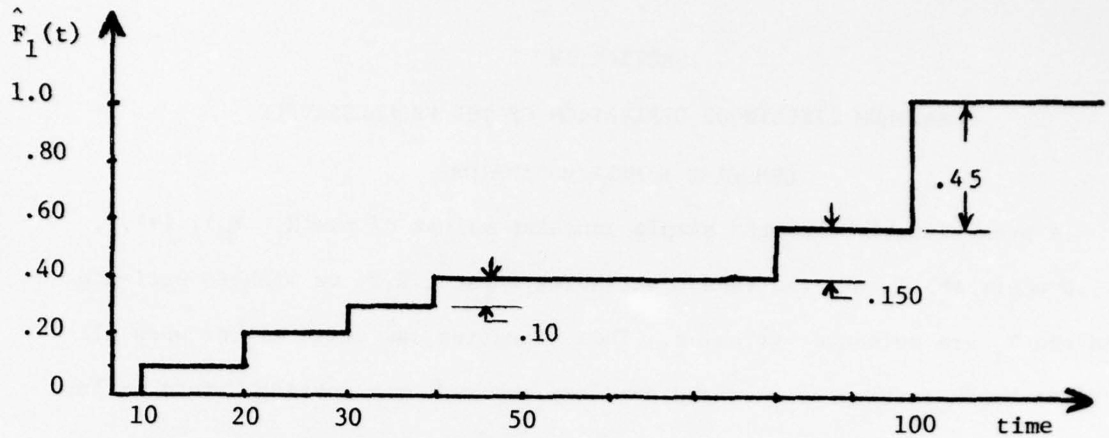

Figure 2. The Maximum Likelihood Estimate of $F_1(t)$.

SECTION IV

MAXIMUM LIKELIHOOD DERIVATION OF THE PROGRESSIVELY CENSORED SAMPLE ESTIMATOR

A progressively censored sample contains values of $\min(X_i, Y_i)$, $i=1,2,$ $..,N$ where the X_i are the random variables whose c.d.f. we wish to estimate and the Y_i are nuisance variables. This situation can arise either when all N items begin testing at once and some are removed from testing before failure at times Y_i or, as in Air Force engine management, the sample data taken at fixed times include some replacement times X_i and some ages Y_i of engines which have not yet been replaced. The age Y_i of the survivors provides additional information about the c.d.f. of the replacement times X_i and should be used in estimation of that c.d.f.

Denote ordered sequences of event times as follows:

$\{t_i\}$ - the sequence of usage replacement times,

$\{t'_j\}$ - the sequence of inspection times and maximum time if any, and

$\{t''_h\}$ - the sequence of ages of surviving engines.

As in part 3, m_i and n_j count usage removals and inspection removals, and $j(t_i)$ is the index of the last inspection prior to a usage removal at t_i. Conversely, $i(t'_j)$ is the index of the last usage removal prior to inspection at time t'_j. Similarly, $k(t''_h)$ and $j(t''_h)$ are indexes of the last usage removal time and inspection time prior to the age of a survivor, t''_h. The underlying c.d.f. of replacement times is again assumed to have mass only at observed removal times $\{t_i\}$ and $\{t'_j\}$, and these masses are denoted by $f(\cdot)$. With this notation, the likelihood function for a progressively censored sample from the multi-risk model (1) is

$$L = C \cdot \prod_{i=1}^{M_{k+1}} [f(t_i) \prod_{j=1}^{j(t_i)} (1-p_j)] \cdot \qquad \text{(usage removals)}$$

$$\cdot \prod_{j=1}^{k} [p_j \prod_{i=1}^{j-1} (1-p_i) (1- \sum_{i=1}^{i(t_j')} f(t_i))]^{n_j} . \qquad \text{(inspection removals)}$$

$$\cdot [(1- \sum_{i=1}^{M_{k+1}} f(t_i)) \prod_{j=1}^{k} (1-p_j)]^{n_{k+1}} . \qquad \text{(max. time removals)}$$

$$\cdot \prod_{h=1}^{N-N_{k+1}-M_{k+1}} [(1- \sum_{k=1}^{k(t_h'')} f(t_k)) \prod_{j=1}^{j(t_h'')} (1-p_j)] \qquad \text{(survivors)}$$

The log likelihood is

$$\ln L = \ln C + \sum_{i=1}^{M_{k+1}} [\ln f(t_i) + \sum_{j=1}^{j(t_i)} \ln(1-p_j)] +$$

$$+ \sum_{j=1}^{k} n_j [\ln p_j + \sum_{i=1}^{j-1} \ln(1-p_i) + \ln(1- \sum_{i=1}^{i(t_j')} f(t_i))] +$$

$$+ n_{k+1}[\ln(1- \sum_{i=1}^{M_{k+1}} f(t_i)) + \sum_{j=1}^{k} \ln(1-p_j)] +$$

$$+ \sum_{h=1}^{N-M_{k+1}-N_{k+1}} [\ln(1- \sum_{k=1}^{k(t_h'')} f(t_k)) + \sum_{j=1}^{j(t_h'')} \ln(1-p_j)]$$

The maximum of the log likelihood function will be derived by calculus subject to the usual constraints on c.d.f.s.

The first result to be observed is that the masses $f(\cdot)$ are all equal between neighboring inspection times and/or survivors' ages (within time intervals where the numbers of survivors and inspection removals remain unchanged). For example, subsequent to the last inspection and last survivor's age, the derivative

$$\frac{\delta \ln L}{\delta f(t_i)} = \frac{1}{f(t_i)} - n_{k+1}/(1-\sum_{i=1}^{M_{k+1}} f(t_i))$$

is the same for <u>all</u> usage removals (if any). Similarly, prior to any inspections and survivors, the derivative

$$\frac{\delta \ln L}{\delta f(t_i)} = \frac{1}{f(t_i)} - \sum_{j=1}^{k+1} \frac{n_j}{1-\sum_{i=1}^{i(t'_j)} f(t_i)} - \sum_{h=1}^{N-M_{k+1}-N_{k+1}} \frac{1}{1-\sum_{k=1}^{k(t''_h)} f(t_k)}$$

is the same for all such usage removals. The generality of the assertion is suggested by the form of the derivative subsequent to the last inspection but prior to the last survivor's age. (Assume no usage removals after the last survivor's age.)

$$\frac{\delta \ln L}{\delta f(t_i)} = \frac{1}{f(t_i)} - \frac{n_{k+1}}{1-\sum_{i=1}^{M_{k+1}} f(t_i)} - \frac{N-M_{k+1}-N_{k+1}+1}{1-\sum_{i=1}^{M_{k+1}} f(t_i)}$$

The last two terms combine to give a numerator which is the number of survivors after the last usage removal, and the combination is the same for all such removals. In order to give the general form of the derivatives, new notation must be introduced; $k(t'_j)$ is the index of the last usage removal prior to inspection at t'_j, $h(t_i)$ is the index of the last survivor time prior to the i^{th} usage removal at time t_i, and $h(t'_j)$ is the index of the last survivor time prior to the j^{th} inspection at time t'_j. The general form of the derivative is

$$\frac{\delta \ln L}{\delta f(t_i)} = \frac{1}{f(t_i)} - \sum_{j=j(t_i)}^{k+1} \frac{n_j}{1-\sum_{k=1}^{k(t'_j)} f(t_k)} - \sum_{h=h(t_i)}^{N-M_{k+1}-N_{k+1}} \frac{1}{1-\sum_{k=1}^{k(t''_h)} f(t_k)} \qquad (10)$$

for all usage removal indexes i between neighboring survivors' ages and/or inspection times.

It is very easy to show that the estimator $\hat{P}_j$ of the inspection time removal probability has the same form as in section 3. Set the derivative with respect to p_j of the log likelihood equal to zero,

$$\frac{\delta \ln L}{\delta p_j} = 0 = \frac{n_j}{p_j} - \sum_{i=j+1}^{k+1} \frac{n_i}{1-p_j} - \sum_{h=h(t_j')}^{N-M_{k+1}-N_{k+1}} \frac{1}{1-p_j}$$

The $1-p_j$ terms factor out of each sum leaving

$$0 = \frac{n_j}{p_j} - [\frac{1}{1-p_j}] \cdot [\text{number of survivors after the } j^{th} \text{ inspection}]$$

The solution is

$$\hat{p}_j = n_j \,/\, (\text{number of survivors to the } j^{th} \text{ inspection})$$

as in section 3.

The relation (10) for the partial derivatives can be rewritten in the same form as equation (3) in the previous section, and consequently, it has the same form of solution. The sample information must be re-expressed in order to show the relation between (10) and (3) and to show the solution, the maximum likelihood estimate of F(t). Combine the ordered sequences of inspection and survivors' ages into a single sequence $\{t_\ell\}$, $\ell = 1,2,\ldots,N-M_{k+1}$. Define the new sequence $\{r_\ell\}$ as the sequence of numbers of usage removal times between successive inspection times and/or survivors' ages.

It was shown earlier that the estimates of $f(t_i)$ are constant between successive survivors' ages and/or inspection times, and these constants will be denoted $\hat{f}_\ell$. The relation (10) can now be rewritten as

$$\frac{1}{\hat{f}_i} = \sum_{\ell=i}^{N-M_{k+1}} \frac{n_\ell}{1-\sum_{k=1}^{\ell} r_k \hat{f}_k}$$

and this is the same as equation (3).

In order to express the solution, a generalized form of the sequence of estimators $\{\hat{p}_\ell\}$ must be defined which includes the sequence of inspection time removal probabilities. Let

$$\hat{p}_\ell = \begin{cases} \dfrac{n_j}{\text{number of survivors to the } j^{th} \text{ inspection}} & \text{for } \ell = j^{th} \text{ inspection index, } j=1,2,\ldots,k+1 \\[2ex] \dfrac{1}{\text{number of survivors to the } j^{th} \text{ survivors' age}} & \text{for } \ell = j^{th} \text{ survivors' age, } j=1,2,\ldots,N-M_{k+1}-K-1 \end{cases}$$

so the maximum likelihood estimators may be expressed as

$$\hat{f}_i = \frac{1}{N} \prod_{\ell=1}^{i} \left[\frac{1}{1-\hat{p}_\ell}\right], \qquad i=1,2,\ldots,N-M_{k+1}. \qquad (11)$$

The numbers of survivors to an inspection or to a usage removal can be expressed as follows. The number of survivors still in operation at the time of the j^{th} inspection is

$$N - \sum_{i=1}^{j-1} n_i - \sum_{i=1}^{j} m_i - \sum_{\ell=1}^{\ell(t_j')} r_\ell\,, \qquad j=1,2,\ldots,k+1$$

where the index ℓ ranges over inspection and survivor's ages up to inspection j or the last survivor's age prior to the j^{th} inspection at time t_j'. The upper limit $l(t_j')$ is the index of the j^{th} inspection in the sequence of inspections and survivor's ages. The number of survivors still in operation at the time of the j^{th} usage removal may be expressed as

$$N - \sum_{i=1}^{i(t_j)} (m_i + n_i) - (j - i(t_j)) - \sum_{\ell=1}^{j+i(t_j)} t_\ell \qquad j=1,2,\ldots,M_{k+1}$$

where $i(t_j)$ is the index of the last inspection prior to the j^{th} usage removal. Again, the index ℓ ranges over both usage removal times and inspections.

A function of a maximum likelihood estimator is also a maximum likelihood estimator, so when $\hat{f}_i$ and $\hat{p}_j$ are substituted into the formula for the c.d.f. F(t) (1), the maximum likelihood estimator should be obtained. To define the result, called the "product limit" estimator by Kaplan and Meier [1], order all usage removal times, inspection times, and survivors' ages into one sequence $\{t_r\}$, r=1,2,...,N. The product limit estimator is

$$\hat{F}(t) = 1 - \prod_r (N-r)/(N-r+1)$$

where r takes on values only at usage removal times and inspection times prior to or at time t. For example, if the following realization had been obtained for a sample

time	t_1	t_2	t_3	t_4	t_5	t_{max}
removal condition	Usage	Survivor	Usage	Inspection	Survivor	Max. Time
number removed	1	0	1	2	0	2
$\hat{F}(t)$	.125	.125	.271	.562	.562	1.0

The product limit estimator of $\hat{F}(t)$ in the Figure 3 is obtained.

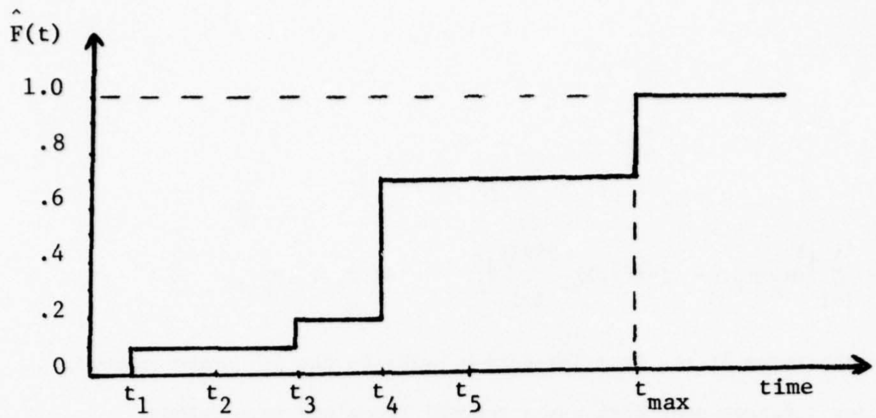

Figure 3. The Product Limit Estimator.

Figure 4 shows the maximum likelihood estimator of $F_1(t)$ the usage removal distribution for the same data. The jumps between inspection and between survivors' ages will be of constant height. The estimator of inspection removal probability is (now carrying index $\ell = 2$ because it is second in the sequence of survivor times <u>and</u> inspection times)

$$\hat{p}_2 = 2/6 = .333.$$

Also needed are the $\hat{p}_\ell$ terms for the two survivor times t_2 and t_5

$$\hat{p}_1 = 1/7 = .143 \text{ and } \hat{p}_3 = 1/3 = .333.$$

From (11) the jumps at usage removals are $\hat{f}_1 = 1/8 = .125$ for the jump at t_1, and $\hat{f}_2 = (1/8)(1-.143)^{-1} = .146$ for the jump at t_3.

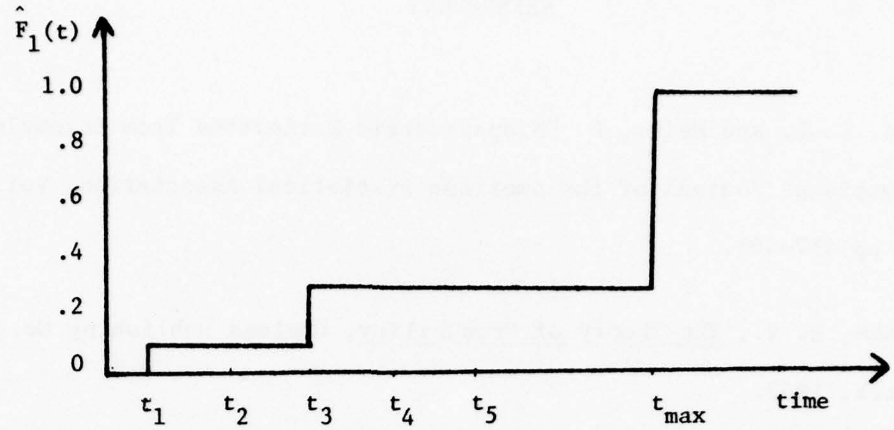

Figure 4. The Maximum Likelihood Estimator of $F_1(t)$

from a Progressively Censored Sample.

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APPENDIX B

VARIANCE REDUCTION FOR RENEWAL AND REPLACEMENT

PROCESS SIMULATION

SUMMARY

SUMMARY

Unless component lifetimes are exponentially distributed, analysis of renewal processes requires transform methods or simulation. If components are not all identical or new, if the process repeats for several period of time, or if there are several simultaneously operating components supplied from the same stock of spares, then determination of replacement requirements can only be done by simulation or normal approximation. The use of complementary antithetic random variates in the generation of component lifetimes reduces the variance of the sample mean number of replacements in a multiple component, multiple period replacement process simulation where the components and spares may be either new or used.

MATHEMATICAL SYMBOLS

SYMBOLS	DEFINITION	PAGE
t, t_i	time value or age value	
$F(t)$	cumulative distribution function of the random variable component lifetime	
$R(t)$	the random variable residual lifetime given age at installation is t time units	
GI/G/1	the queuing notation for a single server queuing system with arbitrary independent arrival distribution and arbitrary service distribution	
$N(t)$	the number of replacements required to obtain t operating hours from some identical components operated sequentially	
$\sum$	summation	
$S(J)$	operating time required of Jth component and its replacements	
M	the number of simultaneously operating components	
N	simulation run size	
$\cap$	set intersection	
X, X_i	the random variable lifetime of a component	
Var	variance	
Cov	covariance	
U, U_i	random numbers between 0 and 1.0	
λ	lambda, the parameter of an exponential distribution	
e	the Naperian base 2.7183	
ln	natural logarithm	
f,g	arbitrary Lebesgue integrable functions	
ϵ	set containment symbol	
$N_j^i(t)$	The replacement counting function with a stock of j spares, i is a run index	
{ }	sequences of functions will be surrounded by parentheses	
$F^{-1}(U)$	the inverse transform of F(t)	

SYMBOL	DEFINITION	PAGE
lim	limit	13
∞	infinity	13
$\int$	integral	13
E	mathematical expectation	13
$\underline{u}$	a particular value of a random number	13
$\vert$	used to separate an event statement about a random variable from a condition on that event in probabilities and expectations	13
$\equiv$	identity	14
Max	maximum	14
$N_{js}^{k}(t)$	the replacement counting function with j operating components and initially s spares, k is a run index	16
$N^{i}(t_j)$	the counting function for period j when t_j operating hours are required, i is a run index	18
M	mean	22
V	variance	22
C	correlation coefficient	22
VSM	variance of sample mean	22
α	a parameter of the Weibull distribution	22
β	a parameter of the Weibull distribution	22
H(J)	the number of replacements	24
N(I)	the cumulative number of replacements on several runs	24
M(I)	the cumulative squared number of replacements on several runs	24
ICOV(I)	the cumulative cross products of numbers of replacements on several runs	24
CCOEF(I)	pooled sample correlation coefficient	25
VREPL(I)	pooled sample variance of the number of replacements	25
VMEAN(I)	pooled sample variance of the mean number of replacements	25

VARIANCE REDUCTION FOR RENEWAL AND
REPLACEMENT PROCESS SIMULATION

1. INTRODUCTION

Many maintenance systems call for the replacement of operating items at failure. It is necessary to predict expected replacement requirements for reordering spares and for planning of replacement workload. It is also useful for planning spares purchase policy to know the upper quantiles of replacement requirements to predict the probability of running out of spares. The usual type of information available is the cumulative life distribution function

$$F(t) = P[\text{operating life} \le t] \qquad t \ge 0 \qquad (1)$$

based on accumulated operating time at failure. If a single operating component is replaced at failure with new identical components which have independently distributed lifetimes, then the replacement counting variable $N(t)$, the number of replacements required to achieve t operating time units, is a "renewal process," and the properties of $N(t)$ are, theoretically at least, computable, Cox [2]. If a number of components are operating simultaneously, then the number of replacements required to achieve t operating time units from each is known as a superposition of renewal processes, and except when the component life distribution is exponential, only asymptotic results are known. In fact, the only computationally convenient result for renewal processes and their superposition is their asymptotic normal distribution. These circumstances often call for simulation of renewal processes for prediction of replacement requirements. Kay [4] has simulated the upper quantiles of a renewal process with Weibull life distribution and compared the results with the normal approximation. The normal approximation did not appear to be very accurate and tended to overestimate actual replacement requirements for fairly small values of operating hours, less than four mean lives. On the other hand, the simulation itself takes a substantial amount of computer time to obtain an accurate value of the expected number of replacements, especially for rare values of the numbers of replacements such as the upper quantiles.

In many maintenance systems, the replacements are not new but used, or the replacements may be a mixture of new and used items. The cumulative distribution function of the residual lifetime R(t) given age at installation t is the conditional distribution

$$P[R(t) \le u] = [F(t + u) - F(t)]\,[1 - F(t)]^{-1} \qquad u, t \ge 0 .$$

There are no analytical results for this more general replacement process unless the component life distribution is exponential, and the memoryless property of the exponential distribution gives the residual life the same exponential distribution as a new item, regardless of age at installation. Simulation of this more general replacement process is necessary to obtain results about the expected number of replacements and its quantiles.

The technique of antithetic variates was first applied to simulation by Hammersley and Handscomb [5] for reducing the variance of the sample average output of the simulation of the mean of a distribution. The technique has since been applied to simulation of GI/G/1 service system by Mitchell [6] and to determinations of the critical path length distribution in project networks with random task times by Burt, Gaver, and Perlas [1], but without the order of magnitude reduction of variance of the sample average experienced when antithetic variates are applied to simulation of replacement processes.

2. Simulation of a Replacement Process

The simulation of a single component replacement process follows easily from the relation between number of replacements W(t) required to achieve t operating hours and the residual lives $R(t_1)$ $R(t)_2$, ... for the initial item and its replacements given their ages at installation t_1, t_2 ... That relation is

$$P[N(t)=0] = P[R(t_1) > t]$$

$$P[N(t)=1] = P[R(t_1) \leq t \cap R(t_1)+R(t_2)>t]$$

.

.

.

etc.,

and in general

$$P[N(t)=n] = P[\sum_{i=1}^{n} R(t_i) \leq t \cap \sum_{i=1}^{n+1} R(t_i) > t] \qquad (2)$$

To simulate the replacement process, generate the residual life of the first item, $R(t_1)$. If it exceeds the operating time t, count N(t)=0. If not, generate the residual life of the second item $R(t_2)$ and count N(t)=1 if $R(t_1)+R(t_2)$ exceeds t. If not, continue until the sum of residual lives exceeds t. This procedure may be repeated as often as desired to obtain an estimate of the mean number of replacements from the average of all simulations and estimates of quantitiles from sample quantiles.

The simulation of a multiple component replacement process is similar except that the total operating time t is a vector of operating time requirements from each component and its replacements. The number of replacements is the sum of all failed components and spares. This simulation is shown in Figure 1.

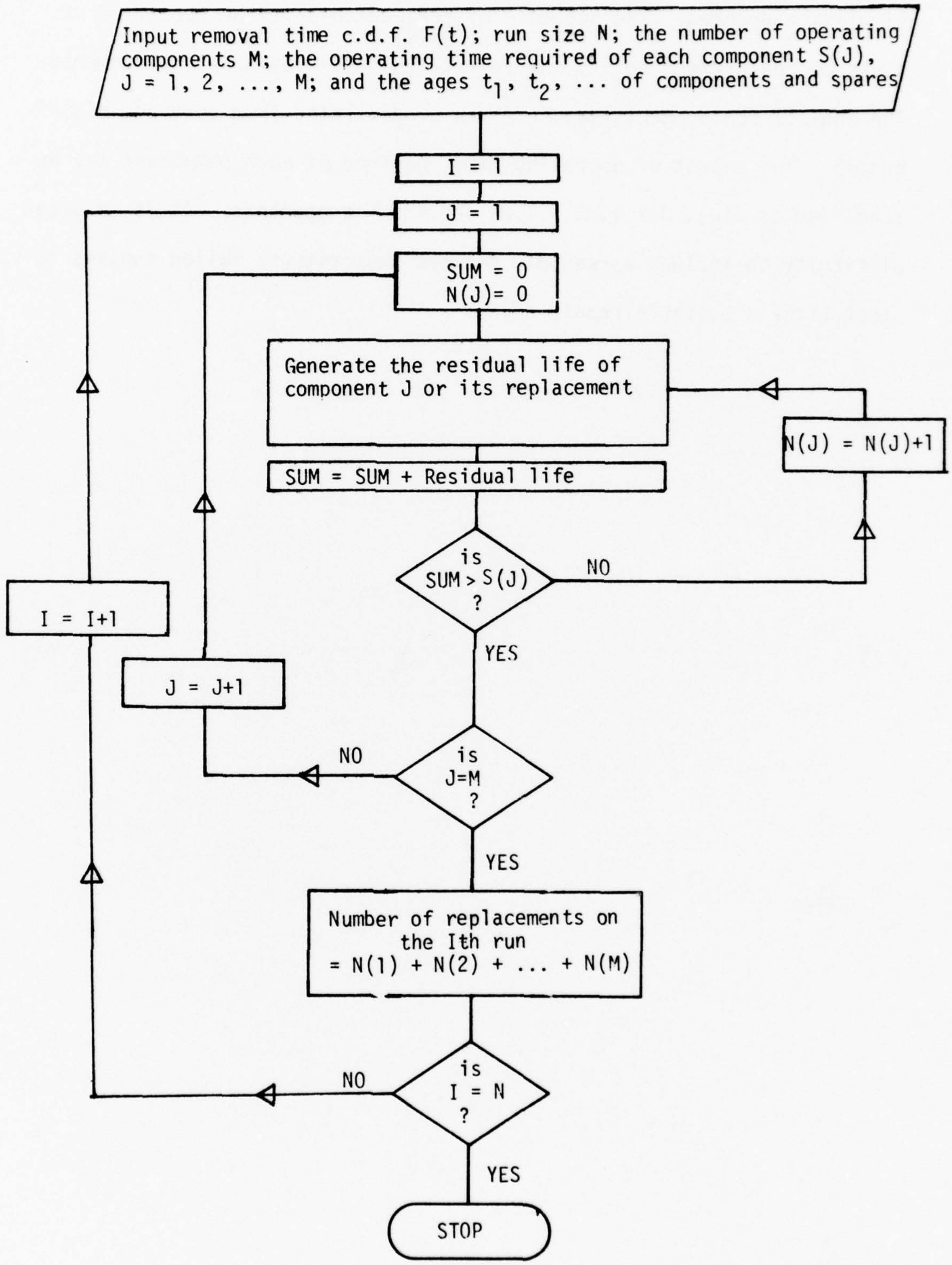

Figure 1: Simulation of a Multiple Component Replacement Process

Several program details may be supplied by the writer of the simulation program. The sequence of replacements may be specified or a random replacement policy may select from all available replacements. The ages of replacements may be given or generated from some age distribution. The amount of operating time required of each component may be specified as S(J), J = 1, 2, ...,M for the M components. It is no great difficulty to include a resupply network that returns failed engines to stock after a suitable repair delay.

3. Application of Antithetic Variates to the Single Component Replacement Process Simulation

The computer program required to simulate the multiple component replacement process with high accuracy may strain the capacity of available computers. One method to reduce the run size required to achieve a fixed level of accuracy is the method of antithetic variates, Hammersley and Handscomb [5]. This method has been used in statistics but has received little use in simulation of replacement processes.

The measure of accuracy that may be improved by the application of antithetic variates is the variance of a sample statistic. For example, the expected value of a random variable may be simulated by generating several values of the random variable X_i, i = 1, 2, ..., N and averaging them. The sample average is an unbiased, consistent, and otherwise well behaved estimator of the expected value. Its variance is

$$\mathrm{Var}\,\overline{X} = \mathrm{Var}\left(\sum_{i=1}^{N} X_i/N\right) = \frac{1}{N^2}\left\{\sum_{i=1}^{N} \mathrm{Var}\,X_i + 2\sum_{i<j}\sum \mathrm{Cov}(X_i,X_j)\right\}. \qquad (3)$$

The covariance term enters into the calculation if the random variables $X_1, X_2, \ldots, X_N$ are generated so that they are correlated. The "trick" of the antithetic variates technique is to generate them so that $\mathrm{Cov}(X_i,X_j) \leq 0$ and thereby reduce the variance of the sample mean below $\sum_{i=1}^{N} \mathrm{Var}\,X_i/N^2$ without biassing the sample mean.

Two methods exist for creating negative correlation among random numbers used to generate random variables. They are to use the complements, (1-random number), and to generate half the required set of random numbers and use them in reverse order for the second half.

Both methods were tested, and the former seems more promising for simulation of renewal and replacement processes. The computer program is shown in Figure 2. One reason for this preference is that many random numbers must be generated to give one value of the number of removals according to the relations (2). Reversing the sequence of random numbers used to generate the $R(t_i)$ in the next run may give independent values of N(t) if the number of replacements is small relative to the number of random numbers generated. For example, suppose a test program generates 50 random numbers for each run, U(1), U(2), ..., U(50), which are used to generate 50 residual lives. If t is small, then the number of replacements may also be small, say 5. On the second run, the sequence of random numbers will be used in reverse order, U(50), U(49), ..., U(1), and N(t) will probably involve only the first few values of the reversed sequence, independent of U(1), U(2), ..., U(5).

The variance reduction property will be proved for the single component replacement process and for several methods of generating the residual life times of the components, the inverse transformation method and a method of acceptance for generating conditional lifetimes. The variance reduction property for multi-component replacement processes will be proved in the next section.

The inverse transformation method is based on the fact that for any strictly increasing cumulative distribution function F(x), the random variable F(X) has the uniform distribution on the interval from zero to

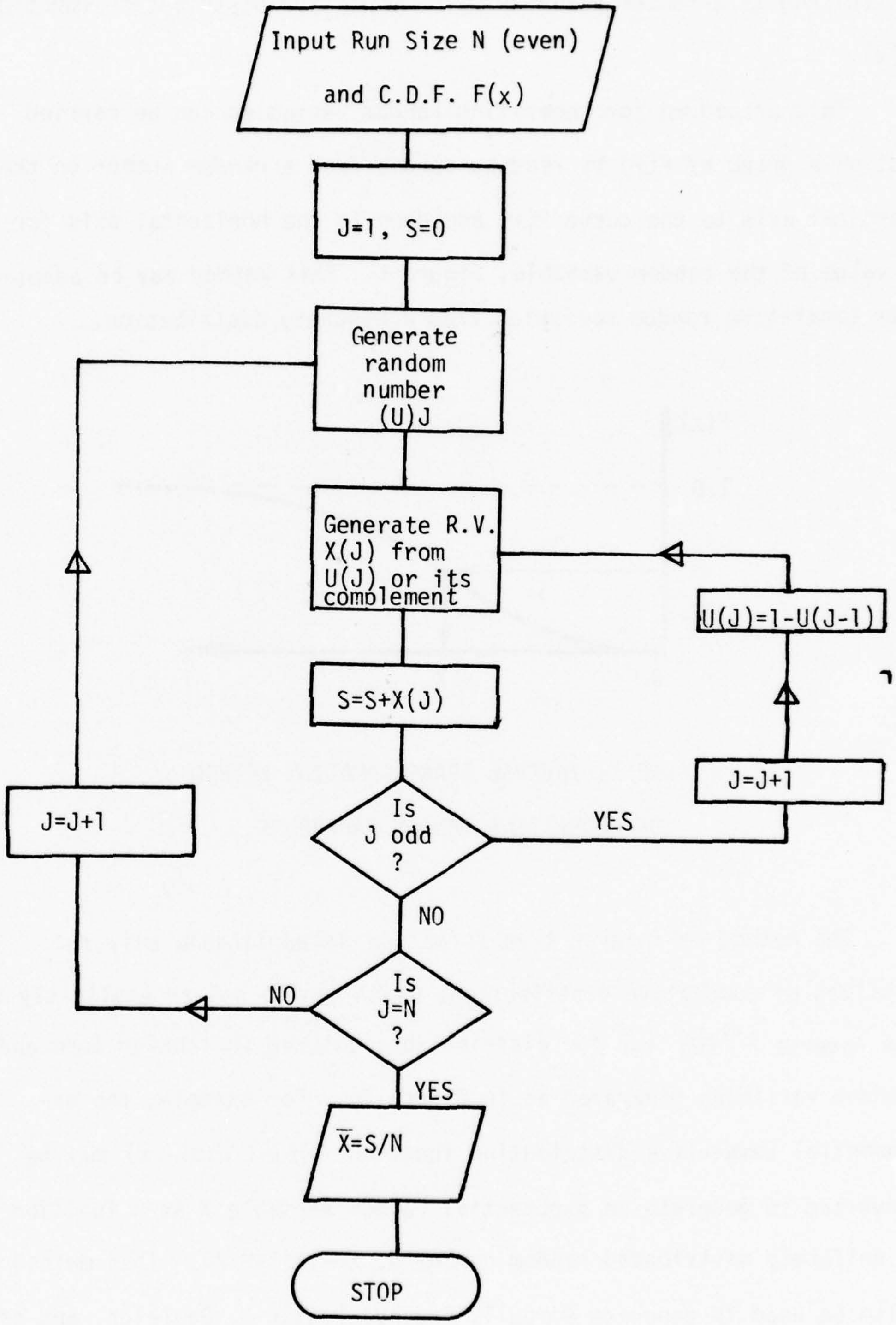

Figure 2: Method of Complements Generation of $\overline{X}$

one. The inverse transform of a uniform random number U is denoted $F^{-1}(U)$ and is a random variable with cumulative distribution function F(x).

This procedure for generating random variables can be carried out on a graph of F(x) by reading across from a random number on the vertical axis to the curve F(x) and down to the horizontal axis for a value of the random variable, Figure 3. This method may be adapted for generating random variables from a discrete distribution.

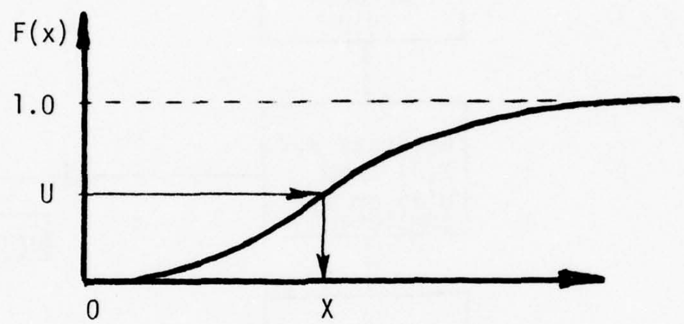

FIGURE 3; INVERSE TRANSFORMATION METHOD OF GENERATING RANDOM VARIABLES

The method of inverse transformation is applicable only for families of cumulative distributions which can be solved explicitly for the inverse $F^{-1}(U)$ (or for distributions entered in tabular form and random variables generated as in Figure 3). For example, the exponential cumulative distribution function $F(x)=1-\exp(-\lambda x)$ may be inverted to generate an exponential random variable X as a function of a uniformly distributed random number U, $X=-\ln(1-U)/\lambda$. This method may also be used to generate Weibull, Cauchy, logistic, Rayleigh, and of course, uniform random variables.

The variance reduction property to be proved is :

Theorem 1: The use of complementary antithetic random variates reduces the variance of the sample mean number of replacements in simulation of a single component replacement process when component lifetimes are generated by the inverse transformation method.

Proof: According to (3), the variance can be reduced by generating correlated replacement counting functions so that for every pair of runs, the covariances of the numbers of replacements $Cov(N^i(t),N^j(t))$ are non-positive thereby reducing the variance of the sample mean from N runs to less than or equal to

$$(1/N^2) \sum_{i=1}^{N} Var\ N^i(t)$$

obtained by generating the replacements in each run from independent random numbers. It will later be shown that for at least one pair of runs, the covariance is strictly negative.

The proof of non-positive covariance is to apply the following result from Mitchell [6] to replacement processes with a limited number of spares: "Let f and g be real valued Lebesgue integrable functions of $X \varepsilon R^k$. Suppose for each argument X_j, $1 \leq j \leq k$, either f is non-increasing and g is non-decreasing or f is non-decreasing and g is non-increasing in X_j. Let $U=(U_1,U_2,...,U_k)$ be uniform on [0,1]. Then $Cov(g(U),f(U)) \leq 0$."

Define the sequences $\{N_j^1(t)\}$ and $\{N_j^2(t)\}$ for j = 1, 2, ... as follows:

$$N_1^1(t) = 0$$

$$N_2^1(t) = \begin{cases} 0 \text{ if } F^{-1}(U_1) > t \\ 1 \text{ if } F^{-1}(U_1) \leq t \end{cases}$$

$$N_3^1(t) = \begin{cases} 0 \text{ if } F^{-1}(U_1) > t \\ 1 \text{ if } F^{-1}(U_1) \leq t \ \cap \ F^{-1}(U_1) + F^{-1}(U_2) \geq t \\ 2 \text{ if } F^{-1}(U_1) + F^{-1}(U_2) \leq t \end{cases}$$

$\vdots$

etc.

and $N_1^2(t) = 0$

$$N_2^2(t) = \begin{cases} 0 \text{ if } F^{-1}(1-U_1) > t \\ 1 \text{ if } F^{-1}(1-U_1) \leq t \end{cases}$$

$$N_3^2(t) = \begin{cases} 0 \text{ if } F^{-1}(1-U_1) \leq t \\ 1 \text{ if } F^{-1}(1-U_1) \leq t \ \cap \ F^{-1}(1-U_1) + F^{-1}(1-U_2) > t \\ 2 \text{ if } F^{-1}(1-U_1) + F^{-1}(1-U_2) \leq t \end{cases}$$

$\vdots$

etc.

The $N_j^i(t)$ are counting functions which tell the number of replacements required given a supply of j-1 replacement items. The $N_j^1(t)$ sequence is generated by inverse transformation $F^{-1}(U)$ from random numbers $U_1, U_2, \ldots, U_{j-1}$ and the $N_j^2(t)$ sequence is generated from the complements $1-U_1, 1-U_2, \ldots, 1-U_{j-1}$.

First it will be shown that

$$\text{Cov}(N_j^1(t), N_j^2(t)) \leq 0$$

for all j and then

$$\lim_{j\to\infty} \mathrm{Cov}(N_j^1(t),N_j^2(t)) = \mathrm{Cov}(N^1(t),N^2(t)) \leq 0$$

follows directly. To apply Mitchell's result, it must be shown that $N_j^1(t)$ is nondecreasing in each argument $U_1, U_2, \ldots, U_{j-1}$ and $N_j^2(t)$ is nonincreasing in the same arguments. This occurs because a large random number U_1 generates a large random lifetime $F^{-1}(U_1)$ resulting in a small $N_j^1(t)$ and conversely for $N_j^2(t)$ because large U_1 results in small $1-U_1$ and small $F^{-1}(1-U_1)$ and a larger number of replacements $N_j^2(t)$.

The fact that all the covariances are bounded allows the result

$$\lim_{j\to\infty} \mathrm{Cov}(N_j^1(t),N_j^2(t)) = \mathrm{Cov}(\lim_{j\to\infty} N_j^1(t),\lim_{j\to\infty} N_j^2(t))$$

$$= \mathrm{Cov}(N^1(t),N^2(t))$$

the covariance of replacement processes with unlimited supplies of replacements. Since every term in the sequence of covariances is non-positive, so is the limit

$$\mathrm{Cov}(N^1(t),N^2(t)) \leq 0 \ .$$

In order to show there will be some improvement when complementary antithetic random variates are used, we express the covariance conditioned on the value of the first random number U_1 which has the uniform density on the unit interval.

$$\mathrm{Cov}(N^1(t),N^2(t)) = \int_0^1 E[(N^1(t) - EN(t))(N^2(t) - EN(t)) \mid U_1 = u]du$$

where $EN(t) = EN^1(t) = EN^2(t)$. Assume there exists a random number $\underline{u}$ close enough to 1.0 so that $F^{-1}(\underline{u})$ is larger than t. Then $N^1(t) \equiv 0$ when $U_1 \geq \underline{u}$ and the covariance integral is

$$Cov(N^1(t),N^2(t)) = \int_{\underline{u}}^{1} -EN(t)E[N^2(t)-EN(t)|U_1 = u]du +$$

$$+ \int_0^{\underline{u}} E[N^1(t)-EN(t))(N^2(t)-EN(t))|U_1=u]du.$$

The first term will be strictly negative because $U_1 \geq \underline{u}$ means $1-U_1$ is small causing $E[N^2(t)|U_1=u]$ to exceed $EN(t)$. The second term will remain non-positive because $N^1(t)$ is larger than usual and $N^2(t)$ smaller under the condition $U_1=u < \underline{u}$. Thus some reduction in the variance of the sample mean can be expected when complementary antithetic variates are used on alternate simulation runs.

This first theorem is contingent on generating the component lives by inverse transformation. When components have already accumulated operating time prior to installation, their residual operating times must be generated from the conditional life distribution

$$P[R(t) \leq u|Life > t_i] = \frac{F(t_i+u) - F(t_i)}{1-F(t_i)}$$

The inverse transformation may again be used and the conditional inverse must be recomputed for each different age t_i. The variance reduction property will still hold when antithetic random variates are generated by inverse transformation from the conditional life distribution above. The same method of prcof yields:

Corollary 1: The use of complementary antithetic random variates reduces the variance of the sample mean number of replacements in simulation of a replacement process when the component residual lifetimes are generated by the inverse transformation method from their conditional life distributions given their ages at installation.

4. Multiple Component Replacement Processes

Often the replacement process will include more than one operating item and a stock of replacements for any item as it fails. If all replacements are new, then the number of replacements is obtained from superposition of renewal processes for each operating item and some information about the distribution of the number replacements is known, Cox, chapter 6 [2]. If the replacement items and perhaps the operating items are not new, simulation is the only way to estimate the distribution of the number of replacements and other properties. The simulation program is shown in Figure 1. It can be adapted to simulate any policy that selects the sequence of installation for spares including a random replacement policy.

Complementary antithetic variates can again be used to reduce the variance of the sample mean number of replacements for the multi-component replacement process. The application is to simulate the replacement process to obtain the replacement requirements to achieve the desired operating time t for all items and replacements. Save the random numbers used in generating item life times and use their complements in the next simulation run to generate component lives in the same order as they are used in the first run. The variance reduction property will be proved under the assumption that the item lives are generated by the inverse transformation method:

Theorem 2: The use of complementary antithetic random variates reduces the variance of the sample mean number of replacements in simulation of a multiple component replacement process when the item lives are generated by the inverse transformation method.

Proof: The method is as in Theorem 1 to limit the number of replacements available and use Mitchell's Theorem to prove the non-positivity of covariance between two *antithetic counting functions.* Then the limit on replacements can be removed.

Define the item lifetimes for a replacement process that starts with j simultaneously operating items and k spares as X_{1i}, and the antithetic sequence of lives as X_{2i}, i = 1, 2, ..., j+s. The counting functions obtained from the two sequences will be obtained as

$$N^k_{js}(t) = \begin{cases} 0 \text{ if } X_{ki} > t \quad i=1, 2, \ldots, j \\ 1 \text{ if only one } X_{ki} \leq t, i=1, 2, \ldots, j \\ \quad \text{and } X_{ki+1} \text{ survives past time t} \\ \vdots \\ j+s \text{ if all s replacements are needed,} \end{cases}$$

k= 1,2. Since the inverse transformation method is used to generate all random lifetimes, the *counting functions can be expressed as functions* of j+s random numbers,

$$N^1_{js}(t) = f(U_1, U_2, \ldots, U_{j+s})$$

and

$$N^2_{js}(t) = g(1-U_1, 1-U_2, \ldots, 1-U_{j+s}).$$

Mitchell's Theorem can again be used to show that

$$Cov(N^1_{js}(t), N^2_{js}(t)) \leq 0$$

and consequently in the limit as s the number of spares increases. The proof of Theorem 1 can be adapted to show these will be strictly negative.

As in the previous proofs, the effect of a decrease in U_1 will be examined. A decrease in U_1 causes X_{11} to decrease because it is generated by an inverse transformation, and this in turn may increase $N^1_{js}(t)$ if X_{11}

decreases to less than t, requiring a replacement. The opposite effect occurs on $N_{js}(t)$ because X_{21} is generated from $1-U_1$ and is larger if U_1 decreases. Then $N^1_{js}(t)$ and $N^2_{js}(t)$ move in opposite directions in response to changes in U_1 or for that matter any U_i $i=1, 2, \ldots, j+s$ although the effect may not be as large for large i because the corresponding spare item may never be needed.

The remainder of the proof is to let the number of spares increase to ∞,

$$\lim_{s\to\infty} \text{Cov}\ (N^1_{js}(t),\ N^2_{js}(t)) = \text{Cov}\ (N^1_j(t),\ N^2_j(t)) \leq 0.$$

for the counting functions of a j item replacement process where all j items or their replacements must operate for t time units and the number of spares is unlimited.

The complimentary antithetic variates improvement in variance of the sample mean number of replacements can be extended to the case where the required operating time for each item is different, t is a vector of j operating times required from j items or their replacements. The proof includes the situation where not all items and spares are new, as long as the inverse transformation is used to generate residual lives.

5. Multiple Period Replacement Processes

Often the replacement process will proceed for many periods and estimates of the replacement requirements in each period are needed. Because items which have been operated in one period are survived have accumulated operating time when put into service in the next period, the situation in every period but the first is as in corollary 1 of theorem 1, the items to be used may already have accumulated a random age depending on their initial age if any and their accumulated operating time in earlier periods. Corollary 1 of theorem 1 may be applied to prove negative covariance is obtained if antithetic random variates are used to generate residual random lives for all components given their age at the beginning of each period where they may be used. This will require regeneration of the residual life of any component used in more than one period from its conditional life distribution given its age at installation in a subsequent period.

For example, let $N^i(t_1)$ and $N^i(t_2)$, i=1,2 be the numbers of replacements in periods one and two requiring operating times t_1 and t_2 when generated from independent random numbers and their complements. Random numbers $U_1, U_2, \ldots, U_{N^1(t_1)+1}$ are used to generate component lives $X_{11}, \ldots, X_{1,N^1(t_1)+1}$ where

$$\sum_{i=1}^{N^1(t_1)+1} X_{1i} > t_1 .$$

Component number $N^1(t_1)+1$ survived to be used in the second period with accumulated age

$$t_1 - \sum_{i=1}^{N^1(t_1)} X_{1i} \equiv a$$

Random number $U_{N^1(t_1)+2}$ is used to generate the residual operating life in period 2 from an inverse transformation on the conditional life distribution

$$\frac{F(a+t)-F(a)}{1-F(a)}$$

and all subsequent replacements will be generated as usual. This means that the replacement processes in several periods given the ages of components that may have survived partial use in earlier periods are conditionally independent because the random numbers used to generate each period's replacement process are different and independent.

The only remaining problem is to insure that the antithetic replacement processes are also conditionally independent. Let $N=\max(N^1(t_1), N^2(t_1))+1$ and use random numbers $U_{N+1}, U_{N+2}, \ldots$ to generate the replacement process and its antithetic process for period 2 independently.

The results of this extension can be adapted for reduction of the variance of the sample mean for a multiple component, multiple period replacement process.

6. EMPIRICAL TESTS OF ANTITHETIC VARIATES

In this section results on several types of replacement process simulations with complementary antithetic variates are reported. Although run sizes are not large and results are somewhat dependent on the seed chosen, the reduction in the variance of the sample mean number of replacements is unexpected. The proofs in previous sections only show there will be a reduction but do not suggest the magnitude of the reduction which in several cases was one tenth! (Mitchell {6} obtained about a 20% reduction.)

Runs were made for several different combinations of new and used components, single and double operating components, and several distributions of ages and lives. In every run, the unconditional expected life was 0.5, and each component and its replacements were to operate for either five or ten hours. Ages for used components were generated from the same distribution as the component lives, but the *choice of age distribution is arbitrary.* The total number of components used was recorded for each run. The simulation program would first make a number of runs (20 or 100) with independently generated random numbers. The results from these runs are listed in column "Base". Then an equal number of runs were made using the complementary antithetic random numbers. A third set of runs was generated from another set of independent random numbers and the result of these runs are tabulated in the column "Independent". The means and variances of the number of replacements in the Base, Complementary and Independent runs are tabulated. Then the correlation coefficient between Base and Complementary and between Base and Independent are listed in columns Complementary and Independent. In nearly every simulation, the correlation between Base and Complementary numbers of replacements is less than -.5 and the correlation between Base and Independent is moderate. The last entry is the variance of the sample means computed from the formula (3) after combining variances and means in pairs, Base with Complement and Base with Independent. The simulation results are in Table 1.

The statistical subroutine which was used may be adapted to any test of effectiveness of any variance reduction technique applied to simulation of any output statistic. It is flowcharted in Figure 4 and listed in FORTRAN G in Figure 5.

TABLE 1 SIMULATIONS OF REPLACEMENT PROCESSES (continued on next page)

LIFE DISTRIBUTION	AGE DISTRIBUTION	NUMBER OF COMPONENTS	OPERATING HOURS	NUMBER OF RUNS	STATISTICS	BASE	COMPLEMENT	INDEPENDENT
Uniform	New	1	10	100	Mean	20.6	20.8	20.8
$F(x)=x$					Variance	28.2	26.8	7.5
EX=0.5					Correlation		-.886	.173
					Variance of Sample Mean		.0044	.0461
Exponential	New	1	10	100	M	20.5	21.5	21.3
$F(x)=1-e^{-\lambda x}$					V	21.2	22.9	14.3
EX=0.5					C		-.532	.221
$\lambda=2$					VSM		.051	.113
Exponential	New	2	10	100	M	42.4	42.9	42.7
$\lambda=2$					V	47.2	47.5	46.6
					C		-.633	.222
					VSM		.0867	.286
Weibull	New	1	10	100	M	21.8	20.9	21.8
$F(x)=1-e^{-(x/\beta)^{\alpha}}$					V	38.9	34.7	32.3
EX=0.5					C		-.428	-.109
$\alpha=0.75$					VSM		.105	.197
(Decreasing Failure Rate)								
Weibull	New	2	10	100	M	41.8	42.4	42.9
EX=0.5					V	57.0	137.3	63.1
$\alpha=0.75$					C		-.327	.164
(Decreasing Failure Rate)					VSM		.341	.349
Weibull	New	1	10	100	M	40.4	40.24	40.43
EX=0.5					V	2.53	6.487	3.400
$\alpha=4$					C		-.605	.147
(Increasing Failure Rate)					VSM		.0103	.0170
Uniform	Uniform	1	5	20	M	22.0	23.4	23.4
EX=0.5	EX=0.5				V	14.4	11.4	15.5
					C		-.846	.192
					VSM		.0515	.446
Uniform	Uniform	2	5	20	M	43.9	44.1	44.7
EX=0.5	EX=0.5				V	20.0	24.2	10.9
					C		-.500	.130
					VSM		.276	.389
Exponential	Exponential	1	5	20	M	10.5	12.1	10.6
EX=0.5	EX=0.5				V	10.6	8.5	8.4
					C		-.540	.055
					VSM		.111	.251

LIFE DISTRIBUTION	AGE DISTRIBUTION	NUMBER OF COMPONENTS	OPERATING HOURS	NUMBER OF RUNS	STATISTICS	BASE	COMPLEMENT	INDEPENDENT
Weibull	Weibull	1	5	20	M	7.5	8.2	10.0
EX=0.5	EX=0.5				V	11.2	8.7	11.7
α=0.75	=0.75				C		-.540	.076
(Decreasing Failure Rate)					VSM		.115	.308
Weibull	Weibull	1	5	20	M	34.4	34.0	34.3
EX=0.5	EX=0.5				V	18.6	23.0	26.7
α=3	=3				C		-.877	-.030
(Increasing Failure Rate)					VSM		.0660	.548
Weibull	Weibull	1	5	20	M	45.6	45.0	45.0
EX=0.5	EX=0.5				V	16.1	13.2	8.8
=4	=4				C		-.860	-.100
					VSM		.053	.280

TABLE 1 SIMULATIONS OF REPLACEMENT PROCESSES
(continued from previous page)

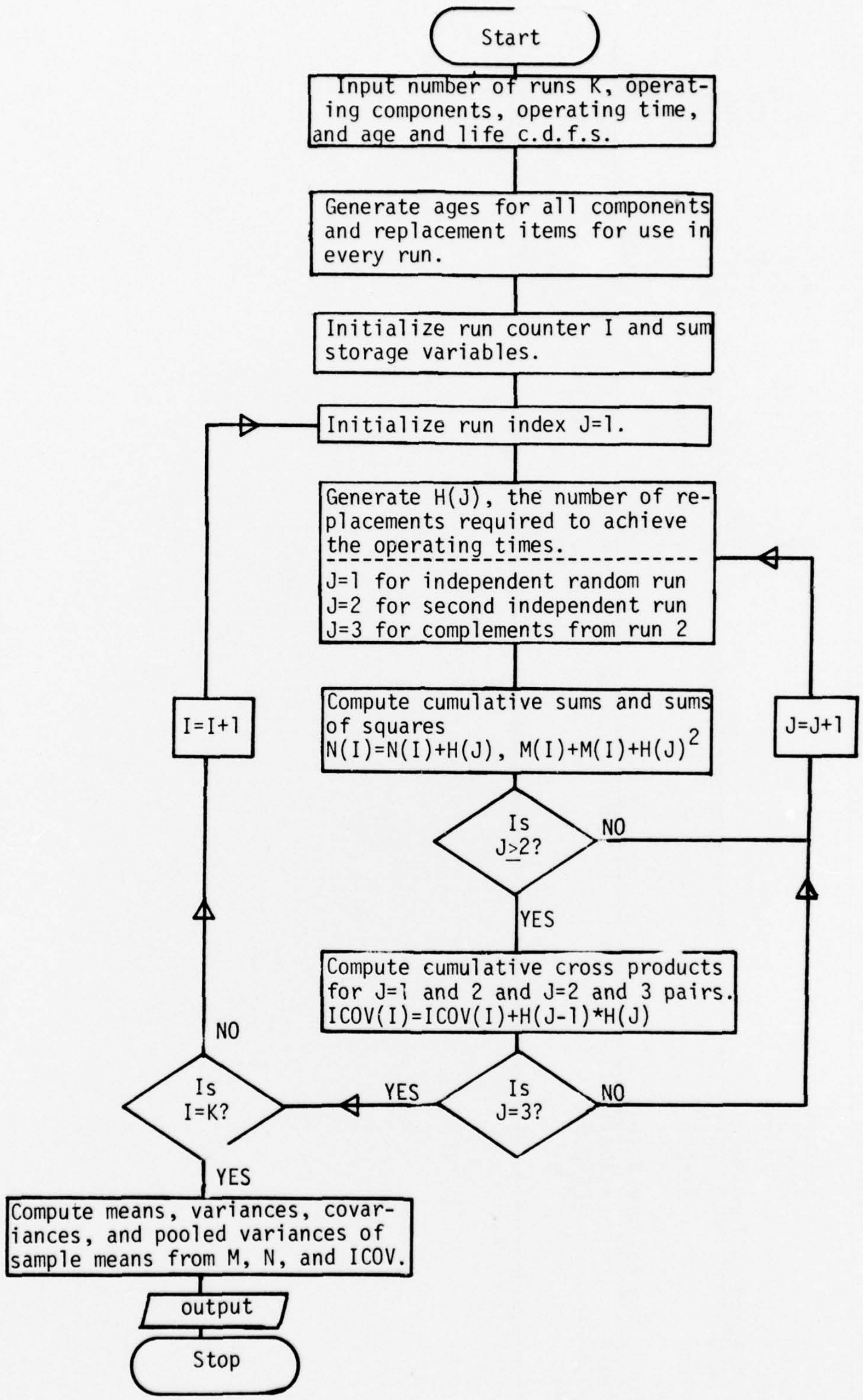

Figure 4, SIMULATION TEST PROGRAM

```
C       H(1), H(2), AND H(3) ARE THE NUMBERS OF REPLACEMENTS
C       REQUIRED ON A RUN USING BASE, INDEPENDENT, AND COMPLE-
C       MENTARY RANDOM NUMBERS.  N(I) AND M(I), I=1,2,3, KEEP
C       RUNNING SUMS AND SUMS OF SQUARES, AND ICOV(I), I=1,2,
C       COLLECTS CROSS PRODUCT SUMS.
C
        DO 100 I=1,3
        N(I)=N(I)+H(I)
        M(I)=M(I)+H(I)*H(I)
 100    CONTINUE
        DO 101 I=2,3
        ICOV(I)=ICOV(I)+H(I-1)*H(I)
 101    CONTINUE
C
C       VAR(I), I=1,2,3, IS THE SAMPLE VARIANCE OF THE NUMBER
C       OF REPLACEMENTS. COV(I),CCOEF(I), VREPL(I), AND VMEAN(I),
C       I=2,3, ARE POOLED COVARIANCES, CORRELATION COEFFICIENTS,
C       SAMPLE VARIANCES, AND VARIANCES OF SAMPLE MEANS FOR
C       BASE/INDEPENDENT AND INDEPENDENT/COMPLEMENTARY PAIRS.
C       VARIABLE "K" IS THE NUMBER OF RUNS IN EACH OF BASE,
C       COMPLEMENT, AND INDEPENDENT SIMULATIONS.
C
        DO 102 I=1,3
 102    VAR(I)=(M(I)-(N(I)*N(I))/K)/(K-1)
        DO 103 I=2,3
        COV(I)=(ICOV(I)-K*MEAN(I-1)*MEAN(I))/(K-1)
        CCOEF(I)=COV(I)/SQRT(VAR(I-1)*VAR(I))
        CMEAN(I)=(MEAN(I-1)+MEAN(I))/2.0
        VREPL(I)=(M(I-1)+M(I)-((2*K)*CMEAN(I)**2))/(2*K-1)
 103    VMEAN(I)=(VAR(I-1)+VAR(I)+(2*COV(I)))/(4*K)
        STOP
```

Figure 5 STATISTICAL SUBROUTINE LISTING

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