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ASYMPTOTIC NORMALITY AND CONVERGENCE RATES OF LINEAR RANK STATISTICS (U)
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Asymptotic Normality and Convergence Rates
of Linear Rank Statistics Under AlternativesBy Madan L. Puri and Navaratna S. Rajaram
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0. Summary. Asymptotic normality of linear rank statistics under alternatives is proved employing techniques of Chernoff and Savage, (1958). Under suitable assumptions the rate of convergence to normality under alternatives is also obtained for such statistics. The results of the paper are related to those of Hájek (1968) and Hoeffding, (1973). Results on the rate of convergence extend those of Jurečková and Puri, (1976), and Bergström and Puri, (1976).

1. Introduction. Let $\{X_{Ni}, i \geq 1\}$ be a sequence of independent random variables with continuous cdfs (cumulative distribution functions) $\{F_{Ni}, i \geq 1\}$ respectively. Consider a linear rank statistic S_N given by

$$(1.1) \quad S_N = \sum_{i=1}^N c_{Ni} a_N(R_{Ni})$$

where R_{Ni} is the rank of X_{Ni} in $(X_{N1}, \dots, X_{NN})$, $(c_{N1}, \dots, c_{NN})$ are known (regression) constants, and $a_N(1), \dots, a_N(N)$ are "scores" generated by a known real-valued function $\varphi(t)$, $0 < t < 1$ in either of the following ways:

$$(1.2) \quad a_N(i) = \varphi(i/(N+1)), \quad 1 \leq i \leq N$$

$$(1.3) \quad a_N(i) = E \varphi^{(i)}(U_N^{(i)}), \quad 1 \leq i \leq N$$

where $U_N^{(i)}$ is the i th order statistic in a sample of size N from the rectangular distribution over $(0,1)$.

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A. D. BLOSE
Technical Information Officer

We make the following assumptions:

$$(I A) \quad \max_{1 \leq i \leq N} |c_{Ni}| / s_N^{-1/2} = O(N^{-1/2})$$

$$(I B) \quad |\varphi^{(i)}(t)| \leq K [t(1-t)]^{\delta \cdot i - 1/2}, \quad i = 0, 1; \quad \delta > 0,$$

K a generic constant.

s_N^2 is approximate variance of S_N and is given by (1.7) and (1.8) below.

Our main results are the following:

Theorem 1.1. Let the scores $a_N(i)$, $1 \leq i \leq N$ be defined as in (1.2).
Then, under the assumptions (I A) and (I B),

$$(1.4) \quad \sup_x |P\left(\frac{S_N - \mu_N}{s_N} \leq x\right) - \Phi(x)| \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad s_N \neq 0$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt,$$

$$(1.5) \quad \mu_N = \sum_{i=1}^N c_{Ni} \int_{-\infty}^{\infty} \varphi(H(x)) dF_{Ni}(x),$$

$$(1.6) \quad H(x) = \frac{1}{N} \sum_{i=1}^N F_{Ni}(x),$$

$$(1.7) \quad s_N^2 = \sum_{i=1}^N s_{Ni}^2; \quad s_{Ni}^2 = \text{Var}(A_{Ni}(X_{Ni})),$$

$$(1.8) \quad A_{Ni}(x) = \frac{1}{N} \sum_{j=1}^N (c_{Ni} - c_{Nj}) \int \{I_{[x \leq y]} - F_{Ni}(y)\} \varphi'(H(y)) dF_N(y),$$

and

$$I_{[x \leq y]} = \begin{cases} 0 & \text{if } y < x \\ 1 & \text{if } y \geq x \end{cases}$$



Corollary 1.1. Let $\varphi(t) = F^{-1}(t)$, where F is a cdf. Let the scores $a_N(i)$, $1 \leq i \leq N$ be given by (1.3). Then, under the assumptions (IA) and (IB), the conclusions of Theorem 1.1 hold.

Theorem 1.2. Let the scores $a_N(i)$, $1 \leq i \leq N$ be given by (1.2) and the assumption (IA) be satisfied. Let

$$(1.10) \quad \sup_x |\varphi'(x)| = \|\varphi'\| < \infty$$

then

$$(1.11) \quad \sup_x \left| P\left(\frac{S_N - \mu_N}{s_N} \leq x\right) - \Phi(x) \right| \leq c_1 \ell_{3,N} + \Delta_N$$

where

$$(1.12) \quad \ell_{3,N} = \rho_N^3 / s_N^3, \quad c_1 = 0.7985$$

$$(1.13) \quad \rho_N^3 = \sum_{i=1}^N \rho_{Ni}^3, \quad \rho_{Ni}^3 = E |A_{Ni}(X_{Ni})|^3, \text{ and } \Delta_N \rightarrow 0$$

Furthermore,

$$(1.14) \quad N^{\frac{1}{2}} \Delta_N = o_p(\|\varphi'\| s_N^{-1} \sum_{i=1}^N |c_{Ni}|)$$

Remark. Theorem 1.1 has been proved by Hájek (1968). His conditions on the score generating function φ are milder than ours but the conclusion of our theorem 1.1 is sharper in the sense that the centering constant μ_N appears naturally in place of ES_N given by Hájek (1968). Corollary 1.1 is an extension of a similar result proved by Chernoff and Savage (1958) for the two sample problem and serves to remove some of the complications encountered in Hoeffding (1973). Theorem 1.2 is related to the results of Bickel (1972), Jurečková-Puri (1975), and Bergström-Puri (1976). However, the bounds obtained in these papers are non-random and thus sharper than ours. On the other hand, our conditions on the score generating function φ are milder. We believe that the theorem 1.2 is true even when the condition (1.10) is replaced by the assumption (IB). At the present time, the theory of asymptotic expansion for sums of dependent random variables is still at a rudimentary stage (Stein (1970)) and it is doubtful if the random term Δ_N can be removed without additional assumptions on the underlying distributions.

Asymptotic Normality of S_N . The proof of theorem 1.1 (and corollary 1.2) will be along the lines of the Chernoff-Savage theorem (1958) as given in Puri and Sen (1971) with some modifications necessitated by greater generality of the present problem.

First we introduce the following notations:

$$(2.1) \quad H_N(x) = \frac{1}{N} \sum_{i=1}^N I_{[X_{Ni} \leq x]}$$

$$(2.2) \quad H(x) = \frac{1}{N} \sum_{i=1}^N F_{Ni}(x)$$

$$(2.3) \quad C_N(x) = \sum_{i=1}^N C_{Ni} I_{[X_{Ni} \leq x]}$$

$$(2.4) \quad C(x) = \sum_{i=1}^N C_{Ni} F_{Ni}(x) .$$

Note that the functions $H_N(x)$ and $C_N(x)$ are stochastic variables whereas $H(x)$ and $C(x)$ are non-random though depending on N .

Then the following inequalities are obvious:

$$(2.5) \quad |C_N(x)| \leq N \max_{1 \leq i \leq N} |C_{Ni}| H_N(x)$$

$$(2.6) \quad |C(x)| \leq N \max_{1 \leq i \leq N} |C_{Ni}| H(x) , \quad -\infty < x < \infty .$$

Proof of Theorem 1.1. We rewrite S_N defined in (1.1) as

$$(2.7) \quad \begin{aligned} S_N &= \int_{-\infty}^{\infty} \varphi\left(\frac{N}{N+1} H_N(x)\right) dC_N(x) \\ &= \mu_N + B_{1N} + B_{2N} + \sum_{i=1}^3 D_{iN} \end{aligned}$$

where μ_N is given by (1.5), and

$$(2.8) \quad B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x)) d(C_N(x) - C(x)) ,$$

$$(2.9) \quad B_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(H(x)) dC(x) ,$$

$$(2.10) \quad D_{1N} = \frac{-1}{N+1} \int_{-\infty}^{\infty} H_N(x) \varphi'(H(x)) dC_N(x) ,$$

$$(2.11) \quad D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(H(x)) d(C_N(x) - C(x)) ,$$

$$(2.12) \quad D_{3N} = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N}{N+1} H_N(x) - H(x)\right) \varphi'(H(x)) \right\} dC_N(x) .$$

The proof will be accomplished if we establish the following:

$$(a) \quad |\mu_N| < \infty ,$$

$$(b) \quad (B_{1N} + B_{2N})/s_N \text{ is asymptotically normal,}$$

$$(c) \quad D_{iN} = o_p(s_N), \quad i = 1, 2, 3 .$$

Proof of (a). Using (2.4) and (2.6), we obtain

$$(2.13) \quad |\mu_N| \leq N \max_{1 \leq i \leq N} |C_{Ni}| \int_{-\infty}^{\infty} |\varphi(H(x))| dH(x) < \infty, \text{ by assumption (I B) .}$$

Proof of (b). To prove (b), we shall verify the Liapunov condition for B_{1N}/s_n and B_{2N}/s_n . Integrating B_{2N} by parts, we obtain

$$(2.14) \quad B_{2N} = [H_N(x) - H(x)] B^*(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} B^*(x) d[H_N(x) - H(x)]$$

where

$$B^*(x) = \int_{x_0}^x \varphi'(H(x)) dC(x)$$

where x_0 is determined arbitrarily such that $H(x_0) > 0$.

We first consider $\int_{-\infty}^{\infty} B^*(x) d[H_N(x) - H(x)]$

$$(2.15) \quad \int_{-\infty}^{\infty} B^*(x) d[H_N(x) - H(x)] = \sum_{i=1}^N \left[B^*(x_{Ni}) - E B^*(x_{Ni}) \right] / N$$

We verify the condition $\frac{1}{s_N^{2+\delta'}} \sum_{i=1}^N E \left[|B^*(x_{Ni}) - E B^*(x_{Ni})| / N \right]^{2+\delta'} \rightarrow 0$

as $N \rightarrow \infty$ for some $\delta' > 0$. To verify this, it suffices to show that

$$(2.16) \quad \frac{1}{s_N^{2+\delta'}} \sum_{i=1}^N E \left[|B^*(x_{Ni})| / N \right]^{2+\delta'} \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Choose $\delta' > 0$ such that $(2 + \delta')(\delta - \frac{1}{2}) > 1$. Then

$$(2.17) \quad \frac{1}{s_N^{2+\delta'}} \sum_{i=1}^N E |B^*(x_{Ni}) / N|^{2+\delta'} = \frac{1}{s_N^{2+\delta'} \cdot N^{2+\delta'}} \sum_{i=1}^N \int_{-\infty}^{\infty} \left| \int_{x_0}^x \varphi'(H(y)) dC(y) \right|^{2+\delta'} dF_{Ni}(x)$$

$$\leq \frac{1}{s_N^{2+\delta'} \cdot N^{2+\delta'}} \sum_{i=1}^N N^{2+\delta'} \max_{1 \leq i \leq N} |C_{Ni}|^{2+\delta'} \int_{-\infty}^{\infty} \left| \int_{x_0}^x \varphi'(H(y)) dH(y) \right|^{2+\delta'} dF_{Ni}(x)$$

$$\leq \left\{ \max_{1 \leq i \leq N} |C_{Ni}| / s_N \right\}^{2+\delta'} \sum_{i=1}^N \int_{-\infty}^{\infty} \left\{ |\varphi(H(x))| + |\varphi(H(x_0))| \right\}^{2+\delta'} dF_{Ni}(x)$$

$$= O(N^{-\delta'/2}) \int_{-\infty}^{\infty} \left\{ |\varphi(H(x))| + |\varphi(H(x_0))| \right\}^{2+\delta'} dH(x).$$

We now show that

$$\frac{\beta(x)}{s_N} \Big|_{-\infty}^{\infty} = o_p(1), \text{ where } \beta(x) = [H_N(x) - H(x)] B^*(x) \quad (\text{see (2.14)}).$$

We note that

$$\begin{aligned}
\left| \frac{F_N(x)}{s_N} \right| &= \frac{1}{N} |H_N(x) - H(x)| \cdot \frac{1}{s_N} \left| \int_{x_0}^x \varphi'(H(y)) dC(y) \right| \\
&\leq \frac{1}{N} |H_N(x) - H(x)| O(1) \left| \int_{x_0}^x \varphi'(H(y)) dH(y) \right| \\
&\leq k \frac{1}{N} |H_N(x) - H(x)| \{H(x)(1-H(x))\}^{\delta-1/2}
\end{aligned}$$

Now since $\forall \epsilon > 0, \delta' > 0, \exists c(\epsilon, \delta') \ni$

$$(2.18) \quad P \left[\sup_x \frac{1}{N} \frac{|H_N(x) - H(x)|}{\{H(x)(1-H(x))\}^{\delta-1/2}} > c(\epsilon, \delta') \right] < \epsilon,$$

it follows that with probability $> 1 - \epsilon$,

$$\left| \frac{F_N(x)}{s_N} \right| \leq k \{H(x)(1-H(x))\}^{\delta-\delta'} c(\epsilon, \delta') \rightarrow 0 \text{ as } x \rightarrow \pm \infty$$

by choosing $\delta' < \delta$.

Thus the Liapunov condition (2.16) is satisfied for B_{2N}/s_N . The verification (of the Liapunov condition) for B_{1N}/s_N is similar, and the same is true for $(B_{1N} + B_{2N})/s_N$ by using the C_r -inequality. This proves (b).

Proof of (c).

$$(2.19) \quad \left| \frac{1_N}{N} \right| \leq \frac{1}{Ns_N} \left| \sum_{i=1}^N \varphi'(H(X_{Ni})) C_{Ni} \right| \leq \frac{1}{N} \sum_{i=1}^N V_{Ni}$$

where

$$(2.20) \quad V_{Ni} = \left| \varphi'(H(X_{Ni})) \frac{C_{Ni}}{s_N} \right|$$

To establish that $\frac{1}{N} \sum_{i=1}^N V_{Ni} \rightarrow 0$ in probability, it suffices to show that

$$(2.21) \quad \frac{1}{N} \sum_{i=1}^N E|V_{Ni}|^\alpha < \infty \text{ for some } 0 < \alpha < 1 \text{ (cf. Loève (1963), page 241).}$$

Taking $\alpha = 2/3$,

$$\begin{aligned} \frac{1}{N^{2/3}} \sum_{i=1}^N E |V_{Ni}|^{2/3} &\leq \frac{K}{N^{2/3}} \max_{1 \leq i \leq N} \left| \frac{C_{Ni}}{s_N} \right|^{2/3} \times \\ &\times \sum_{i=1}^N \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\frac{2\delta}{3}-1} dF_{Ni}(x) \\ &\leq K \int_0^1 \{u(1-u)\}^{\frac{2\delta}{3}-1} du < \infty \text{ uniformly in } N. \end{aligned}$$

Now consider

$$D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(H(x)) d(C_N(x) - C(x))$$

Noting (2.18), it follows that with probability $> 1-\epsilon$,

$$|H_N(x) - H(x)| |\varphi'(H(x))| \leq \frac{K}{N^{1/2}} C(\epsilon, \delta') \{H(x)(1-H(x))\}^{\delta-\delta'-1}$$

Setting $0 < \delta^* = \delta - \delta'$ Choosing $\delta' < \delta$, it suffices to show that

$$\begin{aligned} (2.22) \quad \frac{1}{s_N N^{1/2}} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\delta^*-1} dC_N(x) &= \frac{1}{s_N N^{1/2}} \sum_{i=1}^N C_{Ni} \times \\ &\times \{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1} \end{aligned}$$

Setting

$$(2.23) \quad V_{Ni} = \frac{\frac{1}{2} C_{Ni}}{s_N} \{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1},$$

we have to show that

$$(2.24) \quad \frac{1}{N} \sum_{i=1}^N \{V_{Ni} - E V_{Ni}\} \rightarrow 0 \text{ in probability.}$$

This will follow if we show that for some $\alpha > 0$,

$$(2.25) \quad N^{-(1+\alpha)} \sum_{i=1}^N E |V_{Ni}|^{1+\alpha} \rightarrow 0$$

Choose $\alpha > 0$ such that $(1+\alpha)(\delta^* - 1) > -1$ (i.e. $0 < \alpha < \frac{\delta^*}{1-\delta^*}$). Then

$$\begin{aligned}
 (2.26) \quad N^{-(1+\alpha)} \sum_{i=1}^N E |V_{Ni}|^{1+\alpha} &\leq \frac{1}{\alpha N \cdot N} \left| \frac{\max_{1 \leq i \leq N} |C_{Ni}|}{s_N} \right|^{1+\alpha} \times \\
 &\times \sum_{i=1}^N E \{H(X_{Ni})(1-H(X_{Ni}))\}^{(1+\alpha)(\delta^*-1)} \\
 &= O(1) \frac{1}{N^\alpha} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{(1+\alpha)(\delta^*-1)} dH(x) \rightarrow 0, \text{ as } N \rightarrow \infty.
 \end{aligned}$$

This proves (2.25).

Finally, consider

$$D_{3N} = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N}{N+1} H_N(x) - H(x)\right) \varphi'(H(x)) \right\} dC_N(x)$$

We have to show that $\frac{D_{3N}}{s_N} = o_p(1)$. Denote

$$(2.27) \quad C_{3N} = \frac{D_{3N}}{\frac{1}{2} s_N},$$

and note that

$$(2.28) \quad |C_{3N}| \leq O(1) \int_{-\infty}^{\infty} \left| \varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N}{N+1} H_N(x) - H(x)\right) \varphi'(H(x)) \right|$$

$$\varphi'(H(x)) |dH_N(x), \text{ since } \max_{1 \leq i \leq N} \frac{|C_{Ni}|}{s_N N^{\frac{1}{2}}} = O(N^{-1}).$$

The proof that the right hand side of (2.28) is $o_p(N^{-\frac{1}{2}})$ follows precisely as in Puri and Sen (1971, pages 401-405). Thus the proof of Theorem 1.1 follows.

Proof of Corollary 1.1. Denote

$$\varphi_N(t) = \sum_{i=1}^{[Nt]} a_N(i), \quad 0 \leq t < 1; \quad a_N(i) = E \varphi(U_N^{(i)}), \quad 1 \leq i \leq N$$

where $[x]$ is the greatest integer $\leq x$; and let

$$(2.29) \quad S_N^* = \int_{-\infty}^{\infty} \varphi_N \left(\frac{N H_N(x)}{N+1} \right) dC_N(x)$$

The proof of the corollary will be accomplished if we show that $(S_N - S_N^*)/s_N \rightarrow 0$ in probability, as $N \rightarrow \infty$.

Lemma 2.1. Under the hypotheses of theorem 1.1 and Corollary 1.1,

$$(2.30) \quad \lim_{N \rightarrow \infty} \varphi_N(t) = \varphi(t)$$

$$(2.31) \quad \left| \int_{-\infty}^{\infty} \left\{ \varphi_N \left(\frac{N}{N+1} H_N(x) \right) - \varphi \left(\frac{N}{N+1} H_N(x) \right) \right\} dC_N(t) \right| = o_p(s_N).$$

Proof. The proof of (2.30) is well known (see Puri and Sen (1971), pages 408-409); and so we prove (2.31).

Using (2.5), we obtain

$$(2.32) \quad \begin{aligned} & \left| \int_{-\infty}^{\infty} \left\{ \varphi_N \left(\frac{N H_N(x)}{N+1} \right) - \varphi \left(\frac{N H_N(x)}{N+1} \right) \right\} dC_N(x) \right| \\ & \leq \max_{1 \leq i \leq N} |C_{Ni}| \sum_{i=1}^N \left| \varphi_N \left(\frac{N H_N(X_{Ni})}{N+1} \right) - \varphi \left(\frac{N H_N(X_{Ni})}{N+1} \right) \right| \\ & = \max_{1 \leq i \leq N} |C_{Ni}| \sum_{i=1}^N \left| \varphi_N \left(\frac{i}{N+1} \right) - \varphi \left(\frac{i}{N+1} \right) \right| \end{aligned}$$

Now by assumption I (A),

$$(2.33) \quad \max_{1 \leq i \leq N} \left| \frac{C_{Ni}}{s_N} \right| \sum_{i=1}^N \left| \varphi_N \left(\frac{i}{N+1} \right) - \varphi \left(\frac{i}{N+1} \right) \right| = O(1) N^{-1/2} \sum_{i=1}^N \left| \varphi_N \left(\frac{i}{N+1} \right) - \varphi \left(\frac{i}{N+1} \right) \right|$$

which $\rightarrow 0$ as $N \rightarrow \infty$ (cf. Puri and Sen (1971), pages 409-411).

This proves the Lemma; and hence the corollary 1.1.

To compute the $\text{var}(B_{1N} + B_{2N})$, note that

$$B_{1N} = \sum_{i=1}^N \varphi(H(X_{Ni})) C_{Ni} + \text{constant}.$$

$$B_{2N} = -\frac{1}{N} \sum_{i=1}^N B^*(X_{Ni}) + \text{constant, where}$$

$$B^*(x) = \int_{x_0}^x \varphi'(H(y)) d\mathcal{C}(y) = \sum_{j=1}^N C_{Nj} \int_{x_0}^x \varphi'(H(y)) dF_{Nj}(y)$$

Noting that $\varphi(H(x)) = \int_{x_0}^x \varphi'(H(y)) dH(y) + \varphi(H(x_0))$, and setting

$$\begin{aligned} A_{Ni}(x) &= \frac{C_{Ni}}{N} \sum_{j=1}^N \int_{x_0}^x \varphi'(H(y)) dF_{Nj}(y) - \frac{1}{N} \sum_{j=1}^N C_{Nj} \int_{x_0}^x \varphi'(H(y)) dF_{Nj}(y) \\ &= \frac{1}{N} \sum_{j=1}^N (C_{Ni} - C_{Nj}) \int_{x_0}^x \varphi'(H(y)) dF_{Nj}(y) \end{aligned}$$

we obtain

$$\text{Var}(B_{1N} + B_{2N}) = \sum_{i=1}^N \text{var}(A_{Ni}(X_{Ni})).$$

Proof of Theorem 1.2. Using (2.7), we have

$$(2.34) \quad \frac{S_N - u_N}{s_N} = \frac{T_N}{s_N} + \frac{D_N}{s_N}$$

where

$$(2.35) \quad T_N = B_{1N} + B_{2N}, \text{ and } D_N = D_{1N} + D_{2N} + D_{3N}.$$

Denote

$$(2.36) \quad F_N(x) = P\left(\frac{S_N - u_N}{s_N} \leq x\right), \quad G_N(x) = P\left(\frac{T_N}{s_N} \leq x\right)$$

Then

$$\begin{aligned} (2.37) \quad F_N(x) - \Phi(x) &= G_N\left(x - \frac{D_N}{s_N}\right) - \Phi(x) \\ &= \left[G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)\right] + \left[\Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x)\right] \end{aligned}$$

Since (by Polya's theorem), both $F_N(x) - \Phi(x)$ and

$G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)$ converge to 0 uniformly in x , it

follows that the random quantity

$$(2.38) \quad \Delta_N = \sup_x \left| \Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x) \right| \text{ converges to zero.}$$

We now estimate

$$(2.39) \quad \sup_x \left| G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right) \right| = \sup_x \left| G_N(x) - \Phi(x) \right|$$

Observe that by the Berry-Esseen theorem (cf. Feller (1971), page 544),

$$(2.40) \quad \sup_x \left| G_N(x) - \Phi(x) \right| \leq \frac{C \rho_N^3}{s_N^3}$$

where C can be taken to be .7975 (cf. Van Beek (1972), and Bhattacharya-Ranga Rao (1976)).

From (2.40) and (2.37), we have

$$(2.41) \quad \sup_x \left| F_N(x) - F(x) \right| \leq \frac{C \rho_N^3}{s_N^3} + \Delta_N$$

It remains to show that $N^{\frac{1}{2}} \Delta_N = O_p(\|\varphi'\| s_N^{-1} \sum_{i=1}^N |C_{Ni}|)$

From (2.38), since

$$(2.42) \quad \left| \Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x) \right| = \left| \frac{D_N}{s_N} \right| \left| \Phi'\left(x - \frac{\alpha D_N}{s_N}\right) \right| \text{ for some } \alpha, 0 \leq \alpha \leq 1$$

$$\leq \frac{1}{\sqrt{2\pi}} \left| \frac{D_N}{s_N} \right|$$

it suffices to show that $N^{\frac{1}{2}} \Delta_N = O_p(\|\varphi'\| s_N^{-1} \sum_{i=1}^N |C_{Ni}|)$.

Now re-arranging the terms of D_N , we obtain

$$(2.43) \quad D_N = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) \right\} dC_N(x) - \int_{-\infty}^{\infty} (H_N(x) - H(x)) dC(x)$$

To simplify the proof, we drop the factor $\frac{N}{N+1}$ since it does not affect the conclusion. By the mean-value theorem

$$(2.44) \quad \varphi(H_N(x)) - \varphi(H(x)) = (H_N(x) - H(x)) \varphi'(\xi_N(x)) \text{ for some } \xi_N(x).$$

Hence

$$\left| \frac{N^{\frac{1}{2}}}{s_N} \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(\xi_N(x)) dC_N(x) \right| \leq \frac{\|\varphi'\|}{s_N} \int_{-\infty}^{\infty} N^{\frac{1}{2}} (H_N(x) - H(x)) dC_N(x)$$

Let $\epsilon > 0$ be given. Then from Puri and Sen (1971), there exists a constant $C(\epsilon)$ such that with probability $> 1 - \epsilon$,

$$\sup_x N^{\frac{1}{2}} |H_N(x) - H(x)| < C(\epsilon)$$

Hence, with probability $> 1 - \epsilon$,

$$(2.45) \quad \left| \frac{N^{\frac{1}{2}}}{s_N} \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(\xi_N(x)) dC_N(x) \right| \leq \frac{C(\epsilon)}{s_N} \|\varphi'\| \sum_{i=1}^N |C_{Ni}|$$

The proof of

$$(2.46) \quad \left| \frac{1}{s_N} \int_{-\infty}^{\infty} N^{\frac{1}{2}} (H_N(x) - H(x)) \varphi'(H(x)) dC(x) \right| \leq \frac{C(\epsilon)}{s_N} \|\varphi'\| \sum_{i=1}^N |C_{Ni}|$$

in probability is identical.

(2.45) and (2.46) establish the theorem (1.2).

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20 Abstract

Hoeffding (1973, Ann. Statist., 1, 54-66). Results on the rate of convergence extend those of Jurečkova and Puri (1975, Ann. Prob., 526-533), and Bergström and Puri (1977, Ann. Statist.).

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