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BERNSTEIN'S INEQUALITY IN L^p FOR $0 < p < 1$

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ABSTRACT

Let $0 < p < 1$ and T_n be a trigonometric polynomial of order n. Then

$$\int_{-\pi}^{\pi} |T_n'(t)|^p dt \leq \frac{4e}{p} n^p \int_{-\pi}^{\pi} |T_n(t)|^p dt .$$

A similar inequality is established for algebraic polynomials in weighted L_p spaces.

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BERNSTEIN'S INEQUALITY IN L^p FOR $0 < p < 1$

Paul G. Nevai

One of the most powerful tools in approximation theory is Bernstein's inequality:

$$\int_{-\pi}^{\pi} |T_n'(t)|^p dt \le C(p) n^p \int_{-\pi}^{\pi} |T_n(t)|^p dt \tag{1}$$

which holds for $1 \le p \le \infty$ with $C(p) = 1$. Here T_n is an arbitrary trigonometric polynomial of order n.

The main purpose of the present note is to prove the following

<u>Theorem 1</u>. Let $0 < p < 1$. Then Bernstein's inequality (1) is satisfied with $C(p) = 4ep^{-1}$.

<u>Proof</u>. Let $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$. Then $|D_n(x)| \le D_n(0) = 2n+1$, $|D_n'(x)| \le n(n+1)$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n^2(t) dt = 2n+1 .$$

If T_n is a trigonometric polynomial of order n then convolving T_n with D_n we get T_n, that is $T_n = T_n * D_n$. Hence $T_n' = T_n * D_n'$. Therefore we have the following two inequalities

$$|T_n(x)| \le \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)| dt \tag{2}$$

and

$$|T_n'(x)| \le \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} |T_n(t)| dt . \tag{3}$$

Now let $0 < p < 1$. We obtain from (2)

$$\max_{|x| \le \pi} |T_n(x)| \le \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt \max_{|x| \le \pi} |T_n(x)|^{1-p} ,$$

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that is

$$|T_n(x)|^p \le \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt .$$

Thus by (3)

$$|T_n'(x)| \le \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt \Big[\max_{|x| \le \pi} |T_n(x)|^p \Big]^{\frac{1-p}{p}} \le$$

$$\le \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt (2n+1)^{\frac{1-p}{p}} \Big[\frac{1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt \Big]^{\frac{1-p}{p}} .$$

Hence

$$|T_n'(x)|^p \le n^p (n+1)^p (2n+1)^{1-p} \frac{1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p dt .$$

Now comes the trick. Let $k = [\frac{2}{p}] + 1$ and put here $T_n(t) D_n^k(x-t)$ instead of T_n. $T_n D_n^k$ is of order $n(k+1)$. Therefore

$$|T_n'(x)(2n+1)^k - k(2n+1)^{k-1} D_n'(0) T_n(x)|^p \le$$

$$\le n^p (k+1)^p [n(k+1)+1]^p [2n(k+1)+1]^{1-p} \frac{1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p |D_n(x-t)|^{kp} dt .$$

Using $D_n'(0) = 0$ and $kp \ge 2$ we get

$$|T_n'(x)|^p \le n^p (k+1)^p (2n+1)^{-2} [n(k+1)+1]^p [2n(k+1)+1]^{1-p} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|^p D_n^2(x-t) dt .$$

Integrating this inequality we obtain

$$\int_{-\pi}^{\pi} |T_n'(x)|^p dx \le n^p (k+1)^p (2n+1)^{-1} [n(k+1)+1]^p [2n(k+1)+1]^{1-p} \cdot \int_{-\pi}^{\pi} |T_n(t)|^p dt .$$

Let m be a natural integer. Apply this inequality to $T_n(mx)$ instead of $T_n(x)$, divide by m^p and let $m \to \infty$. The result is

$$\int_{-\pi}^{\pi} |T_n'(x)|^p dx \le (k+1)^{1+p}2^{-p}n^p \int_{-\pi}^{\pi} |T_n(t)|^p dt .$$

Recall that $k \le 2p^{-1} + 1$. Thus the theorem follows.

Let us note that it would be of definite interest to find the exact value of the constant factor $C(p)$. There are several consequences and possible generalizations of our result.

In the following we will establish weighted Bernstein inequalities for algebraic polynomials. Denote $p_n(\alpha, \beta, x) = \gamma_n(\alpha, \beta)x^n + \cdots$ the orthonormed Jacobi polynomials and let

$$K_n(\alpha, \beta, x) = \sum_{j=0}^{n-1} p_j^2(\alpha, \beta, x) .$$

<u>Lemma 2</u>. Let $\alpha > -1$, $\beta > -1$, $\gamma > -1$, $k = 0, 1, \ldots$, $\ell = 0, 1, \ldots$, $m = 0, 1, \ldots$ and $0 < \varepsilon < 1$ be fixed. Put

$$P(x) = n^{-2}x^k(1-x)^\ell(1+x)^m K_n(\alpha, \beta, x)K_n(-\tfrac{1}{2}, \gamma, 2x^2 - 1) \tag{4}$$

for $n = 1, 2, \ldots$. Then

$$|P'(x)| \le C_1 |x|^{-1}(1-x^2)^{-1}|P(x)| \tag{5}$$

for $|x| \le 1$ and

$$0 < C_2 \le |P(x)|\,|x|^{-k+2\gamma+1}(1-x)^{-\ell+\alpha+\frac{1}{2}}(1+x)^{-m+\beta+\frac{1}{2}} \le C_3 < \infty \tag{6}$$

for $\varepsilon n^{-1} \le |x| \le 1 - \varepsilon n^{-2}$ where C_1, C_2 and C_3 do not depend on x and n.

<u>Proof</u>. First let us calculate $K_n'(\alpha, \beta, x)$. By the Christoffel-Darboux formula we have

$$K_n(\alpha, \beta, x) = \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} [p_n'(\alpha, \beta, x)p_{n-1}(\alpha, \beta, x) - p_{n-1}'(\alpha, \beta, x)p_n(\alpha, \beta, x)] .$$

Hence

$$K_n'(\alpha, \beta, x) = \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} [p_n''(\alpha, \beta, x)p_{n-1}(\alpha, \beta, x) - p_{n-1}''(\alpha, \beta, x)p_n(\alpha, \beta, x)] .$$

Note that $p_n(\alpha, \beta, x)$ satisfies the differential equation

$$(1 - x^2)Y'' = -n(n + \alpha + \beta + 1)Y + [\alpha - \beta + (\alpha + \beta + 2)x]Y' .$$

Therefore we obtain

$$K'_n(\alpha, \beta, x) = \frac{\alpha - \beta + (\alpha + \beta + 2)x}{1 - x^2} K_n(\alpha, \beta, x) - \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} \frac{2n + \alpha + \beta}{1 - x^2} p_{n-1}(\alpha, \beta, x) p_n(\alpha, \beta, x) .$$

It has been shown in [2] that

$$n|p_{n-1}(\alpha, \beta, x)p_n(\alpha, \beta, x)| \le \text{const } K_n(\alpha, \beta, x)$$

for $|x| \le 1$. Thus

$$|K'_n(\alpha, \beta, x)| \le \text{const}(1 - x^2)^{-1} K_n(\alpha, \beta, x)$$

for $|x| \le 1$ which yields (5) by a simple computation. Concerning (6) see e.g. [2], § 6.3.

Lemma 3. Let $\alpha > -1$, $\beta > -1$, $\gamma > -1$ and $0 < p < \infty$. Then there exists a number $\delta > 0$ such that for every polynomial π_n of degree at most n

$$\int_{-1}^{1} |\pi_n(t)|^p (1 - t)^\alpha (1 + t)^\beta |t|^\gamma dt \le 2 \int_{\frac{\delta}{n} \le |t| \le 1 - \frac{\delta}{n^2}} |\pi_n(t)|^p (1 - t)^\alpha (1 + t)^\beta |t|^\gamma dt .$$

This lemma has been proved in [2], § 6.3.

Lemma 4. Let $0 < p < \infty$, $0 < \varepsilon < 1$. Let a, b and c be given real numbers. Then there exist two constants $\delta > 0$ and C_4 such that for every polynomial π_n of degree at most n the inequality

$$\int_{\frac{\varepsilon}{n} \le |t| \le 1 - \frac{\varepsilon}{n^2}} |\pi'_n(t)\sqrt{1 - t^2}|^p (1 - t)^a (1 + t)^b |t|^c dt \le C_4 n^p \int_{\frac{\delta}{n} \le |t| \le 1 - \frac{\delta}{n^2}} |\pi_n(t)|^p (1 - t)^a (1 + t)^b |t|^c dt$$

holds.

Proof. If $a = b = -\frac{1}{2}$ and $c = 0$ then the lemma follows from Bernstein's inequality ($1 \le p < \infty$), Theorem 1 ($0 < p < 1$) and Lemma 3. Otherwise we choose α, β, γ, k, ℓ and m so that they satisfy the conditions of Lemma 2, further $a = p(\ell - \alpha - \frac{1}{2}) - \frac{1}{2}$, $b = p(m - \beta - \frac{1}{2}) - \frac{1}{2}$ and $c = p(k - 2\gamma - 1)$. Let P be defined by (4). Then $P\pi_n$ is of degree $5n + k + \ell + m = O(n)$. Applying the case $a = b = -\frac{1}{2}$, $c = 0$ to $P\pi_n$ instead of π_n we easily obtain the lemma.

Lemmas 3 and 4 combined give us the following

Theorem 5. Let $0 < p < \infty$. Let $1 = x_1 > x_2 > \ldots > x_N = -1$, $\gamma_i > -1$ and $\Gamma_i \in \mathbb{R}$ for $i = 1, 2, \ldots, N$. Let

$$W(t) = \prod_{i=1}^{N} |t - x_i|^{\gamma_i}$$

and

$$W_n(t) = \left(\sqrt{1-t} + \frac{1}{n}\right)^{2\Gamma_1} \prod_{i=2}^{N-1} \left(|t - x_i| + \frac{1}{n}\right)^{\Gamma_i} \left(\sqrt{1+t} + \frac{1}{n}\right)^{2\Gamma_N}.$$

Then for every polynomial π_n of degree at most n

$$\int_{-1}^{1} |\pi_n'(t)\sqrt{1-t^2}|^p W_n(t) W(t)\,dt \le C_5 n^p \int_{-1}^{1} |\pi_n(t)|^p W_n(t) W(t)\,dt$$

where C_5 is independent of n.

Let us remark that Theorem 5 is new only for $0 < p < 1$. For $1 \le p < \infty$ it was proved in [2]. Even for the case $1 \le p < \infty$ the present proof is much simpler than that in [2]. There is an extensive literature dealing with $N = 2$, that is when W is a Jacobi weight. We refer the reader to [1] where a great number of works on weighted Bernstein inequalities is mentioned in the references.

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A similar inequality is established for algebraic polynomials in weighted L_p spaces.

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