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ON AN EXTREMAL PROBLEM

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ON AN EXTREMAL PROBLEM

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ABSTRACT

Let $X = (x_1, x_2, \dots, x_N)$, $f: \mathbb{R} \rightarrow \mathbb{C}$ and let \mathbb{P}_n be the class of polynomials of degree at most n . The generalized Christoffel function Λ_n corresponding to the measure $d\alpha$ is defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i=1, 2, \dots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t).$$

It is shown that if α satisfies some rather weak conditions then

$\lim_{n \rightarrow \infty} n \Lambda_n(X; f, N, d\alpha)$ exists and the limit is also evaluated.

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ON AN EXTREMAL PROBLEM

Paul G. Nevai

The purpose of the present note is to investigate the asymptotic behavior of the functions $\Lambda_n : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i=1, 2, \dots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t).$$

Here $X = (x_1, x_2, \dots, x_N)$, $f : \mathbb{R} \rightarrow \mathbb{C}$ is a fixed and almost everywhere finite function, \mathbb{P}_n is the set of all polynomials π of degree at most n and α is a weight function, that is α is nondecreasing on \mathbb{R} , it has infinitely many points of increase and every polynomial π belongs to $L_{d\alpha}^2$. Therefore Λ_n is defined and finite for almost every $X \in \mathbb{R}^N$.

Estimates for Λ_n lead to several results in probability theory, statistics and in the theory of orthogonal polynomials. (See [1], [2] and [4].) In fact, it is not hard to explicitly compute Λ_n ([3]) but the formula for Λ_n is so complicated that it cannot be used to estimate Λ_n when α is not nice. It will be shown that $\lim_{n \rightarrow \infty} n \Lambda_n(X; f, N, d\alpha)$ exists under rather weak assumptions on α and the corresponding limit will also be calculated.

Let $\{p_n(d\alpha)\}_{n=0}^{\infty}$ be the system of polynomials which is orthogonal with respect to $d\alpha$, that is $p_n(d\alpha, x) = \gamma_n x^n + \dots$ with $\gamma_n > 0$ and

$$\int_{-\infty}^{\infty} p_n(d\alpha, t) p_m(d\alpha, t) d\alpha(t) = \delta_{nm}.$$

Let M denote the class of those weights α for which

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} t p_n^2(d\alpha, t) d\alpha(t) = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} t p_{n-1}(d\alpha, t) p_n(d\alpha, t) d\alpha(t) = \frac{1}{2}.$$

Let us remark that M contains many weights. If, for instance, $\text{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos \theta) \in L^1$ then $\alpha \in M$ ([5]).

Furthermore, if α is absolutely continuous, $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha'(t) = \varphi(t) \exp\{-(1-t^2)^{-\frac{1}{2}}\}$ where $0 < c \leq \varphi(t)$ for $-1 \leq t \leq 1$ and φ is Riemann integrable then also $\alpha \in M$ ([4]). Another example is the Pollaczek weight ([5]). In the following Δ denotes an interval.

It is known that if $\alpha \in M$ then $\Delta \subset \text{supp}(d\alpha)$ iff $\Delta \subset [-1, 1]$ ([4]).

THEOREM. Let $\alpha \in M$, $\Delta \subset \text{supp}(d\alpha)$, $1/\alpha' \in L^1(\Delta)$. Let exist a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k \geq 0$, $\varepsilon_k \rightarrow 0$ such that for every fixed k the function Ψ_k , which is defined by $\Psi_k(t) = [(1 - \varepsilon_k)^2 - t^2]^{-\frac{1}{2}} \log \alpha'(t)$, belongs to $L^1(-1 + \varepsilon_k, 1 - \varepsilon_k)$. Then for almost every $X \in \Delta^{\mathbb{N}}$

$$\lim_{n \rightarrow \infty} n \Lambda_n(X; f, N, d\alpha) = \pi \sum_{i=1}^N |f(x_i)|^2 \alpha'(x_i) \sqrt{1 - x_i^2}.$$

Proof. For $N = 1$ the Theorem has been proved in [4]. Now let $N > 1$.

If $\pi \in \mathbb{P}_{n-1}$ then π can be expressed as

$$\pi(x) = \int_{-\infty}^{\infty} \pi(t) K_n(d\alpha, x, t) d\alpha(t)$$

where

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x) p_k(d\alpha, t).$$

If X is given and $\prod_{i < j} (x_i - x_j) \neq 0$ then we can write

$$\begin{aligned} \sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) &= \\ &= \int_{-\infty}^{\infty} \pi(t) \left[\sum_{i=1}^N \text{sign } \pi(x_i)^2 \pi(x_i) \Lambda_n(x_i, 1, 1, d\alpha) K_n(d\alpha, x_i, t) \right] d\alpha(t). \end{aligned}$$

Using Cauchy's inequality, orthogonality relations and the well known fact that

$$(1) \quad K_n(d\alpha, x, x) = \Lambda_n(x; 1, 1, d\alpha)$$

we obtain

$$\begin{aligned} \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 &\leq \\ &\leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \cdot \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) + \right. \\ &\quad \left. + 2 \sum_{j < i} |\pi(x_i)| |\pi(x_j)| \Lambda_n(x_i; 1, 1, d\alpha) \Lambda_n(x_j; 1, 1, d\alpha) |K_n(d\alpha, x_i, x_j)| \right]. \end{aligned}$$

If $\alpha \in M$ then clearly $\text{supp}(d\alpha)$ is compact. Therefore by the Christoffel-Darboux formula there exists a number C depending on X and $\text{supp}(d\alpha)$ such that

$$|K_n(d\alpha, x_i, x_j)| \leq C [|p_n(d\alpha, x_i)p_{n-1}(d\alpha, x_j)| + |p_{n-1}(d\alpha, x_i)p_n(d\alpha, x_j)|] .$$

Hence

$$\begin{aligned} & \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 \leq \\ & \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) + \right. \\ & \quad \left. + 4C \sum_{i=1}^N |\pi(x_i)| |p_n(d\alpha, x_i)| \Lambda_n(x_i; 1, 1, d\alpha) \cdot \right. \\ & \quad \left. \cdot \sum_{i=1}^N |\pi(x_i)| |p_{n-1}(d\alpha, x_i)| \Lambda_n(x_i; 1, 1, d\alpha) \right] . \end{aligned}$$

Formula (1) implies that $p_{n-1}^2(d\alpha, x_i) \Lambda_n(x_i, 1, 1, d\alpha) \leq 1$. Consequently by Cauchy's inequality

$$\begin{aligned} & \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 \leq \\ & \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \cdot \\ & \quad \cdot \{1 + 4C\sqrt{N} \left[\sum_{i=1}^N \Lambda_n(x_i; |p_n(d\alpha, x_i)|, 1, d\alpha) \right]^{\frac{1}{2}}\} . \end{aligned}$$

This inequality holds for each $X \in \mathbb{R}^N$ such that $\prod_{i < j} (x_i - x_j) \neq 0$.

It has been proved in [4] that if $\alpha \in M$ then for every $x \in \text{supp}(d\alpha)$

$$(2) \quad \lim_{n \rightarrow \infty} \Lambda_n(x; |p_n(d\alpha, x)|, 1, d\alpha) = 0 .$$

Consequently if $\alpha \in M$ then for almost every $X \in [\text{supp}(d\alpha)]^N$

$$(3) \quad \sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha) \leq [1 + \sigma(1)] \Lambda_n(X, f, N, d\alpha)$$

where $\lim_{n \rightarrow \infty} \sigma(1) = 0$ for almost every $X \in [\text{supp}(d\alpha)]^N$. Our next

aim is to establish the converse inequality. For a given X let π^X

be defined by

$$\pi^X(t) = \sum_{i=1}^N f(x_i) \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) K_{n-N+1}(d\alpha, t, x_i) \ell_i(t)$$

where $\{\ell_i\}_{i=1}^N$ are the fundamental polynomials of Lagrange interpolation

corresponding to $\{x_i\}_{i=1}^N$. Clearly $\pi^X \in \mathbb{P}_{n-1}$ for almost every $X \in \mathbb{R}^N$.

Furthermore, if $\pi^X \in \mathbb{P}_{n-1}$ then $\pi^X(x_i) = f(x_i)$ for $i = 1, 2, \dots, N$.

Let us compute the $L_{d\alpha}^2$ norm of π^X . Introducing the notation

$$G_n(d\alpha, g, x) = \Lambda_n(x; 1, 1, d\alpha) \int_{-\infty}^{\infty} g(t) K_n^2(d\alpha, x, t) d\alpha(t)$$

we have

$$(4) \quad \int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) = \sum_{i=1}^N |f(x_i)|^2 \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) G_{n-N+1}(d\alpha, \ell_i^2, x_i) \\ + 2 \sum_{i < j} \text{Re}[f(x_i) \overline{f(x_j)}] \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) \Lambda_{n-N+1}(x_j; 1, 1, d\alpha) \cdot \\ \cdot \int_{-\infty}^{\infty} \ell_i(t) \ell_j(t) K_{n-N+1}(d\alpha, t, x_i) K_{n-N+1}(d\alpha, t, x_j) d\alpha(t).$$

If $\alpha \in M$, g is continuous on $\text{supp}(d\alpha)$ and $x \in \text{supp}(d\alpha)$ then

$$\lim_{n \rightarrow \infty} G_n(d\alpha, g, x) = g(x)$$

(see [4]), in particular, if X is such that $\prod_{i < j} (x_i - x_j) \neq 0$ then

$$\lim_{n \rightarrow \infty} G_n(d\alpha, \ell_i^2, x_i) = \ell_i^2(x_i) = 1.$$

Let

$$I_{ij} = \int_{-\infty}^{\infty} \ell_i(t) \ell_j(t) K_{n-N+1}(d\alpha, t, x_i) K_{n-N+1}(d\alpha, t, x_j) d\alpha(t).$$

If $\prod_{k < \ell} (x_k - x_\ell) \neq 0$ then $\ell_i \ell_j \in \mathbb{P}_{2N-2}$. By a direct calculation we obtain that

$$\begin{aligned} I_{ij} &= \int_{-\infty}^{\infty} \ell_i(t) \ell_j(t) \left[K_{n-3N+3}(d\alpha, x_i, t) + \sum_{\ell=n-3N+3}^{n-N} p_k(d\alpha, x_i) p_k(d\alpha, t) \right] \cdot \\ &\quad \cdot K_{n-N+1}(d\alpha, x_j, t) d\alpha(t) = \ell_i(x_j) \ell_j(x_j) K_{n-3N+3}(d\alpha, x_i, x_j) + \\ &\quad + \sum_{k=n-3N+3}^{n-N} p_k(d\alpha, x_i) \sum_{\ell=k-2N+2}^{n-N} p_\ell(d\alpha, x_j) \cdot \\ &\quad \cdot \int_{-\infty}^{\infty} \ell_i(t) \ell_j(t) p_k(d\alpha, t) p_\ell(d\alpha, t) d\alpha(t). \end{aligned}$$

Since $\ell_i(x_j) = 0$ we get

$$(5) \quad |I_{ij}| \leq C_1 \sum_{k=n-5N}^{n-N} |p_k(d\alpha, x_i)| \sum_{\ell=n-5N}^{n-N} |p_\ell(d\alpha, x_j)|$$

where C_1 depends on X and $\text{supp}(d\alpha)$. Using the recurrence formula

which the orthogonal polynomials satisfy it can easily be seen that

for $\alpha \in M$ (5) implies

$$|I_{ij}| \leq C_2 \sum_{k=n-N-1}^{n-N} |p_k(d\alpha, x_i)| \sum_{\ell=n-N-1}^{n-N} |p_\ell(d\alpha, x_j)|$$

where C_2 is independent of n . Applying now (1) and (2) the second sum on the right hand side of (4) can easily be estimated. We get

$$\int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) = [1 + \sigma(1)] \sum_{i=1}^N |f(x_i)|^2 \Lambda_{n-N+1}(x_i; 1, 1, d\alpha)$$

for almost every $X \in [\text{supp}(d\alpha)]^N$. It has been proved in [4] that if $\alpha \in M$ and $x \in \text{supp}(d\alpha)$ then

$$\lim_{n \rightarrow \infty} \Lambda_{n-N+1}(x; 1, 1, d\alpha) \Lambda_n(x; 1, 1, d\alpha)^{-1} = 1$$

for every fixed N . Thus

$$(6) \quad \Lambda_n(X; f, N, d\alpha) \leq [1 + \sigma(1)] \sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha)$$

for almost every $X \in [\text{supp}(d\alpha)]^N$. The Theorem follows now from estimates (3) and (6).

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