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ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY

Ph. Clément[1,4] and J. A. Nohel[1,2,3]

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ABSTRACT

Let X be a real or complex Banach space. We study the Volterra equation

$$\text{(v)} \qquad u(t) + \int_0^t a(t-s)Au(s)ds = f(t) \quad (0 \le t \le T,\ T > 0)\ ,$$

where a is a given kernel, A is a bounded or unbounded linear operator from X to X, and f is a given function with values in X (of particular importance is the case $f = u_0 + a * g$, $u_0 \in X$, $g \in L^1(0,T;X)$, $*$ denotes the convolution). We establish sufficient conditions involving a, A, f which insure that solutions of (v) are positive by using certain representation formulas for solutions of (v). We also discuss the positivity of solutions of (v) when A is a nonlinear (m-accretive) operator and we discuss several examples when A is a partial differential operator.

AMS (MOS) Subject Classifications: 45D05, 45N05, 45G99, 45M99, 47H05, 47H10, 47H15

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ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY

Ph. Clément[1,4] and J. A. Nohel[1,2,3]

1. Introduction and Principal Results.

Let X be a real or complex Banach space. We study the linear Volterra equation

$$u(t) + a * Au(t) = f(t) \qquad (0 \leq t \leq T;\ T > 0) \tag{1.1}$$

where $a * Au(t) = \int_0^t a(t-s)Au(s)ds$, a is a given real kernel, A is a bounded or unbounded linear operator from X to X and f is a given function with values in X.

An important and perhaps the most useful special case of (1.1) for certain applications is the linear equation

$$u(t) + a * Au(t) = u_0 + a * g(t) \qquad (0 \leq t \leq T;\ T > 0) \tag{1.1a}$$

where $u_0 \in X$ and the given function $g \in L^1(0,T;X)$. We will establish conditions on the kernel a and the operator A which insure that the respective solutions operators for (1.1) and (1.1a) preserve a convex cone in X (see Theorems 3 and 4). We then consider in Section 3 a nonlinear problem of the form (1.1) in which A is a m-accretive operator. Finally, in Section 4 we discuss three examples to illustrate the theory. Example 3 was proposed to us by Professor L. A. Peletier. We are grateful to Professor M. G. Crandall for discussing Example 3 with us.

We will suppose throughout that the following assumptions are satisfied.

(H_1) $A : D(A) \subseteq X \to X$ and $-A$ generates a linear continuous contraction semi-group on X, which we shall denote by $e^{-\omega A}$ $(\omega \geq 0)$.

(H_2) $a \in L^1(0,T;\mathbb{R})$

(H_3) $f \in L^1(0,T;X)$

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Definition 1. We say that $u : [0,T] \to X$ is a strong solution of (1.1) if $u \in L^1(0,T;X)$, $u(t) \in D(A)$ a.e. on $[0,T]$, $Au \in L^1(0,T;X)$, and u satisfies (1.1) a.e. on $[0,T]$.

We denote the norm in X by $\|\cdot\|$. If B is a linear unbounded operator on X, we use the notation $X_B = D(B)$; if $u \in X_B$, its graph norm is denoted by $\|u\|_{X_B} = \|u\| + \|Bu\|$. Of particular interest are the spaces X_A and X_{A^2} where A satisfies (H_1). Recall that the space X_A is dense in X and X_{A^2} is dense in X_A; see [15, Theorem 2.9, pg. 8]. If u is a strong solution of (1.1), Definition 1 states that $u \in L^1(0,T;X_A)$.

To discuss solutions of (1.1) and (1.1a) we make use of the operators R and S defined respectively by the equations

$$\text{(R)} \qquad u(t) + a * Au(t) = a(t)x \quad (x \in X_A;\ 0 \le t \le T)$$

$$\text{(S)} \qquad u(t) + a * Au(t) = x \quad (x \in X_A;\ 0 \le t \le T)\ .$$

It follows that under the assumptions of Theorem 1 below, equations (R) and (S) each have a unique strong solution which we write respectively as $R(t)x$ and $S(t)x$. While the operators R and S are so defined for $x \in X_A$, Theorem 1 together with a density argument shows that R and S can be extended uniquely as bounded operators in $L^1(0,T;X)$ and $C(0,T;X)$ respectively.

Our main result for the linear case is

Theorem 1. Let (H_1), (H_2) be satisfied.

(i) Let the kernel a satisfy the following condition

(H_4) For every $\lambda \ge 0$, the unique solution $r(t,\lambda) \in L^1(0,T;\mathbb{R})$ of the scalar equation (resolvent equation)

$$r(t) + \lambda a * r(t) = a(t) \qquad (0 \le t \le T)$$

satisfies $r(t,\lambda) \ge 0$ a.e. on $[0,T]$.

Then for every $x \in X_A$, the equation (R) has a unique strong solution which we denote by $R(t)x$, $0 \le t \le T$. Moreover, for almost every $t \in [0,T]$, there exists a positive measure μ_t on $\mathbb{R}^+$, depending only on the kernel a, such that

$$(1.2)\quad \begin{cases} R(t)x = \int_0^\infty e^{-\omega A} x \, d\mu_t(\omega) \\ a(t) = \int_0^\infty d\mu_t(\omega) \end{cases} \qquad t \in [0,T] \quad \text{a.e.}$$

and the following estimates are satisfied:

$$(1.3)\quad \|Rx\|_{L^1[0,T;Y]} \le \|a\|_{L^1[0,T;\mathbb{R}]} \|x\|_Y ,$$

where $Y = X$ or X_A or X_{A^n} and

$$(1.4)\quad \|R * v\|_{L^p[0,T;Y]} \le \|a\|_{L^1[0,T;\mathbb{R}]} \|v\|_{L^p[0,T;Y]} \quad (1 \le p \le \infty) .$$

(ii) Let the kernel a satisfy the assumptions (H_4) and:

(H_5) For every $\lambda \ge 0$, the unique solution $s(t,\lambda)$ (absolutely continuous on $[0,T]$) of the scalar equation

$s(t) + \lambda a * s(t) = 1 \quad (0 \le t \le T)$

satisfies $s(t,\lambda) \ge 0,\ 0 \le t \le T$.

Then for every $x \in X_A$ the equation (S) has a unique strong solution which we denote by $S(t)x$, $0 \le t \le T$. Moreover, for every $t \in [0,T]$, there exists a probability measure ν_t on $\mathbb{R}^+$ depending only on the kernel a, such that

$$(1.5)\quad S(t)x = \int_0^\infty e^{-\omega A} x \, d\nu_t(\omega) \quad (t \in [0,T]) ,$$

and the following estimates hold:

$$(1.6)\quad \|S(t)x\|_Y \le \|x\|_Y ,$$

$$(1.7)\quad \|S * v\|_{C[0,T;y]} \le \|v\|_{L^1[0,T;Y]} ,$$

where $Y = X$ or X_A or X_{A^n}.

Remark 1.1. If $a \equiv 1$, then $R(t) = S(t) = e^{-tA}$ and $\mu_t = \nu_t$ = the Dirac measure at t.

Assumptions (H_4) and (H_5) require some clarification.

Proposition 1. (i) Let (H_2) be satisfied and let $a \in C(0,T)$ and $a(t) > 0$. If $\log a(t)$ is convex on $(0,T)$ then (H_4) is satisfied on $[0,T]$.

(ii) Let (H_2) be satisfied and let $a(t)$ be nonnegative and nonincreasing on $(0,T)$. Then (H_5) is satisfied on $[0,T]$.

While the content of Proposition 1 is implicitly contained in the literature (see [7], [8], [12] and [14]), we give the proof in Appendix 1. In the literature the results are for t on the infinite interval and under slightly stronger assumptions.

Remark 1.2. It is useful to observe that

$$s(t,\lambda) = 1 - \lambda \int_0^t r(\sigma,\lambda)d\sigma$$

where r and s are defined in (H_4) and (H_5) respectively. This follows from the fact that $a * s = 1 * r$, together with the equation defining s. Thus if (H_4) and (H_5) are satisfied on $[0,T]$ for every $T > 0$ then $\int_0^T r(t,\lambda)dt \leq \int_0^T a(t)dt$ and $0 \leq \int_0^\infty r(t,\lambda)dt \leq \frac{1}{\lambda}$, $\lambda > 0$; in particular, $r(t,\lambda) \in L^1(0,\infty)$, $\lambda > 0$.

Remark 1.3. If $a(t)$ satisfies (H_2) and is completely monotonic on $(0,T)$, then a satisfies (H_4) and (H_5), see [7], [14].

Remark 1.4. We also note that, if $a(t) = e^t$, then (H_4) is satisfied but not (H_5). However, (H_5) does not imply (H_4). To see this, take $a(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$. Then by Proposition 1 (ii), (H_5) is satisfied. But for $\lambda = 1$, as shown by Levin [12; example following Theorem 1.4], $r(1,t) < 0$ for some $1 < t < 2$.

Theorem 1 is used to deduce the following results about solutions of equations (1.1) and (1.1a).

Theorem 2. (i) Let the assumptions (H_1), (H_2), (H_4) and $g \in L^1(0,T;X_A)$ be satisfied. Then the equation

$$u(t) + a * Au(t) = a * g(t) \qquad (0 \leq t \leq T) \tag{1.8}$$

has a unique strong solution u given by

$$u = R * g, \tag{1.9}$$

where R is the solution of equation (R) given by (1.2), and (by (1.3))

$$\|u\|_{L^1(0,T;X)} \leq \|a\|_{L^1(0,T;\mathbb{R})} \|g\|_{L^1(0,T;X)}. \tag{1.10}$$

(ii) Let the assumptions (H_1), (H_2), (H_4), (H_5) and

$$(H_6)\quad \begin{cases} f = f_1 + f_2 \ \text{where} \ f_1 \in L^p(0,T;X_{A^2}) \ 1 \le p \le \infty \\ \text{and} \ f_2 \in W^{1,1}(0,T;X_A), \ \text{where} \ W^{1,1} \ \text{is the usual Sobolev space,} \end{cases}$$

be satisfied. Then equation (1.1) has a unique strong solution $u = u_1 + u_2$ where

$$u_1(t) = f_1(t) - R * Af_1(t) \quad \text{a.e. on } [0,T] , \tag{1.11}$$

and

$$u_2(t) = S(t)f_2(0) + S * f_2'(t) \qquad t \in [0,T] , \tag{1.12}$$

where S is the solution of equation (S) given by (1.5); moreover there is a constant $c = c(T) > 0$ such that

$$\|u\|_{L^1(0,T;X)} \le c\{ \|f_1\|_{L^1(0,T;X_A)} + \|f_2\|_{W^{1,1}(0,T;X)} \} . \tag{1.13}$$

Remark 2.1. If A is any bounded linear operator, then $X = X_A = X_{A^2}$ and the existence and uniqueness of solutions of (1.1), with only $a \in L^1(0,T;\mathbb{R})$, $f \in L^1(0,T;X)$ is well-known. In the case when A is not bounded, existence and uniqueness results for solutions of (1.1) have been obtained by Friedman and Shinbrot [9], even for the case $A(t)$ where $A(t)$ generates an analytic semi-group under different conditions both for the kernel and the function f with, however, different objectives than ours.

Remark 2.2. Formula (1.11) is well-known when A is a bounded operator; formula (1.12) has also been employed in [8], [9] where S is called a fundamental solution of (1.1).

Remark 2.3. In the unbounded case we may define a weak solution of (1.1) as follows: there exist sequences $\{u_n\}$, $\{f_n\}$ where each $f_n \in L^1(0,T;X)$ and each u_n is a strong solution of (1.1) with $f = f_n$ such that $f_n \to f$ and $u_n \to u$ in $L^1(0,T;X)$. From (1.13) it follows that if $f \in L^1(0,T;X_A) + W^{1,1}(0,T;X)$, then equation (1.1) possesses a unique weak solution. (Note that $L^1(0,T;X_{A^2})$ is dense in $L^1(0,T;X_A)$ with respect to the norm in $L^1(0,T;X)$; similarly for $W^{1,1}(0,T;X_A)$ in $W^{1,1}(0,T;X)$. A similar remark applies to (1.8).

Remark 2.4. If $f_1 = 0$, then conclusion (1.13) can be strengthened to:

$$\|u\|_{C(0,T;X)} \le c\|f_2\|_{W^{1,1}(0,T;X)} . \tag{1.14}$$

Remark 2.5. Since the kernel is real, the case when X is a real Banach space can be treated as a special case of the complex case: If $\tilde{X} = X + iX$, the operator $\tilde{A}(x + iy) := Ax + iAy$ satisfies (H_1) whenever A satisfies (H_1). Therefore, we can restrict ourselves to the complex case.

Remark 2.6. If $a(t) = \delta(t)$ where $\delta(t)$ is the Dirac measure, then (1.1) reduces to $u(t) + Au(t) = f(t)$, and

$$S(t) = (I + A)^{-1} = \int_0^\infty e^{-\omega A} e^{-\omega} d\omega \quad [19; \text{p. } 240] \,. \tag{1.15}$$

The kernel $a(t) = \delta(t)$ does not satisfy (H_2). However, $\delta(t)$ can be approximated by kernels $a_\sigma(t) = \frac{1}{\sigma} e^{-\frac{t}{\sigma}}$ $(\sigma \to 0^+)$; each a_σ satisfies (H_2), (H_4), (H_5) so that $a(t) = \delta(t)$ is a limiting case of our theory and the corresponding measures $\nu_t^{(\sigma)}$ approach the measures ν_t in (1.15) of density $e^{-\omega}$, independent of t, as $\sigma \to 0^+$.

By (1.2) and (1.5), $R(t)$ and $S(t)$ are respectively positive and convex "combinations" of contraction semigroups $e^{-\omega A}$. From this observation we obtain the following applications of Theorems 1 and 2 which we state as Theorems 3 and 4.

Theorem 3. *Let (H_1), (H_2), (H_4) be satisfied. Let P be a closed convex cone in X, such that*

$$(I + \lambda A)^{-1} P \subseteq P \quad \textit{for every} \quad \lambda \geq 0 \,. \tag{1.16}$$

Then

$$R(t)P \subseteq P \quad \text{a.e. on} \quad [0,T] \,. \tag{1.17}$$

Moreover, if in equation (1.8) $g(t) \in P$ a.e., then the solution u of (1.8) lies in P a.e. on $[0,T]$. If in (1.1) $f \in L^1(0,T;X_A)$ and $Af(t) \in P$ a.e. on $[0,T]$, then

$$f(t) \in u(t) + P \quad \text{a.e. on} \quad [0,T] \,, \tag{1.18}$$

where u is the (weak) solution of (1.1); in particular, if P is a positive cone in X, the last statement is equivalent to the "maximum principle":

$$u(t) \leq f(t) \quad \text{a.e. on} \quad [0,T] \,. \tag{1.19}$$

The proof of (1.17) in Theorem 3 is an immediate consequence of formula (1.2) for the operator R, together with the standard fact that assumption (1.16) implies that $e^{-\omega A}$ maps P into P for every $\omega \in \mathbb{R}^+$. Having established (1.17), the remaining conclusions of Theorem 3 follow from the representation formula (1.11).

Remark 3.1. If one studies equation (1.8) in the scalar case, one takes $A = \lambda \geq 0$ to satisfy (H_1). If (H_2) is satisfied and if $P = \mathbb{R}^+$, then the condition (H_4) is necessary and sufficient in order to guarantee that the solution u of (1.8) satisfies $u(t) \geq 0$ for every $g \geq 0$. Thus one cannot hope to improve on condition (H_4) in the abstract case.

Theorem 4. Let (H_1), (H_2), (H_4), (H_5) be satisfied. Let P be a closed convex cone in X satisfying (1.16). Then

(1.20) $$S(t)P \subseteq P \quad \text{for} \quad 0 \leq t \leq T .$$

(i) Moreover, if $u_0 \in P$ and if $g(t) \in P$ a.e. in equation (1.1a), then the solution u of (1.1a) lies in P for almost every $t \in [0,T]$.

(ii) If in (1.1), $f \in W^{1,1}[0,T;X]$ where $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0,T]$, then the (weak) solution u of (1.1) lies in P for every $t \in [0,T]$. (The last assertion holds for any closed convex set P in X).

(iii) Moreover, if X is a real Hilbert space, and if the function $\varphi : X \to [0,\infty]$ is convex, lower semicontinuous, proper and satisfies

(1.21) $$\varphi((I + \lambda A)^{-1}x) \leq \varphi(x) \quad \text{for every} \quad \lambda \geq 0 \quad \text{and every} \quad x \in X ,$$

then

(1.22) $$\varphi(S(t)x) \leq \varphi(x) \quad \text{for every} \quad t \in [0,T] \quad \text{and every} \quad x \in X .$$

The proof of (1.20) in Theorem 4 follows from formula (1.5) for the operator S, together with the observation that assumption (1.16) implies that $e^{-\omega A}$ maps P into P for every $\omega \in \mathbb{R}^+$. Then conclusion (i) of Theorem 4 follows from (1.9), (1.12) with $f(t) \equiv u_0$, and the fact that the operators R and S each map P into P. Similarly, conclusion (ii) follows from (1.12). To establish (iii) recall that assumption (1.21) implies that

$$\varphi_\lambda(e^{-\omega A}x) \leq \varphi_\lambda(x) \quad \text{for every} \quad \omega \geq 0,\ \lambda > 0,\ x \in X ,$$

where φ_λ is the Yosida approximation of φ, [3, Proposition 2.11]. Then (1.22) follows from (1.5), Jensen's inequality and $\sup_{\lambda>0} \varphi_\lambda(x) = \varphi(x)$, [3, Proposition 2.11].

Remark 4.1. Conclusion (ii) of Theorem 4 is an abstraction of a result of Levin [12; Lemma 1.3] in $\mathbb{R}^1$. His result is

"Let $a \in L^1_{loc}(0,\infty)$, $a(t)$ nonnegative nonincreasing on $(0,\infty)$. Let $f \in C[0,\infty)$ be nonnegative and nondecreasing on $[0,\infty)$. Then the solution x of the equation

$$x(t) + a * x(t) = f(t) \qquad (0 \le t < \infty)$$

satisfies $0 \le x(t) \le f(t)$".

This result is also an immediate consequence of Proposition 1 (ii) and of the formula

$$x(t) = S(t)f(0) + \int_0^t S(t-\sigma)df(\sigma) .$$

Levin's proof in [12] is different; he improves his result by a smoothing argument which permits him to remove the assumption $f \in C[0,\infty)$. This is also evident from the preceding formula.

In Theorem 4 (ii) both assumptions (H_4) and (H_5) are used. It is of interest to note that in the abstract case the assumption (H_5) (which is satisfied when a is positive and nonincreasing) is not sufficient to insure that S maps P into P when condition (1.16) is satisfied. To see this we consider the following example in $\mathbb{R}^2$.

Let

$$a(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 , \end{cases} \tag{1.23}$$

and consider for $\alpha > 0$ the operator A_α defined by

$$A_\alpha = U^T \Lambda_\alpha U \quad \text{where} \tag{1.24}$$

$$\Lambda_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1+\alpha \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

For every $\alpha > 0$, the real matrix A_α is symmetric and positive definite. Thus $-A_\alpha$ generates a contraction semigroup on $\mathbb{R}^2$, with the usual Euclidean norm. If P is the cone $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$, then it is easily checked that $(I + \lambda A_\alpha)^{-1} P \leq P$ for every $\alpha > 0$, $\lambda > 0$, so that (1.16) is satisfied.

Corresponding to the kernel a defined by (1.23), the function $s(t, \lambda)$ of (H_5) is

$$(1.25) \qquad s(t, \lambda) = \begin{cases} e^{-\lambda t} & \text{if } 0 \leq t < 1 \\ e^{-\lambda t} + \lambda(t - 1)e^{-\lambda(t-1)} & \text{if } 1 \leq t \leq 2 , \end{cases}$$

and clearly (H_5) is satisfied on the interval $0 \leq t \leq 2$.

We next compute the operator S_α corresponding to A_α. Consider the equation

$$(1.26) \qquad u + a * A_\alpha u = x, \quad x \in \mathbb{R}^2 .$$

By setting $v = Uu$, $y = Ux$ equation (1.26) is transformed to the equivalent diagonal form

$$(1.27) \qquad v + a * \Lambda_\alpha v = y ,$$

which by the definition of $s(t, \lambda)$ in (H_5) gives the solution

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} s(t, 1)y_1 \\ s(t, 1 + \alpha)y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s(t, 1)[x_1 + x_2] \\ s(t, 1 + \alpha)[-x_1 + x_2] \end{pmatrix} .$$

Thus the solution of (1.26) is

$$u(t) = \frac{1}{2} \begin{pmatrix} s(t, 1)[x_1 + x_2] - s(t, 1 + \alpha)[-x_1 + x_2] \\ s(t, 1)[x_1 + x_2] + s(t, 1 + \alpha)[-x_1 + x_2] \end{pmatrix} ,$$

and the operator $S_\alpha(t)$ is

$$S_\alpha(t) = \frac{1}{2} \begin{pmatrix} s(t, 1) + s(t, 1 + \alpha) & s(t, 1) - s(t, 1 + \alpha) \\ s(t, 1) - s(t, 1 + \alpha) & s(t, 1) + s(t, 1 + \alpha) \end{pmatrix} .$$

To show that (H_5) is not sufficient to prove that S_α maps $P = R_2^+$ into P, it is sufficient to have $s(t, 1) - s(t, 1 + \alpha) < 0$ for some $t > 0$ and for some $\alpha > 0$. Observe that from (1.25)

$$(1.28) \qquad \frac{\partial s}{\partial \lambda}(t, \lambda) = -e^{-\lambda(t-1)}[\lambda(t - 1)^2 - (t - 1) + te^{-\lambda}] \quad \text{for}$$

$1 \leq t \leq 2$, $\lambda > 0$. Thus $\frac{\partial s}{\partial \lambda}(1 + \frac{1}{10}, 1) > 0$, so that there exists $\alpha > 0$ such that

$s(1 + \frac{1}{10}, 1) - s(1 + \frac{1}{10}, 1 + \alpha) < 0$, which establishes the claim.

We note the above argument also shows that (H_5) does not imply that $s(t, \lambda)$ is completely monotonic in λ. (See remarks following Lemma 2.1 below).

2. Proof of Theorems 1 and 2.

We will prove Theorems 1 and 2 in two main steps. We first consider the case when A is a bounded operator. In this case, by Remarks 2.1 and 2.2, it suffices to prove the representation formulas (1.2) and (1.5); for, having these one immediately has the estimates (1.3), (1.4), (1.6), (1.7) as well as the conclusions of Theorem 2. We then consider the case when A is an unbounded operator as a limiting situation of the bounded case using the Yosida approximation of A. The case where A is bounded is further divided into two parts:

(i) scalar case. We require the following preliminary result.

Lemma 2.1. *If $a(t)$ satisfies assumptions (H_2), (H_4), then $r(t,\lambda)$, defined in (H_4), is completely monotonic in λ for $0 < \lambda < \infty$ for $t \in [0,T]$ a.e. If moreover $a(t)$ satisfies (H_5), then $s(t,\lambda)$, defined in (H_5) is completely monotonic in λ for $0 \leq \lambda < \infty$ for every $t \in [0,T]$.*

Proof of Lemma 2.1. We consider the equations

$$r(t,\lambda) + \lambda a * r(t,\lambda) = a(t) \tag{2.1}$$

$$s(t,\lambda) + \lambda a * s(t,\lambda) = 1 \tag{2.2}$$

of assumptions (H_4) and (H_5) respectively with λ complex rather than $\lambda \geq 0$. Let E denote the spaces $L^1(0,T;\mathbb{C})$ or $C(0,T;\mathbb{C})$. Define the operator $K : E \to E$ by $K_x(t) = a * x(t)$ $(x \in E)$. K is a bounded linear operator with spectrum $\sigma(K) = 0$. Thus for $u \in E$, the function v defined by $v(\lambda) = (I + \lambda K)^{-1}u$, $\lambda \in \mathbb{C}$, is an entire function of λ with values in E. By differentiation and induction one has the formula:

$$(-1)^n \frac{d^n}{d\lambda^n} v(\lambda) = n!\, K_\lambda^n v(\lambda), \qquad n = 0,1,2,\ldots \tag{2.3}$$

where the operator K_λ is defined by

$$K_\lambda = K(I + \lambda K)^{-1}. \tag{2.4}$$

We claim that

$$(2.5) \qquad K_\lambda x(t) = \int_0^t r(t-s,\lambda)x(s)ds \qquad (x \in E) .$$

To prove (2.5) take the convolution product of both sides of (2.1) by $x \in L^1(0,T;\mathbb{C})$, obtaining

$$r(t,\lambda) * x(t) + \lambda a * r(t,\lambda) * x(t) = a * x(t) .$$

Thus $u_\lambda(t) = r(t,\lambda) * x(t)$ satisfies the equation

$$u_\lambda(t) + \lambda a * u_\lambda(t) = a * x(t) ;$$

by uniqueness of solutions of this scalar equation and by the definition of K_λ in (2.4) this shows that $u_\lambda(t) = K_\lambda x(t)$ and proves (2.5).

For $\lambda \geq 0$, assumption (H_4) implies that the operators K_λ map the set of non-negative real functions in E into itself. To prove the first assertion of Lemma 2.1, consider $v_a(\lambda) = (I+\lambda K)^{-1}a$; then $v_a(\lambda)(t) = r(t,\lambda)$ a.e. in $[0,T]$, $r(t,\lambda) \geq 0$ by (H_4), and by (2.3), (2.5) $(-1)^n \frac{\partial^n}{\partial\lambda^n} r(t,\lambda) \geq 0$ a.e. in $[0,T]$ for $0 < \lambda < \infty$. To prove the second assertion of Lemma 2.1, take $v_1(\lambda) = (I+\lambda A)^{-1}1$; then $v(\lambda)(t) = s(t,\lambda) \geq 0$ by (H_5), and complete the proof as above. This completes the proof of Lemma 2.1.

It should be noted that the second assertion of Lemma 2.1 is stated by Friedman [8, lemma 2.7] under only the hypothesis that $a \geq 0$ and nonincreasing. However, his proof also uses (H_4). (He should also require (H_4) for his Theorem 5.2, p. 144). To see that (H_5) is not sufficient for the complete monotonicity of $s(t,\lambda)$ with respect to λ, we consider again the kernel a defined in (1.23). The corresponding function $s(t,\lambda)$ is given by (1.25) and $s(t,\lambda) \geq 0$, for $0 \leq t \leq 2$. However, as seen from (1.28), $\frac{ds}{d\lambda}(1+\frac{1}{10}, 1) > 0$.

We shall next obtain representations of the entire functions $r(t,\lambda)$, $s(t,\lambda)$ for $\mathrm{Re}\,\lambda \geq 0$.

By Lemma 2.1 and Bernstein's theorem [18], there exists a positive finite measure μ_t on $\mathbb{R}^+$ such that

$$(2.6)\quad \begin{cases} r(t,\lambda) = \int_0^\infty e^{-\omega\lambda} d\mu_t(\omega) & (\mathrm{Re}\lambda > 0;\ t \in [0,T] \text{ a.e.}) \\ a(t) = \int_0^\infty d\mu_t(\omega) & (t \in [0,T] \text{ a.e.}) . \end{cases}$$

Similarly, using $s(0,\lambda) = 1$, there exists a probability measure ν_t on $\mathbb{R}^+$ such that

$$(2.7)\qquad s(t,\lambda) = \int_0^\infty e^{-\omega\lambda} d\nu_t(\omega) \quad (\mathrm{Re}\,\lambda \geq 0;\ t \in [0,T]) .$$

Thus (2.6) and (2.7) correspond to formulas (1.2) and (1.5).in the scalar case.

(ii) A is a bounded operator satisfying (H_1). By a standard argument equations (R) and (S) possess for every $x \in X$ a unique solution which we denote by $R(t)x$ and $S(t)x$ respectively. We first prove the representation formulas (1.2) and (1.5) for the opera'ɔrs A_ε defined by

$$(2.8)\qquad A_\varepsilon = \varepsilon I + A \quad (1 > \varepsilon > 0) .$$

Define the operators R_ε and S_ε by the formulas

$$(2.9)\qquad R_\varepsilon(t)x = \frac{1}{2i\pi} \int_{C_\varepsilon} r(t,\lambda)(\lambda I - A_\varepsilon)^{-1} x\, d\lambda \quad (0 \leq t \leq T) ,$$

$$(2.10)\qquad S_\varepsilon(t)x = \frac{1}{2i\pi} \int_{C_\varepsilon} s(t,\lambda)(\lambda I - A_\varepsilon)^{-1} x\, d\lambda \quad (0 \leq t \leq T) ,$$

where $x \in X$, $r(t,\lambda)$, $s(t,\lambda)$ are defined by (2.1) and (2.2) respectively for $\lambda \in \mathbb{C}$. C_ε is a closed contour in the complex λ plane, oriented counterclockwise, consisting of a finite number of rectifiable Jordan arcs and such that $C_\varepsilon = \partial U_\varepsilon$, where U_ε is an open set containing the spectrum of A_ε. The integral in (2.9), (2.10) are the usual Dunford integrals [19, p. 225]. It is shown by Friedman [8, Theorem 3.1] that $S_\varepsilon(t)x$ defined by (2.10) is the unique solution of equation (S) with A replaced by A_ε. An entirely analogous argument shows that $R_\varepsilon(t)x$ defined by (2.9) is the unique solution of equation (R) with A replaced by A_ε.

We next observe that the spectrum $\sigma(A_\varepsilon)$ is contained in the half plane $\mathrm{Re}\,\lambda \geq \varepsilon$, and, if $\epsilon < 1$, in the ball of radius $1 + \|A\|$. Thus we may choose C_ε to be the rectangle bounded by the segments joining the points $(\frac{\varepsilon}{2} - i\,(2 + \|A\|))$, $((2 + \|A\|)(1 - i))$, $((2 + \|A\|)(1 + i))$, $(\frac{\varepsilon}{2} + i(2 + \|A\|))$ oriented counterclockwise. Using the representation (2.6) in (2.9) under assumption (H_4) and the representation (2.7) in (2.10) under assumptions (H_4), (H_5) we obtain

$$R_\varepsilon(t)x = \int_0^\infty e^{-\omega A_\varepsilon} x \, d\mu_t(\omega) \qquad (x \in X)\,, \tag{2.11}$$

$$S_\varepsilon(t)x = \int_0^\infty e^{-\omega A_\varepsilon} x \, d\nu_t(\omega) \qquad (x \in X)\,. \tag{2.12}$$

The proofs of (2.11), (2.12) follow from a theorem on the Dunford integral [19, p. 226], together with Fubini's theorem and the definition of the operator $e^{-\omega A_\varepsilon}$ by

$$e^{-\omega A_\varepsilon} x = \frac{1}{2\pi i} \int_{C_\varepsilon} e^{-\omega\lambda} (\lambda I - A_\varepsilon)^{-1} x \, d\lambda \qquad (x \in X)\,.$$

Thus formulas (2.11), (2.12) establish (1.2) and (1.5) respectively with $A = A_\varepsilon$. We next let $\varepsilon \to 0^+$. We first show that

$$R_\varepsilon(t)x \to z(t) = \int_0^\infty e^{-\omega A} x \, d\mu_t(\omega) \quad \text{in} \quad L^1(0, T; X)\,. \tag{2.13}$$

We then show that $z(t)$ is the unique solution of equation (R). Substituting (2.8) in (2.11) we have

$$\left\| R_\varepsilon(t)x - \int_0^\infty e^{-\omega A} x \, d\mu_t(\omega) \right\| = \left\| \int_0^\infty (e^{-\varepsilon\omega} - 1) e^{-\omega A} d\mu_t(\omega) \right\|\,.$$

Therefore, by a simple application of Lebesgue's dominated convergence theorem

$$\lim_{\varepsilon \to 0^+} \left\| R_\varepsilon(t)x - \int_0^\infty e^{-\omega A} x \, d\mu_t(\omega) \right\| = 0 \quad \text{a.e. on} \quad [0, T]\,.$$

Moreover, since $e^{-\omega A}$ is a contraction semigroup, we have

$$(2.14) \qquad \|R_\varepsilon(t)x\| \le \int_0^\infty \|e^{-\varepsilon\omega}e^{-\omega A}x\|d\mu_t(\omega) \le \|x\|a(t) \quad \text{a.e.}$$

Since $a \in L^1(0,T)$, another application of Lebesgue's theorem establishes (2.13).

We next show that the function z defined in (2.13) is the unique solution of equation (R). We know that $R_\varepsilon(t)x$ is the unique solution of the equation

$$(R_\varepsilon) \qquad u_\varepsilon(t) + a * Au_\varepsilon(t) + \varepsilon a * u_\varepsilon(t) = a(t) \quad \text{a.e.}$$

Observe that by (2.14)

$$\|u_\varepsilon\|_{L^1(0,T;X)} \le \|x\| \int_0^T a(t)dt \, .$$

Consequently $\varepsilon a * u_\varepsilon \to 0$ in $L^1(0,T;X)$ as $\varepsilon \to 0^+$. Since $u_\varepsilon \to z$ in $L_1(0,T;X)$ as $\varepsilon \to 0^+$, one has that $z(t)$ satisfies equation (R) a.e. on $[0,T]$. By uniqueness, $z(t) = R(t)x$, establishing (1.2). An entirely similar argument with $L^1(0,T;X)$ replaced by $C(0,T;X)$ and assuming (H_5) establishes (1.5).

<u>A an unbounded operator satisfying (H_1)</u>. Using the assumptions (H_1), (H_2), (H_4) we define the operator $\tilde{R}$ by the relation

$$(2.15) \qquad \tilde{R}(t)x = \int_0^\infty e^{-\omega A}x \, d\mu_t(\omega) \qquad (x \in X) \, ,$$

for those $t \in [0,T]$ for which $\mu_t(\omega)$ is defined, and define $\tilde{R}(t)x = 0$ $(x \in X)$ otherwise. Similarly, using assumptions (H_1), (H_2), (H_4), (H_5) we define the operator $\tilde{S}$ by the relation

$$(2.16) \qquad \tilde{S}(t)x = \int_0^\infty e^{-\omega A}x \, d\nu_t(\omega) \qquad (x \in X), \; t \in [0,T] \, .$$

The measures μ_t and ν_t in (2.15) and (2.16) are defined in (2.6) and (2.7) respectively. We point out that the operators $\tilde{R}$ and $\tilde{S}$ will be identified with the operators R and S of Theorem 1 after Lemma 2.5 below. By (H_1) and elementary semigroup theory $\tilde{R}$ and $\tilde{S}$ are bounded operators in X, X_A, and X_{A^2}; we have the estimates:

$$(2.17)\qquad \|\tilde{R}(t)x\| \le a(t)\|x\| \qquad (t \in [0,T];\ x \in X)$$

and also

$$(2.18)\qquad \|\tilde{S}(t)x\| \le \|x\| \qquad (t \in [0,T];\ x \in X)\ .$$

Define

$$J_\lambda = (I + \lambda A)^{-1} \qquad (\lambda \ge 0)$$

and the Yosida approximation A_λ of A by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda) \qquad (\lambda > 0)\ ;$$

recall that by (H_1) J_λ is a contraction on X for every $\lambda \ge 0$ and that, see [19; Cor. 2, p. 241] where the notation is different,

$$(2.19)\qquad A_\lambda x = J_\lambda Ax = AJ_\lambda x \qquad (x \in X_A)\ .$$

We also need to define the operators $\tilde{R}_\lambda$ and $\tilde{S}_\lambda$ respectively by the relations

$$(2.20)\qquad \tilde{R}_\lambda(t)x = \int_0^\infty e^{-\omega A_\lambda} x\, d\mu_t(\omega) \qquad (\lambda > 0)\ ,$$

for those $t \in [0,T]$ for which $\mu_t(\omega)$ is defined and $\tilde{R}_\lambda(t)x = 0$ $(x \in X)$ *otherwise*, and

$$(2.21)\qquad \tilde{S}_\lambda(t)x = \int_0^\infty e^{-\omega A_\lambda} x\, d\nu_t(\omega) \qquad (\lambda > 0,\ t \in [0,T],\ x \in X)\ .$$

Since A_λ is a bounded operator for every $\lambda > 0$, it follows from uniqueness in the bounded case and from part (ii) that $\tilde{R}_\lambda(t)x = R_\lambda(t)x$ for $t \in [0,T]$ a.e. and $x \in X$, where $R_\lambda(t)x$ *is the unique solution of* equation (R) with A replaced by A_λ. Similarly, $\tilde{S}_\lambda(t)x = S_\lambda(t)x$ for $t \in [0,T]$ and $x \in X$, where $S_\lambda(t)x$ is the unique solution of equation (S) with A replaced by A_λ. We shall use the following properties of the operators $\tilde{R}$, $\tilde{R}_\lambda$, $\tilde{S}$, $\tilde{S}_\lambda$.

<u>Lemma 2.2</u>. <u>Let</u> (H_1), (H_2), (H_4) <u>be satisfied</u>. <u>Let</u> $\tilde{R}$ <u>and</u> $\tilde{R}_\lambda$ <u>be defined by</u> (2.15) <u>and</u> (2.20) <u>respectively</u>. <u>Then</u>

$$(2.22)\qquad \tilde{R}x \in L^1(0,T;X) \qquad (x \in X)\ ,$$

(2.23) $$\lim_{\lambda \to 0^+} \|\tilde{R}x - \tilde{R}_\lambda x\|_{L^1(0,T;X)} = 0 \quad (x \in X) .$$

Moreover, if $v \in L^1(0,T;X)$, then as a function of s

(2.24) $$\tilde{R}(t-s)v(s) \in L^1(0,T;X) \quad (t \in [0,T] \text{ a.e.}) ,$$

(2.25) $$\int_0^t \tilde{R}(t-s)v(s)ds = \tilde{R} * v(t) \in L^1(0,T;X) ,$$

(2.26) $$\lim_{\lambda \to 0^+} \|\tilde{R} * v - \tilde{R}_\lambda * v\|_{L^1(0,T;X)} = 0 .$$

Finally, if $v_\lambda \to v$ in $L^1(0,T;X)$ as $\lambda \to 0^+$, then

(2.27) $$\lim_{\lambda \to 0^+} \|\tilde{R} * v - \tilde{R}_\lambda * v_\lambda\|_{L^1(0,T;X)} = 0 .$$

Lemma 2.3. *Let* (H_1), (H_2), (H_4), (H_5) *be satisfied. Let* $\tilde{S}$ *and* $\tilde{S}_\lambda$ *be defined by* (2.16) *and* (2.21) *respectively. Then properties* (2.22) - (2.27) *hold with* $\tilde{R}$ *replaced by* $\tilde{S}$ *and* $\tilde{R}_\lambda$ *replaced by* $\tilde{S}_\lambda$.

Remark 2.4. In Lemmas 2.2 and 2.3 the space X can be replaced by X_A or X_{A^2} without changing the proof. Also if $v, v_\lambda \in L^p(0,T;X)$, $p \geq 1$, then the properties (2.24) - (2.27) hold in $L^p(0,T;X)$. Moreover, in Lemma 2.3 one can replace $L^1(0,T;X)$ by $C([0,T];X)$ in the formulas corresponding to (2.22), (2.23), (2.25) - (2.27).

We only give the proof of Lemma 2.2.

First (2.22) is immediate from (2.17) by integration. To prove (2.23) we observe from (2.20) that

(2.28) $$\|\tilde{R}_\lambda(t)x\| \leq a(t)\|x\| \quad (x \in X;\ t \in [0,T]) .$$

Next, we show

(2.29) $$\tilde{R}_\lambda(t)x \to \tilde{R}(t)x \quad (\lambda \to 0^+;\ x \in X;\ t \in [0,T] \text{ a.e.}) .$$

By semigroup theory, [19],

$$e^{-\omega A_\lambda}x \to e^{-\omega A}x \quad (\lambda \to 0^+)$$

uniformly in ω on compact subsets of $\mathbb{R}^+$, and so (2.28) holds by Lebesgue's dominated convergence theorem. Thus (2.23) follows from (2.28), (2.29), (H_2), and Lebesgue's dominated convergence theorem. By (2.22) $\tilde{R}(t-s)v(s)$, as a function of (s,t), is measurable for $0 \le s \le t \le T$ with values in X. By (2.28) one has

$$\|\tilde{R}_\lambda(t-s)v(s)\| \le a(t-s)\|v(s)\| , \tag{2.30}$$

where by (H_2) $a(t-s)\|v(s)\| \in L^1(0,T;\mathbb{R}^+)$ for $t \in [0,T]$ a.e. Thus one obtains (2.24) by letting $\lambda \to 0^+$ and by applying Lebesgue's dominated convergence theorem in (2.30). To prove (2.25) and (2.26) we integrate (2.30) obtaining

$$\int_0^T \int_0^t \|\tilde{R}_\lambda(t-s)v(s)\| ds dt \le \int_0^T a(t)dt \|v\|_{L^1(0,T;X)} .$$

Therefore, (2.25) follows from Fatou's lemma and (2.26) follows by again applying Lebesgue's theorem. Finally, writing

$$\tilde{R} * v - \tilde{R}_\lambda * v_\lambda = (\tilde{R} * v - \tilde{R}_\lambda * v) + (\tilde{R}_\lambda * v - \tilde{R}_\lambda * v_\lambda) ,$$

and using arguments similar to those employed above one obtains (2.27). This completes the proof of Lemma 2.2.

We next establish the uniqueness of solutions of (1.1) when A is an unbounded operator satisfying (H_1).

<u>Lemma 2.5</u>. <u>Let</u> (H_1), (H_2), (H_4) <u>be satisfied and let</u> $u \in L^1(0,T;X_A)$ <u>be a strong solution of the equation</u>

$$u + a * Au = 0 .$$

<u>Then</u> $u = 0$.

<u>Proof of Lemma 2.5</u>. For any $\lambda > 0$ we have from the given equation and from (2.19) that

$$J_\lambda u + a * A_\lambda u = 0 ,$$

or equivalently

$$u + a * A_\lambda u = u - J_\lambda u .$$

By using the fact that A_λ is a bounded operator, together with the representation formula (1.11) where A is replaced by A_λ, f_1 is replaced by $u - J_\lambda u$, and R is replaced by $\tilde{R}_\lambda$, and the uniqueness of solutions of (1.1) in the bounded case, we obtain

$$u = u - J_\lambda u - \tilde{R}_\lambda * A_\lambda(u - J_\lambda u)\,, \tag{2.31}$$

where $\tilde{R}_\lambda$ is defined by (2.20). We wish to show that $u - J_\lambda u$ and $A_\lambda(u - J_\lambda u)$ each tend to zero as $\lambda \to 0^+$ in $L^1(0,T;X)$ for $u \in L^1(0,T;X_A)$. We have

$$\int_0^T \|u - J_\lambda u\|(t)dt = \int_0^T \lambda\|A_\lambda u\|(t)dt \le \lambda \int_0^T \|Au\|(t)dt\,,$$

which tends to zero as $\lambda \to 0^+$. Also

$$\|A_\lambda(u - J_\lambda u)\|(t) = \|A_\lambda \lambda A_\lambda u\|(t) = \|\lambda A_\lambda J_\lambda v\|(t)\,,$$

where $v = Au$; thus

$$\|A_\lambda(u - J_\lambda u)\|(t) = \|J_\lambda v - J_\lambda(J_\lambda v)\|(t) \le \|v - J_\lambda v\|(t)\,.$$

But

$$\|v - J_\lambda v\|(t) \le 2\|v\|(t) = 2\|Au\|(t) \in L^1(0,T)\,;$$

moreover,

$$\|v - J_\lambda v\|(t) \to 0 \quad a.e.\ on\ [0,T]\,,$$

and therefore, by Lebesgue's theorem, $A_\lambda(u - J_\lambda u) \to 0$ as $\lambda \to 0^+$ in $L^1(0,T;X)$ for $u \in L^1(0,T;X_A)$. Letting $\lambda \to 0^+$ in (2.31) and using the above facts together with (2.27) of Lemma 2.2, we obtain $u = 0$. This completes the proof of Lemma 2.5.

We will complete the proof of Theorems 1 and 2 by first noting that Lemma 2.5 establishes the uniqueness assertions in Theorems 1 and 2. To prove Theorem 1, part (i), we shall prove that $\tilde{R}(t)x$ is a solution of equation (R) for $x \in X_A$. We know that $u_\lambda = \tilde{R}_\lambda(t)x$ defined by (2.20) is the unique solution of the approximating equation associated with (R):

$$u_\lambda + a * A_\lambda u_\lambda = a(t)x \quad (0 \le t \le T;\ x \in X_A)\,. \tag{2.32}$$

By Lemma 2.2 and Remark 2.4

$$u_\lambda \to u = \tilde{R}x \ \text{ in } L^1(0,T;X_A) \ \text{ as } \lambda \to 0^+\,, \tag{2.33}$$

where $\tilde{R}$ is defined by (2.15). Thus to pass to the limit in equation (2.32) as $\lambda \to 0^+$, it suffices to show that

$$A_\lambda u_\lambda \to Au \text{ in } L^1(0,T;X) \text{ as } \lambda \to 0^+ .$$

But $A_\lambda u_\lambda = AJ_\lambda u_\lambda$; thus it suffices to show that $J_\lambda u_\lambda \to u$ in $L^1(0,T;X_A)$ as $\lambda \to 0^+$. This is equivalent to showing that

$$\lambda A_\lambda u_\lambda = u_\lambda - J_\lambda u_\lambda \to 0 \text{ in } L^1(0,T;X_A) \text{ as } \lambda \to 0^+. \tag{2.34}$$

But

$$\int_0^T \|A_\lambda u_\lambda(s)\| ds = \int_0^T \|J_\lambda A u_\lambda(s)\| ds \le \int_0^T \|Au_\lambda(s)\| ds \le \|u_\lambda\|_{L^1(0,T;X_A)} \le M ,$$

where $M > 0$ is constant and where the last inequality follows from (2.33). This proves (2.34) and shows that $\tilde{R}(t)x$ is a solution of equation (R) for $x \in X_A$ and for $0 \le t \le T$. By the uniqueness result of Lemma 2.5 we identify $\tilde{R}(t)x$ with $R(t)x$ of Theorem 1 and thereby prove (1.2). The a priori estimates (1.3), (1.4) follow from Lemma 2.2 and Remark 2.4. This completes the proof of Theorem 1 (i).

The proof of Theorem 1 (ii), is similar using the approximating equation associated with (S):

$$u_\lambda + a * A_\lambda u_\lambda = x ,$$

where $u_\lambda = \tilde{S}_\lambda(t)x$ defined by (2.21), and Lemma 2.3. This completes the proof of Theorem 1.

To prove Theorem 2 (i) it is sufficient, by Lemmas 2.2 and 2.5, to show that $u = R * g$ is a strong solution of (1.8). To do this we consider the approximating equation associated with (1.8):

$$u_\lambda + a * A_\lambda u_\lambda = a * g \quad (g \in L^1(0,T;X_A)) . \tag{2.35}$$

We already know, since A_λ is a bounded operator, that $u_\lambda = \tilde{R}_\lambda * g$ is the unique solution of (2.35), and by Lemma 2.2 and Remark 2.4

$$u_\lambda \to u = R * g \text{ in } L^1(0,T;X_A) \text{ as } \lambda \to 0^+ .$$

One completes the proof of Theorem 2(i) by letting $\lambda \to 0^+$ in (2.35) and by observing as before that $A_\lambda u_\lambda \to Au$ in $L^1(0,T;X)$ as $\lambda \to 0^+$. The estimate (1.10) follows immediately from its validity for $u_\lambda = \tilde{R}_\lambda * g$, together with Lemma 2.2.

To prove Theorem 2(ii) we consider the approximating equation associated with (1.1):

$$u_\lambda + a * A_\lambda u_\lambda = f. \tag{2.36}$$

We first take $f = f_1$ in (H_6). Since A_λ is bounded

$$u_{1\lambda} = f_1 - \tilde{R}_\lambda * A_\lambda f_1$$

is the unique solution of (2.36) with $f = f_1$. We have that $u_{1\lambda} \in L^1(0,T;X_A)$ and by Lemma 2.2 and Remark 2.4

$$u_{1\lambda} \to u_1 = f_1 - R * Af_1 \quad \text{in} \quad L^1(0,T;X_A) \quad \text{as} \quad \lambda \to 0^+.$$

As above, $A_\lambda u_{1\lambda} \to Au_1$ in $L^1(0,T;X)$ as $\lambda \to 0^+$. Thus letting $\lambda \to 0^+$ in (2.36) and using Lemma 2.5, shows that u_1 given by (1.11) is a strong solution of (1.1).

We next take $f = f_2$ in (H_6), and we obtain (1.12) by a completely analogous argument. The estimate (1.13) follows from formulas (1.11), (1.12), together with the estimates (1.4), (1.6), (1.7). This completes the proof of Theorem 2.

3. A a Nonlinear Operator.

In this section we give a nonlinear analogue of Theorems 3 and 4. Let X be a real Banach space and let $P \subseteq X$ be a closed convex cone. Let $A : D(A) \subseteq X \to 2^X$ be a given, possibly multivalued, m-accretive operator [6; p. 139] satisfying the condition

$$(I + \lambda A)^{-1} P \subseteq P \qquad (\lambda > 0) . \tag{3.1}$$

Let a satisfy (H_2) and (H_4) and let f satisfy (H_3). Consider the equation

$$u(t) + a * Au(t) \ni f(t) \qquad t \in [0, T] , \tag{3.2}$$

where $T > 0$. We say that $u \in L^1(0, T; X)$ is a solution of (3.2) on $[0, T]$ if there exists $w \in L^1(0, T; X)$, where $w(t) \in Au(t)$ a.e., such that $u(t) + a * w(t) = f(t)$ a.e. for $t \in [0, T]$.

Theorem 5. Let (H_2), (H_4) be satisfied. Let f satisfying (H_3) be such that

$$(H_7) \quad \begin{cases} \text{for every } \lambda > 0, \ v, \ \text{the unique solution of the linear equation} \\ (3.3) \qquad v(t) + \lambda a * v(t) = f(t) \qquad t \in [0, T] \ \text{a.e.}, \\ \text{satisfies } v(t) \in P \ \text{a.e. on } [0, T] . \end{cases}$$

For every $\lambda > 0$ let u_λ be the unique solution of the equation

$$u_\lambda(t) + a * A_\lambda u_\lambda(t) = f(t) \qquad t \in [0, T] \ \text{a.e.}, \tag{3.4}$$

where A_λ is the Yosida approximation of A. If (3.1) is satisfied, then $u_\lambda(t) \in P$ a.e. on $[0, T]$. Moreover, if u is a solution of equation (3.2) such that $u = \text{weak} \lim_{\lambda \to 0} u_\lambda$ in $L^1(0, T; X)$, then $u(t) \in P$ a.e. on $[0, T]$.

Remark 5.1. Under the assumptions of Theorem 5 it follows from Theorems 3 and 4 with $A = \lambda I$ that if $f(t) = a * g(t)$, $g \in L^1(0, T; X)$, then (H_7) is satisfied if $g(t) \in P$ a.e. on $[0, T]$. If $f(t) = u_0 + a * g(t)$, where $u_0 \in P$ and g is as above, then (H_7) is satisfied provided that (H_5) holds. If $f \in W^{1,1}(0, T; X)$, then (H_7) is satisfied provided that (H_5) holds, and that $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0, T]$.

Remark 5.2. If A is linear and satisfies (H_1), equation (3.2) is (1.1); it was shown in section 2 that the unique solution u_λ of (3.4) converges to u, the unique solution of (1.1), under the assumptions of Theorem 2.

Remark 5.3. If $X = H$ a real Hilbert space and if $A = \partial\varphi$, where $\varphi : H \to (-\infty, \infty]$ is convex, l.s.c. and proper, Barbu [1] and Londen [13] establish the existence and uniqueness of the solution u of equation (3.2) as a limit of solutions u_λ of equation (3.4), so that Theorem 5 can be applied to such a nonlinear equation. A generalization to the case when A is a maximal monotone operator on H is carried out by Gripenberg [11]. It should be noted that in the existence theory of [1], [11], and [13] $a(0) > 0$ and finite is essential, while in Theorem 5 $a(0^+) = +\infty$ is permitted.

Proof of Theorem 5. Consider the equation (3.4) written in the equivalent form

$$u_\lambda + \frac{1}{\lambda} a * u_\lambda = f + \frac{1}{\lambda} a * J_\lambda u_\lambda . \tag{3.5}$$

Define $f_\lambda \in L^1(0, T; X)$ to be the unique solution of (3.3) with λ replaced by $\frac{1}{\lambda}$. By (H_7) $f_\lambda(t) \in F$ a.e. on $[0, T]$. It is easily checked using

$$r(t, \tfrac{1}{\lambda}) + \frac{1}{\lambda}\int_0^t a(t-\sigma) r(\sigma, \tfrac{1}{\lambda}) d\sigma = a(t) \quad \text{and} \quad f_\lambda(t) = f(t) - \frac{1}{\lambda}\int_0^t r(t-\sigma, \tfrac{1}{\lambda}) f(\sigma) d\sigma$$

that equation (3.5) is equivalent to the equation

$$u_\lambda = F_\lambda(u_\lambda) , \tag{3.6}$$

where

$$F_\lambda(z)(t) = \frac{1}{\lambda}\int_0^t r(t-\sigma, \tfrac{1}{\lambda}) J_\lambda(z)(\sigma) d\sigma + f_\lambda(t) . \tag{3.7}$$

Observe that F_λ maps $L^1(0, T; X)$ into itself. We prove that some iterate of F_λ is a strict contraction in $L^1(0, T; X)$. Indeed, from (3.7), (H_2) and the contraction property of J_λ (recall A is m-accretive) one has

$$\|F_\lambda(u)(t) - F_\lambda(v)(t)\| \le \frac{1}{\lambda}\int_0^t |r(t-s, \tfrac{1}{\lambda})| \, \|u(s) - v(s)\| ds . \tag{3.8}$$

Define $b_\lambda(t) = \frac{1}{\lambda} r(t, \frac{1}{\lambda})$ and $b_\lambda^n(t) = b_\lambda * b_\lambda * \cdots * b_\lambda(t)$, where the convolution is taken n times. Iterating (3.8) n times we obtain

$$(3.9) \qquad \|F_\lambda^n(u)(t) - F_\lambda^n(v)(t)\| \le b_\lambda^n * \|u - v\|_{L^1(0,T;X)} .$$

For any fixed λ choose n_λ so large that $\int_0^t b_\lambda^{n_\lambda}(\sigma)d\sigma = K_\lambda < 1$; then integrating (3.9) we obtain

$$(3.10) \qquad \|F_\lambda^{n_\lambda}(u) - F_\lambda^{n_\lambda}(v)\|_{L^1(0,T;X)} \le K_\lambda \|u - v\|_{L^1(0,T;X)} .$$

Thus (3.6) (and by the equivalence also (3.4)) has a unique solution $u_\lambda \in L^1(0,T;X)$ given by

$$u_\lambda = \lim_{n\to\infty} F_\lambda^n(u_0), \quad \text{for any } u_0 \in L^1(0,T;X) .$$

In particular if $u_0(t) \in P$ a.e. on $[0,T]$ and if assumptions (H_4) and (H_7) are satisfied, then by (3.1) and (3.7) $F_\lambda(u_0)(t) \in P$ a.e. on $[0,T]$ and the same holds for $F_\lambda^n(u_0)(t)$ for every n. Consequently the unique solution of (3.4) $u_\lambda(t) \in P$ a.e. on $[0,T]$. This completes the proof of Theorem 5.

Remark 5.4. From the proof of Theorem 5 it is clear that Theorem 5 provides an alternative, and in fact simpler, treatment of Theorems 3 and 4 in the linear case. However, in the linear case Theorems 1 and 2 provide explicit representations for the operators R and S and hence more information about the solution. Moreover, the method of proof of Theorem 5 can be used to analyse more general situations. For example, let X be the product of n Banach spaces $X_1, X_2, \ldots, X_n$, and interpret equation (3.2) as a system of n equations with $u(t)$, $f(t) \in X$ *for* $t \in [0,T]$ *and the kernel* a being a $n \times n$ matrix satisfying (H_2) componentwise, and such the associated matrix resolvent $r(t,\lambda) \ge 0$ componentwise (analogue of (H_4)). Let P be a closed convex cone in X and let A be a m-accretive operator on X for a suitable norm satisfying (3.1). If f satisfies (H_3) and (H_7), then the conclusions of Theorem 5 hold.

Remark 5.5. The proof of Theorem 5 is in the same spirit as the proof of Theorem 1 of Weis [17] for the equation

$$x(t) = f(t) + \int_0^t a(t-s)g(s,x(s))ds$$

where x, f, g have values in $\mathbb{R}^n$ and a is a $n \times n$ matrix $\in L^1_{loc}(0,\infty)$ and where g has "separated structure" in the sense that $g(t,x) = \mathrm{col}(g_i(t,x_i))$, $i = 1,\dots,n$, where each g_i is locally Lipschitz with respect to x_i uniformly for t bounded. Weiss gives a condition which corresponds to (H_4) and (H_7) which insures that the solution $x(t) \geq 0$ for as long as it exists.

4. Examples.

Example 1. This example is an application of Theorem 5. Consider the equation

$$u(t,x) + a * (-\nabla^2 u(t,x) + \beta(u(t,x)) \ni f(t,x) , \tag{4.1}$$

$0 < t < \infty$, $x \in \Omega$, a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ with u satisfying Dirichlet boundary conditions on Γ. β is a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. For simplicity we assume that the kernel a is completely monotonic on $[0,\infty)$; thus (see Remark 1.3) assumptions (H_2), (H_4), (H_5) are satisfied on $[0,T]$ for every $T > 0$. We assume $f \in W^{1,2}_{loc}(0,\infty;X)$, $X = L^2(\Omega)$. To see that equation (4.1) is a particular case of (3.2) define

$$Au = -\nabla^2 u + \beta(u) \quad \text{with} \quad D(A) = \{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) : \beta(u) \in L^2(\Omega)\} . \tag{4.2}$$

As is known, see Brézis [4], A is the subdifferential of the convex, l.s.c., proper function $\varphi : L^2(\Omega) \to (-\infty,+\infty]$ defined by

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_\Omega (\text{grad } u)^2 dx + \int_\Omega j(u)dx & \text{if } u \in W^{1,2}_0(\Omega),\ j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} , \end{cases}$$

where j is the unique, convex, l.s.c., proper function mapping $\mathbb{R}$ into $(-\infty,+\infty]$ such that $j(0) = 0$ and $\beta = \partial j$. Thus A is maximal monotone on the Hilbert space $L^2(\Omega)$ and hence A is m-accretive. Thus (4.1) with the boundary condition $u = 0$ on Γ is a particular case of (3.2). Let $f \in W^{1,2}_{loc}(0,\infty;X)$; in particular, $f \in C[0,\infty;X)$ and $f(0)$ is well defined as an element of $L^2(\Omega)$. We assume that $f(0) \in W^{1,2}_0(\Omega)$ and $\int_\Omega j(f(0))dx < \infty$. These assumptions on f imply that (H_3), (H_6) are satisfied. It is now easily checked that all the assumptions Londen [13; Theorem 1] or Barbu [1, Theorem 1] are satisfied and therefore, (4.1) possesses a unique solution u on $[0,T]$ for every $T > 0$ in the sense of the definition given following equation (3.2) above. Moreover, $u = \lim_{\lambda \to 0^+} u_\lambda$ in $L^1(0,T;X)$ (even in $L^2(0,T;X)$) for every $T > 0$, where u_λ is the unique solution of the approximating equation (3.4). We shall apply Theorem 5 with $P = L^2_+(\Omega)$. It is well known that the operator A defined by (4.2) satisfies condition (3.1). Therefore, if we require that condition (H_7) is satisfied - this will be the case. For example, if $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0,\infty)$ (see Remark 5.1), then the solution $u(t)$ of (4.1) is nonnegative a.e. on $(0,\infty)$.

Example 2. This example is an application of Theorem 4 (iii). Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. On Ω we consider the linear second order differential operator

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i u) + Cu$$

where $a_{ij}, a_i \in C^1(\bar{\Omega})$, $C \in L^\infty(\Omega)$,

$$C \geq 0,\ C + \sum_i \frac{\partial a_i}{\partial x_i} \geq 0 \quad \text{a.e.},$$

and for some positive constant α

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a.e.},\quad \xi \in \mathbb{R}^n .$$

We define $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. It is known (see [5]) that A satisfies (H_1) with $X = L^2(\Omega)$. Consider the equation

$$u(t) + a * Au(t) = u_0, \quad t \in [0,T], \tag{4.3}$$

where $u_0 \in L^2(\Omega)$ and where a satisfies assumptions (H_2), (H_4), (H_5) on $[0,T]$. Equation (4.3) has a unique weak solution u (see Remark 2.3); moreover, if $u_0 \in D(A)$, then the solution u is strong. Let j be a convex l.s.c. proper function: $\mathbb{R} \to [0,\infty]$ with $0 \in \partial j(0)$, and we fix $j(0) = 0$. Define $\varphi : X \to [0,\infty]$ by

$$\varphi(v) = \begin{cases} \int_\Omega j(v)dx & \text{if } j(v) \in L^1(\Omega) \\ +\infty & \text{otherwise} . \end{cases}$$

Then by [5, Lemma 2] we have $(A_\lambda x, y) \geq 0$ for every $[x, y] \in \partial\varphi$ and for all $\lambda > 0$. Moreover, by [3; Theorem 4.4] (1.21) is satisfied. Consequently, by Theorem 4(iii), if $j(u_0) \in L^1(\Omega)$, one has

$$\int_\Omega j(u(t))(x)dx \leq \int_\Omega j(u_0)(x)dx, \qquad t \in [0, T] .$$

In particular, if $j(u) = |u|^p$, $1 \leq p < \infty$, one obtains

$$\|u(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)}, \tag{4.4}$$

if $u_0 \in L^p(\Omega)$. Note that the case $p = \infty$ can be obtained by passing to the limit. Inequality (4.4) can be obtained directly from Theorem 1, inequality (1.6), if one uses the known that A satisfies (H_1) with $X = L^p(\Omega)$, $1 \leq p < \infty$ see [5; Theorem 8 and remarks preceding].

Example 3. This example is an application of the linear theory developed in Theorems 1-4 to a nonlinear problem. Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. Let $\gamma : \mathbb{R} \to \mathbb{R}$, $\gamma(0) = 0$, γ continuous and nondecreasing. Assume that the nonlinear elliptic equation

$$-\nabla^2 u = \gamma(u), \quad u|_\Gamma = 0 \tag{4.5}$$

has a nontrivial, positive solution $u_\infty \in L^\infty(\Omega)$. Let a satisfy (H_2), (H_4), (H_5), for every $T > 0$ and conside the nonlinear integral equation

$$\begin{cases} u(t) + a * (-\nabla^2 u - \gamma(u))(t) = u_0 & (0 \leq t < \infty), \\ u_0 \in L^\infty(\Omega), \quad u = 0 \text{ on } \Gamma . \end{cases} \tag{4.6}$$

Let $Au = -\nabla^2 u$ with $D(A) = \{u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)\}$. Let $X = L^2(\Omega)$. Then A satisfies (H_1). If u is a solution of (4.6) in the sense that $g = \gamma(u) \in L^\infty(0, \infty; X)$ and u is a weak solution (in the sense of Remark 2.3) of the equation

$$u(t) + a * Au(t) = u_0 + a * g(t), \quad t \in [0, T] \text{ a.e.}, \quad \forall T > 0 ,$$

then by Theorem 2 it is easily shown that u satisfies the nonlinear functional equation

$$u(t) = F_{u_0}(u)(t) \qquad (0 \leq t < \infty) , \tag{4.7}$$

where

$$F_{u_0}(u)(t) = S(t)u_0 + R * \gamma(u)(t) . \tag{4.8}$$

We prove the following result about solutions of (4.7), (4.8).

Theorem 6. Let (H_2), (H_4), (H_5) be satisfied for every $T > 0$. For every $u_0 \in X$ satisfying $0 \leq u_0 \leq u_\infty$, the equation (4.7), (4.8) has a positive maximal solution $u_M \in L^\infty(0,\infty;X)$ and a positive minimal solution $u_m \in L^\infty(0,\infty;X)$, such that if $u \in L^\infty(0,\infty;X)$ is any solution of (4.7), (4.8), then

$$0 \leq u_m(t) \leq u(t) \leq u_M(t) \leq u_\infty \quad \text{a.e. on} \quad (0,\infty) . \tag{4.9}$$

Remark 6.1. If $u \in L^\infty(0,\infty;X)$ is a solution of (4.7), (4.8), then it is easily checked that u is a solution of (4.6) in the sense defined above. Note that if the solution $u \in L^\infty(0,\infty;X)$ satisfies the estimate (4.9), then $u \in L^\infty(0,\infty;L^\infty(\Omega))$, and thus $\gamma(u) \in L^\infty(0,\infty;X)$, as well as $\gamma(u) \in L^1(0,T;X)$ for every $T > 0$. These observations are needed for the definition of weak solution.

Remark 6.2. Theorem 6 also holds if the requirement γ nondecreasing is replaced $\rho u + \gamma(u)$ nondecreasing for some $\rho > 0$. To see this replace $-\nabla^2 u$ by $-\nabla^2 u + \rho u$ and replace $\gamma(u)$ by $\rho u + \gamma(u)$ in (4.5), (4.6) and apply the above analysis.

Remark 6.3. Comparing equations (4.1) of Example 1 and (4.6) and taking $f(t) \equiv u_0$ in (4.1), $u_0 \in L^2_+(\Omega)$ we note that if β is single valued and continuous, equations (4.1) and (4.6) differ only by the sign of the nonlinearity. For equation (4.1) one has existence and uniqueness of solutions on $(0,\infty)$ for every $u_0 \in L^2_+(\Omega)$. By contrast, for equation (4.6) it is known that if equation (4.5) has $u = 0$ as the only nonnegative solution, then equation (4.6) can have a positive solution only on a finite interval $(0,T)$. For example, if $n = 3$ and $\gamma(u) = u^5$, it follows from [16, Remark 3.27] that if Ω is star shaped, then (4.5) has $u = 0$ as the only nonnegative solution. Taking $a(t) \equiv 1$, applying [10, Theorem 2.6 and Remark 2.7], and assuming that $u_0 \geq 0$ and that

$$\int_\Omega u_0(x)\phi_0(x)dx \geq \lambda_0^{1/4} ,$$

where λ_0 is the smallest eigenvalue and the corresponding unique eigenfunction $\phi_0 > 0$ in Ω:

$$-\nabla^2 \phi_0 = \lambda_0 \phi_0 \quad \text{in} \quad \Omega, \quad \phi_0 \Big|_\Gamma = 0 ,$$

then the unique nonnegative solution u of (4.6) exists only on a finite interval.

<u>Proof of Theorem 6</u>. Let $E = L^\infty(0,\infty;X)$ with the usual ordering (i.e. $u, v \in E$, $u \leq v \iff \tilde{u}(t,x) \leq \tilde{v}(t,x)$ a.e. in $(t,x) \in (0,\infty) \times \Omega$, where $\tilde{u}$ and $\tilde{v}$ are elements of the equivalence classes u and v respectively). In E let I denote the interval $[0, u_\infty]$ in the sense of order in E. It can be shown that I is a complete lattice with respect to this ordering. For every $u_0 \in I$ we define the function $\tilde{F}_{u_0}$ by

$$\tilde{F}_{u_0}(u)(t) = S(t)u_0 + R * \tilde{\gamma}(u)(t)$$

where

$$\tilde{\gamma}(u) = \begin{cases} \gamma(u) & \text{if} \quad \mu \leq \|u_\infty\|_{L^\infty(\Omega)} \\ \|u_\infty\|_{L^\infty(\Omega)} & \text{otherwise} , \end{cases}$$

in place of the function F_{u_0} defined by (4.8). Then $\check{F}_{u_0}$ satisfies

(4.10) $$\tilde{F}_{u_0} : I \to I$$

and

(4.11) $\tilde{F}_{u_0}$ is monotone $(u, v \in I$ and $u \leq v \Rightarrow \tilde{F}_{u_0}(u) \leq \tilde{F}_{u_0}(v))$.

Let $u \in I$. Then, by Theorems 3 and 4, $\tilde{F}_{u_0}(u) \geq 0$. Moreover, by the fact that

$$u_\infty = Su_\infty + R * \gamma(u_\infty), \quad \text{we have}$$

$\tilde{F}_{u_0}(u) = Su_0 + R * \tilde{\gamma}(u) \leq Su_0 + R * \gamma(u_\infty) = S(u_0 - u_\infty) + u_\infty \leq u_\infty$; which proves (4.10).

Clearly, (4.11) is evident from Theorem 3. By [2, Theorem 11, p. 115], the operator $\tilde{F}_{u_0}$ has a least and a greatest fixed point in I, which correspond respectively to the solutions u_m and u_M, since $u_m \leq u_M \leq u_\infty$ and therefore, $\tilde{\gamma}(u_m) = \gamma(u_m)$, $\tilde{\gamma}(u_M) = \gamma(u_M)$, and so $\tilde{F}_{u_0}(u_m) = F_{u_0}(u_m)$, $\tilde{F}_{u_0}(u_M) = F_{u_0}(u_M)$. This completes the proof of Theorem 6.

Appendix 1

An assumption which has been used frequently in the literature concerning the kernel a is

$$(A_1) \quad \begin{cases} a(t) \in C(0,T), \ a(t) > 0, \ t \in (0,T), \ \text{and} \\ \dfrac{a(t)}{a(t+\sigma)} \ \text{nonincreasing as a function of } t \text{ for each} \\ \qquad \sigma > 0, \ 0 < t + \sigma < T, \end{cases}$$

see Friedman [7], Levin [12], Miller [14]. We shall prove that condition (A_1) is equivalent to the condition

(A_2) $\quad a(t) \in C(0,T), \ a(t) > 0 \quad t \in (0,T)$ and $\log a(t)$ convex on $(0,T)$.

Moreover, we first prove a preliminary result.

Lemma 1. *Let assumption (A_2) be satisfied. Then for every $\nu > 0$, there exists a function a_ν satisfying (A_2) and $a_\nu \in C^1[0,T]$, and $a_\nu(t) \uparrow a(t)$ as $\nu \downarrow 0^+$ for $t \in (0,T)$.*

Proof. Define $b : \mathbb{R} \to (-\infty, +\infty]$ by

$$b(t) = \begin{cases} \log a(t) \quad \text{if } t \in (0,T) \\ b(0) = \lim\limits_{t \to 0^+} \log a(t), \ b(T) = \lim\limits_{t \to T^-} \log a(t) \\ +\infty \quad \text{if } t \notin [0,T]. \end{cases}$$

Observe that $a(t) > 0$ on $(0,T)$ and the definition of convexity of $\log a(t)$ on $(0,T)$ excludes $a(0^+) = 0$ and $a(T^-) = 0$. Thus b is convex, l.s.c. and proper on $\mathbb{R}$. Define b, $\nu > 0$, to be the Yosida approximation of b; then, see [3; Proposition 2.11],

$$b_\nu(t) = \min_{y \in \mathbb{R}} \{\frac{1}{2\nu} |y - t|^2 + b(y)\}, \quad t \in \mathbb{R},$$

and $b_\nu \in C^1(\mathbb{R})$, b'_ν satisfies a Lipschitz condition on $\mathbb{R}$ with constant $\frac{1}{\nu}$; moreover $b_\nu(t) \uparrow b(t)$ as $\nu \downarrow 0^+$, $t \in \mathbb{R}$. Define $a_\nu = e^{b_\nu}$ and the result follows. This completes the proof of Lemma 1. Using Lemma 1 we shall prove

Lemma 2. *The conditions (A_1) and (A_2) are equivalent.*

Proof. That $(A_1) \Rightarrow (A_2)$ follows from

$$\frac{a(t)}{a(t+\sigma)} \geq \frac{a(t+\tau)}{a(t+\tau+\sigma)} \quad (0 < t < t+\sigma, \ t+\tau < t+\sigma+\tau < T);$$

using $a(t) > 0$ and putting $\sigma = \tau$ we obtain

$$a(t)a(t + 2\tau) \geq a^2(t + \tau) .$$

Thus putting $t_1 = t$, $t_2 = t + 2\tau$ we have

$$\log a\left(\frac{t_1 + t_2}{2}\right) \leq \frac{1}{2} \log a(t_1) + \frac{1}{2} \log a(t_2) .$$

We note that in [12; calculation following Theorem 1.3] it is only shown that $(A_1) \Rightarrow a(t)$ convex, with the additional assumption that a is nonincreasing, which is not used. Of course, $\log a(t)$ convex implies $a(t)$ convex.

To prove that $(A_2) \Rightarrow (A_1)$, it is sufficient by Lemma 1 to prove $(A_2) \Rightarrow (A_1)$ with the additional assumption $a \in C'[0, T]$. Then $\log a(t)$ convex implies

$$\frac{a'(t)}{a(t)} \leq \frac{a'(t + \sigma)}{a(t + \sigma)} \qquad (0 < t < t + \sigma < T) .$$

Using $a(t) > 0$ we then have

$$\frac{d}{dt} \frac{a(t)}{a(t + \sigma)} = \frac{a(t + \sigma)a'(t) - a'(t + \sigma)a(t)}{a^2(t + \sigma)} \leq 0 ,$$

which completes the proof of Lemma 2.

Proof of Proposition 1. By Lemma 2 it is sufficient to prove Proposition 1(i) under assumptions (A_1) and (H_2). If, in addition, $a \in C[0, T]$, Proposition 1(i) follows directly from [14, Theorem 1] with $h = f = a$ and $g(x, t) = x$.

Let a satisfy assumptions (A_1) and (H_2). Consider the functions a_ν of Lemma 1. Then by the above remark, the functions $r_\nu(t, \lambda) \geq 0$, $t \in [0, T]$, for every $\lambda > 0$, $\nu > 0$, where $r_\nu(t, \lambda)$ is the resolvent kernel associated with $\lambda a_\nu(t)$. The functions a_ν converge to a in $L^1(0, T)$ as $\nu \downarrow 0^+$, since $a_\nu(t) \uparrow a(t)$ as $\nu \downarrow 0^+$ and $a \in L^1(0, T)$. Therefore, it is easily checked that the functions $r_\nu(\cdot, \lambda)$ converge to $r(\cdot, \lambda)$ in $L^1(0, T)$, where $r(t, \lambda)$ is the resolvent kernel corresponding to $\lambda a(t)$, and $r(t, \lambda) \geq 0$ on $[0, T]$ a.e. This completes the proof of part (i).

Part (ii) is proved in [8; Lemma 2.5] with $h = \lambda a$ (see also [12; Lemma 1.3] with $f \equiv 1$), where the proof is carried out on $(0, \infty)$; this can be applied by extending $a(t)$ as a constant on $[T, \infty)$. This completes the proof of Proposition 1.

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Let X be a real or complex Banach space. We study the Volterra equation

$$(v) \qquad u(t) + \int_0^t a(t-s)Au(s)ds = f(t) \qquad (0 \le t \le T,\ T > 0),$$

where a is a given kernel, A is a bounded or unbounded linear operator from X to X, and f is a given function with values in X (of particular

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