

AD-A039 030

BROWN UNIV PROVIDENCE R I LEFSCHETZ CENTER FOR DYNAM--ETC F/G 12/1  
BIFURCATION NEAR FAMILIES OF SOLUTIONS.(U)  
MAR 77 J K HALE

UNCLASSIFIED

AFOSR-TR-77-0571

AF-AFOSR-2078-71

NL

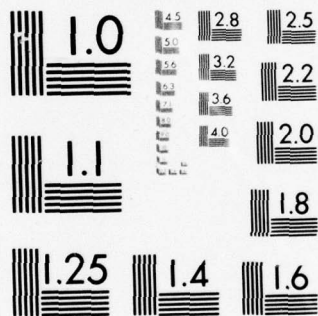
1 OF 1  
AD  
A039030



END

DATE  
FILMED  
5-77





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

AD A 039030



Jack P. Kennedy

by

➤Lefschetz Center for Dynamical Systems  
Division of Applied Mathematics  
Brown University  
Providence, Rhode Island 02912/USA

DDC  
RECEIVED  
MAY 3 1977  
C

+ This research was supported in part by the Air Force Office of Scientific Research under AFOSR 71-2078, in part by the National Science Foundation under MPS 71-02923 and in part by the U.S. Army Research Office under AROD AAG 29-76-6-0052.

|                                                                                                                          |                               |
|--------------------------------------------------------------------------------------------------------------------------|-------------------------------|
| ATIS<br>DBC<br>UNANNOUNCED<br>JUSTIFICATION<br>BY<br>DISTRIBUTION AVAILABILITY STATES<br>GIL<br>APRIL 1967<br>APRIL 1967 | WHITE SECTION<br>EDIT SECTION |
|--------------------------------------------------------------------------------------------------------------------------|-------------------------------|

AD No.                       
JDC FILE COPY

DDO  
RECEIVED  
MAY 8 1951  
PROPRIETARY

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)  
NOTICE OF TRANSMITTAL TO DDC  
This technical report has been reviewed and is  
approved for public release IAW AFR 190-12 (7b).  
Distribution is unlimited.  
A. D. BLOSE  
Technical Information Officer

## BIFURCATION NEAR FAMILIES OF SOLUTIONS

Jack K. Hale

Summary: Many investigations in bifurcation theory are concerned with the following problem. If  $M(0,0) = 0$  and  $\partial M(0,0)/\partial x$  has a nontrivial null space, find all solutions of the equation

$$M(x,\lambda) = 0 \quad (1.1)$$

for  $(x,\lambda)$  in a neighborhood of  $(0,0) \in X \times \Lambda$ .

If  $\dim \Lambda = 1$ ; that is, there is only one parameter involved then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about  $\partial M(0,0)/\partial x$  and  $\partial M(0,0)/\partial x \partial \lambda$ . However, if  $\dim \Lambda \geq 2$ , then the problem is much more difficult and more detailed information is needed about the function  $M$ . A careful examination of the existing literature for  $\dim \Lambda \geq 2$  reveals that the additional conditions imposed on  $M$  imply, in particular, that the solution  $x = 0$  of the equation

$$M(x,0) = 0 \quad (1.2)$$

is isolated (see, for example, the papers on catastrophe theory). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing  $x = 0$ . Such a situation occurs, for example, for  $M(x,\lambda) = Ax + N(x,\lambda)$ , where  $A$  is linear with a nontrivial null space and  $N(x,0) = 0$  for all  $x$ . There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period  $2\pi$  with the family of solutions being

Summary (continued)

obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters  $(\lambda_1, \lambda_2)$  could correspond to a small damping term and a small forcing term of period  $2\pi$ . To the author's knowledge, the first complete investigations of special problems of each of these latter types are contained in papers by Hale, Táboas and Rodrigues.

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in the paper by Hale and Táboas.

|                                    |                                                   |
|------------------------------------|---------------------------------------------------|
| ACCESSION for                      |                                                   |
| NTIS                               | White Section <input checked="" type="checkbox"/> |
| DGC                                | Bull Section <input type="checkbox"/>             |
| UNANNOUNCED                        |                                                   |
| JUSTIFICATION                      |                                                   |
| BY DISTRIBUTION/AVAILABILITY CODES |                                                   |
| Dist.                              | AVAIL. and/or SPECIAL                             |
| A                                  |                                                   |

# BIFURCATION NEAR FAMILIES OF SOLUTIONS

Jack K. Hale  
Lefschetz Center for Dynamical Systems  
Division of Applied Mathematics  
Brown University  
Providence, Rhode Island 02912/USA

## §1. Introduction and statement of problem.

Suppose  $X, Z, \Lambda$  are Banach spaces,  $M: X \times \Lambda \rightarrow Z$  is continuous together with its Fréchet derivatives up through order two. Many investigations in bifurcation theory are concerned with the following problem. If  $M(0,0) = 0$  and  $\partial M(0,0)/\partial x$  has a nontrivial null space, find all solutions of the equation

$$M(x, \lambda) = 0 \quad (1.1)$$

for  $(x, \lambda)$  in a neighborhood of  $(0,0) \in X \times \Lambda$ .

If  $\dim \Lambda = 1$ ; that is, there is only one parameter involved, then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about  $\partial M(0,0)/\partial x$  and  $\partial M(0,0)/\partial x \partial \lambda$ . However, if  $\dim \Lambda \geq 2$ , then the problem is much more difficult and more detailed information is needed about the function  $M$ . A careful examination of the existing literature for  $\dim \Lambda \geq 2$  reveals that the additional conditions imposed on  $M$  imply, in particular, that the solution  $x = 0$  of the equation

$$M(x, 0) = 0 \quad (1.2)$$

is isolated (see, for example, [1], [2] and the papers on catastrophe theory in [3]). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing  $x = 0$ . Such a situation occurs, for example, for  $M(x, \lambda) = Ax + N(x, \lambda)$ , where  $A$  is linear with a nontrivial null space and  $N(x, 0) = 0$  for all  $x$ . There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period  $2\pi$  with the family of solutions being obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters  $(\lambda_1, \lambda_2)$  could correspond to a small damping term and a small forcing term of period  $2\pi$ . To the author's knowledge, the first complete investigations of special problems of each of these latter

types are contained in [4], [5].

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in [5]. More specifically, suppose Equation (1.2) has a one parameter family of solutions  $x = p(t)$ ,  $0 \leq t \leq 1$ ,  $p(0) = p(1)$ , which is continuous together with derivatives up through order two with  $p'(t) = dp(t)/dt \neq 0$ ,  $p'(0) = p'(1)$ ,  $p''(0) = p''(1)$ . Boundary conditions on  $p$  are imposed only to avoid a special discussion at the end points of the curve defined by  $p$ . Since  $M(p(t), 0) = 0$  for all  $t$ , it follows that  $p'(t)$  is a nonzero element of the null space  $\mathcal{N}(\Lambda(t))$  of the linear operator

$$\Lambda(t) = \partial M(p(t), 0) / \partial x \quad (1.3)$$

for  $0 \leq t \leq 1$ .

If  $\Gamma = \{p(t), 0 \leq t \leq 1\} \subseteq X$ , the problem is to characterize the solutions of Equation (1.1) in a neighborhood of  $\Gamma \times \{0\} \subseteq X \times \Lambda$ . Suppose  $\mathcal{R}(\Lambda(t))$  is the range of  $\Lambda(t)$ . For the case in which  $\dim \mathcal{N}(\Lambda(t)) = 1 = \text{codim } \mathcal{R}(\Lambda(t))$ , and  $\Lambda = \mathbb{R}^2$ , we give a solution to this problem under certain hypotheses on  $\partial M(p(t), 0) / \partial \lambda$ . One important implication of the results can easily be stated. Suppose  $\gamma \subseteq \Lambda = \mathbb{R}^2$  is a continuous curve,  $0 \notin \gamma$ ,  $0 \in \text{Cl } \gamma$ , the closure of  $\gamma$ , and suppose  $x(\lambda)$  is a solution of system (1.1) defined and continuous for  $\lambda \in \gamma$ . If the set  $x(\gamma) \subseteq X$  remains in a sufficiently small neighborhood of  $\Gamma$  and the set  $x(\gamma)$  is precompact, then all limit points of  $x(\lambda)$  as  $\lambda \in \gamma$  approaches zero belong to  $\Gamma$ , but  $x(\lambda)$  has a limit as  $\lambda \rightarrow 0$  if and only if  $\cot^{-1}(\lambda_1/\lambda_2)$  approaches a constant as  $\lambda \in \gamma$  approaches zero.

## §2. Statement and implications of results.

For any Banach spaces  $X, Z$ , we let  $C^k(X, Z)$  be the linear space of all functions from  $X$  to  $Z$  which are continuous together with all derivatives up through order  $k$ . If no confusion may arise, we sometimes write  $C^k$  for  $C^k(X, Z)$ . For any finite collection of elements  $q_1, \dots, q_k$  of a Banach space, we let  $[q_1, \dots, q_k]$  denote the linear subspace spanned by  $q_1, \dots, q_k$ . By our boundary condition on  $p$ , we may suppose  $p \in C^2(\mathbb{R}, X)$  and is 1-periodic, that is, periodic of period 1. Suppose  $[p'(t)] = \mathcal{N}(\Lambda(t))$  and there is a  $q \in C^2(\mathbb{R}, Z)$  1-periodic, such that  $[q(t)] \oplus \mathcal{R}(\Lambda(t)) = Z$ . If  $\mathcal{B}(X)$  is the Banach space of bounded linear operators on  $X$ , let  $U \in C^2(\mathbb{R}, \mathcal{B}(X))$  be such that  $U(t)$  is a projection onto  $\mathcal{N}(\Lambda(t))$  and let  $E \in C^2(\mathbb{R}, \mathcal{B}(Z))$ ,  $E(t)$  a projection onto  $\mathcal{R}(\Lambda(t))$ ,  $I - E(t)$  a projection onto  $[q(t)]$ . Also suppose  $U, E$  are 1-periodic.

If  $\Lambda = \mathbb{R}^2$ ,  $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ , define  $\alpha_j \in C^2(\mathbb{R}, \mathbb{R})$ ,  $j = 1, 2$ ,

1-periodic, by the relation

$$\alpha_j(t)q(t) = (I-E(t))DM(p(t),0)/\partial\lambda_j \quad (2.1)$$

Our first hypothesis on  $\alpha(t) = (\alpha_1(t), \alpha_2(t))$  is

$$(H_1) \quad \alpha(t) \neq 0 \text{ for } t \in \mathbb{R}.$$

If  $\beta(t) = (\alpha_2(t), -\alpha_1(t))$ , then  $\beta(t) \neq 0$  by Hypothesis  $(H_1)$  and we can let  $\phi(t)$  be the angle measured in the counterclockwise direction which  $\beta(t)$  makes with the horizontal axis. The function  $\phi \in C^2(\mathbb{R}, \mathbb{R})$  and is 1-periodic. We impose the following hypotheses on  $\phi$ :

$$(H_2) \quad \text{The function } \phi'(t) \text{ has at most a finite set of zeros } \{t^k, k = 1, 2, \dots, n\} \subseteq [0, 1) \text{ and } \phi''(t^k) \neq 0 \text{ for } k = 1, 2, \dots, n.$$

$$(H_3) \quad \phi(t^j) \neq \phi(t^k), j \neq k, j, k = 1, 2, \dots, n.$$

We now state the main results of the paper together with implications. The proofs will be given in Section 3. Suppose  $\gamma$  is a smooth curve in  $\mathbb{R}^2$  through the origin. If for any  $q \in \gamma, q \neq (0,0)$ ,  $L_q^1$  denotes a positively oriented normal to  $L_q$  at  $q$ , we say  $\gamma$  is crossed from right to left at  $q$  if  $\gamma$  is crossed by moving along  $L_q^1$  in the positive direction.

Theorem 2.1. If Hypotheses  $(H_1)$ - $(H_3)$  are satisfied, then there exist neighborhoods  $U$  of  $\Gamma$ ,  $V$  of  $\lambda = (0,0)$ , and  $s_0 > 0$ , such that, for each  $t^j \in \{t^k, k = 1, 2, \dots, n\}$ , there corresponds a unique curve  $\mathcal{L}_j \subseteq V$ , tangent to the line  $\alpha(t^j) \cdot \lambda = 0$  at zero,  $\mathcal{L}_j \cap \partial V \neq \emptyset$ , each  $\mathcal{L}_j$  intersects lines through the origin in at most one nonzero point, these curves intersect only at  $(0,0)$ , the number of solutions of Equation (1.1) increases (or decreases) by exactly two as  $\mathcal{L}_j$  is crossed from right to left if  $t^j$  is a relative minimum (or maximum) of  $\phi$ .

The curves  $\mathcal{L}_j$  can be defined parametrically in the form  $\lambda = s\beta_j(s), 0 \leq s < s_0$  where  $\beta_j \in C^2([0, s_0], \mathbb{R}^2), |\beta_j(s)| = 1, 0 \leq s < s_0$ , and  $\alpha(t^j) \cdot \beta_j(0) = 0$ . If  $t_*, t^*$  are the absolute minimum, maximum, respectively, of  $\phi$  and

$$S(V) = \{\lambda \in V: \lambda \cdot \beta^*(s) < 0 < \lambda \cdot \beta_*(s), 0 \leq s < s_0\},$$

where  $\lambda = s\beta^*(s), \lambda = s\beta_*(s)$  are the curves corresponding to  $t^*$  and  $t_*$ , then there are no solutions of Equation (1.1) in  $U$  for  $\lambda \in S(V)$ , at least two in  $S^C(V) = V \setminus S(V)$  and all solutions are distinct in the interior of  $S^C(V)$ .

The curves  $\mathcal{L}_j$  in Theorem 2.1 are called the bifurcation curves. To see how easy it is to obtain the complete qualitative picture of the bifurcations near  $\lambda = 0$ , let us consider a few special cases. If  $\phi(t)$  has only one maximum at  $t^*$  and one minimum at  $t_*$ , there are only two bifurcation curves  $\mathcal{L}_*, \mathcal{L}^*$  corresponding to  $t_*, t^*$ , respectively. There are no solutions of Equation (1.1) in  $U$  for  $\lambda \in S(V)$  and exactly two solutions in  $S^C(V)$  which are distinct in the interior of  $S^C(V)$  (see Figure 1). If there are two maxima and two minima (there must always be an even number of maxima and minima by periodicity), then the situation is depicted in Figure 2. By changing the function  $\phi$ , one can obtain every possible rotation of these pictures.

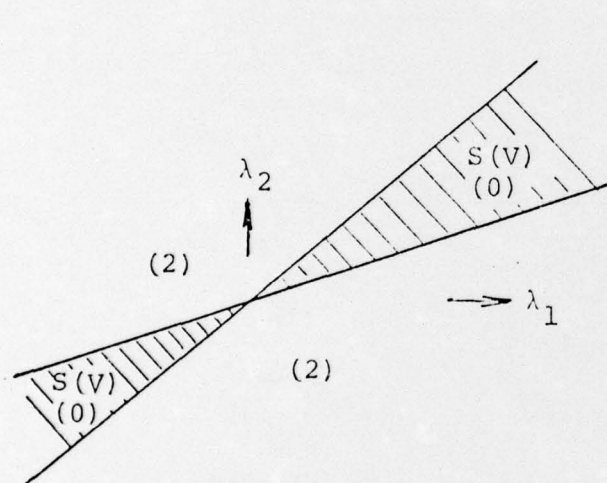


Figure 1.

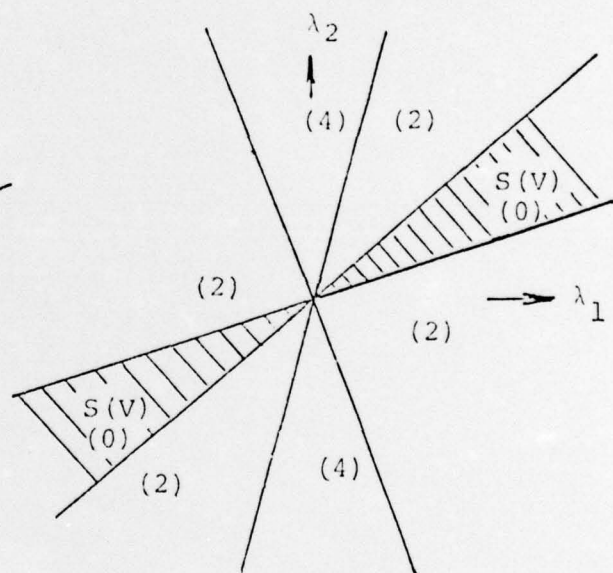


Figure 2.

Another interesting special case is  $\alpha_1(t) = -1$ ,  $t \in \mathbb{R}$ . Hypothesis  $(H_1)$  is always satisfied,  $\beta(t) = (\alpha_2(t), 1)$ ,  $\phi(t) = \cot^{-1} \alpha_2(t)$  and the hypotheses  $(H_1)$ ,  $(H_2)$  are equivalent to

- $(H'_2)$  The function  $\alpha'_2$  has a most finite set of zeros  $\{t^k, k = 1, 2, \dots, n\} \subseteq [0, 1)$  and  $\alpha''_2(t^k) \neq 0$ ,  $k = 1, 2, \dots, n$ .
- $(H'_3)$   $\alpha_2(t^j) \neq \alpha_2(t^k)$ ,  $j \neq k$ ,  $j, k = 1, 2, \dots, n$ .

Theorem 2.1 for this case is essentially contained in [5]. Since  $\alpha_1(t) = -1$ , the set  $S(V)$  must contain the  $\lambda_1$ -axis and, thus, the bifurcation diagram is a rotated version of the ones in Figures 1 and 2.

The following result gives some information about the possible behavior of the solutions of Equation (1.1) as  $\lambda \rightarrow 0$ .

**Theorem 2.2.** Suppose Hypotheses  $(H_1)$ – $(H_2)$  are satisfied,  $U, V$  are the

neighborhoods given in Theorem 2.1, and suppose  $\gamma$  is a continuous curve defined parametrically by  $\lambda_1 = \lambda_1(\tau)$ ,  $\lambda_2 = \lambda_2(\tau)$ ,  $0 \leq \tau \leq 1$ , and  $\lambda_1^2(\tau) + \lambda_2^2(\tau) = 0$  if and only if  $\tau = 0$ . Also, suppose  $\gamma \subseteq V$  and for each point  $(\lambda_1(\tau), \lambda_2(\tau)) \in \gamma$ , there is a solution  $x(\tau) \in U$  of Equation (1.1) which is continuous in  $\tau$  on the half open interval  $(0, 1]$ . If

$$\mathcal{S}(\gamma) = \{x(\tau), 0 < \tau \leq 1\} \subseteq X \quad (2.2)$$

is precompact, and

$$\begin{aligned} \phi_m(\gamma) &= \liminf_{\tau \rightarrow 0} \cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \\ \phi_M(\gamma) &= \limsup_{\tau \rightarrow 0} \cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \end{aligned} \quad (2.3)$$

then there is an interval  $I(\gamma) \subseteq [0, 1]$  such that  $\phi(I(\gamma)) = [\phi_m(\gamma), \phi_M(\gamma)]$  and

$$(cl \mathcal{S}(\gamma)) \setminus \mathcal{S}(\gamma) = \{p(t), t \in I(\gamma)\}. \quad (2.4)$$

A consequence of the above result is the following

Corollary 2.1. If  $\gamma, x(\tau)$  satisfy the conditions of Theorem 2.2, then a necessary and sufficient condition that  $x(\tau)$  have a limit as  $\tau \rightarrow 0$  is that  $\cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau))$  has a limit  $\phi_0$  as  $\tau \rightarrow 0$ . In this case,  $x(\tau) \rightarrow p(t_0)$  where  $t_0 \in [0, 1]$  is a solution of the equation  $\cot^{-1}(\lambda_1(t)/\lambda_2(t)) = \phi_0$ .

The fact that one can obtain solutions which are not continuous in  $\lambda$  at  $\lambda = 0$  is not surprising. Consider the scalar equation  $\lambda_1 x - \lambda_2 = 0$  which has the solution  $x = \lambda_2/\lambda_1$  for  $\lambda_1 \neq 0$ . Along a curve  $\gamma \in \mathbb{R}^2$ , this solution has a limit as  $\lambda \rightarrow 0$  in  $\gamma$  if and only if  $\lambda_1/\lambda_2$  approaches a limit as  $\lambda \rightarrow 0$  in  $\gamma$ .

Let us now make a more interesting application to the second order scalar ordinary differential equation

$$\frac{d^2 x}{ds^2} + g(x) + \lambda_1 h(s) \frac{dx}{ds} - \lambda_2 f(s) = 0 \quad (2.5)$$

where  $h, f$  are continuous and 1-periodic,  $g \in C^2(\mathbb{R}, \mathbb{R})$ ,  $xg(x) > 0$  for  $x \neq 0$ . For  $\lambda_1 = \lambda_2 = 0$ , the equation

$$\frac{d^2 x}{ds^2} + g(x) = 0 \quad (2.6)$$

has a general solution of the form  $x = \psi(\omega(a)s + t, a)$ ,  $(a, t) \in \mathbb{R}^2$ , where  $\psi(\zeta, a) = \psi(\zeta+1, a)$  for all  $(\zeta, a)$ , and  $(a, 0) = (x(0), dx(0)/ds)$ . We suppose Equation (2.6) has a nondegenerate 1-periodic orbit; that is,

$$\text{There is an } a_0 > 0 \text{ such that } \omega(a_0) = 1, d\omega(a_0)/da \neq 0. \quad (2.7)$$

Let  $Z = \{y: \mathbb{R} \rightarrow \mathbb{R} \text{ which are continuous and 1-periodic}\}$  and use the supremum norm on  $Z$ . Let  $X = \{y \in Z: y \text{ has continuous derivatives up through order two}\}$  and use the usual  $C^2$  norm on  $X$ . If we define  $M: X \times \Lambda \rightarrow Z$ ,  $\Lambda = \mathbb{R}^2$  by

$$M(x, \lambda)(s) = \frac{d^2 x(s)}{ds^2} + g(x(s)) + \lambda_1 h(s) \frac{dx(s)}{ds} - \lambda_2 f(s)$$

then we are in a position to apply the previous results. In fact, if  $p(t)(s) = \psi(\omega(a_0)s + t, a_0)$ , then  $p(t) \in X$  and satisfies  $M(p(t), 0) = 0$ ,  $0 \leq t \leq 1$ . Also, Hypothesis (2.7) implies that  $\dim \mathcal{N}(\Lambda(t)) = 1 = \text{codim } \mathcal{R}(\Lambda(t))$ , where  $\Lambda(t) = \partial M(p(t), 0)/\partial x$ . Furthermore, the function  $\dot{p}(t)$  is a basis for  $\mathcal{N}(\Lambda(t))$  and a complement for  $\mathcal{R}(\Lambda(t))$ . It is now an obvious calculation to see that the functions  $\alpha_1(t), \alpha_2(t)$  in (2.1) are given by

$$\alpha_1(t) = -\int_0^1 h(s) \dot{p}(s+t)^2 ds, \quad \alpha_2(t) = \int_0^1 \dot{p}(s+t) f(s) ds. \quad (2.8)$$

If  $(\alpha_1, \alpha_2)$  satisfy  $(H_1)-(H_3)$ , then the above results are directly applicable to the determination of the bifurcation curves for the 1-periodic solutions of Equation (2.5) which lie in a neighborhood of the periodic orbit  $\Gamma \subseteq \mathbb{R}^2$  of Equation (2.6) defined by  $\Gamma = \{(p(s), dp(s)/ds), 0 \leq s \leq 1\}$ . For  $h(s) = 1$ ,  $0 \leq s \leq 1$ , these results were previously obtained in [5]. A detailed explanation of the manner in which the 1-periodic solutions wind onto the cylinder  $\Gamma \times \mathbb{R}$  as  $\lambda \rightarrow 0$  along a curve  $\gamma \in \mathbb{R}^2$  is given in [5]. Also, a reasonable physical explanation for the discontinuities of the solutions at  $\lambda = 0$  is given in [5].

### §3. Proof of the results.

Our objective in this section is to give the essential elements of the proofs of the results of Section 2. The notation of that section will be used without explanation.

By the Implicit Function Theorem and the compactness of  $\Gamma$ , one obtains the following result.

Lemma 3.1. There is a  $\delta > 0$  such that the transformation  $x \mapsto (t, y)$ ,

$$x = p(t) + y, \quad y \in [I - U(t)]X, \quad (3.1)$$

from a neighborhood of  $\Gamma$  to  $[0, 1) \times (I - U(t))X$  is a diffeomorphism for  $t \in [0, 1)$ ,  $|y| < \delta$ .

For the determination of all solutions of Equation (1.1) in a sufficiently small neighborhood of  $\Gamma$ , Lemma 3.1 implies that it is sufficient to consider  $x$  given by Equation (3.1) and  $|y|$  in a sufficiently small neighborhood of zero. If  $x$  is a solution of

Equation (1.1) and  $y$  is defined by Equation (3.1), then  $y$  satisfies the equation

$$0 = M(p(t) + y, \lambda) \stackrel{\text{def}}{=} A(t)y + N(t, y, \lambda) \quad (3.2)$$

where  $N(t, y, \lambda) = M(p(t) + y, \lambda) - A(t)y$ . By the boundary conditions on  $p$ , we may consider this equation for  $t \in \mathbb{R}$ . Decomposing this equation into its components in  $E(t)Z$  and  $[I-E(t)]Z$  and using the fact that  $A(t)$  as a mapping from  $[I-U(t)]X$  onto  $E(t)Z$  has a bounded inverse (this is the method of Liapunov-Schmidt), there exist  $\lambda_0 > 0$ ,  $\delta > 0$  and a unique function  $y^* \in C^2(\mathbb{R} \times \{|\lambda| < \lambda_0\}, (I-U(t))X)$ ,  $y^*(t, 0) = 0$  for all  $t$ , such that Equation (3.2) has a solution for  $t \in \mathbb{R}$ ,  $|y| < \delta$ ,  $|\lambda| < \lambda_0$  if and only if  $y = y^*(t, \lambda)$  and  $(t, \lambda)$  satisfies the bifurcation equation

$$0 = F(t, \lambda) \stackrel{\text{def}}{=} (I-E(t))[M(p(t) + y^*(t, \lambda), \lambda) - A(t)y^*(t, \lambda)] \quad (3.3)$$

If we define the scalar function  $f(t, \lambda)$  by the relation

$$F(t, \lambda) \stackrel{\text{def}}{=} f(t, \lambda)q(t) \quad (3.4)$$

then the bifurcation equation is equivalent to the scalar equation

$$f(t, \lambda) = 0 \quad (3.5)$$

for  $t \in \mathbb{R}$ ,  $|\lambda| < \lambda_0$ . Since  $y^*(t, 0) = 0$  for all  $t$ , it follows that

$$f(t, 0) = 0, \quad t \in \mathbb{R}. \quad (3.6)$$

Relation (3.6) reflects the fact that Equation (1.2) does not have an isolated solution. Equality (3.6) is the basic reason why this problem differs from the usual bifurcation problem.

If  $A = \mathbb{R}^2$ ,  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ , then

$$\partial f(t, 0) / \partial \lambda_j = \alpha_j(t), \quad j = 1, 2 \quad (3.7)$$

where each  $\alpha_j$ ,  $j = 1, 2$ , is defined in Equation (2.1). The function  $f$  can thus be written as

$$f(t, \lambda) = \alpha(t) \cdot \lambda + h(t, \lambda)$$

where  $h(t, 0) = 0$ ,  $\partial h(t, 0) / \partial \lambda = 0$ . For any  $\lambda \neq 0$ , solving Equation (3.5) is equivalent to solving the equation

$$\alpha(t) \cdot (\lambda / |\lambda|) + h(t, \lambda) / |\lambda| = 0. \quad (3.8)$$

If  $\lambda / |\lambda| = \beta$ ,  $H(t, \beta, |\lambda|) = h(t, \beta|\lambda|) / |\lambda|$ , then  $\beta \in S^1 = \{\beta \in \mathbb{R}^2: |\beta| = 1\}$  and  $H$  is  $C^2$  in its arguments. The discussion of the solutions of Equation (3.8) becomes equivalent to the discussion of the equation

$$G(t, \beta, s) \stackrel{\text{def}}{=} \alpha(t) \cdot \beta + H(t, \beta, s) = 0 \quad (3.9)$$

for  $t \in [0,1)$ ,  $\beta \in S^1$ ,  $s$  small and nonnegative. In the following, we always understand  $\beta \in S^1$  even though it may not be said explicitly.

If  $\alpha(t_0) \cdot \beta_0 = 0$ , then  $G(t_0, \beta_0, 0) = 0$ . If  $\alpha'(t_0) \cdot \beta_0 \neq 0$ , then  $\partial G(t_0, \beta_0, 0)/\partial t \neq 0$  and the Implicit Function Theorem implies there is an  $s_0 = s_0(t_0, \beta_0) > 0$  and a unique solution  $t^*(\beta, s)$  of Equation (3.9) for  $|\beta - \beta_0| < s_0$ ,  $0 \leq s < s_0$ ,  $t^*(\beta_0, 0) = t_0$ .

To complete the proof, we need to reformulate Hypotheses  $(H_2)$ ,  $(H_3)$  in an equivalent form. The vector  $\alpha(t)$  defines a continuous linear functional on  $\mathbb{R}^2$  by the relation  $\alpha(t) \cdot \lambda$ ,  $\lambda \in \mathbb{R}^2$ . If the null space of  $\alpha(t)$  is denoted by  $\mathcal{N}(\alpha(t))$ , then  $(\alpha_2(t), -\alpha_1(t))$  is a basis for  $\mathcal{N}(\alpha(t))$ . By computing  $\phi', \phi''$ , one easily observes that  $(H_2)$ ,  $(H_3)$  are equivalent to

$(H_2)$  The vector  $\alpha'(t) \in \mathbb{R}^2$  is orthogonal to  $\mathcal{N}(\alpha(t))$  at most at a finite number of points  $\{t^k, k = 1, 2, \dots, n\} \subseteq [0,1)$  and  $\alpha''(t^k)$  is not orthogonal to  $\mathcal{N}(\alpha(t^k))$  for any  $k = 1, 2, \dots, n$ .

$(H_3)$  The lines through the origin and  $\alpha(t^j)$  and  $\alpha(t^k)$  are not colinear for  $j \neq k$ ,  $j, k = 1, 2, \dots, n$ .

The numbers  $t^k$  here are the same as before.

If  $\alpha'(t_0) \cdot \beta_0 = 0$ , then Hypothesis  $(H_2)$  implies  $\alpha''(t_0) \cdot \beta_0 \neq 0$ . Thus, the Implicit Function Theorem implies there is an  $s_0 = s_0(t_0, \beta_0) > 0$  and a function  $t^*(\beta, s)$ ,  $t^*(\beta_0, 0) = t_0$ , such that

$$\partial G(t^*(\beta, s), \beta, s)/\partial t = 0$$

for  $|\beta - \beta_0| < s_0$ ,  $0 \leq s < s_0$  and  $t^*(\beta, s)$  is unique in the region  $|t - t_0| < s_0$ . Thus, the function  $Q(\beta, s) \stackrel{\text{def}}{=} G(t^*(\beta, s), \beta, s)$  is a maximum or minimum of  $G(t, \beta, s)$  with respect to  $t$  at  $(\beta, s)$ . A few elementary calculation show that  $Q(\beta_0, 0) = 0$ ,  $\partial Q(\beta_0, 0)/\partial \beta = \alpha(t_0)$ . Therefore, the derivative of  $Q(\beta, s)$  with respect to  $\beta$  on the sphere  $S^1$  at  $\beta = \beta_0$ ,  $s = 0$  is the inner product of the vector  $\alpha(t_0)$  with a unit vector orthogonal to  $\beta_0$ . But this vector will be a nonzero constant times  $\alpha(t_0)$  since  $\alpha(t_0) \neq 0$  by Hypothesis  $(H_1)$ . Consequently,  $Q(\beta_0, 0)/\partial \beta$  on  $S^1$  is nonzero. The Implicit Function Theorem implies there is a  $\delta(\beta_0) > 0$  and a function  $\beta^*(s) \in S^1$ ,  $\beta^*(0) = \beta_0$ , such that  $Q(\beta^*(s), s) = 0$  for  $0 \leq s < \delta(\beta_0)$  describes a curve. On one side of this curve, there are two simple solutions of Equation (3.9) and no solutions on the other side. In terms of the original coordinates  $(\lambda_1, \lambda_2)$ , this implies there are two solutions of Equation (3.5) near  $t_0$  on one side of the curve  $\lambda = s\beta^*(s)$ ,  $0 \leq s < \delta(\beta_0)$  and none on the other. This curve in  $\lambda$ -space is a bifurcation curve

and is tangent to the line  $\alpha(t_0) \cdot \beta_0 = 0$  at  $\lambda = 0$ . The fact that the number of solutions increases or decreases as stated in the theorem is clear. 9

The above analysis can be applied to each of the points  $t^j$  in Hypothesis  $(H_2)$  to obtain an  $s_0 > 0$  such that all solutions of Equation (3.9) for  $|t - t^j| < s_0$ ,  $\beta \in S^1$ ,  $0 \leq s < s_0$  are determined by the argument above. We obtain the bifurcation curves as well. The complement of the intervals  $|t - t^j| < s_0$ ,  $j = 1, 2, \dots, n$ , in  $[0, 1]$  is compact and  $\alpha'(t_0) \cdot \beta_0 \neq 0$  for any  $t_0, \beta_0$  satisfying  $\alpha(t_0) \cdot \beta_0 = 0$ . A repeated application of the Implicit Function Theorem shows one can choose  $s_0$  so that no further bifurcations occur in this complement for any  $\beta \in S^1$ ,  $0 \leq s < s_0$ . Returning to the original coordinates  $(\lambda_1, \lambda_2)$ , we see that the complete bifurcation diagram has been obtained for a full neighborhood of  $\lambda = 0$ .

To describe precisely the bifurcation pattern as stated in Theorem 2.1, we need to know that no two bifurcation curves obtained by the above process coincide. This is the only reason Hypothesis  $(H_3)$  is imposed. This proves the first part of Theorem 2.1.

The last part of the theorem is clear from the definitions of the terms involved. This completes the proof of Theorem 2.1.

To prove Theorem 2.2, we first note that the method of Liapunov-Schmidt implies there is a  $\tau_0 > 0$  and a continuous function  $t(\tau) \subseteq [0, 1]$ ,  $0 < \tau < \tau_0$  such that the solution  $x(\tau)$  is given by

$$x(\tau) = p(t(\tau)) + y^*(t(\tau), \lambda(\tau)), \quad 0 < \tau < \tau_0,$$

where  $y^*(t, \lambda)$  is the solution of Equation (3.2),  $(t(\tau), \lambda(\tau))$  satisfy Equation (3.5) or, equivalently, Equation (3.8). Suppose there is a sequence  $\tau_j \rightarrow 0$  such that  $\cot^{-1}(\lambda_1(\tau_j)/\lambda_2(\tau_j)) \rightarrow \phi_0 \in [0, 2\pi]$  as  $j \rightarrow \infty$ , or, equivalently,  $\lambda(\tau_j)/|\lambda(\tau_j)| \rightarrow \beta_0 \in S^1$  as  $j \rightarrow \infty$ . Without loss of generality, we may assume  $t(\tau_j) \rightarrow t_0 \in [0, 1]$  as  $j \rightarrow \infty$ . Then  $x(\tau_j) \rightarrow p(t_0)$ ,  $\alpha(t_0) \cdot \beta_0 = 0$ ,  $\phi(t_0) = \phi_0$ . Since all functions are continuous, the conclusion of Theorem 2.2 follows immediately, and the proof is complete.

To prove Corollary 2.1, suppose the conditions of Theorem 2.2 are satisfied and the interval  $[\phi_m(\gamma), \phi_M(\gamma)]$  consists of more than one point, then  $x(\tau)$  cannot have a limit as  $\tau \rightarrow 0$  although every limit point satisfies Equation (1.2). If  $x(\tau)$  has a limit as  $\tau \rightarrow 0$ , then  $\mathcal{S}(\gamma)$  is precompact and it is, therefore, necessary that  $\cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau))$  approach a limit as  $\tau \rightarrow 0$ . Conversely, if  $\cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \rightarrow \phi_0$  as  $\tau \rightarrow 0$ , then  $\lambda(\tau)/|\lambda(\tau)| \rightarrow \beta_0 \in S^1$  as  $\tau \rightarrow 0$  and  $\alpha(t(\tau)) \cdot \beta_0 \rightarrow 0$  as  $\tau \rightarrow 0$  from the argument used in the

proof of Theorem 2.2. Hypothesis  $(H_2)$  implies the set of  $t \in [0,1]$  such that  $\alpha(t) \cdot \beta_0 = 0$  is isolated. Since  $t(\tau)$  is continuous for  $0 < \tau \leq 1$ , this implies  $t(\tau) \rightarrow t_0 \in \{t^k, k = 1, 2, \dots, n\}$  as  $\tau \rightarrow 0$ . The argument used in the proof of Theorem 2.2 implies  $x(\tau) \rightarrow p(t_0)$  as  $\tau \rightarrow 0$  and the proof of Corollary 2.1 is complete.

#### REFERENCES

- [1] Chow, S., Hale, J.K. and Mallet-Paret, Applications of generic bifurcation I, II. Arch. Rat. Mech. Ana. 59(1975), 159-188; 62(1976), 209-236.
- [2] Thompson, J.M.T. and Hunt, G.W., A General Theory of Elastic Stability. Wiley, 1973.
- [3] Dynamical Systems - Warwick 1974, Lecture Notes in Math., vol. 468, Springer-Verlag.
- [4] Hale, J.K. and Rodrigues, H.M., Bifurcation in the Duffing equation with independent parameters I, II. Proc. Royal Soc. Edinburg, Ser. A. To appear.
- [5] Hale, J.K. and Táboas, P., Effect of damping and forcing in a second order equation. J. Nonlinear Analysis - Theory, Method, Applications (JNA-TMA). To appear.

This research was supported in part by the Air Force Office of Scientific Research under AFOSR 71-2078D, in part by the National Science Foundation under MPS 71-02923 and in part by the U.S. Army Research Office under AROD AAG 29-76-6-0052.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                          | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                |  |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|------------------------------------------------------------|--|
| 1. REPORT NUMBER<br><b>AFOSR-TR-77-0571</b>                                                                                                                                                                                                                                                                                                                                                                                                                                 | 2. GOVT ACCESSION NO.                                                                    | 3. RECIPIENT'S CATALOG NUMBER<br><b>9</b>                  |  |
| 4. TITLE (and Subtitle)<br><b>BIFURCATION NEAR FAMILIES OF SOLUTIONS</b>                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                          | 5. TYPE OF REPORT & PERIOD COVERED<br><b>Interim Rept.</b> |  |
| 7. AUTHOR(s)<br><b>J. K. Hale</b>                                                                                                                                                                                                                                                                                                                                                                                                                                           | 8. CONTRACT OR GRANT NUMBER(s)<br><b>AFOSR 71-2078</b>                                   |                                                            |  |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br><b>Brown University<br/>Lefschetz Center for Dynamical Systems<br/>Providence, RI 02912</b>                                                                                                                                                                                                                                                                                                                                  | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br><b>61102F<br/>2304/A1</b> |                                                            |  |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br><b>Air Force Office of Scientific Research/NM<br/>Bolling AFB DC 20332</b>                                                                                                                                                                                                                                                                                                                                                       | 12. REPORT DATE<br><b>Mar 77</b>                                                         |                                                            |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 13. NUMBER OF PAGES<br><b>10</b>                                                         |                                                            |  |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)<br><b>12/15p.</b>                                                                                                                                                                                                                                                                                                                                                                               | 15. SECURITY CLASS. (of this report)<br><b>UNCLASSIFIED</b>                              |                                                            |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE                                               |                                                            |  |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><b>Approved for public release; distribution unlimited.</b>                                                                                                                                                                                                                                                                                                                                                                  |                                                                                          |                                                            |  |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)<br><b>15 VAF-AFOSR-2078-71,<br/>VNSF-MPS-71-02923</b>                                                                                                                                                                                                                                                                                                                            |                                                                                          |                                                            |  |
| 18. SUPPLEMENTARY NOTES<br><b>16 2304 / 17 A1</b>                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                          |                                                            |  |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                          |                                                            |  |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>Many investigations in bifurcation theory are concerned with the following problem. If $M(0,0) = 0$ and $\partial M(0,0)/\partial x$ has a non-trivial null space, find all solutions of the equation<br>$M(x,\lambda) = 0 \quad (1.1)$ for $(x,\lambda)$ in a neighborhood of $(0,0) \in X \times \Lambda$ .<br>If $\dim \Lambda = 1$ : that is, there is only one parameter involved |                                                                                          |                                                            |  |

401834

## UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about  $M(0,0)/\partial x$  and  $M(0,0)/\partial x \partial \lambda$ . However, if  $\dim \Lambda \geq 2$ , then the problem is much more difficult and more detailed information is needed about the function  $M$ . A careful examination of the existing literature for  $\dim \Lambda \geq 2$  reveals that the additional conditions imposed on  $M$  imply, in particular, that the solution  $x = 0$  of the equation

$$M(x, 0) = 0 \quad (1.2)$$

is isolated (see, for example, the papers on catastrophe theory). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing  $x = 0$ . Such a situation occurs, for example, for  $M(x, \lambda) = Ax + N(x, \lambda)$ , where  $A$  is linear with a nontrivial null space and  $N(x, 0) = 0$  for all  $x$ . There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period  $2\pi$  with the family of solutions being obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters  $(\lambda_1, \lambda_2)$  could correspond to a small damping term and a small forcing term of period  $2\pi$ . To the authors knowledge, the first complete investigations of special problems of each of these latter types are contained in papers by Hale, Táboas and Rodrigues.

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in the paper by Hale and Táboas.