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STOCHASTIC INTEGRALS OF WEIGHTED EMPIRICAL PROCESSES AND AN APP--ETC(U)
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STOCHASTIC INTEGRALS OF WEIGHTED EMPIRICAL PROCESSES
AND AN APPLICATION TO THE LIMIT DISTRIBUTION OF
LINEAR RANK STATISTICS UNDER ALTERNATIVES

Abbreviated Title: STOCHASTIC INTEGRALS OF WEIGHTED EMPIRICAL PROCESSES

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(series a from m1 to mn)

Introduction: The distribution under alternatives of a simple linear rank statistic S_n *S-sub-n* (defined in (2.1)) is governed by the following entities: a set of regression constants $(C_{n1}, \dots, C_{nn})$, scores $(a_{n1}, \dots, a_{nn})$ and distribution functions $(F_{n1}, \dots, F_{nn})$. One usually assumes the scores to be generated by a known function ϕ . *(series C from m1 to mn)* ϕ *(series F from m1 to mn)*

AMS 1970 subject classification: Primary 60H05; secondary 60B10, 62G10.

Key words and phrases: Linear rank statistics; weighted empirical processes; stochastic integrals; reproducing kernel Hilbert spaces.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Asymptotic normality of linear rank statistics under alternatives is obtained by considering them as second order stochastic integrals of weighted empirical processes. Results of this paper are related to those of Hajek (1968, <u>Ann. Math. Statist.</u>, 39, 325-346), Pyke-Shorack (1968, <u>Ann. Math. Statist.</u>, 39, 755-771) and Hoeffding (1973, <u>Ann. Statist.</u>, 1, 54-66). Methods of this paper do not require any sample path analysis and thus differ (Continued on reverse side.)		

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20. Abstract

from the Pyke-Shorack (1968) results. An expression for the centering constant is obtained, thus unifying the Hájek (1968) and Hoëffding (1973) results. Asymptotic degeneracy of such statistics is considered.

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For the important special case arising when one takes for some $m < n$, $C_{n1} = \dots = C_{nm} = \frac{1}{m}$ and $C_{nm+1}, \dots, C_{nn} = 0$; $F_{n1} = \dots = F_{nm} = F$ and $F_{nm+1} = \dots = F_{nn} = G$, Chernoff-Savage (1958) obtained the asymptotic normality of S_n with some restrictions on the growth rates of the first two derivatives of φ . The same results have since been proved by Govindarajulu-Le Cam-Raghavachari (1966) and Puri-Sen (1971) with milder conditions on φ . A far-reaching generalization of these results is due to Hájek (1968). In a remarkable paper, Pyke-Shorack (1968) provided an alternative approach to the Chernoff-Savage theorems by considering the weak convergence of a two sample empirical process to a Gaussian process. This approach has had a very significant impact on research practices in the area of nonparametric statistics and there now exists a substantial body of literature devoted to stochastic processes arising in such problems. In spite of this, however, techniques of neither the Pyke-Shorack kind nor of the Chernoff-Savage kind have been applied to problems of the regression type. The method of attack for such problem has for the most part been along the lines of Hájek (1968), Puri-Sen (1969), Hoeffding (1973), to name a few. We have been able to obtain results of the type given in Hájek (1968) both by the Chernoff-Savage approach and by a method involving convergence of certain stochastic processes. The latter is given in this paper; the former, together with some results on convergence rates is published separately. (See Puri-Rajaram (1977).)

As is well known, the classical weak convergence method, to be found, for example, in Billingsley (1968) or Pyke-Shorack (1968) generally entail securing certain bounds on the sample path fluctuations of the processes involved, usually with the help of certain refined metrics. If the distributions are allowed to vary a great deal, any analysis of the sample paths can become extremely involved. Further, unlike statistics of the

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Kolmogorov-Smirnov type which are measures of the maximum sample path fluctuation, statistics of the regression type are weighted sums involving ranks and any sample path behavior as such is not directly relevant. Taking these factors into consideration, we have adopted an approach which uses only the second order properties of the processes involved, thus circumventing the need for sample path analysis.

It is well known that a second order process, that is to say a stochastic process $X(t)$ whose variance $K(t, t)$ is finite for all values of t , spans a reproducing kernel Hilbert space with kernel $K(s, t) = \text{Cov}(X(s), X(t))$. We exploit the fact that linear rank statistics can be approximated by elements of the Hilbert space spanned by suitably defined empirical processes. In general, such a Hilbert space does not characterize the process spanning it completely. However, the spanning process if Gaussian, is completely determined by its Hilbert space. We establish the asymptotic normality of linear rank statistics by proving their convergence to a point in the Hilbert space spanned by a Gaussian process. Integrals of the type considered here are used in a wide range of applications; Parzen (1958), for example, has used them in his basic work in time series analysis. But the application given here for a problem in nonparametric statistics appears to be new. Brown (1970) has some interesting results on convergence in distribution of stochastic integrals. The basic reference on reproducing kernel Hilbert spaces is Aronszajn (1950). Yosida (1972) also has useful results on reproducing kernels.

Our conditions on the score generating functions are the same as those in Puri-Sen (1971), being somewhat stronger than those of Hájek (1968) but less stringent than in Chernoff-Savage (1958). However, the centering constant, μ_n , is used throughout in the place of $E(S_n)$ used by Hájek. This,

in addition to being easier to compute, serves to unify the results of Hájek (1968) and Hoeffding (1973). We also examine the asymptotic degeneracy of S_n and in this respect our results extend both the Hájek and the Chernoff-Savage, Puri-Sen results.

2. Definitions and Terminology.

Let $X_{n1}, X_{n2}, \dots, X_{nn}$ be a sequence of independent random variables with continuous distribution functions $F_{n1}, F_{n2}, \dots, F_{nn}$ respectively. Let R_{ni} be the rank of X_{ni} among $(X_{n1}, \dots, X_{nn})$, that is

$$R_{ni} = (\text{number of } X_{nj} \leq X_{ni}); j = 1, \dots, n.$$

We define a simple linear rank statistic by

$$(2.1) \quad S_n = \sum_{i=1}^n C_{ni} a_n(R_{ni})$$

where, $(C_{n1}, \dots, C_{nn})$, $(a_n(1), \dots, a_n(n))$ are known constants, C_{ni} being known as regression constants and $a_n(i)$ as scores. One usually assumes $a_n(i)$ to be generated by a known real-valued function φ defined on $(0, 1)$ in either of the following ways:

$$(2.2) \quad (a) \quad a_n(i) = \varphi\left(\frac{i}{n+1}\right); \quad (b) \quad a_n(i) = E \varphi(U_n^{(i)})$$

where $U_n^{(i)}$ is the i th order statistic among n independent random variables distributed uniformly over $(0, 1)$. We shall assume (2.2) (a), namely, $a_n(i) = \varphi\left(\frac{i}{n+1}\right)$. Extension to case (b) can be done in one of the several ways but has little to do with the techniques developed in this paper and need not concern us. Thus, by (2.2) (a), we can write S_n as

$$(2.3) \quad S_n = \sum_{i=1}^n C_{ni} \varphi\left(\frac{R_{ni}}{n+1}\right).$$

For a fixed x , we define the empirical distribution function $H_n(x)$ for the sample $X_{n1}, \dots, X_{nn}$, by

$$(2.4) \quad H_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_{ni} \leq x)$$

where the "indicator function" $I(\cdot)$ is defined by

$$(2.5) \quad \begin{aligned} I(X_{ni} \leq x) &= 1 \text{ if } X_{ni} \leq x \\ &= 0 \text{ if } X_{ni} > x. \end{aligned}$$

We also define the combined sample distribution function by

$$(2.6) \quad F_n(x) = H(x) = \frac{1}{n} \sum_{i=1}^n F_{ni}(x).$$

For the special case, $m < n$

$$\begin{aligned} C_{n1} &= \cdots = C_{nm} = \frac{1}{m}, \quad C_{nm+1} = \cdots = C_{nn} = 0 \\ F_{n1} &= \cdots = F_{nm} = F, \quad F_{nm+1} = \cdots = F_{nn}. \end{aligned}$$

Pyke-Shorack (1968) consider the two-sample empirical process defined by

$$(2.7) \quad L_n(t) = \sqrt{n} \{F_m(H_n^{-1}(t)) - F(H^{-1}(t))\}$$

where $F_m(x)$ is the empirical distribution function corresponding to the sample $(X_{n1}, \dots, X_{nm})$. They obtain the asymptotic normality of S_n

corresponding to the two sample cases (the Chernoff-Savage theorem) by proving the weak convergence of $L_n(t)$ to a Gaussian process and also its convergence in some refined metrics. Thus for the general regression case, that is, when S_n is defined by (2.1) or (2.3), it is natural to consider the process defined by

$$(2.8) \quad L_{n,c}(t) = \sum_{i=1}^n C_{ni} \{I(X_{ni} \leq H_n^{-1}(t)) - F_{ni}(H^{-1}(t))\}.$$

Unfortunately this process is much too complex and classical weak convergence methods entail a detailed study of the sample path fluctuations of the process. Hence, we will avoid this by basing our approach on the second order behavior.

3. Stochastic Integrals of Second Order Stochastic Processes.

Let $\{X(t): t \in T\}$ be a real-valued second order stochastic process with the kernel $K(s, t) = E[X(s) X(t)]$. Let $\{Y(t): t \in T\}$ be another real-valued second order stochastic process with the kernel $R(s, t) = E[Y(s) Y(t)]$. We always assume that T is an interval on the real line.

Let $T = [a, b]$ where $a \leq b$ are real numbers. We define the second order stochastic integral of $X(t)$ with respect to $Y(t)$ as follows. We follow essentially Loève (1963).

Let $D_T: a = t_1 < t_2 < \dots < t_{n+1} = b$.

Define the random step function

$$(3.1) \quad \int_T X_{D_T}^{(t)} d Y(t) = \sum_{k=1}^n X_k \{Y(t_{k+1}) - Y(t_k)\}$$

where

$$X_k = X(t'_k), \quad t_k \leq t'_k \leq t_{k+1}.$$

Then the stochastic integral (second order) of $X(t)$ with respect to $Y(t)$ is defined by

$$(3.2) \quad \int_T X(t) d Y(t) = \lim_{\substack{\text{l.i.m.} \\ \max_{1 \leq k \leq n} |t_{k+1} - t_k| \rightarrow 0}} \int_T X_{D_T}^{(t)} d Y(t)$$

if it exists, where l.i.m. means limit in the quadratic mean.

Often we will need to define it as an improper integral on $T = (a, b)$. Then we take $T' = [\alpha, \beta]$, $a < \alpha < \beta < b$ and first define as before

$$(3.3) \quad \int_{T'} X(t) d Y(t)$$

and then define the improper integral

$$(3.4) \quad \int_T X(t) d Y(t) = \lim_{\substack{\text{l.i.m.} \\ \alpha \rightarrow a \\ \beta \rightarrow b}} \int_{T'} X(t) d Y(t)$$

if it exists.

Remark 1. These integrals depend on the "increments" of $Y(t)$ and not directly on $Y(t)$ itself.

The following theorem ensures the existence of such an integral.

Theorem 3.1. Let the second order stochastic process $X(t)$ with the kernel $K(s, t)$ be independent of the "increments" of the second order process $Y(t)$ with the kernel $R(s, t)$. Then the stochastic integral $\int_T X(t) dY(t)$ exists in the rank of (3.2) (or (3.4)) if and only if

$$\int_T \int_T K(s, t) d_{12} R(s, t)$$

exists as a Riemann-Stieltjes (perhaps improper) integral.

Proof. See Loève (1963).

Remark 2. Very often in applications, $Y(t)$ will be a nonrandom function $g(t)$. Then theorem 3.1 reduces to the following

Corollary 3.2. The stochastic integral $\int_T X(t) d g(t)$ exists if and only if $\int_T \int_T K(s, t) d g(s) d g(t)$ exists as a Riemann-Stieltjes (perhaps improper) integral.

Proof. Immediate from theorem 3.1.

The following property of Gaussian processes proves very useful.

Theorem 3.3: If $Y(t)$ is a nonrandom function integration of a Gaussian process with respect to $Y(t)$ preserves normality.

Proof. See Loève (1963).

4. Second Order Properties of Weighted Empirical Processes.

Let $X_{n1}, X_{n2}, \dots, X_{nn}, \dots$ be independent random variables with the corresponding distribution functions $F_{n1}, F_{n2}, \dots, F_{nn}, \dots$ which are continuous. We define

$$(4.1) \quad F_n = \frac{1}{n} \sum_{i=1}^n F_{ni}.$$

Let $C_{n1}, \dots, C_{nn}$ be known constants. We shall investigate the second order properties of the two processes,

$$(4.2) \quad \begin{aligned} X_n(t) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \{I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \{I(X_{ni} \leq F_n^{-1}(t)) - t\}. \end{aligned}$$

$$(4.3) \quad Y_n(t) = \frac{1}{s_n} \sum_{i=1}^n C_{ni} \{I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))\}, \quad 0 < t < 1,$$

where $s_n > 0$ is a normalizing constant and $I(\cdot)$ is defined by,

$$(4.4) \quad I(X_{ni} \leq y) = \begin{cases} 1 & \text{if } y \geq X_{ni} \\ 0 & \text{otherwise.} \end{cases}$$

The process $X_n(t)$ is, of course, the well known empirical process and has been extensively studied. We shall closely study the process $Y_n(t)$. The process $Y_n(t)$ as well as the process $L_{n,c}(t)$ defined in (2.8) have been considered by Koul (1970) and Koul and Staudte (1972) respectively. Our approach, however, is significantly different.

Lemma 4.1: $E X_n(t) = E Y_n(t) = 0$

$$(4.5) \quad \begin{aligned} E[X_n(s) X_n(t)] &= K_n(s, t) \\ &= s \wedge t - \frac{1}{n} \sum_{i=1}^n F_{ni}(F_n^{-1}(s)) \cdot F_{ni}(F_n^{-1}(t)) \end{aligned}$$

$$(4.6) \quad \begin{aligned} E(Y_n(s) Y_n(t)) &= R_n(s, t) \\ &= \frac{1}{s_n^2} \sum_{i=1}^n C_{ni}^2 \{F_{ni}(F_n^{-1}(s)) \wedge F_{ni}(F_n^{-1}(t)) - F_{ni}(F_n^{-1}(s)) F_{ni}(F_n^{-1}(t))\} \end{aligned}$$

where for any two real numbers a and b , $a \wedge b = \min(a, b)$.

Proof. Proof of $E X_n(t) = E Y_n(t) = 0$ is obvious. Further (4.5) is a special case of (4.6) and hence it suffices to prove the latter alone.

Without loss of generality let $s \leq t$. Let $u = F_n^{-1}(s) \leq v = F_n^{-1}(t)$.

Then we have

$$\begin{aligned} Y_n(s)Y_n(t) &= \frac{1}{2} \sum_{s_n}^n c_{ni}c_{nj} \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{nj} \leq v) - F_{nj}(v)\} \\ &= \frac{1}{2} \sum_{s_n}^n c_{ni}^2 \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{ni} \leq v) - F_{ni}(v)\} \\ &\quad + \frac{1}{2} \sum_{s_n}^n c_{ni}c_{nj} \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{nj} \leq v) - F_{nj}(v)\}. \end{aligned}$$

On taking expectations, the expectation of the second sum vanishes and is the first sum

$$\begin{aligned} E[\{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{ni} \leq v) - F_{ni}(v)\}] \\ = F_{ni}(u) \wedge F_{ni}(v) - F_{ni}(u) F_{ni}(v). \end{aligned}$$

This establishes (4.6).

Next we examine the L_2 -convergence of the processes $Y_n(t)$.

Definition. A sequence $Y_n(t)$ of second order processes converges in L_2 to another (second order) process $Y(t)$ if and only if

$$\|Y_n(t) - Y(t)\| = \{\text{Var}(Y_n(t) - Y(t))\}^{1/2} \rightarrow 0$$

as $n \rightarrow +\infty$, for each t .

We then write

$$Y_n(t) \xrightarrow{L_2} Y(t)$$

or

$$\|Y_n(t) - Y(t)\| \rightarrow 0.$$

We next obtain criteria for L_2 convergence of the processes $Y_n(t)$ defined in (4.3).

Theorem 4.2. The second order processes $Y_n(t)$ with $R_n(t)$ as their kernel (defined by (4.3) and (4.6) respectively) converge to a second order process $Y(t)$ if and only if, for each t ,

$$(4.7) \quad \lim_{\substack{m \rightarrow +\infty \\ n \rightarrow +\infty}} \frac{1}{s_n s_m} E \left\{ \sum_{i=1}^n \sum_{j=1}^m C_{ni} C_{mj} [I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))] \right. \\ \left. \times [I(X_{mj} \leq F_m^{-1}(t)) - F_{mj}(F_m^{-1}(t))] \right\} \\ = f(t) \text{ exists.}$$

Further, if (4.7) holds, the kernel of $Y(t)$ is given by

$$(4.8) \quad E[Y(s) Y(t)] = R(s, t) = \lim_{n \rightarrow +\infty} R_n(s, t).$$

In statistical applications, we have a continuous sampling situation; that is to say if $m \leq n$, we have, $X_{n1} = X_{m1}, \dots, X_{nm} = X_{mm}$. Then the above theorem becomes a more precise

Corollary 4.3. Let $n \geq m$ and the sequences be such that,

$X_{nj} = X_{mj}, j \leq m$. Then the process $Y_n(t)$ defined as before converges to a process $Y(t)$ with kernel $R(s, t) = \lim_{n \rightarrow \infty} R_n(s, t)$ if and only if

$$(4.9) \quad \lim_{\substack{m \rightarrow +\infty \\ n \rightarrow +\infty}} \frac{1}{s_n s_m} \sum_{i=1}^m C_{ni} C_{mi} \{F_{ni}(F_m^{-1}(t)) \wedge F_{ni}(F_n^{-1}(t)) \\ - F_{ni}(F_m^{-1}(t)) F_{ni}(F_n^{-1}(t))\}$$

exists and is finite for each t .

Theorem 4.2 is proved by a direct application of theorem A, page 469 of Loève (1963). Therefore, we shall only prove corollary 4.3.

Proof of Corollary 4.3. Note that by the hypothesis,

$F_{n1} = F_{m1}, \dots, F_{nm} = F_{mm}, m \leq n$. But $F_n = \frac{1}{n} \sum_{i=1}^n F_{ni}$ and $F_m = \frac{1}{m} \sum_{i=1}^m F_{ni}$ could be different. Also, C_{ni} may not equal C_{mi} and, s_n and s_m could also be different.

In order to simplify the notation, we shall write $u = F_n^{-1}(t)$ and $v = F_m^{-1}(t)$. Then

$$\begin{aligned}
 (4.10) \quad Y_n(t) Y_m(t) &= \frac{1}{s_n s_m} \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} C_{ni} C_{mj} \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{mj} \leq v) - F_{nj}(v)\} \\
 &= \frac{1}{s_n s_m} \sum_{i=1}^m C_{ni} C_{mi} \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{mi} \leq v) - F_{mi}(v)\} \\
 &\quad + \frac{1}{s_n s_m} \sum_{i \neq j} C_{ni} C_{mj} \{I(X_{ni} \leq u) - F_{ni}(u)\} \{I(X_{mj} \leq v) - F_{mj}(v)\}
 \end{aligned}$$

where we have used the fact that $X_{ni} = X_{mi}$ and $F_{ni} = F_{mi}$ for $i \leq m$.

Observe that:

(i) When $i \neq j$, $I(X_{mi} \leq \cdot)$ and $I(X_{mj} \leq \cdot)$ are independent. Hence

$$E\{[I(X_{ni} \leq u) - F_{ni}(u)] [I(X_{nj} \leq v) - F_{mj}(v)]\} = 0.$$

(ii) When $i = j$,

$$\begin{aligned}
 (4.11) \quad &[I(X_{ni} \leq u) - F_{ni}(u)] [I(X_{ni} \leq v) - F_{ni}(v)] \\
 &= I(X_{ni} \leq u) I(X_{ni} \leq v) - F_{ni}(v) I(X_{ni} \leq u) \\
 &= F_{ni}(u) I(X_{ni} \leq v) + F_{ni}(u) F_{ni}(v).
 \end{aligned}$$

In (4.11) for the first term,

$$E\{I(X_{ni} \leq u) I(X_{ni} \leq v)\} = F_{ni}(u \wedge v) = F_{ni}(u) \wedge F_{ni}(v).$$

Taking expectations in (4.11), we get

$$\begin{aligned}
(4.12) \quad & E[[I(X_{ni} \leq u) - F_{ni}(u)][I(X_{ni} \leq v) - F_{ni}(v)]] \\
&= F_{ni}(u) \wedge F_{ni}(v) - F_{ni}(u) F_{ni}(v) \\
&= F_{ni}(F_n^{-1}(t)) \wedge F_{ni}(F_m^{-1}(t)) - F_{ni}(F_n^{-1}(t)) F_{ni}(F_m^{-1}(t)).
\end{aligned}$$

Thus by taking expectations of (4.10) and substituting (4.12) and (i) we get the expression in (4.9). An application of theorem 4.2 proves the corollary.

5. Limit Distributions of the Processes $X_n(t)$ and $Y_n(t)$.

Since $X_n(t)$ is a special case of $Y_n(t)$, it suffices to establish the results for the latter. We have the following theorem:

Theorem 5.1. For the processes $Y_n(t)$, let

- (i) conditions of theorem 4.2 (or of corollary 4.3) hold; and let
- (ii) $(\max_{1 \leq i \leq n} |c_{ni}|) s_n^{-1} = o(\frac{1}{\sqrt{n}}).$

Then $Y_n(t) \xrightarrow{L_2} Y(t)$ where $Y(t)$ is Gaussian with the kernel

$$R(s, t) = \lim_{n \rightarrow \infty} R_n(s, t).$$

Proof. L_2 -convergence and the convergence of the kernels are immediate consequences of condition (i). It suffices to prove the asymptotic normality of the finite dimensional distributions of $Y_n(t)$.

Let $t_1, t_2, \dots, t_k$ be any fixed elements in T . Consider the random vector $(Y_n(t_1), \dots, Y_n(t_k))$.

We use the Cramer-Wold criterion.

Let $a_1, \dots, a_k$ be arbitrary constants. It suffices to show that

$$(5.1) \quad \sum_{j=1}^k a_j Y(t_j) \text{ is asymptotically normal.}$$

To simplify the notation, we write

$$u_j = F_n^{-1}(t_j), j = 1, \dots, k.$$

Then

$$\begin{aligned} (5.2) \quad \sum_{j=1}^k a_j Y(t_j) &= \sum_{j=1}^k a_j \left\{ \sum_{i=1}^n \frac{C_{ni}}{s_n} [I(X_{ni} \leq u_j) - F_{ni}(u_j)] \right\} \\ &= \sum_{i=1}^n \frac{C_{ni}}{s_n} \left\{ \sum_{j=1}^k a_j [I(X_{ni} \leq u_j) - F_{ni}(u_j)] \right\}. \end{aligned}$$

Let $\alpha_{ni} = \frac{\sqrt{n} C_{ni}}{s_n}$. Observe that

$$(5.3) \quad \alpha_{ni} = O(1) \text{ in } n.$$

Thus we can write,

$$(5.4) \quad \sum_{j=1}^k a_j Y(t_j) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha_{ni} \left\{ \sum_{j=1}^k a_j [I(X_{ni} \leq u_j) - F_{ni}(u_j)] \right\}.$$

In (5.4), observe that by (5.3), each random variable

$$\alpha_{ni} \sum_{j=1}^k a_j [I(X_{ni} \leq u_j) - F_{ni}(u_j)]$$

is bounded.

Let

$$(5.5) \quad \sigma_{ni}^2 = \alpha_{ni}^2 \text{Var} \left\{ \sum_{j=1}^k a_j [I(X_{ni} \leq u_j) - F_{ni}(u_j)] \right\}$$

and

$$(5.6) \quad \sigma_n^2 = \sum_{i=1}^n \sigma_{ni}^2.$$

Then by the bounded Liapunov theorem, $\sum_{j=1}^k a_j Y(t_j)$ is asymptotically normal provided $\sigma_n^2 \rightarrow +\infty$; if not, it is degenerate. In any event, $Y_n(t)$ is asymptotically normal, possibly degenerate. We shall examine degeneracy more closely in section 8.

6. Asymptotic Normality of Certain Stochastic Integrals of the Processes $X_n(t)$ and $Y_n(t)$.

In this section we consider the stochastic integrals X_n and Y_n given by

$$(6.1) \quad X_n = \frac{1}{\sqrt{n} s_n} \int_0^1 X_n(t) \varphi'(t) dC(t)$$

and

$$(6.2) \quad Y_n = \int_0^1 Y_n(t) \varphi'(t) dt$$

where

$$(6.3) \quad C(t) = \sum_{i=1}^n C_{ni} F_{ni} (F_n^{-1}(t))$$

and φ is a known function.

In the next theorem we obtain bounds for the variances of X_n and Y_n . This in turn will enable us to prove the asymptotic normality of S_n defined in (2.1). The variance bound for Y_n is also of independent interest because it gives a condition on φ for the process $\varphi'(t) Y_n(t)$ to possess kernels of the so-called Hilbert-Schmidt type, uniformly in n .

Theorem 6.1. Let $X_n(t)$, $Y_n(t)$ and their corresponding integrals X_n and Y_n be as defined. Let the following conditions hold:

$$(a) \quad \max_{1 \leq i \leq n} \left| \frac{C_{ni}}{s_n} \right| = O\left(\frac{1}{\sqrt{n}}\right)$$

(b) there exists $\delta > 0$ such that for some K (generic)

$$|\varphi'(t)| \leq K\{t(1-t)\}^{\delta-3/2}, \quad 0 < t < 1.$$

Then

$$(6.4) \quad \text{Var } Y_n \leq O(1) \left[\int_0^1 |\varphi'(t)| \{R_n(t, t)\}^{1/2} dt \right]^2 \leq \frac{K^2}{\delta^2} 2^{5-4\delta}$$

and

$$(6.5) \quad \text{Var } X_n \leq O(1) \left[\int_0^1 |\varphi'(t)| \{K_n(t, t)\}^{1/2} dt \right]^2 \leq \frac{K^2}{\delta^2} 2^{5-4\delta}.$$

In (6.8) observe that,

$$0 \leq F_{ni}(F_n^{-1}(t)) \leq 1$$

and

$$0 \leq 1 - F_{ni}(F_n^{-1}(t)) \leq 1.$$

Thus

$$(6.9) \quad R_n(t, t) \leq O(1) \frac{1}{n} \sum_{i=1}^n F_{ni}(F_n^{-1}(t)) = O(1)t$$

and

$$(6.10) \quad R_n(t, t) \leq O(1) \frac{1}{n} \sum_{i=1}^n [1 - F_{ni}(F_n^{-1}(t))] = O(1)(1 - t).$$

Also,

$$(6.11) \quad \int_0^1 \{R_n(t, t)\}^{1/2} |\varphi'(t)| dt \\ = \int_0^{1/2} \{R_n(t, t)\}^{1/2} |\varphi'(t)| dt + \int_{1/2}^1 \{R_n(t, t)\}^{1/2} |\varphi'(t)| dt.$$

First consider

$$\begin{aligned} \int_0^{1/2} \{R_n(t, t)\}^{1/2} |\varphi'(t)| dt &\leq O(1) \int_0^{1/2} t^{1/2} K[t(1-t)]^{\delta-3/2} dt \\ &\leq K \int_0^1 t^{\delta-1} \left(\frac{1}{2}\right)^{\delta-3/2} dt = \frac{K}{\delta} 2^{3/2-2\delta} \end{aligned}$$

using (6.9) and assumption (b).

Similarly, using (6.10) we get

$$\int_{1/2}^1 |R_n(t, t)|^{1/2} |\varphi'(t)| dt \leq \frac{K}{\delta} 2^{3/2-2\delta}.$$

Combining the two, we have (6.4). The proof of (6.5) is similar.

We are now in a position to prove the following theorem.

Theorem 6.2. Let $X_n(t)$, $Y_n(t)$, X_n and Y_n be as defined earlier. Let
the following conditions be satisfied:

(a) $\lim_{m, n \rightarrow +\infty} E(X_m(t) X_n(t))$ and $\lim_{m, n \rightarrow +\infty} E(Y_m(t) Y_n(t))$ exist and also

finite for almost all t .

$$(b) \max_{1 \leq i \leq n} \frac{|c_{ni}|}{s_n} = o\left(\frac{1}{\sqrt{n}}\right).$$

$$(c) |\varphi'(t)| \leq K [t(1-t)]^{\delta-3/2} \text{ for some } \delta > 0 \text{ and a generic constant } K.$$

Then, given $\epsilon > 0$, there exist Gaussian random variables X and Y and a positive integer n_0 such that, $n \geq n_0$ entails

$$(6.12) \quad \|X_n - X\| < \epsilon$$

and

$$(6.13) \quad \|Y_n - Y\| < \epsilon.$$

Remark. Condition (a) can be replaced by $\lim_{n \rightarrow \infty} K_n(s, t) = K(s, t)$ where $K_n(s, t) = E(X_n(s) X_n(t))$ and condition of corollary 4.2 for $Y_n(t)$. Then it follows from Shorack (1973) and our corollary 4.2 respectively that $X_n(t)$ and $Y_n(t)$ converge in L_2 to Gaussian processes $X(t)$ and $Y(t)$. We examine these conditions more closely in section 8.

Proof of Theorem 6.2. Consider $Y_n(t)$. By theorem 4.1 (or corollary 4.2) and theorem 5.1, $Y_n(t) \xrightarrow{L_2} Y(t)$ where $Y(t)$ is Gaussian with kernel

$$(6.14) \quad R(s, t) = \lim_{n \rightarrow \infty} R_n(s, t) \\ = \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{i=1}^n c_{ni}^2 \{F_{ni}(F_n^{-1}(s))(1 - F_{ni}(F_n^{-1}(t)))\}, s \leq t.$$

Let

$$(6.15) \quad Y = \int_0^1 Y(t) \varphi'(t) dt. \\ \text{Var } Y = \int_0^1 \int_0^1 \varphi'(s) \varphi'(t) R(s, t) ds dt \\ \leq \left\{ \int_0^1 |\varphi'(t)| (R(t, t))^{1/2} dt \right\}^2 \\ = \left\{ \int_0^1 \lim_{n \rightarrow \infty} |\varphi'(t)| (R_n(t, t))^{1/2} dt \right\}^2$$

$$\leq \{ \liminf_{n \rightarrow \infty} \int_0^1 |\varphi'(t)| \{R_n(t, t)\}^{1/2} dt \}^2 < +\infty.$$

Observe that by theorem 6.1, $\text{Var } Y_n$ is finite uniformly in n .

Thus

$$\begin{aligned} \|Y_n - Y\| &= \left\| \int_0^1 (Y_n(t) - Y(t)) \varphi'(t) dt \right\| \\ &\leq \int_0^1 \|Y_n(t) - Y(t)\| |\varphi'(t)| dt \rightarrow 0 \text{ as } n \rightarrow +\infty \end{aligned}$$

because $\|Y_n(t) - Y(t)\| |\varphi'(t)| \rightarrow 0$ and is dominated by the integrable sequence,

$$(\|Y_n(t)\| + \|Y(t)\|) |\varphi'(t)|.$$

Thus,

$$\|Y_n - Y\| < \epsilon \text{ for } n \text{ sufficiently large.}$$

The proof of $\|X_n - X\| < \epsilon$ is similar. X can be taken to be,

$$X = \frac{1}{\sqrt{n} s_n} \int_0^1 X(t) \varphi'(t) dC(t)$$

where $X(t)$ is Gaussian with $K(s, t) = \lim_{n \rightarrow \infty} K_n(s, t)$ and

$$C(t) = \sum_{i=1}^n c_{ni} F_{ni}^{-1}(F_n^{-1}(t)).$$

7. Asymptotic Normality of Simple Linear Rank Statistics.

We are now in a position to prove the asymptotic normality of the statistic, S_n , defined in (2.1), namely

$$S_n = \sum_{i=1}^n c_{ni} \varphi\left(\frac{R_{ni}}{n+1}\right).$$

We make the following assumptions:

(a) $|\varphi^{(i)}(t)| \leq K[t(1-t)]^{\delta-i-1/2}$, $i = 0, 1$; K, δ being generic constants.

(b) $X_n(t)$ and $Y_n(t)$ satisfy condition of theorem 4.2 (or corollary 4.3).

$$(c) \max_{1 \leq i \leq n} \frac{|c_{ni}|}{(\text{Var } S_n)^{1/2}} = o\left(\frac{1}{\sqrt{n}}\right).$$

We then have the following theorem:

Theorem 7.1. Let the independent random variables $X_{n1}, \dots, X_{nn}$ with
continuous distribution functions $F_{n1}, \dots, F_{nn}$ be such that conditions
(a), (b), and (c) above are satisfied. Then there exists a Gaussian random
variable S with mean 0 and variance 1 such that

$$(7.1) \quad \frac{S_n - \mu_n}{(\text{Var } S_n)^{1/2}} \rightarrow S \text{ in probability, } \text{Var } S_n \neq 0,$$

where

$$(7.2) \quad \mu_n = \int_0^1 \varphi(t) dC(t),$$

$$C(t) = \sum_{i=1}^n c_{ni} F_{ni}(F_n^{-1}(t)).$$

Further, variance of S_n can be replaced by the approximate variance,

$$(7.3) \quad s_n^2 = \sum_{i=1}^n \text{Var } A_{ni}(X_{ni}) = \sum_{i=1}^n s_{ni}^2$$

where

$$(7.4) \quad A_{ni}(x) = \frac{1}{n} \sum_{j=1}^n (c_{nj} - c_{ni}) \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_{nj}(y).$$

Proof. We can express the statistic S_n as

$$(7.5) \quad S_n = \int_0^1 \varphi\left(\frac{n}{n+1} H_n(t)\right) dC_n(t)$$

where

$$(7.6) \quad C_n(t) = \sum_{i=1}^n c_{ni} I(X_{ni} \leq F_n^{-1}(t))$$

$$(7.7) \quad H_n(t) = \frac{1}{N} \sum_{i=1}^n I(X_{ni} \leq F_n^{-1}(t)).$$

It is easily verified that by linear approximation, S_n is expressible as

$$(7.8) \quad S_n = \mu_n + B_{1n} + B_{2n} + D_{1n} + D_{2n} + D_{3n}$$

where μ_n is as in (7.2),

$$(7.9) \quad B_{1n} = \int_0^1 \varphi(t) d(C_n(t) - C(t))$$

$$(7.10) \quad B_{2n} = \int_0^1 (H_n(t) - t) \varphi'(t) dC(t)$$

$$(7.11) \quad D_{1n} = \frac{-1}{n+1} \int_0^1 H_n(t) \varphi'(t) dC(t)$$

$$(7.12) \quad D_{2n} = \int_0^1 (H_n(t) - t) \varphi'(t) d(C_n(t) - C(t))$$

$$(7.13) \quad D_{3n} = \int_0^1 \left\{ \varphi\left(\frac{n}{n+1} H_n(t)\right) - \varphi(t) - \left(\frac{n}{n+1} H_n(t) - t\right) \varphi'(t) \right\} dC_n(t).$$

We will show that,

- (i) $|\mu_n| < +\infty$,
- (ii) $\frac{1}{s_n} (B_{1n} + B_{2n})$ converges in probability to a normal random variable.
- (iii) $D_{in} = o_p(s_n)$, $i = 1, 2, 3$.

The following inequalities are immediate.

$$(7.14) \quad |C_n(t)| \leq n \max_{1 \leq i \leq n} |C_{ni}| H_n(t)$$

$$(7.15) \quad |C(t)| \leq n \max_{1 \leq i \leq n} |C_{ni}| F_n(t).$$

$$\begin{aligned} \text{Thus } |\mu_n| &\leq n \max_{1 \leq i \leq n} |C_{ni}| \frac{1}{n} \int_0^1 \varphi(t) \sum_{i=1}^n dF_{ni}(F_n^{-1}(t)) \\ &\leq n \max_{1 \leq i \leq n} |C_{ni}| \int_0^1 |\varphi(t)| dt < +\infty \text{ by assumption (a). This proves (i).} \end{aligned}$$

Proof of (ii): Consider $B_{1n} = \int_0^1 \varphi(t) d(C_n(t) - C(t))$.

Integration by parts yields,

$$(7.16) \quad B_{1n} = \{C_n(t) - C(t)\}\varphi(t) \Big|_0^1 - \int_0^1 \varphi'(t) \{C_n(t) - C(t)\} dt.$$

Claim: $\frac{1}{s_n} (C_n(t) - C(t)) \varphi(t) \rightarrow 0$ in probability as $t \rightarrow 0$ or 1 .

Because an argument similar to that in the proof of Theorem 6.1 yields

$$(7.17) \quad \text{Var} \left\{ \frac{\varphi(t)(C_n(t) - C(t))}{s_n} \right\} \leq O(1) \min [t \varphi^2(t), (1-t) \varphi^2(t)] \\ \leq K^2 \min [t^\delta (1-t)^{\delta-1}, t^{1-\delta} (1-t)^\delta], \\ + 0 < t < 1 \text{ which proves the claim.}$$

Next, observe that $\frac{1}{s_n} (C_n(t) - C(t)) = Y_n(t)$ and $\sqrt{n}(H_n(t) - t) = X_n(t)$.

Hence,

$$\frac{B_{1n}}{s_n} = -Y_n \quad \text{and}$$

$$\frac{B_{2n}}{s_n} = X_n$$

where X_n and Y_n have been defined in the preceding section.

By theorem 6.1, for each $\epsilon > 0$, there exist X and Y Gaussian such that,

$$\|X_n - X\| < \epsilon/2 \quad \text{and} \quad \|Y_n - Y\| < \epsilon/2.$$

Thus, $\|(X_n - Y_n) - (X - Y)\| < \epsilon$.

It remains to show that $X - Y$ is Gaussian. We can write

$$Z_n(t) = X_n(t) - Y_n(t).$$

By rearranging terms of $X_n(t)$ and $Y_n(t)$, $Z_n(t)$ can be written as

$$\sum_{i=1}^n \left(\frac{1}{\sqrt{n}} - \frac{C_{ni}}{s_n} \right) \{I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))\}$$

and conditions of the bounded Liapunov theorem are satisfied. Thus $Z_n(t)$ is asymptotically Gaussian.

Further, $X_n - Y_n$ is a linear mapping on $Z_n(t)$ and since $X - Y$ is the limit of $X_n - Y_n$, must be Gaussian. This establishes the asymptotic normality of $X_n - Y_n$.

We shall examine the behavior of $X - Y$ more closely in section 8. This proves (ii).

Here we have taken $s_n = \text{Var}(X_n - Y_n)$. Later we find an approximation for s_n .

Proof of (iii): We shall simplify the notation by writing $x = F_n^{-1}(t)$ and

$$H_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_{ni} \leq x)$$

$$C_n(x) = \frac{1}{n} \sum_{i=1}^n C_{ni} I(X_{ni} \leq x)$$

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n F_{ni}(x)$$

$$C(x) = \sum_{i=1}^n C_{ni} F_{ni}(x)$$

$$\frac{D \ln}{s_n} = \frac{-1}{(n+1)s_n} \int_{-\infty}^{\infty} H_n(x) \varphi'(F_n(x)) dC_n(x).$$

Hence

$$(7.18) \quad \left| \frac{D \ln}{s_n} \right| \leq \frac{1}{ns_n} \left| \sum_{i=1}^n \varphi'(F_n(X_{ni})) C_{ni} \right| \leq \frac{1}{n} \sum_{i=1}^n |V_{ni}|$$

(as $C_n(x)$ assigns measure C_{ni} when $x = X_{ni}$ and since $H_n(x) \leq 1$).

Where,

$$(7.19) \quad V_{ni} = \frac{C_{ni}}{s_n} \varphi'(F_n(X_{ni})).$$

Then, it suffices to show that

$$\frac{1}{n} \sum_{i=1}^n |V_{ni}| \rightarrow 0 \text{ in probability as } n \rightarrow +\infty.$$

This follows from "Particular case 1°," Loève (1963), p. 241, if we show

that $\sum_{i=1}^n \frac{E|V_{ni}|^\alpha}{n^\alpha} < +\infty$, uniformly in n for some $0 < \alpha < 1$. Take

$\alpha = 2/3$; then

$$\begin{aligned} \sum_{i=1}^n \frac{E|V_{ni}|^{2/3}}{n^{2/3}} &= \frac{1}{n^{2/3}} \sum_{i=1}^n \frac{|C_{ni}|^{2/3}}{s_n^{2/3}} E\{F_n(X_{ni})(1 - F_n(X_{ni}))\}^{2/3(\delta-3/2)} \\ &\leq \max_{1 \leq i \leq n} \frac{|C_{ni}|^{2/3}}{s_n^{2/3}} \cdot \frac{1}{n^{2/3}} \sum_{i=1}^n \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{2\delta/3-1} dF_{ni}(x) \\ &= O(1) \frac{1}{n} \sum_{i=1}^n \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{2\delta/3-1} dF_{ni}(x) \\ &= O(1) \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{2\delta/3-1} dF_n(x) < +\infty \text{ uniformly in } n. \end{aligned}$$

We have used the fact

$$\max_{1 \leq i \leq n} \left| \frac{C_{ni}}{s_n} \right|^{2/3} = O\left(\frac{1}{n^{1/3}}\right).$$

Thus $D_{1n} = o_p(s_n)$.

Next consider,

$$D_{2n} = \int_{-\infty}^{\infty} (H_n(x) - F_n(x)) \varphi'(F_n(x)) d(C_n(x) - C(x)).$$

Note that, given $\epsilon > 0$, $0 < \delta' < 1/2$, there exists a constant $C(\epsilon, \delta')$ such that,

$$(7.19) \quad P\left(\sup_x \frac{|H_n(x) - F_n(x)|}{\{F_n(x)(1-F_n(x))\}^{\delta'-1/2}} > \frac{C(\epsilon, \delta')}{\sqrt{n}}\right) < \epsilon.$$

Thus on a set of probability $> 1 - \epsilon$,

$$|H_n(x) - F_n(x)| |\varphi'(F_n(x))| \leq \frac{K C(\epsilon, \delta')}{\sqrt{n}} \{F_n(x)(1 - F_n(x))\}^{\delta^*-1}$$

where $\delta^* = \delta - \delta'$ and $\delta' < \delta$ is chosen.

Thus it suffices to show that,

$$(7.20) \quad \frac{K C(\epsilon, \delta')}{\sqrt{n} s_n} \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{\delta^*-1} d(C_n(x) - C(x)) \rightarrow 0$$

in probability.

We use the Liapunov criterion for degenerate convergence, (p. 275, B(i), Loève (1963)),

$$\begin{aligned} \frac{K}{s_n \sqrt{n}} \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{\delta^*-1} d C_n(x) \\ = \frac{K}{\sqrt{n} s_n} \sum_{i=1}^n C_{ni} \{F_n(X_{ni})(1 - F_n(X_{ni}))\}^{\delta^*-1}. \end{aligned}$$

Let

$$(7.21) \quad V_{ni} = \frac{\sqrt{n} C_{ni}}{s_n} \{F_n(X_{ni})(1 - F_n(X_{ni}))\}^{\delta^*-1}.$$

Then,

$$\begin{aligned} (7.22) \quad \frac{1}{\sqrt{n} s_n} \sum_{i=1}^n C_{ni} \{F_n(X_{ni})(1 - F_n(X_{ni}))\}^{\delta^*-1} \\ = \frac{1}{n} \sum_{i=1}^n V_{ni}. \end{aligned}$$

It remains to show $\frac{1}{n} \sum_{i=1}^n (V_{ni} - E V_{ni}) \rightarrow 0$ in probability. This will be done if we show that for some $\alpha > 0$,

$$(7.23) \quad \frac{1}{n^{1+\alpha}} \sum_{i=1}^n E |V_{ni}|^{\alpha+1} \rightarrow 0.$$

We choose an $\alpha > 0$ such that, $(1 + \alpha)(\delta^* - 1) > -1$. (I.e., $0 < \alpha < \frac{\delta^*}{1 - \delta^*}$).

$$\text{Then } \frac{1}{n^{1+\alpha}} \sum_{i=1}^n E |V_{ni}|^{1+\alpha}$$

$$\begin{aligned}
&\leq \frac{1}{n^\alpha} \cdot \frac{1}{n} \left| \frac{\sqrt{n} \max_{1 \leq i \leq n} |C_{ni}|}{s_n} \right|^{1+\alpha} \sum_{i=1}^n E\{F_n(X_{ni})(1 - F(X_{ni}))\}^{(1+\alpha)(\delta^*-1)} \\
&= O\left(\frac{1}{n^\alpha}\right) \frac{1}{n} \sum_{i=1}^n \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{(1+\alpha)(\delta^*-1)} dF_{ni}(x) \\
&= O\left(\frac{1}{n^\alpha}\right) \int_{-\infty}^{\infty} \{F_n(x)(1 - F_n(x))\}^{(1+\alpha)(\delta^*-1)} dF_n(x) \rightarrow 0 \text{ as } n \rightarrow +\infty
\end{aligned}$$

because the integral is bounded uniformly for all n . This proves (7.23).

Again we have used the fact

$$\sqrt{n} \max_{1 \leq i \leq n} |C_{ni}|/s_n = O(1).$$

This establishes $D_{2n} = o_p(s_n)$. Consider,

$$D_{3n} = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{n}{n+1} H_n(x)\right) - \varphi(F_n(x)) - \left(\frac{n}{n+1} H_n(x) - F_n(x)\right) \varphi'(F_n(x)) \right\} dC_n(x).$$

We note the following: with

$$(7.24) \quad C_{3n} = \frac{D_{3n}}{\sqrt{n} s_n}$$

it suffices to prove

$$(7.25) \quad C_{3n} = o_p(1/\sqrt{n}).$$

Observe that

$$\begin{aligned}
(7.26) \quad |C_{3n}| &\leq O(1) \int_{-\infty}^{\infty} \left| \varphi\left(\frac{n}{n+1} H_n(x)\right) - \varphi(F_n(x)) - \left(\frac{n}{n+1} H_n(x) \right. \right. \\
&\quad \left. \left. - F_n(x)\right) \varphi'(F_n(x)) \right| dH_n(x)
\end{aligned}$$

because of the fact

$$\max_{1 \leq i \leq n} \left| \frac{C_{ni}}{s_n \sqrt{n}} \right| = O\left(\frac{1}{n}\right).$$

The proof that the right hand side of (7.26) is $o_p(\frac{1}{\sqrt{n}})$ can be found in Puri and Sen (1971), pp. 401-405. This proves theorem 7.1.

Variance Computation: Since by (ii) $D_{in} = o_p(s_n)$, $i = 1, 2, 3$, we can take s_n to be the variance of $B_{1n} + B_{2n}$.

In order to simplify our notation, we write $y = F_n^{-1}(t)$. Observe that,

$$C(t) = \sum_{j=1}^n C_{nj} F_{nj}(F_n^{-1}(t)) = \sum_{j=1}^n C_{nj} F_{nj}(y) = C(y), \text{ say.}$$

With these transformations, we can write

$$\begin{aligned} (7.27) \quad B_{1n} + B_{2n} &= \frac{1}{n} \int_{-\infty}^{\infty} \sum_{i=1}^n \{I(X_{ni} \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dC(y) \\ &\quad - \int_{-\infty}^{\infty} \sum_{i=1}^n C_{ni} \{I(X_{ni} \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_n(y) \\ &= \sum_{i=1}^n A_{ni}(X_{ni}) \end{aligned}$$

where

$$\begin{aligned} A_{ni}(x) &= \frac{1}{n} \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dC(y) \\ &\quad - C_{ni} \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_n(y) \\ &= \frac{1}{n} \sum_{j=1}^n C_{nj} \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_{nj}(y) \\ &\quad - \frac{1}{n} \sum_{j=1}^n C_{ni} \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_{nj}(y). \end{aligned}$$

Thus,

$$(7.28) \quad A_{ni}(x) = \frac{1}{n} \sum_{j=1}^n (C_{nj} - C_{ni}) \int_{-\infty}^{\infty} \{I(x \leq y) - F_{ni}(y)\} \varphi'(F_n(y)) dF_{nj}(y).$$

Thus, we can express $B_{1n} + B_{2n}$ as the sum of independent random variables, $A_{ni}(X_{ni})$. If we write,

$$s_{ni}^2 = \text{Var } A_{ni}(X_{ni}),$$

we can then take

$$s_n^2 = \sum_{i=1}^n s_{ni}^2.$$

8. Asymptotic Degeneracy of S_n :

From the expression (7.8) for S_n , namely,

$$S_n = \mu_n + B_{1n} + B_{2n} + D_{1n} + D_{2n} + D_{3n},$$

it is clear that we need only examine the limiting behavior of the terms $B_{1n} + B_{2n}$. With the help of the expression (7.4) for $A_{ni}(\cdot)$ we can express $B_{1n} + B_{2n}$ as a sum of independent random variables which would enable standard tools from stability theory to be used. But a glance at the expression for $A_{ni}(\cdot)$ suffices to convince us of the impracticality of the situation. We shall adopt a different procedure using the variance inequalities proved in section 6.

Clearly, s_n are only normalizing constants and we can replace C_{ni}/s_n by C_{ni} themselves but now with the condition,

$$(8.1) \quad \max_{1 \leq i \leq n} |C_{ni}| = o\left(\frac{1}{\sqrt{n}}\right).$$

Then,

$$(8.2) \quad X_n = \frac{1}{\sqrt{n}} \int_0^1 X_n(t) \varphi'(t) dC(t)$$

$$(8.3) \quad Y_n = \int_0^1 Y_n(t) \varphi'(t) dt$$

where now

$$(8.4) \quad Y_n(t) = \sum_{i=1}^n c_{ni} \{I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))\}.$$

In view of (8.1), $Y_n(t)$ is asymptotically either Gaussian or degenerate.

We next examine conditions for the degeneracy (or non-degeneracy) of

$X_n - Y_n$ in the limit.

Assuming that the distribution functions F_{ni} possess densities

$f'_{ni} = f_{ni}$, we can express

$$(8.5) \quad X_n = \int_0^1 \varphi'(t) \frac{C'(t)}{\sqrt{n}} X_n(t) dt$$

where

$$(8.6) \quad C'(t) = \left\{ \frac{1}{n} \sum_{i=1}^n f_{ni}(F_n^{-1}(t)) \right\}^{-1} \left\{ \sum_{i=1}^n c_{ni} f_{ni}(F_n^{-1}(t)) \right\}.$$

We then have the following proposition.

Proposition. Let condition (8.1) be satisfied. With X_n and Y_n defined by (8.2) and (8.3) respectively, $S_n = X_n - Y_n$ is asymptotically degenerate whenever

$$(8.7) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{C'(t)}{n} - c_{ni} \right)^2 F_{ni}(F_n^{-1}(t))(1 - F_{ni}(F_n^{-1}(t))) = 0$$

almost everywhere on $(0, 1)$.

It is, of course, assumed that φ and φ' satisfy the same conditions as before.

Proof. It is easy to verify that

$$S_n = X_n - Y_n = \int_0^1 \varphi'(t) \sum_{i=1}^n \left(\frac{C'(t)}{n} - c_{ni} \right) I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t)) dt.$$

Hence

$$\begin{aligned} \text{Var } S_n &= \int_0^1 \int_0^1 \varphi'(s) \varphi'(t) \sum_{i=1}^n \left| \frac{C'(t)}{n} - c_{ni} \right|^2 [F_{ni}(F_n^{-1}(s) \wedge F_n^{-1}(t)) - \\ &\quad - F_{ni}(F_n^{-1}(s)) F_{ni}(F_n^{-1}(t))] ds dt \\ &\leq \left[\int_0^1 |\varphi'(t)| \left(\sum_{i=1}^n \left| \frac{C'(t)}{n} - c_{ni} \right|^2 F_{ni}(F_n^{-1}(t)) (1 - F_{ni}(F_n^{-1}(t))) \right)^{1/2} dt \right]^2. \end{aligned}$$

By theorem 6.1, (variance inequalities) the last quantity above is uniformly bounded in N . Now it is easily verified that (8.7) entails $\text{Var } S_n \rightarrow 0$. This completes the proof.

Remark: A consequence of condition (8.7) is that whether S_n is asymptotically degenerate or not depends very heavily on the underlying distributions and it may not be possible to secure nondegeneracy by any choice of the regression constants c_{ni} and the score generating function φ . The difficulty is that the so-called Kolmogorov-Smirnov bounds, namely

$$P\left(\sup_t \frac{1}{\sqrt{n}} \left| \sum_{i=1}^n \{I(X_{ni} \leq F_n^{-1}(t)) - F_{ni}(F_n^{-1}(t))\} \right| > \alpha\right)$$

can be very small, when the underlying distributions are nonidentical. An example following theorem 2.11.8, p. 41 of Puri-Sen (1971), illustrates this point.

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