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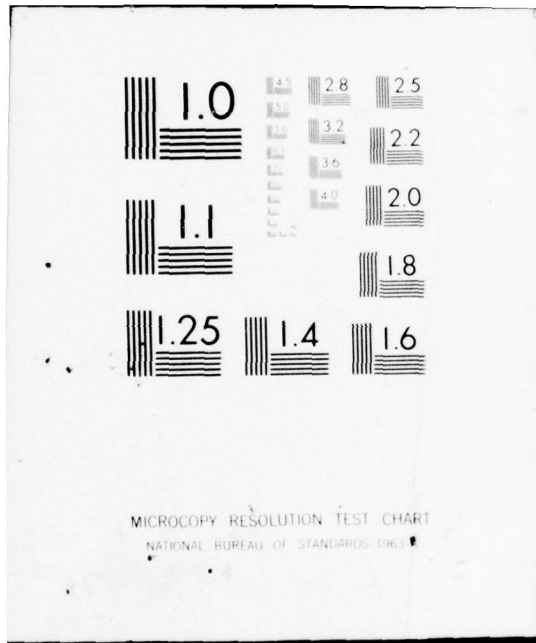
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REPORT R-742 SEPTEMBER, 1976

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# LINEAR SMOOTHING OF SINGULARLY PERTURBED SYSTEMS

DEAN ALTSHULER

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) <u>6</u> LINEAR SMOOTHING OF SINGULARLY PERTURBED SYSTEMS. ✓		5. TYPE OF REPORT & PERIOD COVERED <u>9</u> Technical Report,
7. AUTHOR(s) <u>10</u> Dean/Altshuler		6. PERFORMING ORG. REPORT NUMBER <u>14</u> R-742, UIIU-ENG-76-2230 ✓
9. PERFORMING ORGANIZATION NAME AND ADDRESS Coordinated Science Laboratory ✓ University of Illinois at Urbana-Champaign Urbana, Illinois 61801		8. CONTRACT OR GRANT NUMBER(s) <u>15</u> DAAB-07-72-C-0259, DAAG-29-76-G-0154 ✓
11. CONTROLLING OFFICE NAME AND ADDRESS Joint Services Electronics Program		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE <u>11</u> Sep <del>1975</del> , 1976
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		13. NUMBER OF PAGES 64 <u>12</u> 73p.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
18. SUPPLEMENTARY NOTES		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Singular Perturbation Filtering Smoothing		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) ↙ A near optimal fixed-interval smoother for singularly perturbed systems is derived using the two filter (forward and backward) formulation. This intuitively-pleasing approach simplifies the fast mode boundary layer complications encountered previously, and results in a much simpler implementation than the exact solution. Application of similar singular perturbation techniques as for the (forward) filter, to the backward filter, results in a near optimal $O(n^{1/2})$ smoother with a similar two time-scale solution →		

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20. ABSTRACT (continued)

cont

→ for the fast and slow modes. The alternative use of the unrealizable Wiener filter for the fast mode is justified in the interior of the smoothing interval. Finally, a numerical example serves to illustrate how well the near optimal design approximates the exact for various values of the perturbation parameter  $\mu$ .

$\mu$  → ↑

UILU-ENG 76-2230

LINEAR SMOOTHING OF SINGULARLY  
PERTURBED SYSTEMS

by

Dean Altshuler

This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Contract DAAB-07-72-C-0259 and in part by the Army Research Office under Contract DAAG-29-76-G-0154.

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LINEAR SMOOTHING OF SINGULARLY PERTURBED SYSTEMS

BY

DEAN ALTSHULER

B.S., University of Massachusetts, 1974

THESIS

Submitted in partial fulfillment of the requirements  
for the degree of Master of Science in Electrical Engineering  
in the Graduate College of the  
University of Illinois at Urbana-Champaign, 1976

Thesis Advisor: Professor A. H. Haddad

Urbana, Illinois

## ACKNOWLEDGMENTS

The author wishes to express his vast appreciation to Professor A. H. Haddad for his unfailing guidance and great interest sustained throughout the preparation of this thesis. The problem, in fact, was suggested by him and the many stimulating discussions along the way helped to insure its final solution.

Special thanks are extended to Phyllis Young, Trudy Williams and, most of all, Hazel Corray for their expert typing of a most difficult manuscript.

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## I. INTRODUCTION

### 1.1. General

This thesis considers the limiting behavior of the fixed-interval smoothing problem for singularly perturbed linear systems. Its primary objective is to derive a near optimal smoother for a system containing both fast and slow modes by employing the methods of singular perturbation theory. Recently [1], the linear filtering problem for singularly perturbed systems has been thoroughly treated using these methods. Considerable attention has been given to singularly perturbed systems as applied to control problems in recent years [2-8]. Haddad and Kokotović [9] have established the duality between estimation and control for such systems. The one fundamental difference, however, is that while in most control applications the problem is deterministic, in the estimation problem the presence of white noise inputs complicates the behavior of the resulting fast variables. It should be noted that the work presented here is an extension of the methods and results of the linear filtering problem [1,9,10]. While the smoothing problem has already been treated in [10], the approach used here is more suitable for the decomposition of the smoother into its fast and slow components, and in deriving the uncoupled boundary layers of the fast mode filters at both ends of the observation interval.

Singularly perturbed systems are represented by state variable equations, where a small perturbation parameter  $\mu > 0$  multiplies some of the derivatives. This general form is characteristic of many practical systems, where  $\mu$  may represent the smaller time constants of the system. Therefore, such a formulation is representative of systems with fast and slow modes, e.g., velocity and acceleration of a maneuvering vehicle, or an electro-mechanical system where the electrical response, as is typical, is much faster than the mechanical response.

The theory of singular perturbations has arisen from a desire to simplify the higher order problem as a set of two lower order problems. This not only allows a simplification of the solution, but in addition, provides for non-interactive estimation or control of the fast and slow states. It also generally results in a great reduction in computational complexity and removes ill-conditioning. In this thesis, the behavior of the optimal smoother for singularly perturbed systems as  $\mu \rightarrow 0$  will be investigated and near optimal lower order smoothers will be designed.

## 1.2. Problem Statement

### 1.2.1. General Problem

The problem is to derive a near optimal MMSE (Minimum-Mean-Squared Error) fixed-interval smoother for the singularly perturbed system represented by state equations

$$\dot{x} = A_1 x + A_{12} z + B_1 u, \quad x(0) = x_0 \quad (1.1)$$

$$\mu \dot{z} = A_{21}x + A_2z + B_2u, \quad z(0) = z_0 \quad (1.2)$$

and observations

$$y = C_1x + C_2z + v, \quad 0 \leq t \leq T \quad (1.3)$$

Here  $x$ ,  $y$ , and  $z$  are  $n$ -,  $r$ -, and  $m$ -dimensional, respectively. The vectors  $u$  and  $v$  are  $p$ - and  $r$ -dimensional uncorrelated, white Gaussian noise processes with zero means and respective covariances  $Q$  and  $R$ , i.e.,

$$\begin{aligned} E[u(t_1)u'(t_2)] &= Q(t)\delta(t_1 - t_2) \\ E[v(t_1)v'(t_2)] &= R(t)\delta(t_1 - t_2) \\ E[u(t_1)v'(t_2)] &= 0 \end{aligned} \quad (1.4)$$

where the prime is used to denote transpose. The states  $x(t)$  and  $z(t)$  represent the slow and fast modes of the system, respectively. This characterization is explained in terms of the eigenvalues of the corresponding equations (1.1) and (1.2). The latter have eigenvalues that behave as  $1/\mu$  and hence tend to be very large as  $\mu \rightarrow 0$ .

As previously mentioned, this thesis is mainly concerned with the behavior of the smoothed estimates of the states of (1.1) and (1.2) as  $\mu \rightarrow 0$ , and obtains, as a result, a near optimal lower order smoother in two time-scales as in the filtering problem [1,9,10]. The interpretation of near optimality, in this case, is in the sense that the solution tends to the optimal as  $\mu$  approaches zero. Formally, such limiting behavior will

be denoted by using the following definition: A function  $f(\mu)$  is denoted  $O(\mu)$  if there exist positive constants  $\mu^*$  and  $C$  such that the norm  $\|f\|$  satisfies  $\|f\| < C\mu$  for all  $\mu \in (0, \mu^*]$ . Thus  $x(t) = \bar{x}(t) + O(\mu)$  implies  $x(t) - \bar{x}(t) = O(\mu)$ , i.e.,  $\bar{x}(t)$  is an  $O(\mu)$  approximation of  $x(t)$ .

### 1.2.2. Mode Separation for Singularly Perturbed Systems

Inspection of the system (1.1)-(1.3) reveals coupling between the slow and fast states,  $x(t)$  and  $z(t)$ . Algebraic complexity in the sequel can be reduced by an initial non-singular linear transformation due to Chang [11]. This allows consideration of systems of block diagonal form, thus simplifying the intermediate derivations. If the matrices of (1.1)-(1.3) are all continuous and bounded, and  $A_2(t)$  is also stable, then this transformation can be employed to yield new state vectors,  $\eta(t)$  and  $\xi(t)$ , defined by

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} I_n & -\mu HL & -\mu H \\ & L & I_m \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (1.5)$$

where  $I_k$  is a  $k \times k$  identity matrix, and  $L$  and  $H$  are deterministic matrices satisfying

$$\begin{aligned} \mu \dot{L} &= A_2 L - A_{21} - \mu L(A_1 - A_{12} L) \\ \mu \dot{H} &= -H(A_2 + \mu L A_{12}) + A_{12} + \mu(A_1 - A_{12} L)H \end{aligned} \quad (1.6)$$

It can be shown that the stability of  $A_2(t)$  implies

$$L = A_2^{-1}A_{21} + O(\mu) \triangleq L_0 + O(\mu)$$

$$H = A_{12}A_2^{-1} + O(\mu) \triangleq H_0 + O(\mu)$$
(1.7)

The new state equations and associated observations become

$$\dot{\eta} = \alpha_0 \eta + \beta_0 u, \quad \eta(0) = \eta_0$$
(1.8)

$$\mu \dot{\xi} = \alpha_2 \xi + \beta_2 u, \quad \xi(0) = \xi_0$$
(1.9)

$$y = c_0 \eta + c_2 \xi + v, \quad 0 \leq t \leq T$$
(1.10)

where

$$\alpha_0 = A_1 - A_{12}L = A_1 - A_{12}A_2^{-1}A_{21} + O(\mu) \triangleq A_0 + O(\mu)$$

$$\beta_0 = B_1 - \mu H L B_1 - H B_2 = B_1 - A_{12}A_2^{-1}B_2 + O(\mu) \triangleq B_0 + O(\mu)$$

$$c_0 = C_1 - C_2 L = C_1 - C_2 A_2^{-1} A_{21} + O(\mu) \triangleq C_0 + O(\mu)$$
(1.11)

$$\alpha_2 = A_2 + \mu L A_{12} = A_2 + O(\mu)$$

$$\beta_2 = B_2 + \mu L B_1 = B_2 + O(\mu)$$

$$c_2 = C_2 + \mu (C_1 - C_2) H = C_2 + O(\mu)$$

Since only linear estimators are considered here, the estimates of the original states can be obtained via the inverse transformation

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I_n & \mu H \\ -L & I_m - \mu L H \end{bmatrix} \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$
(1.12)

where H and L may be replaced by (1.7), depending on the desired approximation. Therefore, without loss of generality, the discussion will be restricted to the uncoupled system (1.8)-(1.10). Finally, the inverse transformation (1.12) will be employed to find the estimates of the original states.

### 1.2.3. The Reduced-Order Problem

Observation of the general smoothing problem reveals that if  $\mu = 0$ , the result is a lower order system whose solution is much simpler than that of the original, higher order system. A basic problem in singular perturbation theory is the relationship of the complete or exact solution to the solution of the reduced-order problem. Hence, a reduced-order problem is defined in this section by formally setting  $\mu = 0$  in (1.1)-(1.3) or equivalently, in (1.8)-(1.10). Setting  $\mu = 0$  in the latter yields

$$\xi_0 = -A_2^{-1}B_2u(t) \quad (1.13)$$

and thus,

$$\dot{\eta}_0 = A_0\eta_0 + B_0u \quad (1.14)$$

$$y = C_0\eta_0 + D_0u + v \quad (1.15)$$

where  $D_0 \triangleq -C_2A_2^{-1}B_2$ .

A major objective of this work is to investigate the limit of the smoothed estimate of the complete system as  $\mu \rightarrow 0$  and its relation to the smoothed estimate for the reduced-order problem. It will be shown that while the resulting limit is identical to the reduced-order problem for the slow modes, it is not valid (as is expected) for the fast modes.

#### 1.2.4 General Assumptions

The following assumptions will be needed throughout this thesis:

- (i) The matrices  $A_2(t)$ ,  $B_2(t)$ ,  $C_2(t)$ ,  $Q(t)$  and  $R^{-1}(t)$  are continuous, bounded, and have bounded first derivatives for all  $t \in [0, T]$ .
- (ii) The eigenvalues of  $A_2(t)$  have negative real parts (a stable matrix) for all  $t \in [0, T]$ .

#### 1.3 Thesis Preview

Chapter II investigates the proposed formulation of the solution to the fixed-interval smoothing problem. The exact solution for the complete system is outlined and the reduced-order problem is explicitly solved.

Chapters III and IV consider the limiting behavior of the forward and backward filters respectively, for the exact smoothing problem and investigate its relation to the solution of the reduced problem.

The near optimal smoother as a two time-scale lower order estimator is derived in Chapter V. It is shown that the slow mode smoother tends to the smoother for the reduced system. The fast mode smoother, it is seen, involves the forward and backward fast mode filters both of which are implemented in a stretched time-scale.

Chapter VI is concerned with the general stationary second-order case, in which comparisons can be made relatively easily between the fast mode estimators and the corresponding Wiener filters. Finally a numerical example is considered to illustrate the validity of the near optimal design.

Chapter VII concludes this thesis with a summary of the results and conclusions of the previous chapters.

## II. GENERAL SOLUTION OF THE SMOOTHING PROBLEM

### 2.1. Smoothing as a Combination of Two Filters

Consider the MMSE smoothing problem for a system of the form

$$\dot{x} = Ax + Bu \quad (2.1)$$

$$y = Cx + v, \quad 0 \leq t \leq T \quad (2.2)$$

The optimal fixed-interval smoother for this system can be expressed as a weighted sum of two filters [12,13]. The estimate of  $x(t)$ , given all observations in the interval  $[0,T]$ , is given by

$$\hat{x}(t|T) = F(t)\hat{x}(t) + (I-F(t))\hat{x}_b(t) \quad (2.3)$$

where  $\hat{x}(t)$  is the output of the usual Kalman filter,  $\hat{x}_b(t)$  is the output of a backward Kalman filter, and  $F(t)$  will be chosen to minimize the MSE. The forward filter processes the observations in the interval  $[0,t)$  and the backward filter processes the remaining ones from  $T$  back to  $t$ . The smoother error covariance is given by

$$P(t|T) \triangleq E\{[\hat{x}(t|T)-x(t)][\hat{x}(t|T)-x(t)]'\} = FP(t)F' + (I-F)P_b(t)(I-F)' \quad (2.4)$$

where  $P$  and  $P_b$  are the error covariances of the forward and backward filters, respectively. Choosing  $F(t)$  to minimize the trace of  $P(t|T)$  results in

$$F = P_b(P+P_b)^{-1}, \quad I-F = P(P+P_b)^{-1} \quad (2.5)$$

with corresponding error covariance

$$P(t|T) = [P^{-1}(t) + P_b^{-1}(t)]^{-1} \triangleq [P^{-1}(t) + M(t)]^{-1} \quad (2.6)$$

The resulting optimal smoother becomes

$$\hat{x}(t|T) = P(t|T)[P^{-1}(t)\hat{x}(t) + M(t)\hat{x}_b(t)] \quad (2.7)$$

The backward filter equations given by

$$\frac{d}{d\sigma} \hat{x}_b(t) = -A\hat{x}_b + P_b C' R^{-1}(y - C\hat{x}_b), \sigma = T - t \geq 0 \quad (2.8)$$

$$\frac{d}{d\sigma} P_b = -AP_b - P_b A' + BQB' - P_b C' R^{-1} C P_b, P_b^{-1}(T) = 0 \quad (2.9)$$

are unstable however, and in fact,  $P_b(T)$  is not finite, a consequence of filter and smoother equivalence at the end of the observation interval. Hence, a stabilizing transformation  $s(t) \triangleq M(t)\hat{x}_b(t)$  is used, resulting in

$$\frac{ds(t)}{d\sigma} = (A' - MBQB')s + C' R^{-1} y, s(T) = 0 \quad (2.10)$$

It should be noted that the error covariance of  $s(t)$  is  $M(t)$ , which satisfies the Riccati equation

$$\frac{dM(t)}{d\sigma} = MA + A'M - MBQB'M + C'R^{-1}C, M(T) = 0 \quad (2.11)$$

In what follows  $\sigma = T - t$  will always be used to denote the backward time variable. However, all matrices will be expressed as functions of  $t$  for consistency. The smoothed estimate (2.7) therefore becomes

$$\hat{x}(t|T) = [I + P(t)M(t)]^{-1}[\hat{x}(t) + P(t)s(t)] \quad (2.12)$$

While the quantities  $s(t)$  and  $M(t)$  do not have the physical meaning of their forward counterparts  $\hat{x}(t)$  and  $P(t)$ , they do suggest a convenient mathematical

representation for the implementation of  $\hat{x}(t|T)$ . Equations (2.10) - (2.12) above will be applied directly to the reduced-order smoothing problem and later, also to the exact smoothing problem using the complete system.

## 2.2. The Reduced-Order Smoothing Problem

The smoothing problem for the reduced system (1.14) - (1.15) will now be considered by directly applying the results of section 2.1 with the modification required due to the correlation of the input and measurement noises. The resulting fixed-interval smoothing solution for this system becomes

$$\hat{\eta}_0(t|T) = (I + P_0 M_0)^{-1} [\hat{\eta}_0(t) + P_0 s_0(t)] \quad (2.13)$$

where  $\hat{\eta}_0$  and  $s_0$  are the outputs of the forward and backward filters with covariances  $P_0$  and  $M_0$ , respectively. The expression for the forward filter is given by the usual Kalman filter equation

$$\dot{\hat{\eta}}_0 = A_0 \hat{\eta}_0 + K_0 (y - C_0 \hat{\eta}_0), \quad \hat{\eta}_0(0) = E(\eta_0) \quad (2.14)$$

where

$$K_0 \triangleq (P_0 C_0' + B_0 Q D_0') R_0^{-1} \quad (2.15)$$

$$R_0 \triangleq (R + D_0 Q D_0') \quad (2.16)$$

with the error covariance  $P_0(t)$  satisfying

$$\dot{P}_0 = A_0 P_0 + P_0 A_0' + B_0 Q B_0' - (P_0 C_0' + B_0 Q D_0') R_0^{-1} (C_0 P_0 + D_0 Q B_0'), \quad P_0(0) = \text{cov}(\eta_0) \quad (2.17)$$

The backward filter equation, obtained from (2.10) with the noted modification,

is expressed as

$$\frac{ds_0}{d\sigma} = [A'_0 - (M_0 B_0 + K_b D_0) Q B'_0] s_0 + K_b y, \quad s_0(T) = 0 \quad (2.18)$$

where

$$K_b(t) \triangleq (C'_0 - M_0 B_0 Q D'_0) R_0^{-1} \quad (2.19)$$

The corresponding covariance equation for  $M_0$  is obtained similarly from (2.11) as

$$\frac{dM_0}{d\sigma} = M_0 A_0 + A'_0 M_0 - M_0 B_0 Q B'_0 M_0 + K_b R_0 K'_b, \quad M_0(T) = 0 \quad (2.20)$$

In summary, the optimal smoother for the reduced system is given by (2.13), together with eqs. (2.14), (2.17), (2.18), and (2.20).

### 2.3. Exact Solution of the Smoothing Problem

The exact solution of the smoothing problem for the system (1.8) - (1.10) can be found by applying the results of Section 2.1 directly, with the appropriate substitutions

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_2/\mu \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_2/\mu \end{bmatrix},$$

$$C = [C_0 \quad C_2], \quad \hat{x} = \begin{bmatrix} \hat{\eta} \\ \hat{\xi} \end{bmatrix},$$

where the  $\mu$  terms have been neglected in the coefficients of the right-hand side of (1.8) - (1.10). If  $\mu$  is not to be neglected, the script matrices of (1.11) should be used. In accordance with eq. (2.12), the smoothed estimate will therefore be given by

$$\begin{bmatrix} \hat{\eta}(t|T) \\ \hat{\xi}(t|T) \end{bmatrix} = [I + P(t)M(t)]^{-1} \begin{bmatrix} \hat{\eta}(t) \\ \hat{\xi}(t) \end{bmatrix} + P(t) \begin{bmatrix} s_1(t) \\ \mu s_2(t) \end{bmatrix} \quad (2.21)$$

where  $P \triangleq \begin{bmatrix} P_1 & P_{12} \\ P'_{12} & P_2/\mu \end{bmatrix}$  is the covariance of the forward filter output:  $\begin{bmatrix} \hat{\eta} \\ \hat{\xi} \end{bmatrix}$

and  $M \triangleq \begin{bmatrix} M_1 & \mu M_{12} \\ \mu M'_{12} & \mu M_2 \end{bmatrix}$  is the covariance of the backward filter output:

$\begin{bmatrix} s_1 \\ \mu s_2 \end{bmatrix}$ . The reason for the particular partitionings of  $s(t)$  and  $M(t)$  will be apparent in the sequel.

The equations for the forward filters and their covariances, obtained from the usual Kalman filter equations with the above partitioning, are as follows

$$\dot{\hat{\eta}} = A_0 \hat{\eta} + (P_1 C'_0 + P_{12} C'_2) R^{-1} (y - C_0 \hat{\eta} - C_2 \hat{\xi}), \quad \hat{\eta}(0) = E(\eta_0) \quad (2.22)$$

$$\mu \dot{\hat{\xi}} = A_2 \hat{\xi} + (\mu P'_{12} C'_0 + P_2 C'_2) R^{-1} (y - C_0 \hat{\eta} - C_2 \hat{\xi}), \quad \hat{\xi}(0) = E(\xi_0) \quad (2.23)$$

$$\dot{P}_1 = A_0 P_1 + P_1 A_0' + B_0 Q B_0' - (P_1 C_0' + P_{12} C_2') R^{-1} (C_0 P_1 + C_2 P_{12}'), \quad P_1(0) = \text{cov}(\eta_0) \quad (2.24)$$

$$\mu \dot{P}_{12} = \mu A_0 P_{12} + P_{12} A_2' + B_0 Q B_2' - (P_1 C_0' + P_{12} C_2') R^{-1} (\mu C_0 P_{12} + C_2 P_2), \quad P_{12}(0) = \text{cov}(\eta_0, \xi_0) \quad (2.25)$$

$$\mu \dot{P}_2 = A_2 P_2 + P_2 A_2' + B_2 Q B_2' - (\mu P_{12} C_0' + P_2 C_2') R^{-1} (C_2 P_2 + \mu C_0 P_{12}'), \quad P_2(0) = \mu \text{cov}(\xi_0) \quad (2.26)$$

Note that since the covariance of the fast state  $\xi(t)$  behaves as  $O(1/\mu)$ , the error covariance of its estimate has been denoted by  $P_2/\mu$ .

Similarly, applying equations (2.10)-(2.11), after the appropriate partitioning, yields the backward filter equations for  $s_1, s_2$  and their covariances as given below

$$\frac{ds_1}{d\sigma} = (A_0' - M_1 B_0 Q B_0' - M_{12} B_2 Q B_0') s_1 - (M_1 B_0 + M_{12} B_2) Q B_2' s_2 + C_0' R^{-1} y, \quad s_1(T) = 0 \quad (2.27)$$

$$\mu \frac{ds_2}{d\sigma} = - (\mu M_{12}' B_0 + M_2 B_2) Q B_0' s_1 + (A_2' - \mu M_{12}' B_0 Q B_2' - M_2 B_2 Q B_2') s_2 + C_2' R^{-1} y, \quad s_2(T) = 0 \quad (2.28)$$

$$\frac{dM_1}{d\sigma} = A_0' M_1 + M_1 A_0 + C_0' R^{-1} C_0 - (M_1 B_0 + M_{12} B_2) Q (B_0' M_1 + B_2' M_{12}'), \quad M_1(T) = 0 \quad (2.29)$$

$$\mu \frac{dM_{12}}{d\sigma} = \mu A_0' M_{12} + M_{12} A_2 + C_0' R^{-1} C_2 - (M_1 B_0 + M_{12} B_2) Q (\mu B_0' M_{12} + B_2' M_2), \quad M_{12}(T) = 0 \quad (2.30)$$

$$\mu \frac{dM_2}{d\sigma} = A_2' M_2 + M_2 A_2 + C_2' R^{-1} C_2 - (\mu M_{12}' B_0 + M_2 B_2) Q (B_2' M_2 + \mu B_0' M_{12}'), \quad M_2(T) = 0 \quad (2.31)$$

The form adopted for  $s(t)$  as  $\begin{bmatrix} s_1 \\ \mu s_2 \end{bmatrix}$  is required to yield for  $s_2(t)$  the same role in (2.28) as  $\hat{\xi}(t)$  in (2.23). This also produces  $M^*(t) = \begin{bmatrix} M_1 & M_{12} \\ M'_{12} & M_2/\mu \end{bmatrix}$  as the covariance of the vector  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ , similar to  $P(t)$ .

Inspection of (2.22) - (2.31) reveals that the backward filtering solution is a dual to the forward filtering solution, where the roles of  $A'$ ,  $B'$ ,  $C'$ , and  $R^{-1}$  are interchanged with those of  $A$ ,  $C$ ,  $B$ , and  $Q$  respectively. Comparison of (2.29) - (2.31) to (2.24) - (2.26), for instance, reveals that  $M_1$ ,  $M_{12}$ , and  $M_2$  are the exact duals of  $P_1$ ,  $P_{12}$ , and  $P_2$  for the forward filter. Also, if the innovation term of (2.22) and (2.23) is broken up into its components  $y$ ,  $\hat{\eta}$ , and  $\hat{\xi}$ , it is easily seen that the coefficients of  $s_1(t)$  and  $s_2(t)$  of (2.27) and (2.28) are perfect duals to those of  $\hat{\eta}(t)$  and  $\hat{\xi}(t)$  respectively. The fact that, of all these equations, only the coefficients of  $y(t)$  are not duals is essentially a result of the absence of an innovation term in the backward filter. Any attempt to form such a term in a filter equation describing  $s(t)$  would involve using the unstable quantity  $P_b(t)$ . Overall, this duality property will prove to be extremely useful as a time saving tool, as will be seen in the following chapters. In what follows, the limiting behavior of the Riccati equations (2.24) - (2.26) and (2.29) - (2.31) will be considered. Subsequently, the limiting behavior of the resulting filters (2.22) - (2.23) and (2.27) - (2.28) will be derived.

#### 2.4. Remarks

In summary of the preceding Section 2.3, it should be noted that substitution of the exact filters and covariances as given by (2.22) - (2.26) and (2.27) - (2.31) in equation (2.21) would yield the optimal fixed-interval smoother for the system (1.8) - (1.10).

The purpose of this chapter was to introduce the proposed formulation and indicate the tremendous complexity involved in solving for the optimal smoother using the full system, in addition to the ill-conditioning of such a problem. The reduced-order smoother has also been studied. It was seen to be quite simple to solve and thus implement and, as will be shown in Chapter V, the smoother for the slow states,  $\hat{\eta}(t|T)$ , tends to this smoother for the reduced system as  $\mu \rightarrow 0$ . Thus, this simple smoother can be used as a near optimal (within  $O(\mu^{\frac{1}{2}})$  approximation) estimator for the slow variable  $\eta(t)$ .

The proposed smoothing formulation, as a weighted sum of near optimal filters, will have a limiting behavior which is extremely simple to analyze in terms of the limiting behavior of these composite filters.

### III. THE FORWARD FILTER

#### 3.1. Limits of the Riccati Equations

In this chapter the limits of the forward filter as  $\mu \rightarrow 0$  are investigated. For that purpose, it is necessary to first study the limits of the Riccati equations. These will be needed in determining the limiting behavior of both the filter and the smoother.

It has been shown [1,9,10,14] that the filter error covariance matrices  $P_{12}(t)$  and  $P_2(t)$  as given by eqs. (2.25) and (2.26) can be expressed to within an  $O(\mu)$  approximation, by the sum of quasi steady-state and boundary layer terms [2-8], i.e.,

$$P_{12}(t) = \bar{P}_{12}(t) + \tilde{P}_{12}(\theta) + O(\mu) \quad (3.1)$$

$$P_2(t) = \bar{P}_2(t) + \tilde{P}_2(\theta) + O(\mu) \quad (3.2)$$

where  $\theta = t/\mu$ . The quasi steady-state terms  $\bar{P}_1(t)$ ,  $\bar{P}_{12}(t)$ , and  $\bar{P}_2(t)$  are obtained by setting  $\mu = 0$  in (2.24) - (2.26) and thus, satisfy the following equations:

$$\bar{P}_2 A_2' + A_2 \bar{P}_2 + B_2 Q B_2' - \bar{P}_2 C_2' R^{-1} C_2 \bar{P}_2 = 0 \quad (3.3)$$

$$\bar{P}_{12} = - [B_0 Q B_2' - \bar{P}_1(t) C_0' R^{-1} C_2 \bar{P}_2] A_2'^{-1} \quad (3.4)$$

$$\dot{\bar{P}}_1 = A_0 \bar{P}_1 + \bar{P}_1 A_0' + B_0 Q B_0' - (\bar{P}_1 C_0' + \bar{P}_{12} C_2') R^{-1} (C_0 \bar{P}_1 + C_2 \bar{P}_{12}'), \quad \bar{P}_1(0) = \text{cov}(\eta_0) \quad (3.5)$$

where

$$\bar{A}_2(t) \triangleq A_2 - \bar{P}_2 C_2' R^{-1} C_2 \quad (3.6)$$

Note that  $\bar{P}_2(t)$  is the positive semidefinite root of (3.3). The boundary layer terms  $\tilde{P}_2(\theta)$  and  $\tilde{P}_{12}(\theta)$  satisfy the homogeneous equations

$$\begin{aligned} \frac{d}{d\theta} \tilde{P}_2(\theta) &= \bar{A}_2(0) \tilde{P}_2(\theta) + \tilde{P}_2(\theta) \bar{A}_2'(0) - \tilde{P}_2(\theta) C_2' R^{-1} C_2 \tilde{P}_2(\theta) + o(\mu), \\ \tilde{P}_2(0) &= \mu \text{cov}(\xi_0) - \bar{P}_2(0) \end{aligned} \quad (3.7)$$

$$\begin{aligned} \frac{d}{d\theta} \tilde{P}_{12}(\theta) &= \tilde{P}_{12}(\theta) [\bar{A}_2'(0) - C_2' R^{-1} C_2 \tilde{P}_2(\theta)] - [\bar{P}_1(0) C_0' + \bar{P}_{12}(0) C_2'] R^{-1} C_2 \tilde{P}_2(\theta) + o(\mu). \\ \tilde{P}_{12}(0) &= \text{cov}(\eta_0, \xi_0) - \bar{P}_{12}(0) \end{aligned} \quad (3.8)$$

Since  $A_2(0)$  is stable, for any  $t > 0$  the boundary layer terms tend to zero exponentially fast as  $\mu \rightarrow 0$ . Hence for  $t \geq \epsilon > 0$ , both terms become  $O(\mu)$ . Consequently, on a subinterval  $t \in [\epsilon, T]$ ,  $\epsilon > 0$ , the quasi steady-state terms alone may be used, i.e.,

$$P_2(t) = \bar{P}_2(t) + O(\mu) \quad (3.9)$$

$$P_{12}(t) = \bar{P}_{12}(t) + O(\mu) \quad (3.10)$$

It is also easily seen that since the boundary layer of  $P_{12}$  appears only in the forcing term of (2.24), its effect on  $P_1(t)$  is  $O(\mu)$ , and thus the covariance  $P_1$  can be approximated for all  $t \in [0, T]$  by

$$P_1(t) = \bar{P}_1(t) + O(\mu) \quad (3.11)$$

To avoid any future confusion, it should be noted that  $\bar{P}_1(t)$ ,  $\bar{P}_{12}(t)$ , and  $\bar{P}_2(t)$  were found by setting  $\mu = 0$  in (2.24) - (2.26) which, in general, is not equivalent to the steady-state obtained by setting the derivatives to zero in the same equations. Hence, the term "quasi steady-state" has been used.

### 3.2. Limiting Behavior of the Fast Mode Filter

In considering the limit of the fast mode filter (2.23), two cases are of interest. The first is when the fast mode estimate is of interest in itself. In this case, the limits of the Riccati equations for  $P_{12}$  and  $P_2$ , together with the results of Theorem 1 of [1], allow the implementation of the fast mode filter, to within an  $O(\mu^{1/2})$  approximation, as a stationary filter in a stretched time-scale. That is,

$$\hat{\xi} = \bar{\xi} + O(\mu^{1/2}) \quad (3.12)$$

where  $\bar{\xi}(t)$  satisfies

$$\frac{d}{d\tau} \bar{\xi} = A_2(t_i) \bar{\xi} + \bar{P}_2(t_i) C_2' R^{-1} (y - C_0 \hat{\eta} - C_2 \bar{\xi})$$

for all  $t_{i+1} \geq t \geq t_i \geq \epsilon$ ;  $\epsilon > 0$ , and  $\tau = \frac{t-t_i}{\mu} \geq 0$ . (3.13)

Note that the small subintervals  $[t_i, t_{i+1}]$  form a subdivision of the observation interval  $(0, T]$ . Since the covariance of  $\hat{\xi}$  behaves as  $1/\mu$ ,  $\hat{\xi}$  is  $O(\mu^{-1/2})$ , and thus  $\mu \hat{\xi}$  is  $O(\mu^{1/2})$ , hence the  $O(\mu^{1/2})$  approximation. Near the initial point ( $t=0$ ) of the observation interval,  $P_2(t)$  must be augmented by the

boundary layer term  $\tilde{P}_2(\theta)$ , so that the filter is no longer time-invariant.

The second case to be considered is when  $\hat{\xi}(t)$  is to be used only as an input to the slow mode filter. Under such circumstances,  $\hat{\xi}$  can be replaced by its white noise approximation (Theorem 1 of [1]) obtained by setting  $\mu = 0$  in (2.23) and (2.26). This substitution, valid for  $t \geq \ell > 0$ , is given by

$$C_2 \hat{\xi}_0 = -C_2 \bar{A}_2^{-1} \bar{P}_2 C_2' R^{-1} (y - C_0 \hat{\eta}) \quad (3.14)$$

It should be noted that (3.14) does not yield an approximate estimate of the fast variable, but is only a valid approximation when  $\hat{\xi}$  is to be used as an input to slow systems.

### 3.3. Limiting Behavior of the Slow Mode Filter

The limit of the slow mode filter is obtained from (2.22) by substituting  $\bar{P}_1$ ,  $\bar{P}_{12}$ , and  $\hat{\xi}_0$  as given by (3.5), (3.4) and (3.14) respectively. The resulting filter becomes

$$\dot{\hat{\eta}} = A_0 \hat{\eta} + \bar{K}_1 (y - C_0 \hat{\eta}) + o(\mu^{1/2}), \quad 0 \leq t \leq T \quad (3.15)$$

where

$$\bar{K}_1 \triangleq (\bar{P}_1 C_0 + \bar{P}_{12} C_2') R^{-1} (I - C_2 \bar{A}_2^{-1} \bar{P}_2 C_2' R^{-1})^{-1} \quad (3.16)$$

It can be easily shown (with some lengthy matrix algebra as in [1]) that the slow mode filter tends to the filter for the reduced system as  $\mu \rightarrow 0$ , namely that

$$\bar{K}_1 = K_0 \quad \text{and} \quad \bar{P}_1 = P_0 ,$$

where  $K_0$  and  $P_0$  are given in section 2.2 for the reduced problem. This result is quite significant in that it justifies the solution of a much simpler reduced-order problem for estimating the slow mode, and can be summarized by

$$\hat{\eta} = \hat{\eta}_0 + O(\mu^{1/2}) \quad , \quad 0 \leq t \leq T \quad (3.17)$$

### 3.4. Boundary Layers

The notion of "boundary layers" has been introduced in this chapter for the forward filter. To yield some insight as to why the boundary layer correction terms were proposed for the fast mode filter and their subsequent role in the fast mode smoother (to be derived in Chapter V), a discussion is in order.

Inspection of equations (2.23), (2.25), and (2.26) representing  $\hat{\xi}$ ,  $P_{12}$ , and  $P_2$  respectively, reveals that as  $\mu \rightarrow 0$ , the associated derivatives tend to become quite large. A quasi steady-state solution, as was found for  $P_{12}$  and  $P_2$ , ignores the required boundary conditions of these equations. Thus, there is a small initial time interval  $[0, \mathcal{E})$  where the system experiences the rapid decay of these fast transients. The boundary layer terms, which die off quickly as time increases slightly, are required to account for the large discrepancy between the exact and reduced solutions near the initial point of the observation interval. Since the equation (2.23) for  $\hat{\xi}$  is not reduced (that is, when the behavior of the fast mode is of interest), the only correction terms needed are those associated with the covariances  $P_{12}$  and  $P_2$  in the input term of that equation. However, since the  $P_{12}$  term

is  $O(\mu)$  in (2.23), only  $\bar{P}_2$  need be augmented. It can be easily seen [1] that just as the "white noise approximation"  $\hat{\xi}_0$  was valid as an input to the slow mode filter, so is the quasi steady-state covariance multiplier  $\bar{P}_{12}(t)$ , i.e., there are no boundary layers for the slow mode filter.

The duality of the backward filter will present an essentially identical problem with respect to the boundary layers (this time occurring only near  $t = T$ ) and thus, in view of the forward filtering results, the boundary layer compensation will be obvious. Finally, the near optimal smoothing solution, as will be seen in Chapter V, will not offer much additional difficulty due to the uncoupled boundary layers of the composite forward and backward filters.

### 3.5. Summary

In this chapter it has been shown that a near optimal fast mode filter can be designed using a stretched time-scale implementation with innovation input. A time-invariant filter suffices for any small subinterval of  $[\mathcal{E}, T]$ ,  $\mathcal{E} > 0$ . This filter uses  $\bar{P}_2$  and therefore, if the filter is to be valid near  $t = 0$ , the appropriate boundary layer term  $\tilde{P}_2(\theta)$  must be added so that the filter becomes time-varying.

For the behavior of the slow mode filter, it was seen that the "white noise approximation" to  $\hat{\xi}(t)$  was valid. As might have been suspected, the slow mode filter was equivalent, in the limit, to the filter for the reduced system.

As a final note to this chapter, if only the near optimal filtering solution is desired, direct application of (1.12) yields for the original

states

$$\hat{x}(t) = \hat{\eta}_0(t) + o(\mu^{1/2}) \quad (3.18)$$

$$\hat{z}(t) = -L_0 \hat{\eta}_0(t) + \hat{\xi}(t) + o(\mu^{1/2}) \quad (3.19)$$

where the linearity of the filters has been used. However, for later use in the smoothing solution, it is preferable to express  $\hat{\xi}(t)$ , as given by (3.13), as a sum of uncoupled filter outputs, i.e.,

$$\hat{\xi} = \hat{\zeta} + \bar{A}_2^{-1} \bar{P}_2 C_2' R^{-1} C_0 \hat{\eta}_0 + o(\mu^{1/2}) \quad (3.20)$$

and thus,  $\bar{\zeta}$  satisfies the decoupled equation

$$\frac{d}{d\tau} \bar{\zeta} = A_2(t_i) \bar{\zeta} + \bar{P}_2(t_i) C_2' R^{-1} (y - C_2 \bar{\zeta}), \quad \tau = \frac{t - t_i}{\mu} \geq 0 \quad (3.21)$$

Similarly,  $\tilde{\zeta}$  ( $\bar{\zeta}$  with  $\tilde{P}_2(\theta)$  added) would be the time-varying uncoupled filter output valid over the entire interval.

## IV. THE BACKWARD FILTER

4.1. Limits of the Riccati Equations

In this chapter the techniques used in Chapter III for the forward filter will be applied to the backward filter as well. As was the case with the forward filter, the limits of the corresponding Riccati equations, (2.29) - (2.31) here, must first be examined before deriving the limiting behavior of the backward filter.

Section 2.3 established the duality relationship between the forward and backward filters and covariances, thus simplifying the analysis to a great extent. In accordance with this duality, the backward filter covariances can be approximated to within  $O(\mu)$  for all  $t \in [0, T]$  as follows

$$M_1(t) = \bar{M}_1(t) + O(\mu) \quad (4.1)$$

$$M_{12}(t) = \bar{M}_{12}(t) + \tilde{M}_{12}(\rho) + O(\mu) \quad (4.2)$$

$$M_2(t) = \bar{M}_2(t) + \tilde{M}_2(\rho) + O(\mu) \quad (4.3)$$

where  $\rho = \frac{\sigma}{\mu} = \frac{T-t}{\mu}$ . The quasi steady-state terms  $\bar{M}_2$ ,  $\bar{M}_{12}$ , and  $\bar{M}_1$ , obtained by setting  $\mu = 0$  in (2.29) - (2.31), are the solutions of

$$\bar{M}_2 A_2 + A_2' \bar{M}_2 + C_2' R^{-1} C_2 - \bar{M}_2 B_2 Q B_2' \bar{M}_2 = 0 \quad (4.4)$$

$$\bar{M}_{12} = -[C_0' R^{-1} C_2 - \bar{M}_1(t) B_0 Q B_2' \bar{M}_2] \tilde{A}_2^{-1} \quad (4.5)$$

$$\frac{d\bar{M}_1}{d\sigma} = A_0' \bar{M}_1 + \bar{M}_1 A_0 + C_0' R^{-1} C_0 - N(t) Q N'(t), \quad \bar{M}_1(T) = 0 \quad (4.6)$$

respectively, where

$$\tilde{A}'_2(t) \triangleq A'_2 - \bar{M}_2 B_2 Q B'_2 \quad (4.7)$$

$$N(t) \triangleq \bar{M}_1 B_0 + \bar{M}_{12} B_2 \quad (4.8)$$

and  $\bar{M}_2(t)$  is the positive semidefinite root of (4.4). The boundary layer terms  $\tilde{M}_2(\rho)$  and  $\tilde{M}_{12}(\rho)$  satisfy the homogeneous equations

$$\begin{aligned} \frac{d\tilde{M}_2(\rho)}{d\rho} &= \tilde{A}'_2(T)\tilde{M}_2(\rho) + \tilde{M}_2(\rho)\tilde{A}'_2(T) - \tilde{M}_2(\rho)B_2QB'_2\tilde{M}_2(\rho) + O(\mu), \\ \rho &\geq 0, \tilde{M}_2(\rho=0) = -\bar{M}_2(T) \end{aligned} \quad (4.9)$$

$$\begin{aligned} \frac{d\tilde{M}_{12}(\rho)}{d\rho} &= \tilde{M}_{12}(\rho)[\tilde{A}'_2(T) - B_2QB'_2\tilde{M}_2(\rho)] - [\bar{M}_1(T)B_0 + \bar{M}_{12}(T)B_2]QB'_2\tilde{M}_2(\rho) + O(\mu), \\ \tilde{M}_{12}(\rho=0) &= -\bar{M}_{12}(T) \end{aligned} \quad (4.10)$$

Again, due to the stability of  $A_2(T)$  and hence  $\tilde{A}'_2(T)$ , the boundary layer terms decay exponentially fast as  $\mu \rightarrow 0$  for any  $t < T$ . Hence, for all  $t \in [0, T-\epsilon]$ ,  $\epsilon > 0$ , the approximations given by

$$M_1(t) = \bar{M}_1(t) + O(\mu) \quad (4.11)$$

$$M_{12}(t) = \bar{M}_{12}(t) + O(\mu) \quad (4.12)$$

$$M_2(t) = \bar{M}_2(t) + O(\mu) \quad (4.13)$$

will be valid. It remains to be seen that  $\bar{M}_1$  is equivalent to the reduced covariance  $M_0$  of Section 2.2. This problem will be considered in Section 4.3.

#### 4.2. Limiting Behavior of the Fast Mode Filter

The exact expressions for the backward filter are given by equations (2.27) and (2.28). However, in order to obtain the behavior of the fast mode filter alone, these filters should be decoupled by employing a transformation as in Section 1.2.

Let new variables  $w(t)$  and  $r(t)$  be defined by

$$\begin{bmatrix} w(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} I_n - \mu H_1 L_1 & -\mu H_1 \\ L_1 & I_m \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \quad (4.14)$$

where

$$L_1 \triangleq -\tilde{A}_2'^{-1} \tilde{M}_2' B_2' Q B_0' + O(\mu) \quad (4.15)$$

$$H_1 \triangleq -N(t) Q B_2' \tilde{A}_2'^{-1} + O(\mu) \quad (4.16)$$

This is a non-singular transformation which when inverted yields

$$s_1(t) = w(t) + O(\mu^{1/2}) \quad (4.17)$$

$$s_2(t) = \tilde{A}_2'^{-1} \tilde{M}_2' B_2' Q B_0' w(t) + r(t) + O(\mu^{1/2}) \quad (4.18)$$

where the  $O(\mu^{1/2})$  results from  $\mu s_2$  since  $s_2$  is  $O(\mu^{-1/2})$ . The uncoupled filters are now given by

$$\frac{dw(t)}{d\sigma} = [A_0' - N(t) Q (I + B_2' \tilde{A}_2'^{-1} \tilde{M}_2' B_2' Q) B_0'] w + [C_0' R^{-1} + N(t) Q B_2' \tilde{A}_2'^{-1} C_2' R^{-1}] y + O(\mu^{1/2}) \quad (4.19)$$

$$\frac{\mu dr(t)}{d\sigma} = \tilde{A}_2' r + C_2' R^{-1} y + O(\mu^{1/2}) \quad (4.20)$$

The equation for  $r(t)$  is a standard singularly perturbed filter, which is a strictly fast system with input  $y(t)$  containing white noise. Hence, its

limiting behavior is similar to that of  $\hat{\xi}$ , since  $\tilde{A}_2(t)$  is stable for all  $t \in [0, T]$ . Consequently,  $r(t)$  may be approximated by the output of a stationary filter implemented in a stretched time-scale  $\lambda = \frac{\sigma - \sigma_i}{\mu}$ , for any small subinterval  $[\sigma_i, \sigma_{i+1}] \subset [0, T]$ . The resulting filter, valid for all  $T - \sigma_{i+1} \leq t \leq T - \sigma_i$  may be expressed by

$$\frac{d\bar{r}(t)}{d\lambda} = \tilde{A}_2'(T - \sigma_i) \bar{r}(t) + C_2'R^{-1}y + O(\mu^{1/2}), \quad \lambda \geq 0 \quad (4.21)$$

Near the right endpoint ( $t = T$ ) of the observation interval, it is necessary to augment  $\bar{M}_2$  with  $\tilde{M}_2(\rho)$  in  $\tilde{A}_2'(t)$  above, so that this fast filter is no longer time-invariant.

#### 4.3. Limiting Behavior of the Slow Mode Filter

The limiting behavior of the filter output  $w(t)$  is found by setting  $\mu = 0$  in the right-hand side of (4.19) (the script matrices neglected earlier are actually just now being replaced). This results in

$$\frac{d\bar{w}(t)}{d\sigma} = [A_0' - N(t)Q(I + B_2'\tilde{A}_2'^{-1}\bar{M}_2B_2Q)B_0'] \bar{w} + [C_0' + N(t)QB_2'\tilde{A}_2'^{-1}C_2']R^{-1}y \quad (4.22)$$

Note that (4.17) guarantees that  $s_1(t)$  also satisfies (4.22) in the limit as  $\mu \rightarrow 0$ .

Simple matrix algebra must now be performed to show that the filter of equation (4.22) is identical to the reduced-order filter, namely,  $\bar{w}(t) = s_0(t)$  as given by (2.18). This is essentially the same task as for the forward filtering case, except that the lack of an innovation term in the

backward filter makes it more tedious. To accomplish this, it is first necessary to prove the equivalence of  $\bar{M}_1$  and  $M_0$  as given by equations (4.6) and (2.20), respectively. The substitution of (4.4) and (4.5) in (4.8), after lengthy algebra, yields the dual to equations (63) and (64) for the forward filter in [1]. These expressions are

$$N(t) = (\bar{M}_1 B_0 + C_0' R^{-1} D_0) (I - B_2' A_2'^{-1} \bar{M}_2 B_2 Q)^{-1} \quad (4.23)$$

and

$$N(t) Q (I - B_2' A_2'^{-1} \bar{M}_2 B_2 Q)^{-1} = (\bar{M}_1 B_0 + C_0' R^{-1} D_0) (I + Q D_0' R^{-1} D_0)^{-1} Q \quad (4.24)$$

The further substitution of (4.23) and (4.24) in (4.6) yields the equation

$$\frac{d\bar{M}_1}{d\sigma} = \bar{M}_1 A_0 + A_0' \bar{M}_1 + C_0' R^{-1} C_0 - (\bar{M}_1 B_0 + C_0' R^{-1} D_0) (I + Q D_0' R^{-1} D_0)^{-1} Q (B_0' \bar{M}_1 + D_0' R^{-1} C_0) \quad (4.25)$$

If the following two identities are now substituted in (4.25):

$$(I + Q D_0' R^{-1} D_0)^{-1} = I - Q D_0' R_0^{-1} D_0 \quad (4.26)$$

$$R^{-1} D_0 (I + Q D_0' R^{-1} D_0)^{-1} = R_0^{-1} D_0, \quad (4.27)$$

the resulting equation for (4.25) becomes

$$\begin{aligned} \frac{d\bar{M}_1}{d\sigma} = & \bar{M}_1 A_0 + A_0' \bar{M}_1 + C_0' R^{-1} C_0 - \bar{M}_1 B_0 Q B_0' \bar{M}_1 + \bar{M}_1 B_0 Q D_0' R_0^{-1} D_0 Q B_0' \bar{M}_1 - \bar{M}_1 B_0 Q D_0' R_0^{-1} C_0 \\ & - C_0' R_0^{-1} D_0 Q B_0' \bar{M}_1 - C_0' R_0^{-1} D_0 Q D_0' R^{-1} C_0 \end{aligned} \quad (4.28)$$

Additional algebraic manipulation of (4.28), together with the use of the identity

$$(I - R_0^{-1} D_0 Q D_0') R^{-1} = R_0^{-1} \quad (4.29)$$

results in

$$\frac{d\bar{M}_1}{d\sigma} = \bar{M}_1 A_0 + A_0' \bar{M}_1 - \bar{M}_1 B_0 Q B_0' \bar{M}_1 + (C_0' - \bar{M}_1 B_0 Q D_0') R_0^{-1} (C_0 - D_0 Q B_0' \bar{M}_1) \quad (4.30)$$

which is identical to (2.20).

It now only remains to be shown that the filter coefficients of (4.22) are identical to those of (2.18). The gain coefficient of the filter is given by

$$\begin{aligned} (C_0' + N(t) Q B_2' \tilde{A}_2^{-1} C_2') R^{-1} &= [C_0' - N(t) Q (I - B_2' A_2'^{-1} \bar{M}_2 B_2 Q)^{-1} D_0'] R^{-1} \\ &= [C_0' - (M_0 B_0 + C_0' R^{-1} D_0) (I + Q D_0' R^{-1} D_0)^{-1} Q D_0'] R^{-1} = (C_0' - M_0 B_0 Q D_0') R_0^{-1} \equiv K_b \end{aligned} \quad (4.31)$$

where the identities (4.24), (4.26), and (4.29) have been used. Similarly, the system matrix of (4.22) is given by

$$\begin{aligned} A_0' - N(t) Q (I + B_2' \tilde{A}_2^{-1} \bar{M}_2 B_2 Q) B_0' &= A_0' - N(t) Q [I + (I - B_2' A_2'^{-1} \bar{M}_2 B_2 Q)^{-1} B_2' A_2'^{-1} \bar{M}_2 B_2 Q] B_0' \\ &= A_0' - N(t) Q (I - B_2' A_2'^{-1} \bar{M}_2 B_2 Q)^{-1} B_0' = A_0' - (M_0 B_0 + C_0' R^{-1} D_0) (I + Q D_0' R^{-1} D_0)^{-1} Q B_0' \\ &= A_0' - M_0 B_0 Q (I - D_0' R_0^{-1} D_0 Q) B_0' - C_0' R_0^{-1} D_0 Q B_0' \equiv A_0' - (M_0 B_0 + K_b D_0) Q B_0' \end{aligned} \quad (4.32)$$

where the identities (4.24), (4.26), and (4.27) have been used. It is thus seen from (4.31) and (4.32), that the limit of the slow mode filter is identical to that of the reduced problem, i.e.,

$$s_1(t) = s_0(t) + O(\mu^{1/-}), \quad 0 \leq t \leq T. \quad (4.33)$$

#### 4.4. Summary

In summary, the results for the limits of the backward filter are dual to those of the forward filter, with the difference that the boundary layers now occur at the right endpoint of the observation interval. It should be noted that there is no innovation in the backward filter and thus, the observations are used directly as inputs. Specifically, the fast backward filter tends to a stationary stretched time-scale filter and the slow mode filter tends to the reduced-order backward filter. These filter outputs will now be appropriately combined with the forward filter outputs to obtain the desired smoothed estimate.

## V. NEAR OPTIMAL SMOOTHING SOLUTION

### 5.1. The Uncoupled System

#### 5.1.1. The Near Optimal Slow Mode Smoother

From Section 2.3, the exact smoothing solution is given by (2.21). The partitioned matrix representations of  $P(t)$  and  $M(t)$ , as given in that same section, are now used to obtain the multiplicative terms of (2.21) as follows:

$$\begin{aligned}
 (I + PM)^{-1} &= \left[ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} P_1 & P_{12} \\ P'_{12} & P_2/\mu \end{pmatrix} \begin{pmatrix} M_1 & \mu M_{12} \\ \mu M'_{12} & \mu M_2 \end{pmatrix} \right]^{-1} \\
 &= \begin{bmatrix} I + P_1 M_1 + O(\mu) & \mu(P_1 M_{12} + P_{12} M_2) \\ P'_{12} M_1 + P_2 M'_{12} & I + P_2 M_2 + O(\mu) \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} (I + P_1 M_1)^{-1} + O(\mu) & -\mu(I + P_1 M_1)^{-1} (P_1 M_{12} + P_{12} M_2) (I + P_2 M_2)^{-1} + O(\mu^2) \\ -(I + P_2 M_2)^{-1} (P'_{12} M_1 + P_2 M'_{12}) (I + P_1 M_1)^{-1} + O(\mu) & (I + P_2 M_2)^{-1} + O(\mu) \end{bmatrix} \quad (5.1)
 \end{aligned}$$

Thus, it follows that  $(I + PM)^{-1} P =$

$$\begin{bmatrix} (I + P_1 M_1)^{-1} P_1 + O(\mu) & (I + P_1 M_1)^{-1} [P_{12} - (P_1 M_{12} + P_{12} M_2) (I + P_2 M_2)^{-1} P_2] + O(\mu) \\ -(I + P_2 M_2)^{-1} [(P'_{12} M_1 + P_2 M'_{12}) (I + P_1 M_1)^{-1} P_1 - P'_{12}] + O(\mu) & (I + P_2 M_2)^{-1} P_2 / \mu + O(1) \end{bmatrix} \quad (5.2)$$

For the slow mode smoother, the quasi steady-state covariance terms can be used since the boundary layer terms are  $O(\mu)$ . The forward and backward filters, as given by (2.14), (2.18), (3.13), and (4.21) together with (5.1) and (5.2), may be substituted in (2.21) to yield the slow mode smoother as

$$\begin{aligned}\hat{\eta}(t|T) &= (I + P_0 M_0)^{-1} [\hat{\eta}_0(t) + P_0 s_0(t)] + O(\mu^{1/2}) \\ &= \hat{\eta}_0(t|T) + O(\mu^{1/2}), \quad 0 \leq t \leq T\end{aligned}\quad (5.3)$$

It is now clear that the slow mode smoother tends to the reduced-order smoother. Consequently, if the fast variable is of no interest, a near optimal smoother for the slow variables may be obtained by solving the smoothing problem for the reduced-order system.

#### 5.1.2. The Near Optimal Fast Mode Smoother

In a similar manner, the expressions for the fast mode smoother may be obtained in the interior of the interval  $[0, T]$ , by substituting the quasi steady-state values of the covariances and the filters in (5.1), (5.2), and (2.21) to yield

$$\hat{\xi}(t|T) = (I + \bar{P}_2 \bar{M}_2)^{-1} [\bar{\xi}(t) + \bar{P}_2 \bar{r}(t)] + \hat{\xi}_0(t|T) + O(\mu^{1/2}) \quad (5.4)$$

where

$$\hat{\xi}_0(t|T) \triangleq (I + \bar{P}_2 \bar{M}_2)^{-1} [(N_1 + N_2 M_0) \hat{\eta}_0(t|T) + N_2 s_0(t)] \quad (5.5)$$

and

$$N_1(t) \triangleq (\bar{A}_2^{-1} \bar{P}_2 + \bar{P}_2 \bar{A}_2^{-1}) C_2' R^{-1} C_0 \quad (5.6)$$

$$N_2(t) \triangleq (\bar{A}_2^{-1} - \bar{P}_2 \bar{A}_2^{-1} \bar{M}_2^{-1}) B_2' Q B_0' \quad (5.7)$$

It should be emphasized that  $\hat{\xi}_0(t|T)$  is an output of a slow system, hence its covariance relative to that of  $\hat{\xi}$  is  $O(\mu)$ . Furthermore, in view of its being a slow variable, it need not be performed in the stretched time-scale, and merely represents a slowly varying "mean" about which the fast components of the variable  $\hat{\xi}(t|T)$ , namely  $\bar{\zeta}(t)$  and  $\bar{r}(t)$ , fluctuate.

### 5.1.3. Boundary Layers

If a near optimal solution for the fast mode smoother is needed during the entire observation interval, the boundary layer terms need to be used to augment the fast covariances and outputs. Since the boundary layers at both ends of the observation interval were shown to be independent of each other, the results are greatly simplified. Augmentation of the fast forward covariance  $\bar{P}_2$  in both the forward filter gain and multiplicative terms of eq. (5.4), yields the near optimal fast mode smoother valid in the left boundary layer. This simple modification of the time-invariant smoother results in

$$\hat{\xi}_L(t|T) = [I + (\bar{P}_2 + \tilde{P}_2(\theta))\bar{M}_2]^{-1}[\tilde{\zeta}(t) + (\bar{P}_2 + \tilde{P}_2(\theta))\bar{r}(t)] + \hat{\xi}_0(t|T) + O(\mu^{1/2}),$$

$$0 \leq t \leq T-\epsilon \quad (5.8)$$

where  $\tilde{\zeta}(t)$  is the time-varying fast filter described in Section 3.5.

Similarly, the time-varying smoother valid in the right boundary layer is given by

$$\hat{\xi}_R(t|T) = [I + \bar{P}_2(\bar{M}_2 + \tilde{M}_2(\rho))]^{-1}[\bar{\zeta}(t) + \bar{P}_2\tilde{r}(t)] + \hat{\xi}_0(t|T) + O(\mu^{1/2}),$$

$$\epsilon \leq t \leq T \quad (5.9)$$

where  $\tilde{r}(t)$  is the time-varying backward fast filter described in Section 4.2. Finally, if both endpoints may be needed and the estimator structure is not to be changed, the appropriate smoother is given by

$$\begin{aligned} \hat{\xi}_{L,R}(t|T) = & [I + (\bar{P}_2 + \tilde{P}_2(\theta))(\bar{M}_2 + \tilde{M}_2(\phi))]^{-1} [\tilde{r}(t) + (\bar{P}_2 + \tilde{P}_2(\theta))\tilde{r}(t)] \\ & + \hat{\xi}_0(t|T) + o(\mu^{1/2}), \quad 0 \leq t \leq T \end{aligned} \quad (5.10)$$

It should be noted that, as was the case with the forward fast mode filter, the fast mode smoother requires no boundary layer augmentations for the cross covariances  $P_{12}(t)$  and  $M_{12}(t)$ , i.e., their boundary layer terms always appear where they are  $O(\mu)$  in comparison with the others.

The two filter formation adopted in this work is more suitable for analyzing the boundary layer behavior. In the previous formulation [10], instability problems may occur from the interaction of the two boundary layers. The two filter approach avoids these problems by allowing two separate filters, each with its own independent boundary layer at the two opposing ends of the observation interval, respectively.

## 5.2. Smoothing for the Original States

The near optimal fixed-interval smoother for the original states  $x(t)$  and  $z(t)$  defined by (1.1)-(1.3) can now be summarized, using the earlier results in conjunction with the inverse transformation (1.12), in the following theorem:

**Theorem 1:** If the assumptions (i) and (ii) of Section 1.2.4 hold for the system (1.1)-(1.3), the smoothing solution for the original states satisfies

the relations

$$1) \quad \hat{x}(t|T) = \hat{x}_0(t|T) + o(\mu^{1/2}), \quad 0 \leq t \leq T$$

where  $\hat{x}_0(t|T)$  is the smoother for the reduced-order problem obtained by setting  $\mu = 0$  in (1.1)-(1.3) (note that  $\hat{x}_0(t|T) \equiv \hat{\eta}_0(t|T)$ ),

$$2) \quad \hat{z}(t|T) = -A_2^{-1} A_{21} \hat{x}_0(t|T) + \hat{\xi}(t|T) + o(\mu^{1/2}), \quad \epsilon \leq t \leq T - \epsilon$$

where  $\hat{\xi}(t|T)$  is the solution for the fast uncoupled state given by (5.4).

If a smoother is desired which is valid near the ends of the observation interval, one needs only to use the appropriately modified versions of  $\hat{\xi}(t|T)$  as discussed in the previous section.

The proof of this theorem, as noted in Section 1.2, follows from the results of the previous sections together with the linearity of the estimators considered in the sequel.

As a final note, it should be pointed out that the resulting four filters involved in the near optimal smoothing solution are of lower order than the original smoothing solution, and can be implemented in two time-scales to avoid unnecessary computational burden and ill-conditioning in the problem.

## VI. SPECIAL CASE AND EXAMPLE

6.1. General Stationary Second-Order System

Consider the second-order, time-invariant, uncoupled system represented by

$$\dot{\eta} = a_0 \eta + b_0 u, \quad \eta(0) = \eta_0 \quad (6.1)$$

$$\mu \dot{\xi} = a_2 \xi + b_2 u, \quad \xi(0) = \xi_0 \quad (6.2)$$

$$y = c_0 \eta + c_2 \xi + v, \quad t \geq 0 \quad (6.3)$$

where  $u(t)$  and  $v(t)$  are uncorrelated white Gaussian noise processes with zero means and covariances  $q$  and  $r$ , respectively. The transformation of Chapter I allows consideration of systems of the above form (uncoupled) without loss of generality. Note that stability requires  $a_0, a_2 < 0$ .

The exact filtering solution is given by

$$\hat{\eta} = a_0 \hat{\eta} + \frac{1}{r}(p_1 c_0 + p_{12} c_2)(y - c_0 \hat{\eta} - c_2 \hat{\xi}), \quad \hat{\eta}(0) = E(\eta_0) \quad (6.4)$$

$$\mu \hat{\xi} = a_2 \hat{\xi} + \frac{1}{r}(\mu c_0 p_{12} + c_2 p_2)(y - c_0 \hat{\eta} - c_2 \hat{\xi}), \quad \hat{\xi}(0) = E(\xi_0) \quad (6.5)$$

where  $p_1$ ,  $p_{12}$ , and  $p_2$  satisfy the Riccati equations

$$\dot{p}_1 = 2a_0 p_1 + q b_0^2 - (c_0 p_1 + c_2 p_{12})^2 / r, \quad p_1(0) = \text{cov}(\eta_0) \quad (6.6)$$

$$\mu \dot{p}_{12} = (\mu a_0 + a_2) p_{12} + q b_0 b_2 - (c_0 p_1 + c_2 p_{12})(\mu c_0 p_{12} + c_2 p_2) / r, \quad p_{12}(0) = \text{cov}(\eta_0, \xi_0) \quad (6.7)$$

$$\mu \dot{p}_2 = 2a_2 p_2 + q b_2^2 - (\mu c_0 p_{12} + c_2 p_2)^2 / r, \quad p_2(0) = \mu \text{cov}(\xi_0) \quad (6.8)$$

The near optimal design can now be described quite easily using the results of the previous chapters with the appropriate simplifications due to this second-order system with scalar coefficients. Setting  $\mu = 0$  in (6.6)-(6.8), using the results of Section 3.1, yields the quasi steady-state covariances

$$\bar{p}_{12} = \frac{qb_0 b_2 - c_0 c_2 \bar{p}_1 \bar{p}_2 / r}{c_2^2 \bar{p}_2 / r - a_2} \quad (6.9)$$

$$\bar{p}_2 = a_2 r (1 - \sqrt{r_0 / r}) / c_2^2 \quad (6.10)$$

$$\bar{p}_1 \equiv p_0 = \frac{(b-\beta) \exp\{-\beta t\} - (b+\beta)}{2c(1 - k \exp\{-\beta t\})} \quad (6.11)$$

where

$$k \triangleq \frac{2c\bar{p}_1(0) + (b+\beta)}{2c\bar{p}_1(0) + (b-\beta)}, \quad a \triangleq qb_0^2 / r_0 \quad (6.12)$$

$$b \triangleq 2(a_0 - qd_0 b_0 c_0 / r_0), \quad c \triangleq -c_0^2 / r_0, \quad \beta \triangleq \sqrt{b^2 - 4ac},$$

and

$$d_0 \triangleq -c_2 b_2 / a_2, \quad r_0 \triangleq r + qd_0^2 \quad (6.13)$$

are the scalar counterparts of the matrices (upper case)  $D_0$  and  $R_0$  of the previous chapters.

The solutions of the scalar boundary layer equations corresponding to (3.7) and (3.8) of Chapter III are given by

$$\tilde{p}_2(\theta) = \frac{-2r\bar{a}_2 k_2 \exp\{2\bar{a}_2 \theta\}}{c_2^2 (1 - k_2 \exp\{2\bar{a}_2 \theta\})}, \quad \theta = t / \mu \quad (6.14)$$

$$\tilde{p}_{12}(\theta) = \frac{2k_2[\bar{p}_1(0)c_0 + \bar{p}_{12}(0)c_2]}{c_2(1 - k_2 \exp\{2\bar{a}_2\theta\})} [\exp\{2\bar{a}_2\theta\} - \exp\{\bar{a}_2\theta\}] - \frac{(1-k_2)\bar{p}_{12}(0)\exp\{\bar{a}_2\theta\}}{(1-k_2 \exp\{2\bar{a}_2\theta\})} \quad (6.15)$$

where

$$\bar{a}_2 \triangleq a_2 - \bar{p}_2 c_2^2 / r, \quad k_2 \triangleq \frac{c_2^2 \tilde{p}_2(0)}{c_2 \tilde{p}_2(0) - 2r\bar{a}_2} \quad (6.16)$$

The near optimal fast mode filter is given (in a stretched time-scale) by

$$\frac{d}{d\tau} \bar{\xi} = a_2 \bar{\xi} + \frac{1}{r} \bar{p}_2 c_2 (y - c_0 \hat{\eta}_0 - c_2 \bar{\xi}), \quad \tau = \frac{t-t_i}{\mu} \geq 0 \quad (6.17)$$

where  $\tilde{p}_2(\theta)$  must be added to  $\bar{p}_2$  to account for the boundary layer behavior near  $t=0$ .

The near optimal slow mode filter, which is found from the reduced system as seen in Chapter III, is described by

$$\dot{\hat{\eta}}_0 = a_0 \hat{\eta}_0 + k_0 (y - c_0 \hat{\eta}_0) \quad (6.18)$$

where

$$k_0 \triangleq (p_0 c_0 + q d_0 b_0) / r_0 \quad (6.19)$$

Similarly, the exact backward filtering solution is given by

$$\frac{ds_1(t)}{d\sigma} = (a_0 - m_1 q b_0^2 - m_{12} q b_0 b_2) s_1 - (m_1 b_0 + m_{12} b_2) q b_2 s_2 + \frac{c_0}{r} y, \quad s_1(T) = 0 \quad (6.20)$$

$$\mu \frac{ds_2(t)}{d\sigma} = -(\mu m_{12} b_0 + m_2 b_2) q b_0 s_1 + (a_2 - \mu m_{12} q b_0 b_2 - m_2 q b_2^2) s_2 + \frac{c_2}{r} y, \quad s_2(T) = 0 \quad (6.21)$$

where  $m_1$ ,  $m_{12}$ , and  $m_2$  satisfy the Riccati equations

$$\frac{dm_1}{d\sigma} = 2a_0m_1 + c_0^2/r - q(m_1b_0 + m_{12}b_2)^2, \quad m_1(T) = 0 \quad (6.22)$$

$$\mu \frac{dm_{12}}{d\sigma} = (\mu a_0 + a_2)m_{12} + c_0c_2/r - q(m_1b_0 + m_{12}b_2)(\mu m_{12}b_0 + m_2b_2), \quad m_{12}(T) = 0 \quad (6.23)$$

$$\mu \frac{dm_2}{d\sigma} = 2a_2m_2 + c_2^2/r - q(\mu m_{12}b_0 + m_2b_2)^2, \quad m_2(T) = 0 \quad (6.24)$$

As was shown more generally for the  $(n+m)$ -th order case in Section 2.3, the Riccati equations (6.22)-(6.24) are the exact duals of their forward counterparts (6.6)-(6.8). Thus, the quasi steady-state solutions, trivially found by merely substituting the dual coefficients in (6.9)-(6.11), are

$$\bar{m}_{12} = \frac{c_0c_2/r - qb_0b_2\bar{m}_1\bar{m}_2}{qb_2^2\bar{m}_2 - a_2} \quad (6.25)$$

$$\bar{m}_2 = a_2(1 - \sqrt{1 + qd_0^2/r})/qb_2^2 \quad (6.26)$$

$$\bar{m}_1 \equiv m_0 = \frac{2c[1 - \exp\{-\beta\sigma\}]}{[(b-\beta) - (b+\beta)\exp\{-\beta\sigma\}]} \quad (6.27)$$

where the boundary condition  $m_0(\sigma = 0) = 0$  has been used.

The solutions to the boundary layer equations, using duality, are given by

$$\tilde{m}_2(\rho) = - \frac{2\bar{v}_2\bar{m}_2 \exp\{2\bar{v}_2\rho\}}{2a_2 - qb_2^2\bar{m}_2(1 + \exp\{2\bar{v}_2\rho\})} \quad (6.28)$$

$$\tilde{m}_{12}(\rho) = \frac{2\bar{m}_{12}(0)\exp\{\bar{v}_2\rho\}[qb_2^2\bar{m}_2 \exp\{\bar{v}_2\rho\} - a_2]}{2a_2 - qb_2^2\bar{m}_2[1 + \exp\{2\bar{v}_2\rho\}]} \quad (6.29)$$

where

$$\bar{v}_2 \triangleq a_2 - qb_2 \bar{m}_2 \quad (6.30)$$

The near optimal stretched time-scale backward filter is described by

$$\frac{d\bar{r}(t)}{d\lambda} = (a_2 - qb_2 \bar{m}_2) \bar{r} + \frac{c_2}{r} y, \quad \lambda = \frac{\sigma - \sigma_i}{\mu} \geq 0 \quad (6.31)$$

where  $\tilde{m}_2(\rho)$  is to be added to  $\bar{m}_2$  if the boundary layer behavior is desired.

The near optimal slow mode backward filter satisfies

$$\frac{ds_0(t)}{d\sigma} = [a_0 - (m_0 b_0 + k_b d_0) qb_0] s_0 + k_b y \quad (6.32)$$

where

$$k_b(t) \triangleq (c_0 - m_0 qb_0 d_0) / r_0 \quad (6.33)$$

Finally, the near optimal fixed-interval smoothers are given

by

$$\hat{\eta}(t|T) = \hat{\eta}_0(t|T) = [\hat{\eta}_0(t) + p_0 s_0(t)] n_1(t) \quad (6.34)$$

$$\hat{\xi}_{L,R}(t|T) = \frac{[\tilde{\zeta}(t) + (\bar{p}_2 + \tilde{p}_2(\theta)) \tilde{r}(t)]}{[1 + (\bar{p}_2 + \tilde{p}_2(\theta)) (\bar{m}_2 + \tilde{m}_2(\theta))]} + \hat{\xi}_0(t|T), \quad 0 \leq t \leq T \quad (6.35)$$

where

$$\hat{\xi}_0(t|T) \triangleq [(n_1 + n_2 m_0) \hat{\eta}_0(t|T) + n_2 s_0(t)] / (1 + \bar{p}_2 \bar{m}_2) \quad (6.36)$$

and

$$n_1(t) \triangleq (\bar{p}_2 / \bar{a}_2 + \bar{p}_2 / \bar{v}_2) c_2 c_0 / r, \quad n_2(t) \triangleq (1 / \bar{a}_2 - \bar{p}_2 \bar{m}_2 / \bar{v}_2) qb_0 b_2 \quad (6.37)$$

and  $\tilde{\zeta}(t)$  and  $\tilde{r}(t)$  are the time-varying filters of (6.17) and (6.31).

## 6.2 Relation to the Wiener Filter

### 6.2.1. Realizable Wiener Filter

In light of the boundary layer behavior of the filter for the fast mode  $\xi(t)$ , as given by (6.17), it would seem reasonable that the realizable Wiener filter should be considered as a possible substitute. More specifically, if the smoothing interval is at least a few fast time-constants long, the estimate  $\hat{\xi}(t/T)$ , as given by (6.35), could incorporate this Wiener filter in the interior of the interval. Assuming that  $T$  represents a small delay, a sufficiently small  $\mu$  is required. Note that a Wiener filter would provide a very poor estimate for the slow mode, in general. However, as was previously seen, the simple reduced system filter (6.18) can be easily implemented to estimate  $\eta(t)$ . The above statements apply similarly, to the backward filters also. While the covariance of the realizable Wiener filter can never be achieved for finite  $T$  and non-zero  $\mu$ , it does serve as a useful lower bound which can hopefully be obtained in the limit as  $\mu$  (and subsequently, the boundary layer width) tends to zero.

It has been shown in the previous chapters that  $p_2(t)$  can be expressed as

$$p_2(t) = \bar{p}_2 + O(\mu) \quad (6.38)$$

for  $t \geq \ell > 0$ , and thus the covariance of the optimal fast mode filter of (6.5) can be written as

$$E(e^2(t)) = \bar{p}_2/\mu + O(1), \quad t \geq \ell \quad (6.39)$$

i.e., as  $\mu \rightarrow 0$ , the error covariance tends to infinity asymptotically in such a way that the product  $\mu E(e^2(t))$  tends to a constant, namely  $\bar{p}_2$ .

The error covariance of the realizable Wiener filter  $H_0(j\omega)$  [19] for estimating  $\xi(t)$  from the noisy observations  $y(t)$ , as given by (6.1)-(6.3), will now be investigated. The model for the channel and filter is shown in Figure 1. The two signals,  $s_1(t) = c_0\eta(t)$  and  $s_2(t) = c_2\xi(t)$ , are modeled by white noise (with covariance  $q$ ) thru linear filters, as shown in Figure 2. Note that, for the purpose of estimating  $\xi(t)$ ,  $\eta(t)$  is essentially an additional noise process. From these two figures, it is seen that the power spectra satisfy

$$\phi_{s_1}(\omega) = \frac{qb_0^2c_0^2}{\omega^2 + a_0^2}, \quad \phi_{s_2}(\omega) = \frac{qb_2^2c_2^2/\mu^2}{\omega^2 + a_2^2/\mu^2}, \quad (6.40)$$

$$\phi_n(\omega) = \phi_{s_1}(\omega) + \phi_v(\omega) = \frac{qb_0^2c_0^2}{\omega^2 + a_0^2} + r, \quad (6.41)$$

and

$$\phi_y(\omega) = \phi_{s_2}(\omega) + \phi_n(\omega) + 2\text{Re}\phi_{s_2n}(j\omega) \quad (6.42)$$

where

$$\phi_{s_2n}(j\omega) = \phi_{s_2s_1}(j\omega) = \frac{qb_0b_2c_0c_2/\mu}{(a_0 + j\omega)(a_2/\mu - j\omega)} \quad (6.43)$$

Substituting (6.40), (6.41), and (6.43) in (6.42) yields

$$\phi_y(\omega) = \frac{r\omega^4 + [q(b_0c_0 + b_2c_2/\mu)^2 + r(a_0^2 + a_2^2/\mu^2)]\omega^2 + [q(a_2b_0c_0 + a_0b_2c_2)^2 + ra_0^2a_2^2]/\mu^2}{(\omega^2 + a_0^2)(\omega^2 + a_2^2/\mu^2)} \quad (6.44)$$

Using spectral factorization,  $\phi_y(\omega)$  becomes

$$\phi_y(\omega) = \phi_y^+ \phi_y^- = \frac{\sqrt{r}(j\omega + \omega_3)(j\omega + \omega_4)}{(j\omega - a_0)(j\omega - a_2/\mu)} \frac{\sqrt{r}(\omega_3 - j\omega)(\omega_4 - j\omega)}{(j\omega + a_0)(j\omega + a_2/\mu)} \quad (6.45)$$

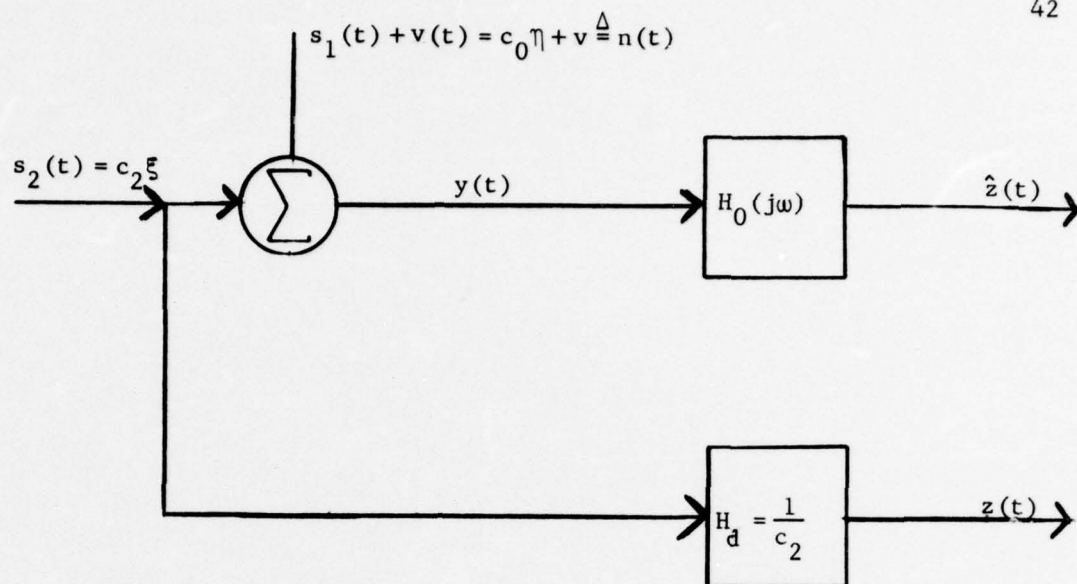


Figure 1. Channel and Filter Model

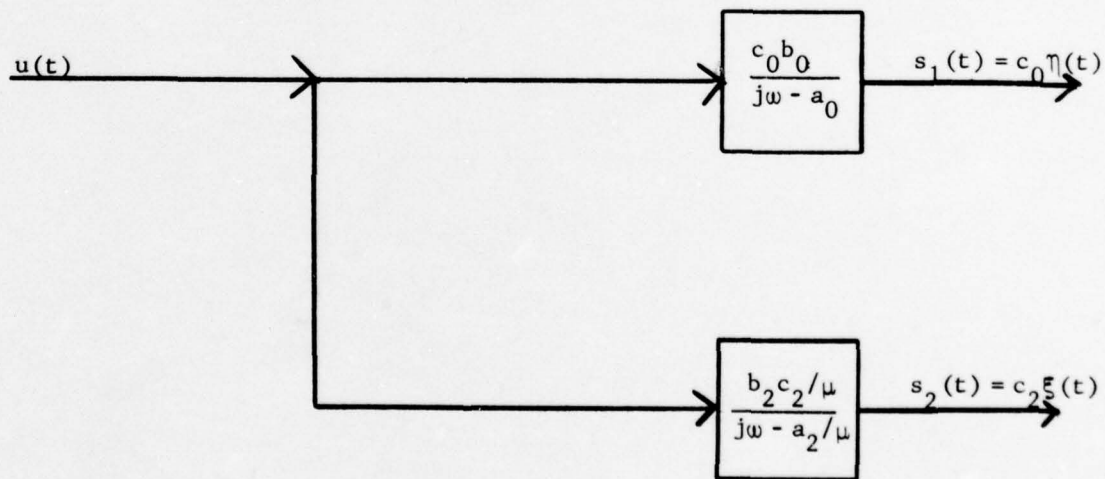


Figure 2. State Model

where  $\omega_3, \omega_4 > 0$  and  $a_0, a_2 < 0$ .

Thus,

$$G(j\omega) \triangleq \frac{\phi_{zy}(j\omega)}{\phi_y^-(j\omega)} = \frac{H_d(\omega)\phi_{s_2y}(j\omega)}{\phi_y^-(j\omega)} = \frac{H_d(\omega)(\phi_{s_2}(\omega) + \phi_{s_2s_1}(j\omega))}{\phi_y^-(j\omega)} \quad (6.46)$$

Substitution of the indicated quantities, together with a partial fraction expansion, yields,

$$G_+(j\omega) = \frac{qb_2k_4}{\mu\sqrt{r}(j\omega - a_2/\mu)} \quad (6.47)$$

where

$$k_4 = \frac{d_0}{1 + \sqrt{r_0}/r} + O(\mu) \quad (6.48)$$

The error covariance is given by

$$\begin{aligned} E(e^2(t)) &= E(z^2(t)) - E(\hat{z}^2(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\phi_z(\omega) - |G_+(j\omega)|^2] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_{s_2}(\omega)}{c_2} d\omega - \int_0^{\infty} g_+^2(t) dt \end{aligned} \quad (6.49)$$

using Parseval's theorem. From (6.47),

$$g_+(t) = \frac{qb_2}{\mu\sqrt{r}} k_4 \exp\{-a_2 t/\mu\} \quad (6.50)$$

Simple integration yields

$$\mu E(e^2(t)) = -\frac{qb_2^2}{2a_2} (1 - qd_0^2/r(1 + \sqrt{r_0}/r)^2) + O(\mu) \quad (6.51)$$

which after algebra reduces to

$$\mu E(e^2(t)) = ra_2(1 - \sqrt{r_0/r})/c_2^2 + O(\mu) \equiv \bar{p}_2 + O(\mu) \quad (6.52)$$

Thus, as  $\mu$  tends to zero, the error covariances of the fast mode filter (outside the boundary layer) and the realizable Wiener filter agree.

### 6.2.2. Unrealizable Wiener Filter

A similar argument to the one made to justify replacing the fast mode filter with the realizable Wiener can be given to justify the use of the unrealizable Wiener filter in place of the fast mode smoother, given by (6.35). In the interior of the smoothing interval, for  $\mu$  sufficiently small, the effects of the two boundary layers have decayed to zero. The fast mode smoother, in other words, experiences a long past and future of essentially steady-state behavior. The unrealizable filter error covariance cannot be obtained, but it provides a lower bound for the fixed-interval smoother. As before, it is hoped that as  $\mu$  tends to zero, the respective covariances will agree.

Equation (2.6) gives the smoother error covariance matrix  $P(t|T)$  as

$$P(t|T) = [P^{-1}(t) + M(t)]^{-1} \quad (6.53)$$

For the second-order case in the interior of the interval, the quasi steady-state covariances are valid to within  $O(\mu)$ . Thus,

$$P(t|T) = \left[ \begin{pmatrix} \bar{p}_1 & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_2/\mu \end{pmatrix}^{-1} + \begin{pmatrix} \bar{m}_1 & \mu \bar{m}_{12} \\ \mu \bar{m}_{12} & \mu \bar{m}_2 \end{pmatrix}^{-1} \right]^{-1} \quad (6.54)$$

The covariance for the fast mode smoother is then given by

$$E[(\hat{\xi}(t|T) - \xi(t))^2] = \frac{(\bar{m}_1 + \bar{p}_2/\mu d_1)}{(\bar{m}_1 + \bar{p}_2/\mu d_1)(\mu \bar{m}_2 + \bar{p}_1/d_1) - (\mu \bar{m}_{12} - \bar{p}_{12}/d_1)^2} \quad (6.55)$$

where

$$d_1 \triangleq \bar{p}_1 \bar{p}_2 / \mu - \bar{p}_{12}^2 \quad (6.56)$$

Simple algebra reveals that

$$\begin{aligned} \mu E[(\hat{\xi}(t|T) - \xi(t))^2] &= \bar{p}_2 / (1 + \bar{p}_2 \bar{m}_2) + O(\mu) \\ &= \frac{q r a_2 b_2^2 (1 - \sqrt{r_0/r})}{[q b_2^2 c_2^2 + r a_2^2 (1 - \sqrt{r_0/r})^2]} + O(\mu) = - \frac{q b_2^2}{2 a_2} \left[ \frac{1 - \sqrt{r_0/r}}{\sqrt{r_0/r} (1 - \sqrt{r_0/r})} \right] + O(\mu) \\ &= - \frac{q b_2^2}{2 a_2 \sqrt{r_0/r}} + O(\mu) \end{aligned} \quad (6.57)$$

The model for the unrealizable filter is given also by Figures 1 and 2. This filter, to be denoted  $H_{0u}(j\omega)$ , is however different from the realizable  $H_0(j\omega)$ . Equations (6.40)-(6.44) are still applicable, but the error covariance is given here by

$$\begin{aligned} \text{MMSE} &= E(z^2(t)) - E(\hat{z}^2(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_z(\omega) d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_{0u}(j\omega)|^2 \phi_y(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_d(j\omega)|^2 \left[ \phi_{s_2}(\omega) - \frac{\phi_{s_2}(\omega) |\phi_{s_2}(\omega) + \phi_{s_2 s_1}(\omega)|^2}{\phi_{s_2}(\omega) + \phi_{s_1}(\omega) + \phi_v + 2 \text{Re} \phi_{s_2 s_1}} \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{q b_2^2 (\omega^2 + a_0^2) / \mu^2}{\omega^4 + [a_0^2 + a_2^2 / \mu^2 + q (b_0 c_0 + b_2 c_2 / \mu)^2 / r] \omega^2 + [a_0^2 a_2^2 + q (a_0 b_2 c_2 + a_2 b_0 c_0)^2 / r] / \mu^2} d\omega \end{aligned} \quad (6.58)$$

Factoring the denominator polynomial yields

$$\begin{aligned} \text{MMSE} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{qb_2^2 (\omega^2 + a_0^2) / \mu^2}{(\omega^2 + s_1^2)(\omega^2 + s_2^2)} d\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{qb_2^2}{\mu^2} \left[ \frac{\frac{a_0^2 - s_1^2}{s_2^2 - s_1^2} \left( \frac{1}{\omega^2 + s_1^2} \right) + \frac{-a_0^2 - s_2^2}{s_2^2 - s_1^2} \left( \frac{1}{\omega^2 + s_2^2} \right)}{\omega^2 + s_2^2} \right] d\omega \end{aligned} \quad (6.59)$$

by employing a partial fraction expansion and noting the even symmetry.

Integration results in

$$\text{MMSE} = \frac{qb_2^2}{2\mu^2 (s_2^2 - s_1^2)} \left[ \frac{a_0^2 - s_1^2}{s_1} + \frac{(s_2^2 - a_0^2)}{s_2} \right] \quad (6.60)$$

where

$$s_1 = \sqrt{\frac{ra_0^2 a_2^2 + q(a_0 b_2 c_2 + a_2 b_0 c_0)^2}{ra_2^2 + qb_2^2 c_2^2}} + o(\mu) \quad (6.61)$$

$$s_2 = (-a_2 \sqrt{r_0/r}) / \mu + o(1)$$

Thus, it is easily seen that

$$\mu \text{MMSE} = \frac{qb_2^2}{2\mu s_2} + o(\mu) = -\frac{qb_2^2}{2a_2 \sqrt{r_0/r}} + o(\mu) \quad (6.62)$$

which is identical to (6.57), proving that as  $\mu \rightarrow 0$  the fast smoother performance tends to that of the unrealizable Wiener filter.

### 6.3 Numerical Example

#### 6.3.1. Near Optimal Design

The system to be studied in this section is of the form of equations (6.1)-(6.3) with  $a_0 = -1$ ,  $a_2 = -2$ ,  $b_0 = b_2 = c_0 = q = 1$ ,  $c_2 = 2$ ,  $r = 0.5$ ,  $T = 1$ , i.e.,

$$\dot{\eta} = -\eta + u, \quad \eta(0) = \eta_0 \quad (6.63)$$

$$\mu \dot{\xi} = -2\xi + u, \quad \xi(0) = \xi_0 \quad (6.64)$$

$$y = \eta + 2\xi + v, \quad 0 \leq t \leq 1 \quad (6.65)$$

where  $\eta_0$  and  $\xi_0$  are random variables with zero means and covariances satisfying

$$\text{cov}(\eta_0) = 2, \quad \text{cov}(\eta_0, \xi_0) = 0, \quad \text{cov}(\xi_0) = 1/\mu \quad (6.66)$$

Application of (6.4)-(6.8) to the above system yields the exact filtering solution as

$$\dot{\hat{\eta}} = -\hat{\eta} + 2(p_1 + 2p_{12})(y - \hat{\eta} - 2\hat{\xi}), \quad \hat{\eta}(0) = 0 \quad (6.67)$$

$$\mu \dot{\hat{\xi}} = -2\hat{\xi} + 2(\mu p_{12} + 2p_2)(y - \hat{\eta} - 2\hat{\xi}), \quad \hat{\xi}(0) = 0 \quad (6.68)$$

$$\dot{p}_1 = -2p_1 + 1 - 2(p_1 + 2p_{12})^2, \quad p_1(0) = 2 \quad (6.69)$$

$$\mu \dot{p}_{12} = -(2 + \mu)p_{12} + 1 - 2(p_1 + 2p_{12})(\mu p_{12} + 2p_2), \quad p_{12}(0) = 0 \quad (6.70)$$

$$\mu \dot{p}_2 = -4p_2 + 1 - 2(\mu p_{12} + 2p_2)^2, \quad p_2(0) = 1 \quad (6.71)$$

The quasi steady-state covariances are given by

$$\bar{p}_2 = (\sqrt{3} - 1)/4 \quad (6.72)$$

$$\bar{p}_1 = p_0 = \frac{(3+2\sqrt{3}) \exp\{-2\sqrt{3}t\} - (3-2\sqrt{3})}{(-3+\sqrt{3}) \exp\{-2\sqrt{3}t\} + (3+\sqrt{3})} \quad (6.73)$$

$$\bar{p}_{12} = \frac{\exp\{-2\sqrt{3}t\} + (1-\sqrt{3})}{(\sqrt{3}-1) \exp\{-2\sqrt{3}t\} - (1+\sqrt{3})} \quad (6.74)$$

The boundary layer terms (substituting in (6.14) and (6.15)) become

$$\tilde{p}_2(\theta) = \frac{(5\sqrt{3}-3) \exp\{-4\sqrt{3}\theta\}}{2[(\sqrt{3}-5) \exp\{-4\sqrt{3}\theta\} + (5+\sqrt{3})]} , \theta = t/\mu \quad (6.75)$$

$$\tilde{p}_{12}(\theta) = \frac{(5\sqrt{3}-3) \exp\{-4\sqrt{3}\theta\} - 3\sqrt{3} \exp\{-2\sqrt{3}\theta\}}{(\sqrt{3}-5) \exp\{-4\sqrt{3}\theta\} + (5+\sqrt{3})} \quad (6.76)$$

From equation (6.17), the time varying fast filter equation is given by

$$\frac{d}{d\tau} \tilde{\zeta} = -2\tilde{\zeta} + [\sqrt{3}-1 + 4\tilde{p}_2(\theta)](y-2\tilde{\zeta}) \quad (6.77)$$

and the slow filter satisfies

$$\dot{\hat{\eta}}_0 = -\hat{\eta}_0 + \frac{2}{3} (1+p_0)(y-\hat{\eta}_0) \quad (6.78)$$

Similarly, the exact backward filtering solution satisfies

$$\frac{ds_1}{d\sigma} = -(1+m_1+m_{12})s_1 - (m_1+m_{12})s_2 + 2y, \quad s_1(T) = 0 \quad (6.79)$$

$$\mu \frac{ds_2}{d\sigma} = -(\mu m_{12} + m_2)s_1 - (2 + \mu m_{12} + m_2)s_2 + 4y, \quad s_2(T) = 0 \quad (6.80)$$

$$\frac{dm_1}{d\sigma} = -2m_1 + 2 - (m_1 + m_{12})^2, \quad m_1(T) = 0 \quad (6.81)$$

$$\mu \frac{dm_{12}}{d\sigma} = -(\mu + 2) m_{12} + 4 - (m_1 + m_{12})(\mu m_{12} + m_2), \quad m_{12}(T) = 0 \quad (6.82)$$

$$\mu \frac{dm_2}{d\sigma} = -4m_2 + 8 - (\mu m_{12} + m_2)^2, \quad m_2(T) = 0 \quad (6.83)$$

The quasi steady-state covariances are given by

$$\bar{m}_2 = 2(\sqrt{3} - 1) \quad (6.84)$$

$$\bar{m}_1 = m_0 = \frac{2[1 - \exp\{-2\sqrt{3}\sigma\}]}{(5 + 3\sqrt{3}) + (3\sqrt{3} - 5) \exp\{-2\sqrt{3}\sigma\}}, \quad \sigma = T - t \quad (6.85)$$

$$\bar{m}_{12} = \frac{4[(3 + \sqrt{3}) + (2\sqrt{3} - 3) \exp\{-2\sqrt{3}\sigma\}]}{[(5\sqrt{3} + 9) + (9 - 5\sqrt{3}) \exp\{-2\sqrt{3}\sigma\}]} \quad (6.86)$$

The boundary layer terms, as given by (6.28)-(6.29), become

$$\tilde{m}_2(\rho) = \frac{-4(3 - \sqrt{3}) \exp\{-4\sqrt{3}\rho\}}{2 + (\sqrt{3} - 1)[1 + \exp\{-4\sqrt{3}\rho\}]} \quad (6.87)$$

$$\tilde{m}_{12}(\rho) = -\frac{4\sqrt{3}[(\sqrt{3} - 1)\exp\{-4\sqrt{3}\rho\} + \exp\{-2\sqrt{3}\rho\}]}{6 + 3(\sqrt{3} - 1)[1 + \exp\{-4\sqrt{3}\rho\}]} \quad (6.88)$$

The near optimal, time-varying fast mode filter is described

by

$$\frac{d\tilde{r}(t)}{d\lambda} = -[2\sqrt{3} + m_2(\rho)] \tilde{r} + 4y \quad (6.89)$$

and the slow filter satisfies

$$\frac{ds_0(t)}{d\sigma} = - \frac{(5+m_0)}{3} s_0 + \frac{2}{3} (1-m_0)y \quad (6.90)$$

with  $m_0(t)$  as given by (6.85).

The near optimal smoothing solution is now given by equations (6.34)-(6.37) of Section 6.1, in terms of the four near optimal filters ((6.77), (6.78), (6.89), (6.90)) just obtained, and the time-varying coefficients explicitly evaluated in this section.

#### 6.3.2. Performance Comparison

In this section, several graphs are presented indicating the results of various performance comparisons among the near optimal and optimal estimators, as just computed in Section 6.3.1. These results illustrate how closely the near optimal design can approximate the optimal. While the results of one numerical example do not constitute a proof, they do suggest the general requirements to obtain a close approximation, as will be noted in the summary. Computer solutions of the exact Riccati equations, as given by (6.69)-(6.71), were employed to obtain the optimal performance. The near optimal design, as seen earlier in this section, can be explicitly evaluated as a function of both  $t$  and  $\mu$ , given the remaining system parameters. Having obtained the performance of the optimal estimates, the relative proximity (to the exact) of the near optimal design can be tested. In addition, various convergence results, as  $\mu$  tends to zero, for both the fast and slow modes can be illustrated.

Figure 3 shows the uniform convergence of the optimal error covariance for the slow mode,  $p_1(t, \mu)$  to  $p_0$  as  $\mu$  tends to zero. The effect of changes in  $\mu$  on the optimal slow mode covariance is seen to be small, as in the steady state,  $p_1(\infty, 0.1) = 0.116143$ ,  $p_1(\infty, 0.05) = 0.107479$ ,  $p_1(\infty, 0.01) = 0.100016$ ,  $p_0(\infty) = 0.098070$ .

The error covariance  $v_1(t)$ , of the near optimal slow mode filter (6.78) using the (actual) full system, i.e., (6.63)-(6.65), can be found by forming

$$\dot{e} = Ae + Bn$$

where

$$e = \begin{bmatrix} \eta - \hat{\eta}_0 \\ \xi \end{bmatrix}, \quad A = \begin{bmatrix} -(1+k_0) & -2k_0 \\ 0 & -2/\mu \end{bmatrix}, \quad (6.91)$$

$$B = \begin{bmatrix} 1 & -k_0 \\ 1/\mu & 0 \end{bmatrix}, \quad n = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

The related equation

$$\dot{V} = AV + VA' + BQB' \quad (6.92)$$

where  $V(t) = E(e(t)e'(t))$  and  $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$  is the covariance of  $n(t)$ , thus has as its first element  $v_1(t, \mu) = E[(\eta - \hat{\eta}_0)^2]$ . Using this information together with the optimal covariances  $p_1(t, \mu)$  from Figure 3, the magnitude of the  $O(\mu)$  term is found. Figure 4 displays this difference,  $e_1(t)$ , between the near optimal, slow mode filter error covariance and the optimal covariance as a function of  $t$  for  $\mu = 0.1, 0.05$ , and  $0.01$ . Inspection

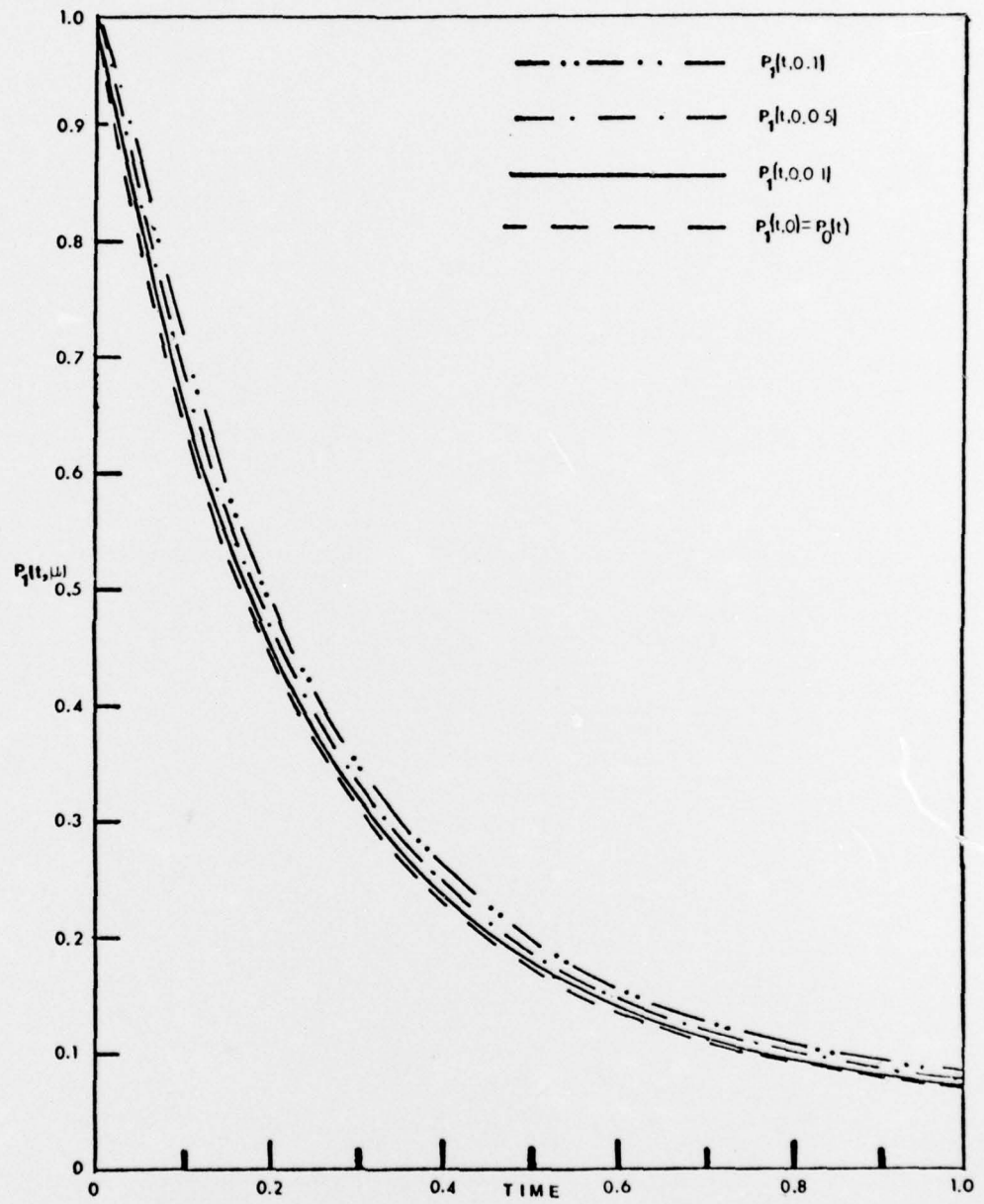


Figure 3. Convergence of Error Covariance-Slow Mode

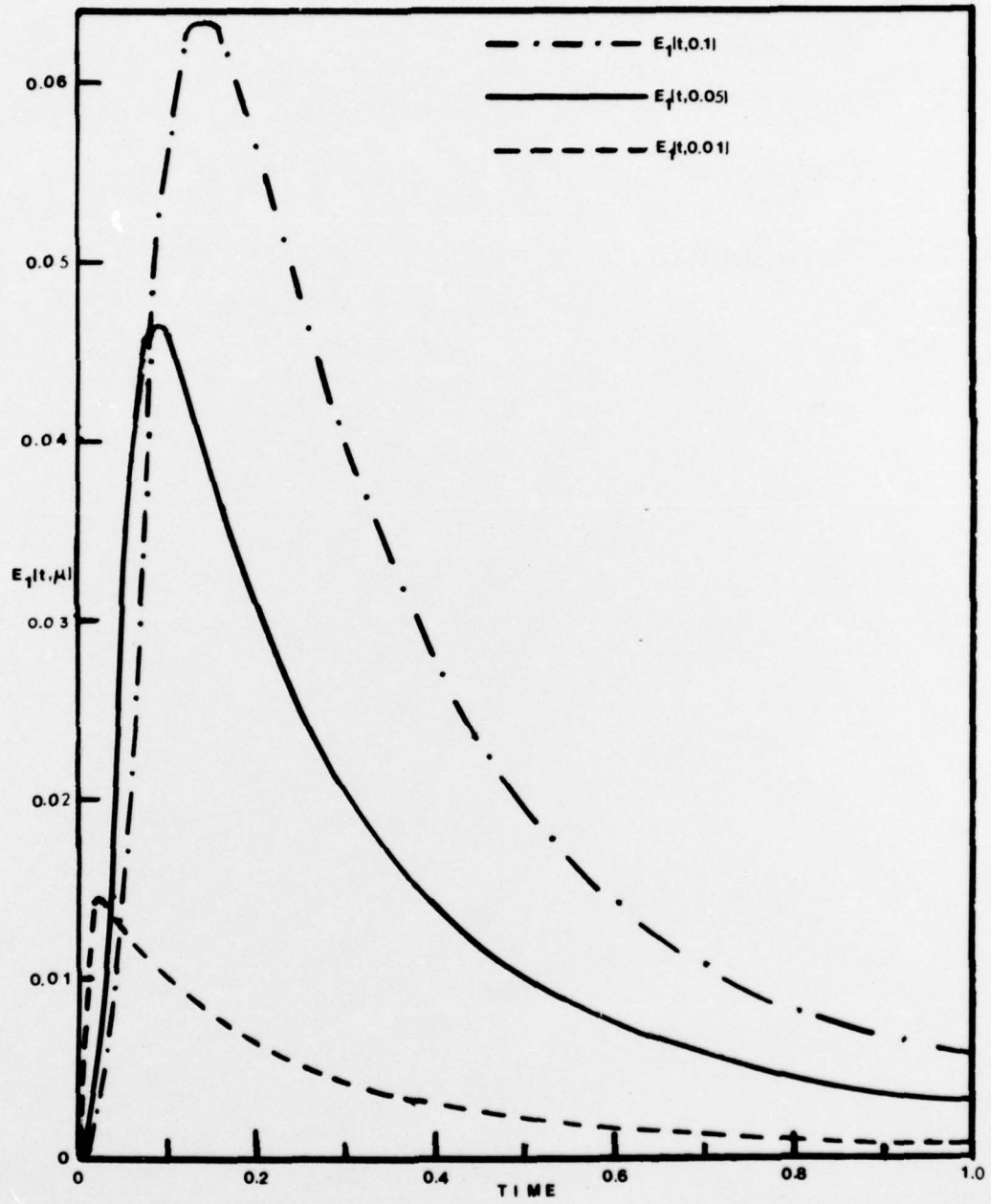


Figure 4. Proximity of Near Optimal Solution-Slow Mode

of the three curves reveals an interesting relationship among them. For  $t > .05$ , the performance of the near optimal reduced system filter  $\hat{\eta}_0(t)$  converges to the optimal (as would be expected, since  $\mu$  is ignored) as  $\mu \rightarrow 0$ . However, in an initial boundary layer, the fast modes dominate the system ever increasingly as  $\mu \rightarrow 0$  and subsequently, the observations also. The reduced system filter, which has no provision for estimating the fast mode  $\xi(t)$ , is essentially misled by the initial observations, which actually convey relatively little information about  $\eta(t)$ . In the steady-state, the reduced system filter performs extremely well as  $e_1(\infty, 0.1) = 3.29 \times 10^{-3}$ ,  $e_1(\infty, 0.05) = 1.72 \times 10^{-3}$ ,  $e_1(\infty, 0.01) = 3.61 \times 10^{-4}$  and, as Figure 4 indicates, converges to the optimal as  $\mu \rightarrow 0$ .

The behavior of the optimal covariance  $p_2(t, \mu)$  in the boundary layer is exhibited by Figure 5, as  $p_2(t)$  is plotted for  $t \leq 0.05$  for three values of the parameter  $\mu$ . What these graphs show is the convergence of  $p_2(t, \mu)$  to  $\bar{p}_2$  as  $\mu \rightarrow 0$ , except in an initial boundary layer (where the initial condition is constrained to be 1). Also they illustrate the rapidity at which the boundary layers decay as  $\mu \rightarrow 0$ . What cannot be seen on such a short time plot as this, is that the three curves later cross and in the steady state  $p_2(\infty, 0) = \bar{p}_2 = 0.183013$ ,  $p_2(\infty, 0.01) = 0.182449$ ,  $p_2(\infty, 0.05) = 0.180255$ , and  $p_2(\infty, 0.1) = 0.177634$ . The convergence would be uniform if not for the small neighborhood where the curves cross the constant value  $\bar{p}_2$ .

It should be noted that these last results can be interpreted in terms of the fast filter covariance. The filter error covariance of  $\hat{\xi}(t)$  as given by (6.68), is  $p_2(t, \mu)/\mu$ . Although  $p_2(t, \mu_2) > p_2(t, \mu_1)$  for

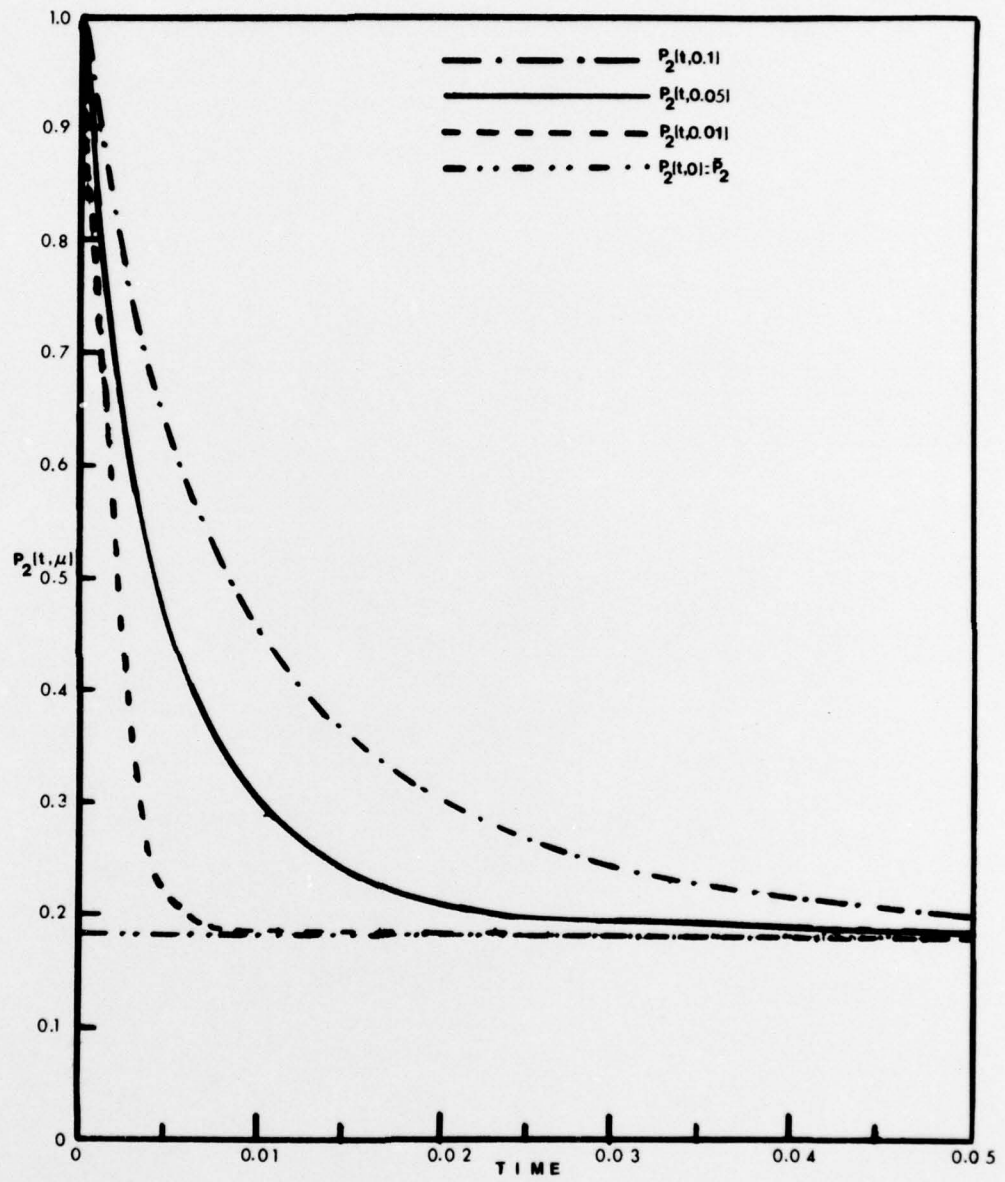


Figure 5. Boundary Layer Convergence-Fast Mode

$\mu_2 > \mu_1$  in the boundary layer, the covariance for the smaller value  $\mu_1$  is still significantly larger than with  $\mu_2$ , i.e., the filter performance is worse. Of course, this is due solely to the fact that the fast state  $\xi(t)$  is tending to a white noise process. Thus, while  $p_2(t, \mu)$  converges to  $\bar{p}_2$  as  $\mu \rightarrow 0$  outside the boundary layer, no such statement can be made about the fast filter error covariance.

The actual error covariance of the near optimal fast mode filter (6.77) can be found, as was done for the slow mode, by forming

$$\dot{e} = Ae + Bn \quad (6.93)$$

where

$$e = \begin{bmatrix} \eta - \hat{\eta}_0 \\ \xi \\ \xi - \hat{\xi} \end{bmatrix}, \quad A = \begin{bmatrix} -(1+k_0) & -2k_0 & 0 \\ 0 & -2/\mu & 0 \\ (1-\sqrt{3})/\mu & 0 & -2\sqrt{3}/\mu \end{bmatrix} \quad (6.94)$$

$$B = \begin{bmatrix} 1 & -k_0 \\ 1/\mu & 0 \\ 1/\mu & (1-\sqrt{3})/\mu \end{bmatrix}, \quad n = \begin{bmatrix} u \\ v \end{bmatrix}$$

Then  $v_{33}(t) = E[(\xi - \hat{\xi})^2]$ , where

$$\dot{V} = AV + VA' + BQB' \quad (6.95)$$

The resulting error covariance  $v_{33}(t)$  for this particular example agrees to within a maximum deviation of  $10^{-3}$  of the optimal for all three values

of  $\mu$ , as given in Figure 5. They were not plotted since, for all intents and purposes, the graphs would appear the same as those of Figure 5.

In Figure 6, a comparison is made between the optimal covariance  $p_2(t)$  and the near optimal gain terms  $\bar{p}_2$  and  $\bar{p}_2 + \tilde{p}_2(\theta)$  for  $\mu = 0.1$ . Note that the  $\mu p_{12}$  term was neglected in the optimal fast mode filter gain,  $(\mu p_{12} + 2p_2)$ . It is easily seen that, as expected, the selection of  $\mu = 0.1$  (results were obtained for  $\mu = 0.01, 0.05, 0.1$ ) produces the greatest discrepancy among the three values of  $\mu$ . That is, as  $\mu \rightarrow 0$ , the approximation continually improves. Inside the boundary layer, the addition of  $\tilde{p}_2(\theta)$  to  $\bar{p}_2$  is needed. The approximation (only shown for  $t \leq 0.05$ ) remains close throughout the entire interval as in the steady-state  $p_2(\infty, 0.1) = 0.177634$ ,  $\bar{p}_2 = 0.183013$ . This near-perfect approximation of the optimal gain term accounts for the essentially optimal performance as just noted above.

Nearly identical results for the near optimality of the lower-order backward filters were obtained. Since the same singular perturbation methods were used, this was to be expected. Equation (2.6) of Section 2.1 then guarantees the near optimality of the smoother. That is, near optimal filtering, both backward and forward, assures a near optimal smoothing solution. Figure 7 illustrates the closeness of the near optimal backward fast mode filter gain term  $[\bar{M}_2 + \tilde{M}_2(\rho)]$  to the optimal  $M_2$  ( $\mu \bar{M}_{12}$  neglected by duality) for  $t \geq 0.95$  and  $\mu = 0.1$ .

Figure 8 illustrates the improvement in terms of error covariance, of smoothing, relative to filtering, for the slow mode. The highest value,  $\mu = 0.1$ , was again chosen, since the advantage of smoothing decreases as  $\mu \rightarrow 0$ .

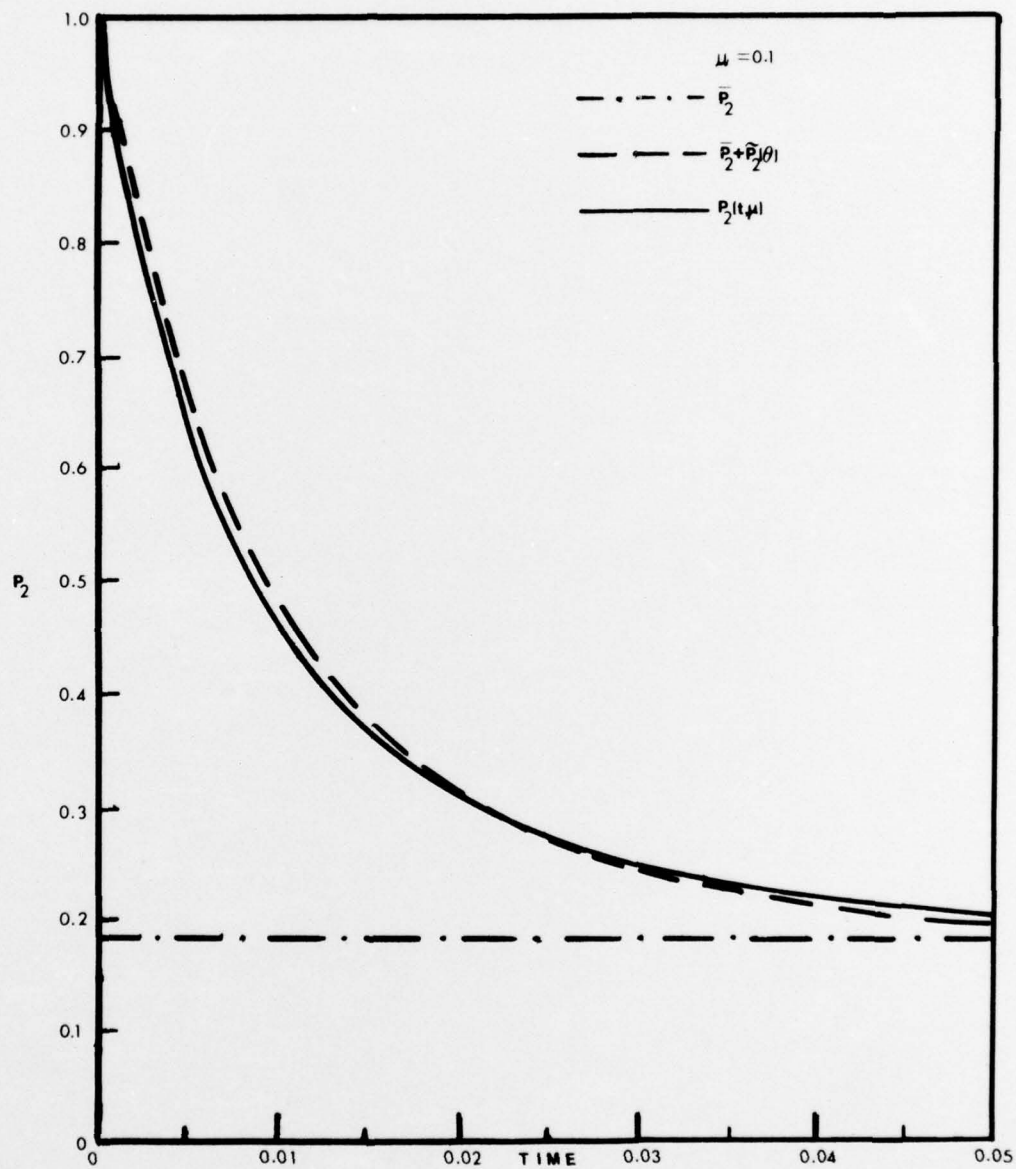


Figure 6. Filter Gain Comparison-Fast Mode

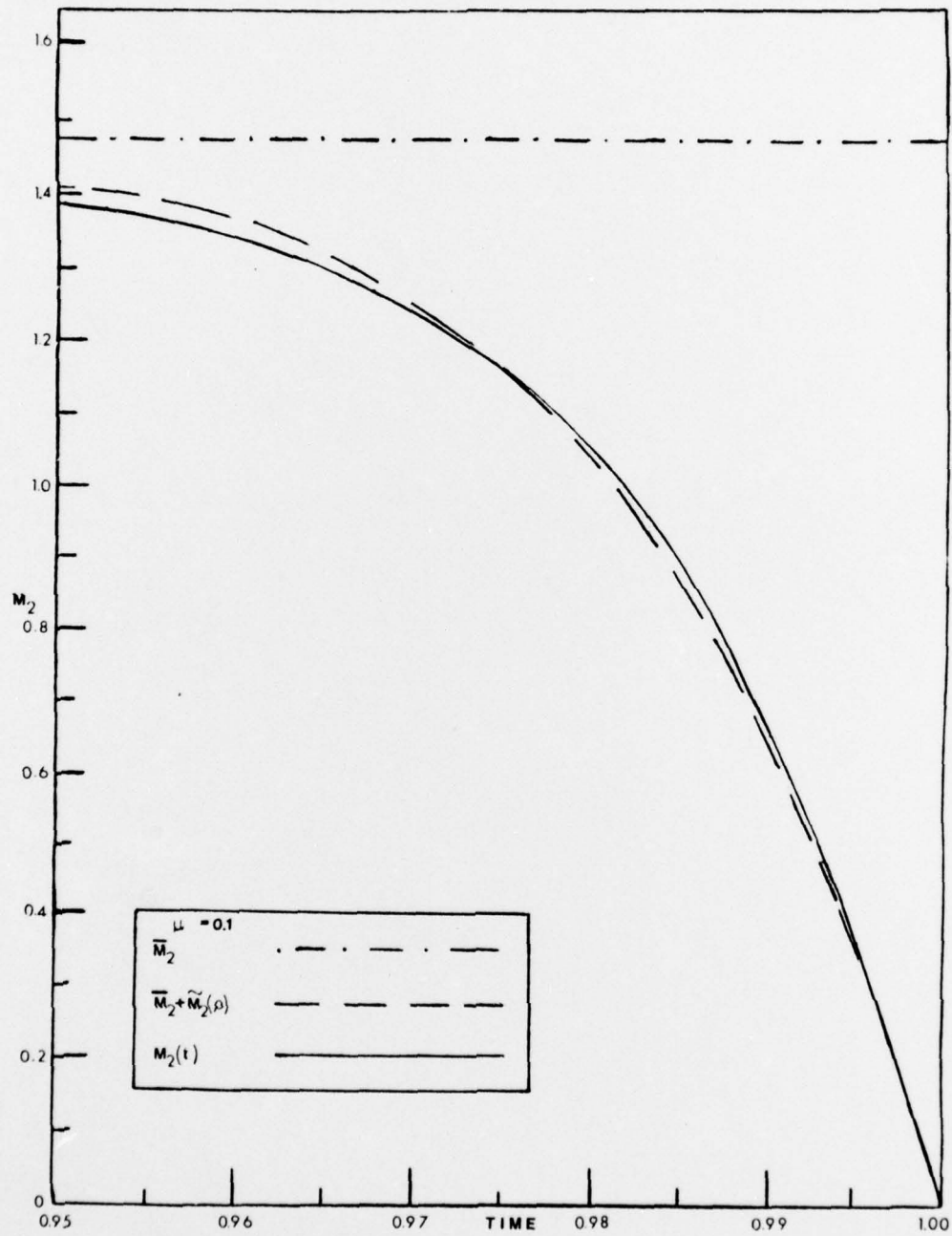


Figure 7. Backward Filter Gain Comparison-Fast Mode

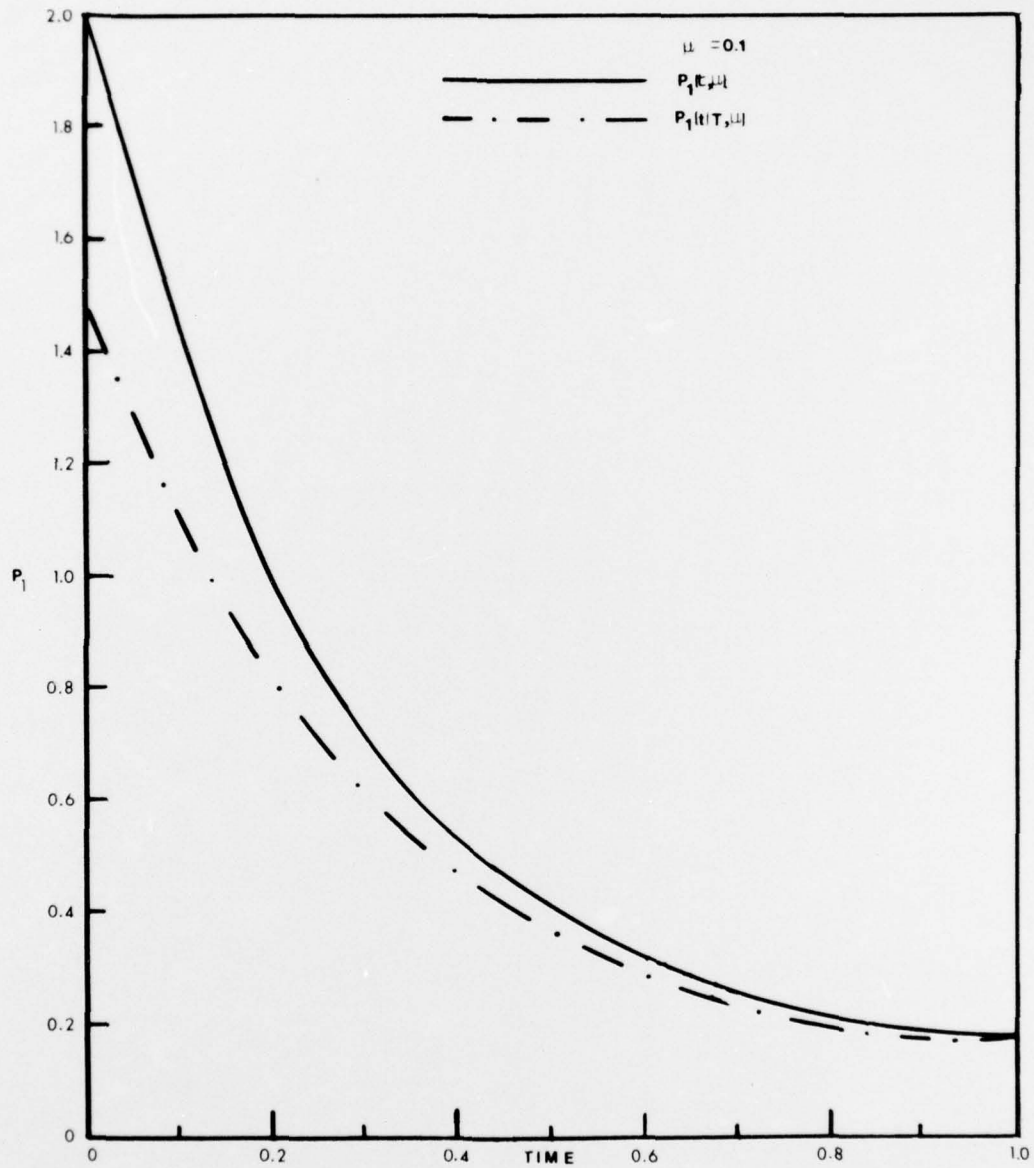


Figure 8. Advantage of Smoothing for the Slow Mode

Two comparisons have been made for the fast mode estimators in Figure 9 using  $\mu = 0.1$ . In the smaller plot, the comparison of smoothing and filtering error covariances is displayed for  $0 \leq t \leq T$ , and in addition, it is shown that the smoother (in the interior of the interval) performs nearly as well as the unrealizable Wiener filter  $H_{ou}(j\omega)$ . Note that the smoothing error covariance  $P_2(t|T)$  exhibits boundary layers at both ends of the interval. The required terminal condition  $M_2(T) = 0$ , due to filter and smoother performance equivalence at  $t = T$ , allows a relatively small right boundary layer decay. The left boundary layer, however, is indicative of a large discrepancy between the initial condition and steady-state value of  $P_2(t)$ . This transient behavior, which is difficult to distinguish on the smaller plot is magnified for easier comparison ( $t \leq 0.05$ ) on the larger graph. Note that while smoothing and filtering performance are identical at the end of the interval, smoothing represents a drastic improvement over filtering initially for this particular singularly perturbed system.

The results of this numerical example are pleasing. Not only has convergence of the optimal covariances been illustrated, but the relative performance of the near optimal design has proved to be incredibly good for values of  $\mu$  as large as 0.1. The advantage of smoothing over filtering for both the slow and fast mode has also been shown. Finally, as predicted for  $\mu$  sufficiently small, the close approximation of the smoother covariance (in the interior of the interval) to that of the unrealizable Wiener filter has been demonstrated.

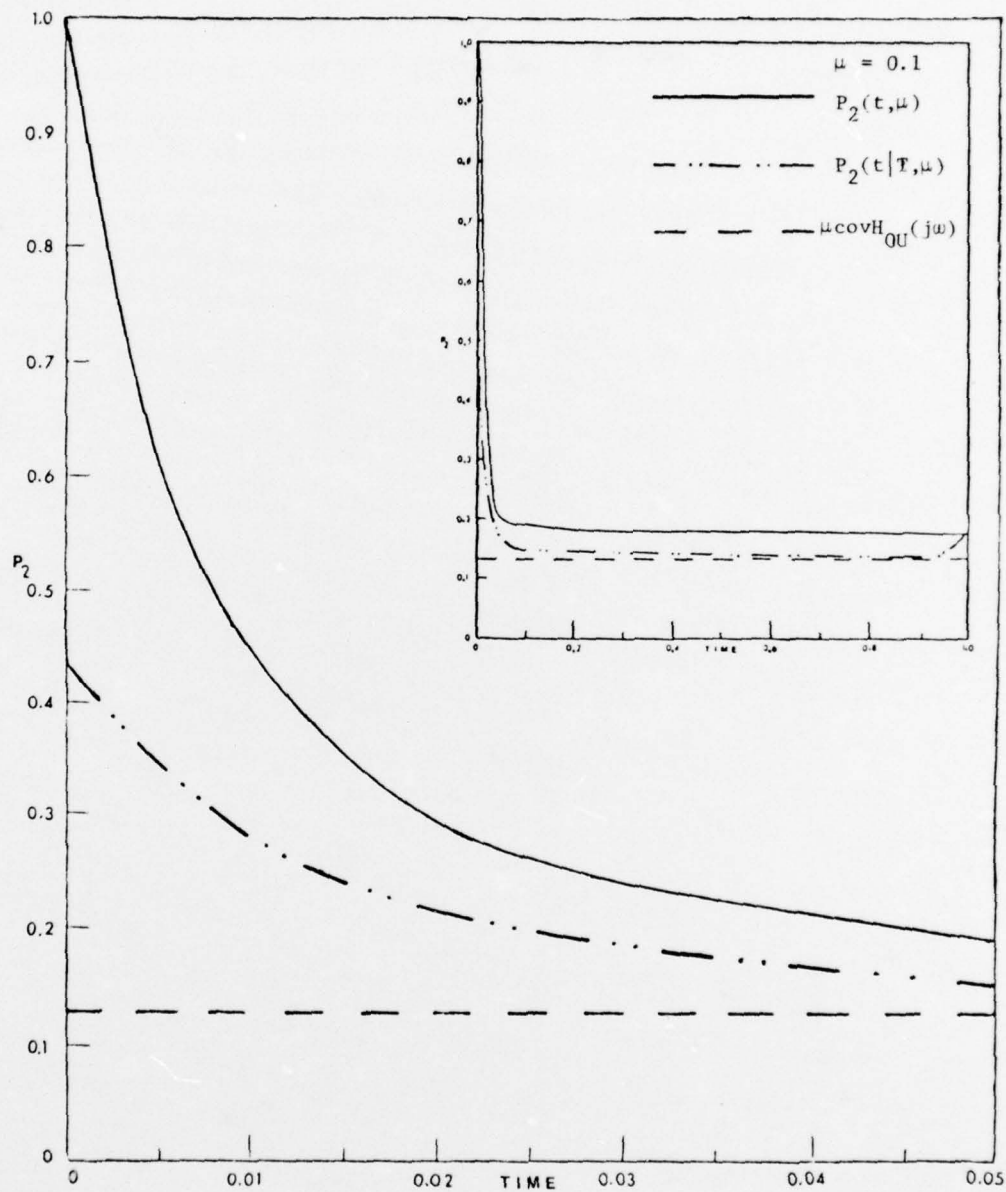


Figure 9. Comparison of Filtering, Smoothing and Unrealizable Wiener Filter Error Covariances for the Fast Mode

## VII. SUMMARY AND CONCLUSIONS

A near optimal fixed-interval smoother for singularly perturbed systems has been derived using the two filter (forward and backward) formulation. This intuitively-pleasing approach simplifies the fast mode boundary layer complications encountered previously, in [10] and results in a much simpler implementation than the exact solution. Application of similar singular perturbation techniques as for the (forward) filter in [1,9,10], to the backward filter, results in a near optimal  $O(\mu^{1/2})$  smoother with a similar two time-scale solution for the fast and slow modes. The alternative use of the unrealizable Wiener filter for the fast mode was justified in the interior of the smoothing interval. Finally, a numerical example served to illustrate how well the near optimal design approximates the exact for various values of the perturbation parameter  $\mu$ .

In conclusion, it can be inferred from the analysis of Chapter VI, that the near optimal design will yield a close approximation (for  $\mu$  sufficiently small), when the system parameters, excluding  $\mu$ , agree to within approximately an order of magnitude. This restriction can be progressively relaxed as  $\mu \rightarrow 0$ . It should lastly be noted that the methods and results of this thesis lend themselves quite easily to systems with a hierarchy of modes. The implementation necessary to obtain the estimates of the system states similarly, would be multi-time-scale utilizing a reduced-order solution for the slow modes.

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