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TO DIFFERENTIAL INCLUSIONS

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Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

March 1977

(Received November 29, 1976)



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MONOTONE INVARIANT SOLUTIONS TO DIFFERENTIAL INCLUSIONS

Frank H. Clarke[†] and J.-P. Aubin[‡]

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ABSTRACT

Let a given set be endowed with a preference preordering, and consider the problem of finding a solution to the differential inclusion

$$\dot{x}(t) \in S(x(t))$$

which remains in the given set and evolves monotonically with respect to the preordering. We give sufficient conditions for the existence of such a trajectory, couched in terms of a notion of tangency developed by Clarke. No smoothness or convexity is involved in the construction, which uses techniques of Filippov.

AMS (MOS) Subject Classifications: Primary 49E10; Secondary 34H05, 34A10

Key Words: invariant trajectory, monotone trajectory, preference ordering, existence, differential inclusion, generalized gradients, generalized tangent cone

Work Unit Number 1 (Applied Analysis)

[†]Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada V6T 1W5.

[‡]Mathématiques de la Décision, Université de Paris IX, 75775 Paris, Cedex 16, France.

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MONOTONE INVARIANT SOLUTIONS TO DIFFERENTIAL INCLUSIONS

Frank H. Clarke[†] and J.-P. Aubin[‡]

1. Statement of the problem

Let X be a compact subset of $\mathbb{R}^n$, S a multifunction (set-valued function) from X into $\mathbb{R}^n$ with non-empty compact values. We regard X as the state set of a dynamical system and $S(x)$ as the set of feasible velocities of the system when its state is x . We introduce a preordering " $y \succ x$ " (y is better than x) on X (i.e., a relation which is both reflexive and transitive).

Let $[0, T]$ be any finite interval ($T > 0$). We say that an absolutely continuous function x from $[0, T]$ into $\mathbb{R}^n$ is a "monotone invariant trajectory" for S starting at $x_0 \in X$ if

$$(1.1) \quad \left\{ \begin{array}{l} \text{i) } \frac{dx}{dt} \in S(x(t)) \text{ for almost all } t \text{ in } [0, T] \\ \text{ii) } x(0) = x_0 \\ \text{iii) } x(t) \in X \text{ for all } t \in [0, T] \\ \text{iv) if } t \geq s, x(t) \text{ is better than } x(s). \end{array} \right.$$

Our main result which we state in this section, gives reasonable sufficient conditions implying the existence of at least one monotone invariant trajectory. In §2 we discuss a notion of tangency for arbitrary

[†]Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada V6T 1W5.

[‡]Mathématiques de la Décision, Université de Paris IX, 75775 Paris, Cedex 16, France.

closed sets [5] which is central in our results, while § 3 is devoted to the proof of the main theorem. We define the set

$$P(x) = \{y \in X \text{ such that } y \succ x\}$$

of elements y better than x . The multifunction P satisfies the properties:

$$(1.2) \quad \begin{cases} \text{i) } \forall x \in X, x \in P(x) \\ \text{ii) } \forall x \in X, \forall y \in P(x), P(y) \subset P(x) . \end{cases}$$

(Conversely, if P is a multifunction from X to X satisfying (1.2), the relation $y \succ x$ defined by $y \in P(x)$ is a preordering on X .) We recall that P is said to be Lipschitz if and only if there exists $L > 0$ such that $P(x) \subset P(y) + L\|x - y\|B$, where B is the unit ball. We recall also that S is upper (resp. lower) semicontinuous if for any $\varepsilon > 0$, there exists a neighborhood $N_\varepsilon(x)$ of x such that,

$$\forall y \in N_\varepsilon(x), S(y) \subset S(x) + \varepsilon B \text{ (resp. } S(x) \subset S(y) + \varepsilon B \text{)} .$$

We will define in the next section the tangent cone $T(P(x);x)$ to the set $P(x)$ at x .

Theorem 1. Let us assume that P satisfies (1.2), that X is compact,
that the images $S(x)$ and $P(x)$ are non-empty and compact and that

$$\begin{cases} \text{i) } S \text{ is continuous} \\ \text{ii) } P \text{ is Lipschitz.} \end{cases}$$

Let us suppose also that

$$(1.3) \quad \forall x \in X, S(x) \subset T(P(x);x) .$$

Then, for any $x_0 \in X$, $v_0 \in S(x_0)$, and $T > 0$, there exists a monotone invariant trajectory starting at x_0 and satisfying

$$\begin{cases} \text{i) } \frac{dx}{dt}(0) = v_0 \\ \text{ii) } \frac{dx}{dt} \text{ is regulated (i.e., is a uniform limit of step functions) .} \end{cases}$$

Remark 1.1. In [2], the following was proved: if $X, S(x)$ and $P(x)$ are non-empty, convex and compact, if S is upper semicontinuous and P is continuous, and if $S(x) \cap T(P(x);x) \neq \emptyset$ for all $x \in X$, then there exists a monotone invariant trajectory.

Remark 1.2. If we neglect the monotonicity requirement (1.1) (iv), (i.e. set $P(x) = X$ for each x) a solution of (1.1) (i), (ii), (iii) is called an invariant trajectory. When S is a continuous function, existence of invariant trajectories was obtained by Nagumo [12], Crandall [7], Martin [10], [11], Yorke [15], Hartman [9]; for S a Lipschitz function, see Brezis [4], Bony [3], Redheffer [14]; for S a Lipschitz multi-function see Clarke [5]. Theorem 1 implies the existence of (at least) one invariant trajectory when S is a continuous multifunction; local existence of the differential inclusion (1.1) (i), (ii) was proved by Filippov [8]. We make use of the techniques introduced in this paper. See also Antosiewicz-Cellina [1], Olech [13] and the references of the latter paper.

2. Tangent Cones

We recall that any locally Lipschitz function $f: \mathbb{R}^l \rightarrow \mathbb{R}$ has a generalized directional derivative, defined in [5], [6] by

$$(2.1) \quad f^0(x;v) = \limsup_{\substack{y \rightarrow x \\ \theta \rightarrow 0+}} [f(y + \theta v) - f(y)]/\theta$$

which is convex positively homogeneous and continuous with respect to v . It coincides with $\langle \nabla f(x), v \rangle$ if f is continuously differentiable and with the derivative from the right if f is convex and continuous. By definition, the generalized gradient $\partial f(x)$ of f at x is the set of $p \in \mathbb{R}^{l*}$ such that

$$\langle p, v \rangle \leq f^0(x;v) \text{ for all } v \text{ in } \mathbb{R}^l.$$

It is a non-empty convex compact set whose support function is $f^0(x; \cdot)$.

Let X be a closed subset of $\mathbb{R}^l$. The distance function

$$y \mapsto d(X;y) = \inf_{z \in X} \|y - z\|, \text{ being a Lipschitz function, admits a}$$

generalized directional derivative $d^0(X;y;v)$ and a generalized gradient

$\partial d(X;y)$ at any point y . If $y \in X$, we shall say that the normal cone

$N(X;y)$ to X at y is the closed convex cone spanned by $\partial d(X;y)$.

We define the tangent cone $T(X;y)$ to X at y to be the (negative) polar cone of $N(X;y)$. Thus

$$(2.2) \quad T(X;y) = \{v \in \mathbb{R}^l \text{ such that } d^0(X;y,v) \leq 0\}.$$

If the interior $\text{int } X$ of X is non-empty and if $y \in \text{int } X$, then

$T(X;y) = \mathbb{R}^l$. When X is a closed convex set, or a smooth manifold, these definitions coincide with the usual ones (see [5]). We recall the following characterization of the tangent cone (see [5, p. 256]).

Proposition 1. A vector v belongs to $T(X;y)$ if and only if

$$(2.3) \quad \lim_{\substack{z \in X \\ z \rightarrow y}} \liminf_{\lambda \rightarrow 0+} d(X;z + \lambda v)/\lambda = 0 .$$

This implies obviously that

$$(2.4) \quad \liminf_{\lambda \rightarrow 0+} d(X;y + \lambda v)/\lambda = 0 ,$$

but this condition does not necessarily imply that $v \in T(X;y)$.

Remark 2.1. It is easy to show from the above (see [5, Proposition 3.7])

that the tangent cone may be defined directly as follows: v belongs to $T(X;x)$ iff for every sequence $\{x_n\}$ in X converging to x and every sequence $\{\lambda_n\}$ of positive numbers converging to 0, there exists a sequence $\{v_n\}$ converging to v such that $x_n + \lambda_n v_n$ belongs to X infinitely often.

Lemma 1. For any $x_0 \in X$ and $v_0 \in R^l$, there exist neighborhoods $N(x_0)$ of x_0 , $M(v_0)$ of v_0 and $h(x_0, v_0) > 0$ such that

$$(2.5) \quad d(X;x + hv)/h \leq d^0(X;x_0;v_0) + \varepsilon$$

when $x \in N(x_0)$, $v \in M(v_0)$ and $0 < h \leq h(x_0, v_0)$.

Proof. Let us set $g(t) = d(X;x + tv)$. Since $g(0) = d(X;x) = 0$ when $x \in X$, we can write

$$(2.6) \quad d(X;x + hv)/h = \frac{1}{h} \int_0^h g'(t)dt .$$

[Indeed g , being a Lipschitz function of t , is differentiable almost everywhere]. But we have

$$(2.7) \quad g'(t) \leq d^0(X; x + tv; v)$$

since, if g is differentiable at t

$$\begin{aligned} g'(t) &= \lim_{\theta \rightarrow 0+} [d(X; x + tv + \theta v) - d(X; x + tv)] / \theta \\ &\leq \lim_{\theta \rightarrow 0+} \sup_{y \rightarrow x + tv} [d(X; y + \theta v) - d(X; y)] / \theta = d^0(X; x + tv; v) . \end{aligned}$$

Since the function $(y, v) \mapsto d^0(X; y; v)$ is obviously upper semicontinuous, we can associate with any $\varepsilon > 0$, a neighborhood $N(x_0)$ of x_0 , $M(v_0)$ of v_0 and a number $h(x_0, v_0) > 0$ such that when $x \in N(x_0)$, $v \in M(v_0)$ and $t \leq h(x_0, v_0)$, then

$$(2.8) \quad d^0(X; x + tv; v) \leq d^0(X; x_0; v_0) + \varepsilon .$$

The result then follows from this, combined with (2.7) and (2.6). Q.E.D.

Let a multifunction S from X to R^l be given. We shall now consider various consequences of the hypothesis that $S(x)$ is contained in $T(X; x)$ for each x (this is (1.3) in the case $P(x) = X$ for each x ; the connection to Theorem 1 is made in Proposition 4 below). Besides developing some machinery that will be needed in § 3, the relationships between the various hypotheses made in the papers on invariant trajectories cited above will be clarified (see Corollary 1).

Proposition 2. Let us assume that S is upper semicontinuous and that

$$\forall x \in X, S(x) \subset T(X; x) .$$

Then the function

$$a(x, h) = \sup_{v \in S(x)} d(X; x + hv)/h$$

converges to 0 with h uniformly for x in compact subsets ($h > 0$).

Proof. Let K be a compact subset of X . Since S is upper semi-continuous, the graph $G_K(S)$ of the restriction of S to K is compact.

Consequently, it can be covered by a finite number p of neighborhoods

$N(x_i) \times M(v_i)$, where $(x_i, v_i) \in G_K(S)$ and the neighborhoods N and

M have the properties of the preceding lemma. If we set $h(K, \epsilon)$

$$= \min_{i=1, \dots, p} h(x_i, v_i), \text{ we deduce that}$$

$$\forall x \in K, \forall v \in S(x), \forall h < h(K, \epsilon), d(X; x + hv)/h \leq \epsilon.$$

So the proposition is proved. Q.E.D.

For any $y \in R^l$, let $\pi(y)$ denote the set of closest points in X to y .

Proposition 3. If we assume that

$$(2.9) \quad \forall x \in X, \forall v \in S(x), \liminf_{h \rightarrow 0+} d(X; x + hv)/h = 0,$$

then

$$(2.10) \quad \forall y \in R^l, \forall x \in \pi(y), \forall v \in S(x), \langle y - x, v \rangle \leq 0.$$

Conversely, if we assume that S is lower semicontinuous property

(2.10) implies that $S(x) \subset T(X; x)$ for all $x \in X$.

Remark. There are examples where (2.10) does not imply (2.9).

Proof. Suppose that (2.10) is false. There exist $y \in R^l$, $x \in \pi(y)$ and $v \in S(x) \cap T(X; x)$ such that $\langle y - x, v \rangle > 0$. Let C be the complement of the open ball of center y and radius $d(X; y)$. Then

$d(X; x + tv) \geq d(C; x + tv)$ and, since $\langle y - x, v \rangle > 0$, $d(C; x + tv) \geq \delta t + o(t)$

where $\delta > 0$. Hence $\liminf_{\theta \rightarrow 0+} d(X; x + \theta v)/\theta \geq \delta > 0$; this contradicts

the fact that $v \in T(X; x)$. Conversely, let us prove that if S is

continuous, then (2.10) implies that $S(x) \subset T(X; x)$. By [5, p. 254],

we know that any element of $N(X; x)$ can be written as a convex combina-

tion of elements $p = \lim_{n \rightarrow \infty} s_n(y_n - z_n)$ where $s_n > 0$, $y_n \rightarrow x$ and

$z_n \in \pi(y_n)$. Since $x \in X$, we have $\|x - z_n\| \leq \|x - y_n\| + \|y_n - z_n\|$

$\leq 2\|x - y_n\|$. Since S is lower semicontinuous, any $v \in S(x)$ is

the limit of some sequence of elements $v_n \in S(z_n)$. So

$\langle p, v \rangle = \lim_{n \rightarrow \infty} s_n \langle y_n - z_n, v_n \rangle \leq 0$. This implies that $v \in N(X; x)^- = T(X; x)$. ■

Corollary 1. Let us assume that

$$(2.11) \quad \left\{ \begin{array}{l} \text{i) } X \text{ is a compact set.} \\ \text{ii) } S \text{ is a continuous multifunction with non-empty} \\ \text{compact values.} \end{array} \right.$$

Then the following statements are equivalent

$$(a) \quad \forall x, S(x) \subset T(X; x)$$

$$(b) \quad \forall x \in X, \forall v \in S(x), \lim_{\substack{y \rightarrow x \\ y \in X}} \liminf_{\theta \rightarrow 0+} d(X; y + \theta v)/\theta = 0$$

$$(c) \quad \lim_{h \rightarrow 0} \sup_{\substack{x \in X \\ v \in S(x)}} d(X; x + hv)/h = 0$$

$$(d) \quad \forall x \in X, \forall v \in S(x), \lim_{h \rightarrow 0+} d(X; x + hv)/h = 0$$

$$(e) \quad \forall x \in X, \forall v \in S(x), \liminf_{h \rightarrow 0+} d(X; x + hv)/h = 0$$

$$(f) \quad \forall y \in R^l, \forall x \in \pi(y), \forall v \in S(x), \langle y - x, v \rangle \leq 0.$$

Proof. That each condition implies the next, and that (f) implies (a), is either self-evident or a consequence of the preceding results. Q.E.D.

Remark 2.2. Conditions (a) and (b) are found in [5], while (d) appears in [4], [7], [9], (e) in [14], [15], [12] and (f) in [3], [7].

Let us consider now another multifunction P mapping X into X .

Proposition 4. Let X be a compact subset of R^n .

Let us assume that S is a continuous multifunction with non-empty compact values and that P is a closed multifunction satisfying (1.2).

If we assume

$$\forall x \in X, S(x) \subset T(P(x); x),$$

then

$$\lim_{h \rightarrow 0+} \sup_{\substack{x \in X \\ v \in S(x)}} d(G(P); (x, x + hv)) / h = 0$$

where $d(G(P); (\cdot, \cdot))$ is the distance to the graph $G(P)$ of P .

Proof. Let $\tilde{S} : G(P) \rightarrow R^{2l}$ be the continuous multifunction defined as follows:

$$\tilde{S}(x, w) = \{0\} \times S(w).$$

Let (x, w) be a point in $G(P)$, and let $(0, v)$ belong to $\tilde{S}(x, w)$.

We claim that

$$(2.12) \quad \liminf_{h \rightarrow 0+} d(G(P); (x, w + hv)) / h = 0.$$

To see this, note that since x belongs to $P(x)$,

$$d(G(P); (x, w + hv)) \leq d(P(x); w + hv).$$

Also, $w \in P(x)$ implies $P(w) \subset P(x)$, so that in turn

$$d(P(x); w + hv) \leq d(P(w); w + hv) .$$

However,

$$\liminf_{h \rightarrow 0+} d(P(w); w + hv)/h = 0$$

by (2.4), since $v \in S(w) \subset T(P(w); w)$. Thus (2.12) ensues. We now apply Corollary 1 (with X replaced by $G(P)$ and S by $\tilde{S}$) and deduce that condition (c) of that result holds. But that implies the desired result, since $(x, x) \in G(P)$. Q.E.D.

3. Proof of the Theorem

By Proposition 4, we know that our assumptions imply that

$$b(h) = \sup_{\substack{x \in X \\ v \in S(x)}} d(G(p); (x, x + hv))/h$$

converges to 0 with h . We will use this fact instead of assumption

(1.3) in the proof of the theorem.

We consider a decreasing sequence of partitions $\rho(h_m)$ of $[0, T]$ made of intervals $[qh_m, (q+1)h_m]$ where q is an integer and where

$$(3.1) \quad T/h_1 \text{ and } \frac{h_{m-1}}{h_m} \text{ are integers.}$$

We shall denote by $\tau_m = qh_m$ (where $q \in \mathbb{N}$) any node of the partition $\rho(h_m)$. We shall construct a sequence of piecewise linear functions x_m on the partition $\rho(h_m)$ defined by

$$(3.2) \quad \forall t \in [\tau_m, \tau_m + h_m], x_m(t) = x_m(\tau_m) + (t - \tau_m)v_m(\tau_m)$$

whose derivative $\dot{x}_m(t)$ is a step function:

$$(3.3) \quad \forall t \in [\tau_m, \tau_m + h_m[, \dot{x}_m(t) = v_m(\tau_m).$$

For that purpose, we define the map $j : \tau_m \rightarrow j(\tau_m)$ associating with any node $\tau_m \in \rho(h_m)$ the smallest index $j = j(\tau_m)$ such that $\frac{\tau_m}{h_{j+1}}$ is an integer. [Indeed, if $1 \leq k \leq m$ is such that $\frac{\tau_m}{h_k}$ is an integer, then $\frac{\tau_m}{h_l}$ is also an integer if $k \leq l \leq m$]. We also define the

map $\psi : \tau_m \rightarrow \psi(\tau_m)$ associating with any positive node $\tau_m \in P(h_m)$ the largest node $\psi(\tau_m)$ of the partition $P(h_{j(\tau_m)})$ strictly smaller than τ_m (if $j = 0$, $\psi(\tau_m) = 0$). We know that $S(X)$ is compact (and thus, contained in a ball of radius $c - 1 > 0$). Since S is uniformly continuous, we can choose δ_m such that

$$(3.4) \quad \|x - y\| \leq c\delta_m \text{ implies that } S(y) \subset S(x) + \varepsilon_m B,$$

where we set $\varepsilon_m = 2^{-m}$. We denote by L the Lipschitz constant of P :

$$P(y) \subset P(x) + L\|x - y\|B.$$

We shall choose h_m small enough in order that

$$h_m \leq \delta_m, \quad 2(L+1)b(h_m) \leq \varepsilon_m,$$

where

$$b(h) = \sup_{(x, v) \in G(S)} d(G(P); (x, x + hv))/h.$$

Construction of x_m on $[0, h_m]$. We consider $v_0 \in S(x_0)$. By

definition of $b(h_m)$, there exist $(y, z) \in G(P)$ such that

$$\max(\|x_0 - y\|, \|x_0 + h_m v_0 - z\|) \leq 2b(h_m)h_m. \text{ Since } P \text{ is Lipschitz,}$$

then $z \in P(y) \subset P(x_0) + L\|x_0 - y\|B \subset P(x_0) + 2Lb(h_m)h_m B$. Therefore,

there exists $u \in P(x_0)$ such that $\|z - u\| \leq 2Lb(h_m)h_m$ and thus,

such that

$$\left\| \frac{x_0 - u}{h_m} + v_0 \right\| \leq 2(L+1)b(h_m) \leq \varepsilon_m.$$

If we define x_m on $[0, \tau_m]$ by setting

$$x_m(0) = x_0 \text{ and } v_m(0) = \frac{u - x_0}{h_m}$$

we obtain the properties

$$\begin{cases} \text{i) } v_m(0) \in S(x_0) + \varepsilon_m \\ \text{ii) } \|v_m(0) - v_0\| \leq \varepsilon_m \\ \text{iii) } x_m(h_m) = u \in P(x_0) . \end{cases}$$

Construction of x_m on $[\tau_m, \tau_m + h_m]$. Let us assume that we have constructed x_m on $[0, \tau_m]$ satisfying, for any node $\sigma_m < \tau_m$,

$$(3.5) \quad \begin{cases} \text{i) } v_m(\sigma_m) \in S(x_m(\sigma_m)) + \varepsilon_m \\ \text{ii) } \|v_m(\sigma_m) - v_m(\psi(\sigma_m))\| \leq 2\varepsilon_{j(\sigma_m)} \\ \text{iii) } x_m(\sigma_m + h_m) \in P(x_m(\sigma_m)) . \end{cases}$$

We shall construct x_m on the interval $[\tau_m, \tau_m + h_m]$ satisfying properties (3.5) where σ_m is replaced by τ_m . Let us set $j = j(\tau_m) < m$ and $\tau_j = \psi(\tau_m) < \tau_m$. Since the node $\tau_j = \sigma_m$ is also a node σ_m of the partition $P(h_m)$, we know that, by (3.5) (i),

$$(3.6) \quad v_m(\tau_j) \in S(x_m(\tau_j)) + \varepsilon_m .$$

Furthermore, $\|x_m(\tau_m) - x_m(\tau_j)\| \leq \int_{\tau_j}^{\tau_m} \|v_m(t)\| dt \leq c|\tau_m - \tau_j| \leq ch_j$.

We deduce from the uniform continuity of S that (see (3.4))

$$(3.7) \quad S(x_m(\tau_j)) \subset S(x_m(\tau_m)) + \varepsilon_j B .$$

Hence there exists $w \in S(x_m(\tau_m))$ such that

$$(3.8) \quad \|v_m(\tau_j) - w\| \leq \varepsilon_j + \varepsilon_m .$$

Now, by definition of $b(h_m)$, there exists $(y, z) \in G(P)$ such that

$$(3.9) \quad \max(\|x_m(\tau_m) - y\|, \|x_m(\tau_m) + h_m w - z\|) \leq 2b(h_m)h_m.$$

Therefore, since P is Lipschitz, we deduce that

$$(3.10) \quad z \in P(y) \subset P(x_m(\tau_m)) + L\|x_m(\tau_m) - y\|B.$$

Then, there exists $u \in P(x_m(\tau_m))$ such that

$$\|z - u\| \leq L\|x_m(\tau_m) - y\| \leq 2Lb(h_m)h_m$$

and thus, such that

$$\left\| \frac{x_m(\tau_m) - u}{h_m} + w \right\| \leq 2(L+1)b(h_m) \leq \varepsilon_m.$$

If we set

$$(3.10) \quad v_m(\tau_m) = \frac{u - x_m(\tau_m)}{h_m}$$

we thus have shown that $v_m(\tau_m) \in w + \varepsilon_m B \subset S(x_m(\tau_m)) + \varepsilon_m B$, that

$$\|v_m(\tau_j) - v_m(\tau_m)\| \leq \|v_m(\tau_j) - w\| + \|w - v_m(\tau_m)\| \leq \varepsilon_j + 2\varepsilon_m \leq 2\varepsilon_j$$

(since $j < m$) and that $u = x_m(\tau_m + h_m) \in P(x_m(\tau_m))$. So x_m is constructed on $[x_m, x_m + \tau_m]$ and satisfies properties (3.5) with

$$\sigma_m = \tau_m.$$

Convergence of the sequence of approximate solutions. We shall prove

now that the sequence $\{\dot{x}_m\}$ is totally bounded in the space $\mathcal{B}(0, T; \mathbb{R}^l)$

of bounded functions from $[0, T]$ into $\mathbb{R}^l$. Let ε_k be fixed. Since

$S(X)$ is compact, it can be covered by p balls $u_j + \varepsilon_k B$ where

$u_j \in S(X)$. Let us consider any interval $[\tau_k, \tau_k + h_k[$ of the partition

$\rho(h_k)$. There exists u_j^k such that

$$(3.11) \quad \|u_j^k - \dot{x}_m(\tau_k)\| \leq \epsilon_k.$$

If $m \leq k$, $\dot{x}_m(t) = \dot{x}_m(\tau_k)$ is constant on this interval. Let $m > k$ and $t \in [\tau_k, \tau_k + h_k[$. Then we shall prove that

$$\|\dot{x}_m(t) - \dot{x}_m(\tau_k)\| \leq 4\epsilon_k.$$

Indeed, there exists a node $\tau_m \in \rho(h_m)$ such that

$$t \in [\tau_m, \tau_m + h_m[\subset [\tau_k, \tau_k + h_k[. \text{ If } \tau_m = \tau_k, \text{ then } \dot{x}_m(t) = \dot{x}_m(\tau_k).$$

If $\tau_k < \tau_m$, then there exists $j_1 = j(\tau_m)$ such that $k \leq j_1 < m$ and $\tau_k \leq \psi(\tau_m) < \tau_m$. If $k < j(\tau_m)$, then there exists $j_2 = j^2(\tau_m)$ such that $k \leq j_2 < j_1$ and $\tau_k \leq \psi^2(\tau_m) < \psi(\tau_m) < \tau_m$. Proceeding in this way, we can eventually write $\tau_k = \psi^\ell(\tau_m)$ where $\ell \leq m - k$. Hence, since $\dot{x}_m(t) = v_m(\tau_m)$ and $\dot{x}_m(\tau_k) = v_m(\psi^\ell(\tau_m))$, inequalities (3.11) imply that

$$(3.12) \quad \|\dot{x}_m(t) - \dot{x}_m(\tau_k)\| \leq 2(\epsilon_k + \epsilon_{k+1} + \dots + \epsilon_m) \leq 4\epsilon_k.$$

So, we deduce from (3.11) and (3.12) that for any m , for any

$t \in [\tau_k, \tau_k + h_k[$, we have

$$(3.13) \quad \|\dot{x}_m(t) - u_j^k\| \leq 5\epsilon_k.$$

This implies that for any ϵ_k , each function $\dot{x}_m(t)$ is in a ball of radius $5\epsilon_k$ whose center is a step function u_m such that $u_m(t) = u_j^k$ if $t \in [\tau_k, \tau_k + h_k[$. Since there is a finite number $(p \frac{T}{h_k})$ of such step functions we have proved that the sequence of derivatives $\dot{x}_m$ is totally bounded.

Consequently, we can extract a subsequence (still denoted by) $\dot{x}_m$ converging uniformly to a function $v \in \mathcal{B}(0, T; \mathbb{R}^l)$. Since $x_m(t) = x_0 + \int_0^t \dot{x}_m(\tau) d\tau$, the sequence x_m converges uniformly to a continuous function x such that

$$x(t) = x_0 + \int_0^t v(\tau) d\tau.$$

Therefore, for any $t \in [0, T]$, $x(t)$ and $v(t)$ are respectively the limits of sequences $x_m(\tau_m)$ and $\dot{x}_m(\tau_m) = v_m(\tau_m)$. So, by the upper semicontinuity of S , we deduce that

$$v(t) \in \dot{x}_m(\tau_m) + \frac{\varepsilon}{2} B \subset S(x_m(\tau_m)) + \frac{\varepsilon}{2} B \subset S(x(t)) + \varepsilon B$$

when m is large enough. Hence $v(t) = \dot{x}(t) \in S(x(t))$. Furthermore, $x(t) \in X$. Finally, property (1.2) (ii) of P , combined with (3.5) (iii), implies that in any of our partitions, larger nodes are better than smaller ones. We deduce from this that $P(x(t)) \subset P(x(s))$ when $t > s$.

Thus we have proved the existence of a monotone trajectory. Q.E.D.

Remark 3.1. The above proof remains valid when X is a compact subset of a Banach space U , S is a continuous multifunction from X into U with non-empty compact images, P a Lipschitz multifunction from X into X satisfying (1.2) with non-empty compact images and $b(h)$ converges to 0 with h ; under these assumptions, there exists a regulated monotone invariant trajectory satisfying $x(0) = x_0$ and $\dot{x}(0) = v_0 \in S(x_0)$.

Remark 3.2. We can generalize Theorem 1 to the case of time-dependent systems, by assuming that the multifunctions $S: [0, T] \times X \rightarrow U$ and $P: [0, T] \times X \rightarrow X$ are continuous, that for any t , $x \mapsto P(t, x)$ is Lipschitz and satisfies (1.2), and that $S(t, x) \subset T(P(t, x); x)$ for any $(t, x) \in [0, T] \times X$. The above proof then needs no modifications. By using techniques of Olech [13], the case where $t \mapsto S(t, x)$ is measurable for any $x \in X$ and where $x \mapsto S(t, x)$ is continuous for almost all t can also be treated.

Remark 3.3. We can consider also the case where X is no longer compact, but closed. Let the intersection of X with the ball of center x_0 and radius d be denoted X_d , and associate with d the scalar

$$c(d) = \sup_{\substack{x \in X_d \\ v \in S(x)}} \|v\| + 1,$$

it is easy to check that the approximate solutions x_m satisfying (3.5) remain in X_d whenever $t \leq T(d) = d/c(d)$. Therefore, by replacing X by X_d in the proof of Theorem 1, we obtain the existence of a monotone trajectory which remains in X_d when $t \leq T(d)$.

Remark 3.4. If the images $S(x)$ are convex, we can replace assumption (1.3) by

$$(3.14) \quad a(h) = \sup_{x \in X} \inf_{v \in S(x)} d(G(P); (x, x + hv))/h \rightarrow 0 \text{ as } h \downarrow 0.$$

We can prove that (3.14) holds whenever we assume

$$(3.15) \quad \forall x \in X, S(x) \cap T(P(x); x) \neq \emptyset \text{ (cf. remark 1.1) .}$$

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 14 MRC-TSR-1729	2. GOVT ACCESSION NO. 9	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) MONOTONE INVARIANT SOLUTIONS TO DIFFERENTIAL INCLUSIONS,		5. TYPE OF REPORT & PERIOD COVERED Technical Summary Report, no specific reporting period
7. AUTHOR(s) 10 Frank H. Clarke and J. P. Aubin		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 (Applied Analysis)
12. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12) 22 P.		12. REPORT DATE 11 Mar 1977
		13. NUMBER OF PAGES 19
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) invariant trajectory differential inclusion monotone trajectory generalized gradients preference ordering generalized tangent cone existence		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Let a given set be endowed with a preference preordering, and consider the problem of finding a solution to the differential inclusion $\dot{x}(t) \in S(x(t))$ which remains in the given set and evolves monotonically with respect to the preordering. We give sufficient conditions ^{are given} for the existence of such a trajectory, couched in terms of a notion of tangency developed by Clarke. No smoothness or convexity is involved in the construction, which uses techniques of Filippov.		