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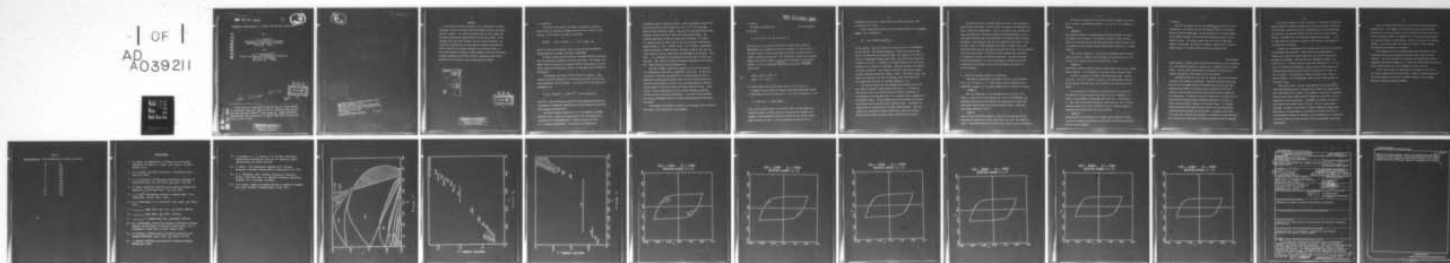
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FREQUENCY ENTRAINMENT OF A FORCED VAN DER POL OSCILLATOR*

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0. Introduction.

The van der Pol equation is typical of components arising in a wide variety of applications ranging from electrical circuitry to physiology. In this paper, we study it in the form

$$d^2u/dt^2 + k(u^2 - 1) du/dt + u = \mu k B \cos(\mu t + \alpha)$$

where k is the tuning parameter, and B , μ and α are the (normalized) forcing amplitude, frequency and phase, respectively.

Periodic solutions of this equation, and their stability properties have been extensively studied in the past fifty years. The regular case ($k \ll 1$) has been studied with averaging methods by Bogoliuboff, et.al.[1], and by analytic and topological methods by Cartwright and Littlewood and others[2,3].

The singular case ($k \gg 1$) is more difficult to analyze. Early studies of the free problem ($B = 0$) provided descriptions of the relaxation oscillation and its period[4]. The period of the free oscillation was shown to be

$$T = (3 - 2 \log 2) k + 7.014 k^{-1/3} - 1.325 + O(\log k / k)$$

for $k \gg 1$. Later studies gave methods for approximating the relaxation oscillation by complicated combinations of matched asymptotic expansions and averaging procedures [4,5].

The forced singular problem ($k \gg 1$, $B \neq 0$) exhibits remarkable solutions, and it serves as the main point of the investigations by Cartwright and Littlewood[3,6-10]. A similar problem, but with $u^2 - 1$ replaced by $\text{signum}(u^2 - 1)$ was studied by Levinson [11]

by patching together explicit solutions. These investigations showed that there are parameter values among $0 < B < 2/3$, $k \gg 1$, for which two stable periodic solutions coexist. These are distinct subharmonics having different periods, say $2n \pm 1$ times the forcing period $2\pi/\mu$. One striking implication of this was pointed out by Levinson: For fixed B , k and μ in such an overlap region, there are two stable subharmonics having periods $T_0 = (2n - 1)2\pi/\mu$ and $T_1 = (2n + 1)2\pi/\mu$, respectively. Given any sequence of zeros and ones, let marks be made on the time axis using lengths T_0 and T_1 , successively as prescribed by the sequence. Then there is a solution of the equation having zeros near and only near the marks. Thus, there is a kind of randomness exhibited by the solution set. These solutions will be discussed later.

Two other studies of the forced singular case that are related to our investigation were made by Hayashi[12] and in [13]. In the first, a numerical calculation was performed on an analog device for fixed k . Subharmonics having periods 2,3,4,5 and 6 times the forcing period were found for various values of B and μ , and an overlap region for solutions having periods one and three times the forcing period was found. In the second, a formal construction was used to derive necessary conditions for the existence of certain subharmonics. It was shown that conditions for various subharmonics could be satisfied simultaneously by some values of B and k .

Our methods are described in section 1 of this paper, and in section 2, the results of our calculations are presented.

1. Methods.

The model is studied here for the parameters satisfying

$$\alpha = 0, \mu = 1, 0 < B < .8, 0 < 1/k < .2.$$

The choice $\alpha = 0$ of course is unimportant since this can always be accomplished by a translation of the time variable. Our choice of $\mu = 1$ was made in order to more easily relate our results to those of Cartwright, Littlewood and Levinson. The calculations carried out by Hayashi suggest that similar results to ours would be obtained for any fixed μ . Setting $\epsilon = 1/k$ and integrating the ^{van der Pol} equation once leads to the ^{equivalent} first order system

$$(1) \quad \begin{aligned} dy/dt &= (1/\epsilon)(y - y^3/3 - v) \\ dv/dt &= \epsilon y - B \cos t \end{aligned}$$

It is this system that we study here for $0 < B < .8, 0 < \epsilon < .2$.

Solutions of this system for various initial data $(y(0), v(0))$ define a transformation of the (y, v) -plane at $t = 0$ to the (y, v) -plane at $t = 2\pi$,

$$P : (y(0), v(0)) \rightarrow (y(2\pi), v(2\pi)).$$

This is the Poincaré mapping, and invariant sets for this mapping can describe periodic and almost periodic solutions of the equation. For example, a solution(subharmonic)having period $3 \cdot 2\pi$ will define a three point invariant set for P . If this subharmonic is stable, then a

neighborhood of any one of these points will approach the point under iterations of $P^3 = P \circ P \circ P$.

A given oscillatory solution can be described further by its rotation number. This is defined by

$$\rho = \lim (1/2n\pi) | \arg(P^n \xi) | ,$$

for any point $\xi = (y_0, v_0)$ lying on the invariant set of P corresponding to the solution. The notation $\arg(y_0, v_0)$ denotes the positive angle this vector makes with the positive y -axis. For example, if the oscillation is a subharmonic having period $3 \cdot 2\pi$, then its rotation number can be either $1/3$ or $2/3$. In general, the rotation number of an oscillatory solution can be rational or irrational. When ρ is rational, the denominator gives the period. Also, when ρ is rational, there may be several distinct solutions having that rotation number. For example, if $\rho = 1/3$, there is a three point set that is invariant under the Poincaré map, and through each one there is a solution having $\rho = 1/3$.

The rotation number was first introduced by Poincaré, and it has been used and studied in many contexts since then. It is known that the rotation number of a stable oscillation depends continuously on the parameters in the problem, and so it is constant over a neighborhood of parameter values. In our problem there are both stable and unstable oscillations present for most of the parameter values. Therefore, the system cannot be characterized by a single rotation number. Even when attention is restricted to stable oscillations, there need not be a unique rotation number since two stable oscillations can coexist.

Our methods consist of fixing values for B and ϵ , then selecting an initial point $(y(0), v(0))$. Next, we solve (1) using an implementation of Gear's code due to Hindmarsh[15]. This is a variable step, variable order numerical integrator. When this solution equilibrates (usually to six or seven place accuracy over several periods), its rotation number is easily calculated. Two techniques were used for selecting initial data. First, we allow ϵ and B to change systematically along a straight line in the (B, ϵ) -plane, with each new calculation using the last value from the previous one as an initial point. In this way, overlap regions were determined by passing over them in two different directions. The other technique involved fixing B and ϵ , and then calculating the rotation number for a sequence of initial data lying near the relaxation oscillation of the free problem.

2. Numerical evaluation of ρ and its implications.

Both kinds of calculations described in section 1 were performed for $0 < B < .8$ and $0 < \epsilon < .2$. The results of these calculations are summarized in Figure 1 by a contour mapping of ρ as a function of B and ϵ .

(FIGURE 1)

The free problem ($B = 0$) is described along the line $B = 0$ in Figure 1. Since the forcing frequency is 1 in the forced problem, the candidates for solutions that develop into subharmonics are those free oscillations whose periods are integer multiples of 2π . These occur at ρ values indicated in Table 1.

(TABLE 1)

These are also indicated on Figure 1, and we see that appropriate subharmonics are observed near these points for B near zero. The values in Table 1 come from evaluation of the first few terms in T given in section 0.

For positive values of B , we see that there are regions over which ρ is constant. The profile of ρ for $B = .1$, $.01 < \epsilon < .1$, is shown in Figure 2.

(FIGURE 2)

This indicates that ρ is a monotonically increasing function of ϵ which is constant over intervals. Such behavior is known in other problems [16], and it illustrates the well known phenomenon of phase locking, where the frequency of the response remains constant when parameters are changed slightly.

The profile of ρ for $B = .475$ is more complicated since discontinuities in ρ occur. Figure 3 indicates that ρ is continuous for $\epsilon > .04$, but that it is double valued over intervals below that.

(FIGURE 3)

The overlap intervals describe parameter values where there is not a unique stable response. The implications of overlaps will be discussed later.

We did not attempt to calculate irrational values of ρ . However, they presumably occur, and they correspond to almost periodic solutions of the equation that have quite small parameter sets over which they are stable.

An indication of the behavior of unstable solutions is given by the behavior of stable solutions near where they lose stability. In Figure 4, five examples are presented which show this. B is fixed at $B = .475$, and ϵ is decreased through the region where $\rho = 1/3$. A stable solution having period $3 \cdot 2\pi$ is plotted for each of several ϵ values by projecting the solution back onto the (y, v) -plane.

(FIGURE 4)

These show that as ϵ decreases, the solution comes successively closer to the free relaxation oscillation, at the same time layering itself over the two vertical segments.

3. Summary

The main motivation for this investigation is the study of stable oscillations in the forced van der Pol equation. ^{Since} this is a difficult problem to study analytically, we have calculated the rotation numbers of the stable responses of a van der Pol relaxation oscillator to sinusoidal forcing. The results suggest that the rotation number (ρ) defines a continuous, but piece-wise constant surface except in overlap regions where it is double valued, having what resemble folds.

The parameter ranges where ρ is single valued illustrate the phenomenon of phase locking; i.e., the response frequency is constant over a neighborhood of parameter values. Various phase locking regions of subharmonic and ultra-subharmonic responses are described by these calculations.

Calculations done near the start of overlaps show that the surface steepens as the overlap is approached. While the overlap regions might be described by simple folds in the surface with the upper and lower branches being stable, our methods give no indication of the actual behavior of the unstable solutions having rotation numbers between the two stable ones. In addition, there may be more than two stable oscillations having distinct ρ values coexisting, although we were able to detect only two stable responses in any overlap region. For example, stable solutions having high periods ($> 100 \cdot 2\pi$) and small domains of attraction would be beyond our calculations. Certainly, if the form of van der Pol's equation were modified, as in studies of multi-vibrators, then coexistence of more than two stable oscillations would be expected.

The overlap regions are quite interesting. Since stable oscillations having different frequencies coexist near the free relaxation oscillation, their domains of attraction must be intertwined in a very complicated way. In particular, the system's behavior is very sensitive to small perturbations. Presumably the random behavior suggested by Levinson is to be found among the solution set having initial values on the boundary of the domains of attraction.

Finally, our calculations are consistent with the results established by Cartwright and Littlewood. Their results can be demonstrated in Figure 1 by fixing ϵ at a small value, say .01, and then observing the ρ values as B changes. They show that for sufficiently small ϵ , the interval $0 < B < .8$ can be broken up into three disjoint sets. One consists of intervals over which ρ is single valued, another consists of intervals over which ρ is double valued, and the third is a small set that is ignored. The intervals of single and double values of ρ are clear in Figure 1, and the remainder consists of sets where ρ takes on irrational values.

From another point of view, the profile of the ρ surface, ^{Say} for B fixed and ϵ decreasing, can be interpreted as a bifurcation diagram where only stable branches are plotted. For example, in Figure 2 as ϵ decreases from 0.1, the solution having period $3 \cdot 2\pi$ loses its stability with the appearance of other oscillations. Presumably the periodic solution is still present for smaller values of ϵ , but it is no longer stable. Therefore, if Figure 2 were viewed as being a bifurcation diagram, then each point on the graph should be extended as a straight line to $\epsilon = 0$. If the unstable branches are included, then the diagram will be solid above the lowest stable value since periodic solutions presumably persist for decreasing ϵ .

Other interesting oscillators can be constructed using information compiled here. For example, if B is allowed to vary slowly back and forth across an overlap region, then the solution would have one ρ value until the overlap boundary is crossed, when it would equilibrate rapidly to a solution having the other ρ value. Then as B reverses, the opposite happens. On the other hand, if B and ξ vary slowly in other regions, the solution could appear to be chaotic.

Some indication of the unstable solutions was obtained here from calculated solutions for parameter values near where they lose their stability. This indicates that the solutions are layered near the free relaxation oscillation, suggesting an onion skin configuration reminiscent of that found in [14] for a two dimensional strange attractor.

We have not attempted in this paper to rigorously derive any of the results suggested by our calculations. However, the calculations do give an indication of the nature of stable responses in this system, which pose difficult analytic problems.

TABLE 1

Protosubharmonics: Free oscillations($B = 0$) having period $2\pi n$.

n	ξ
2	.16
3	.095
4	.069
5	.054
6	.044
7	.037
8	.033
9	.029
10	.026
11	.024
12	.022

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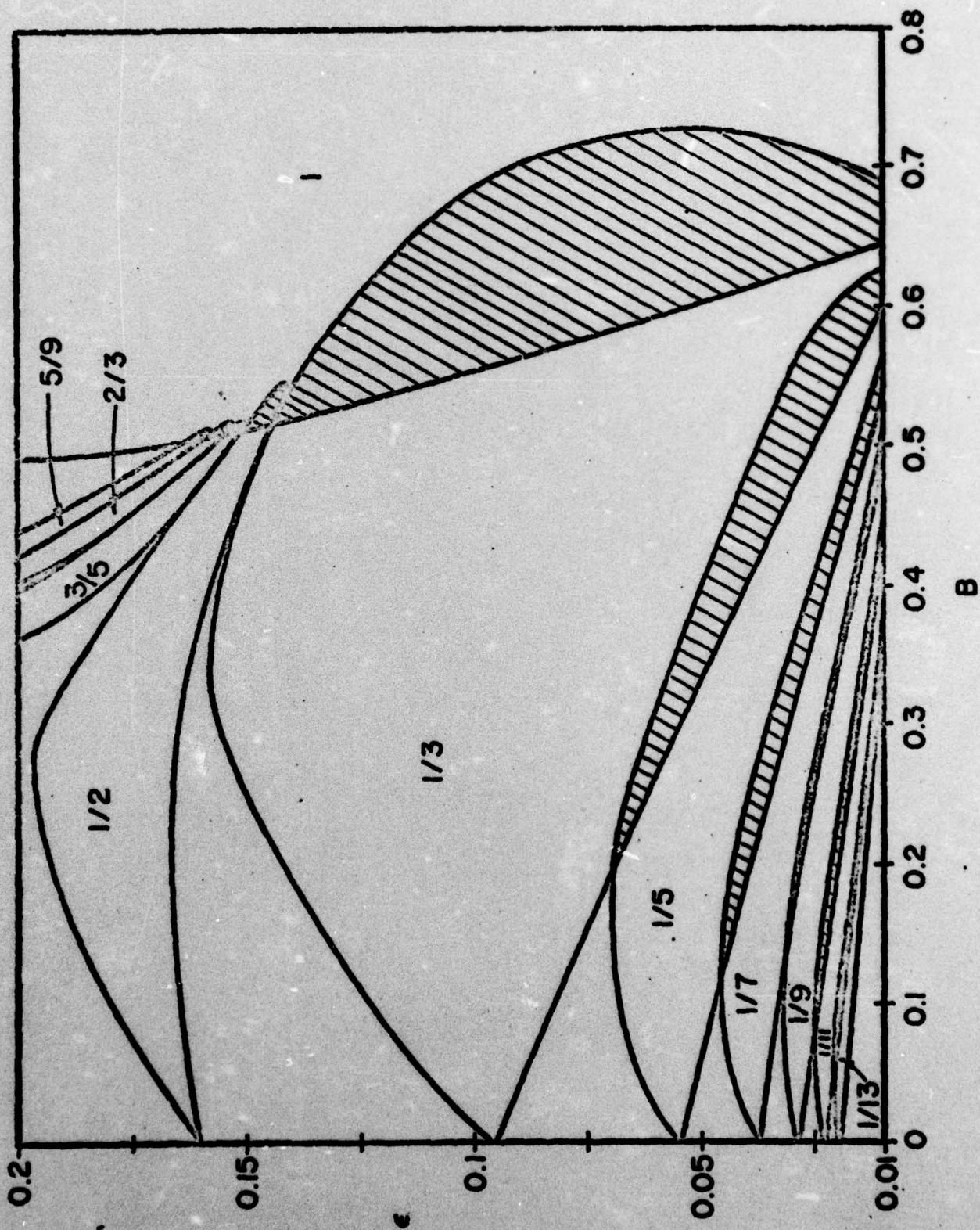


Figure 1

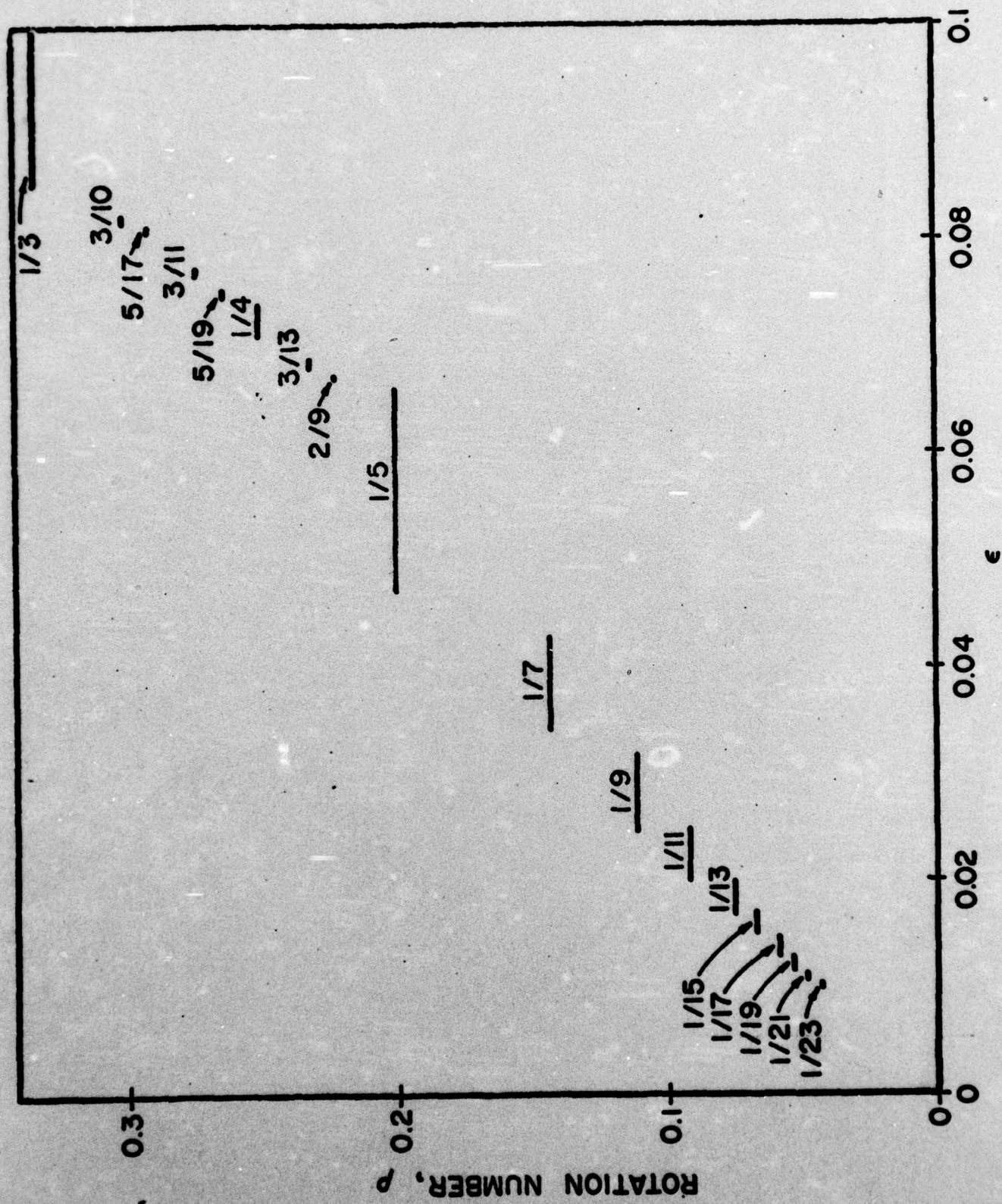


Figure 2

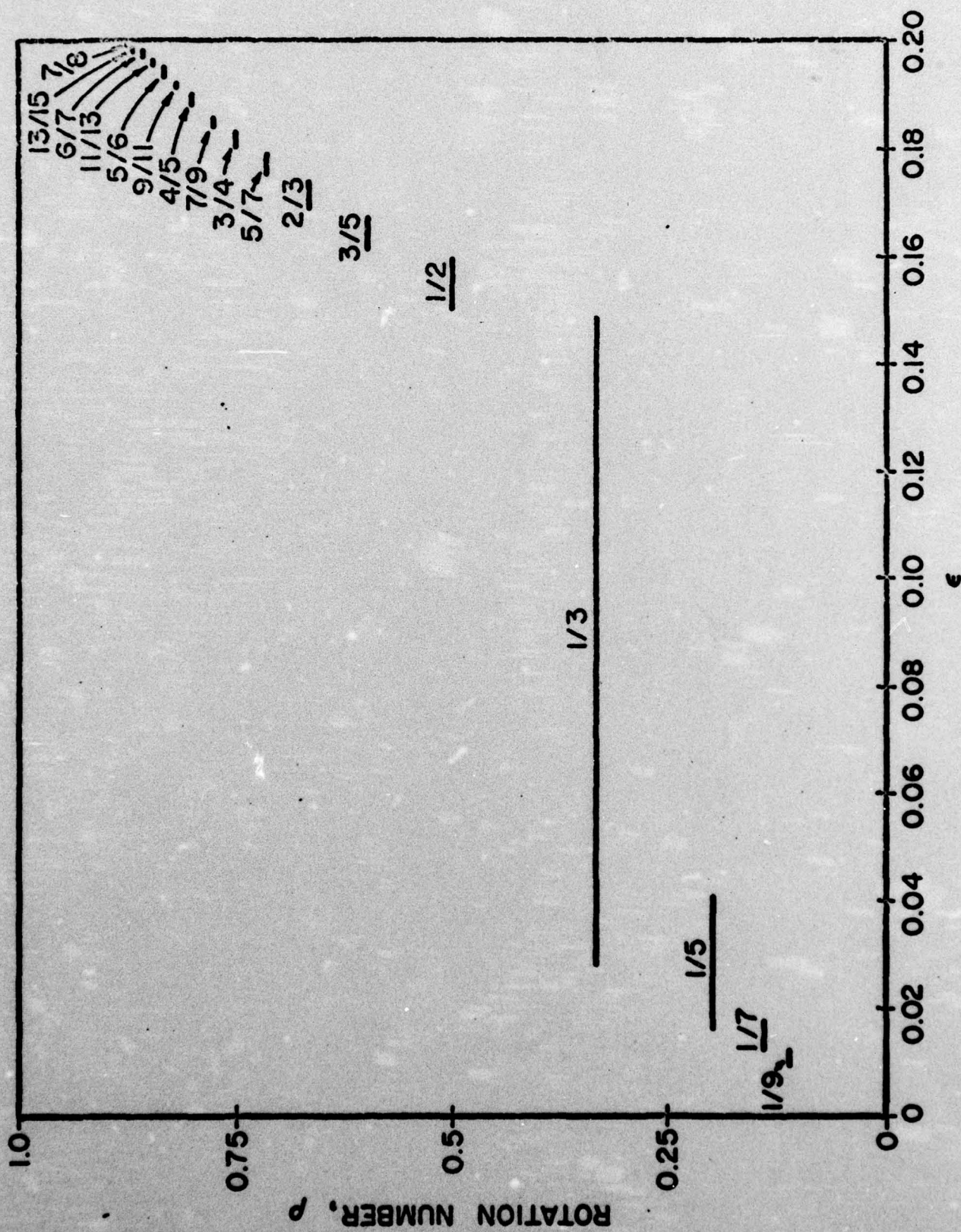
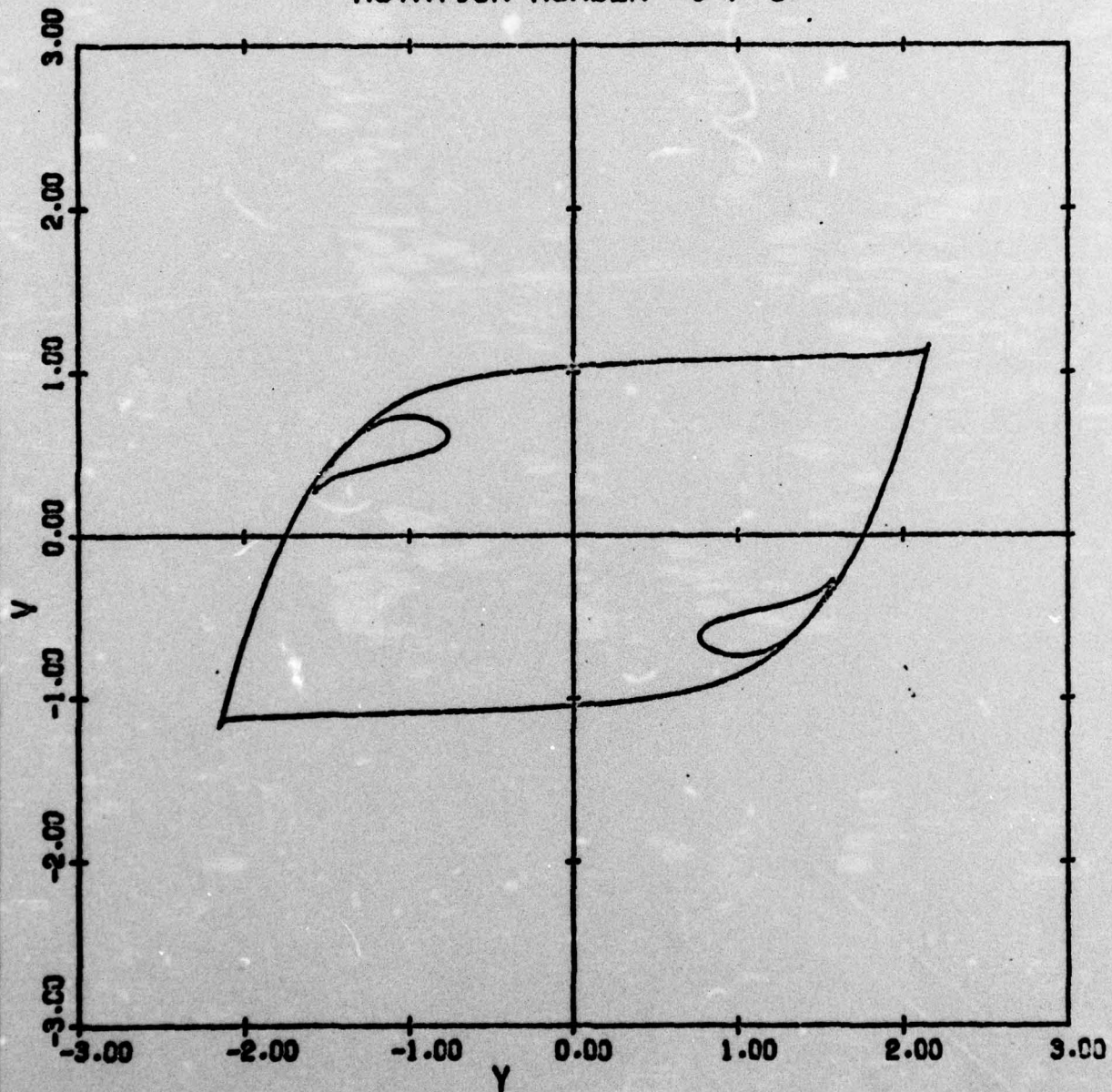
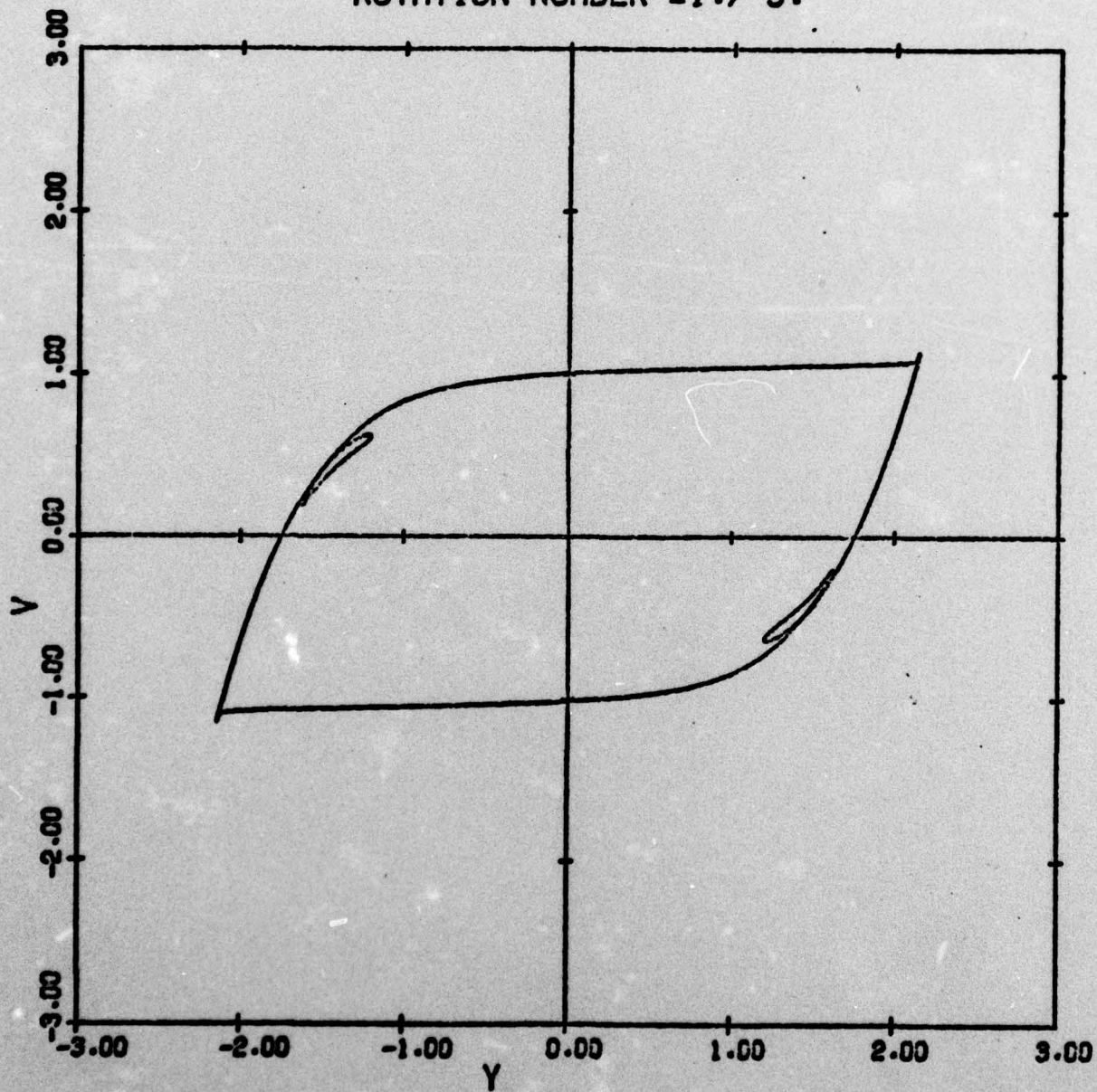


FIGURE 3

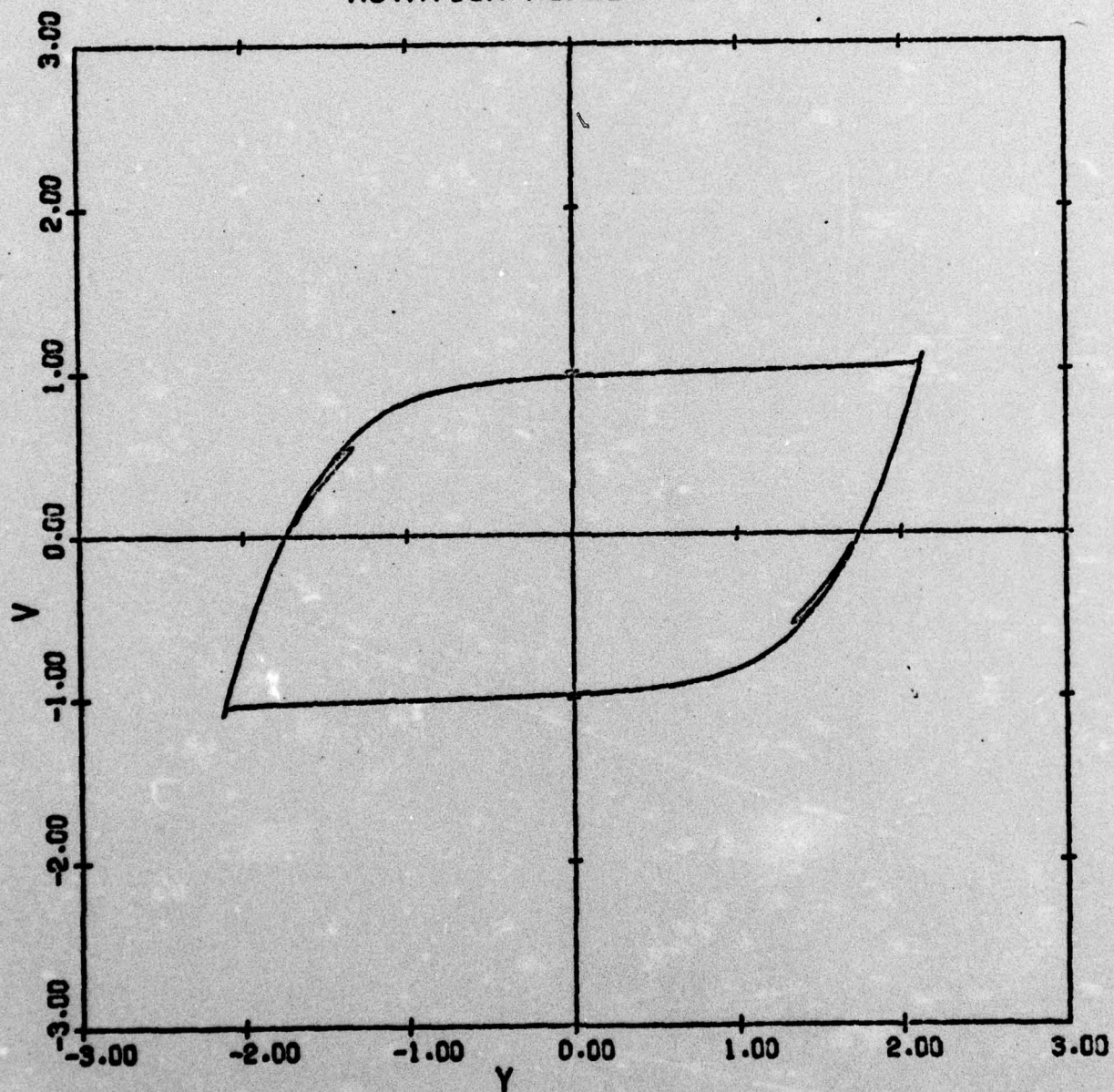
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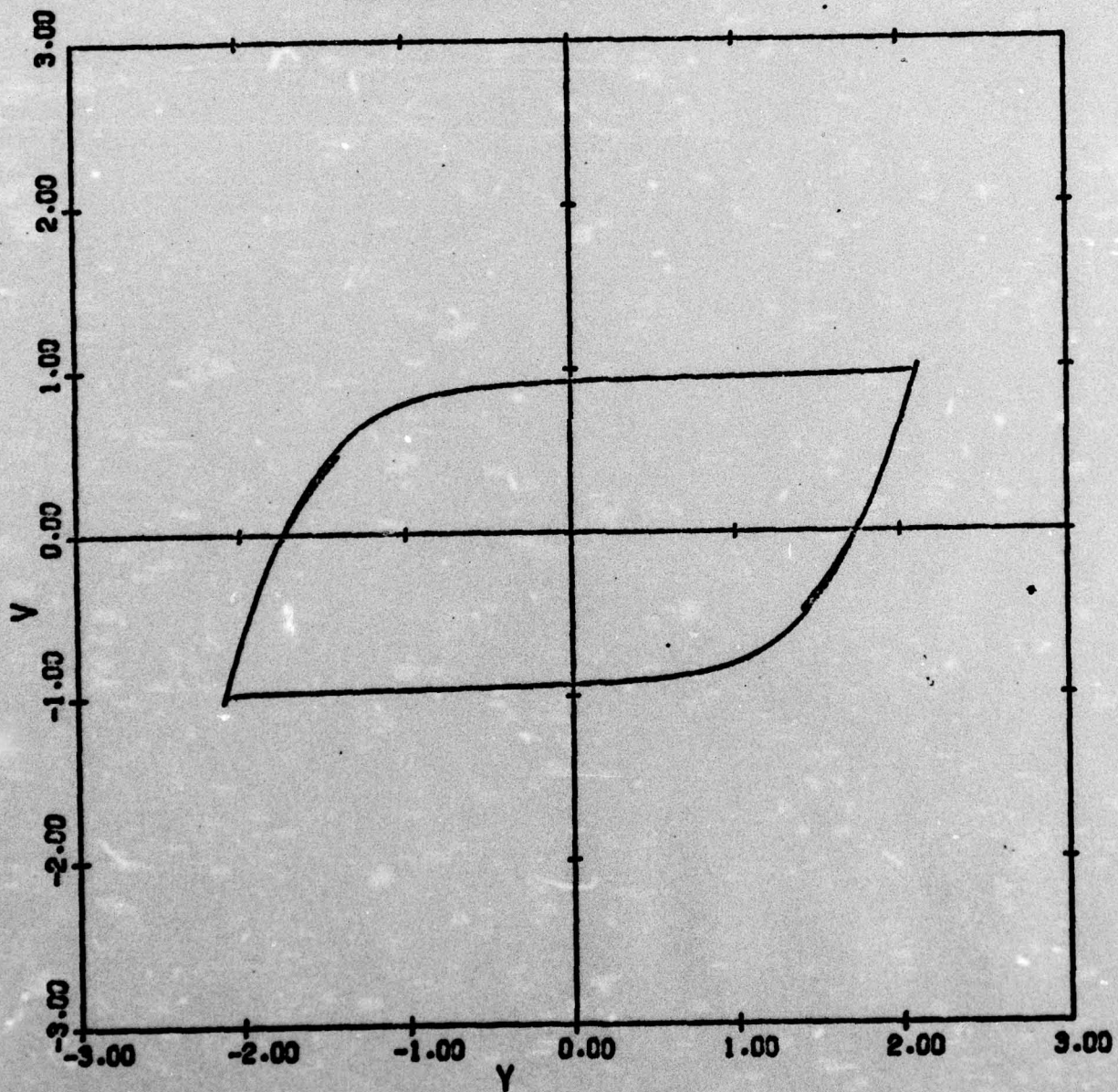
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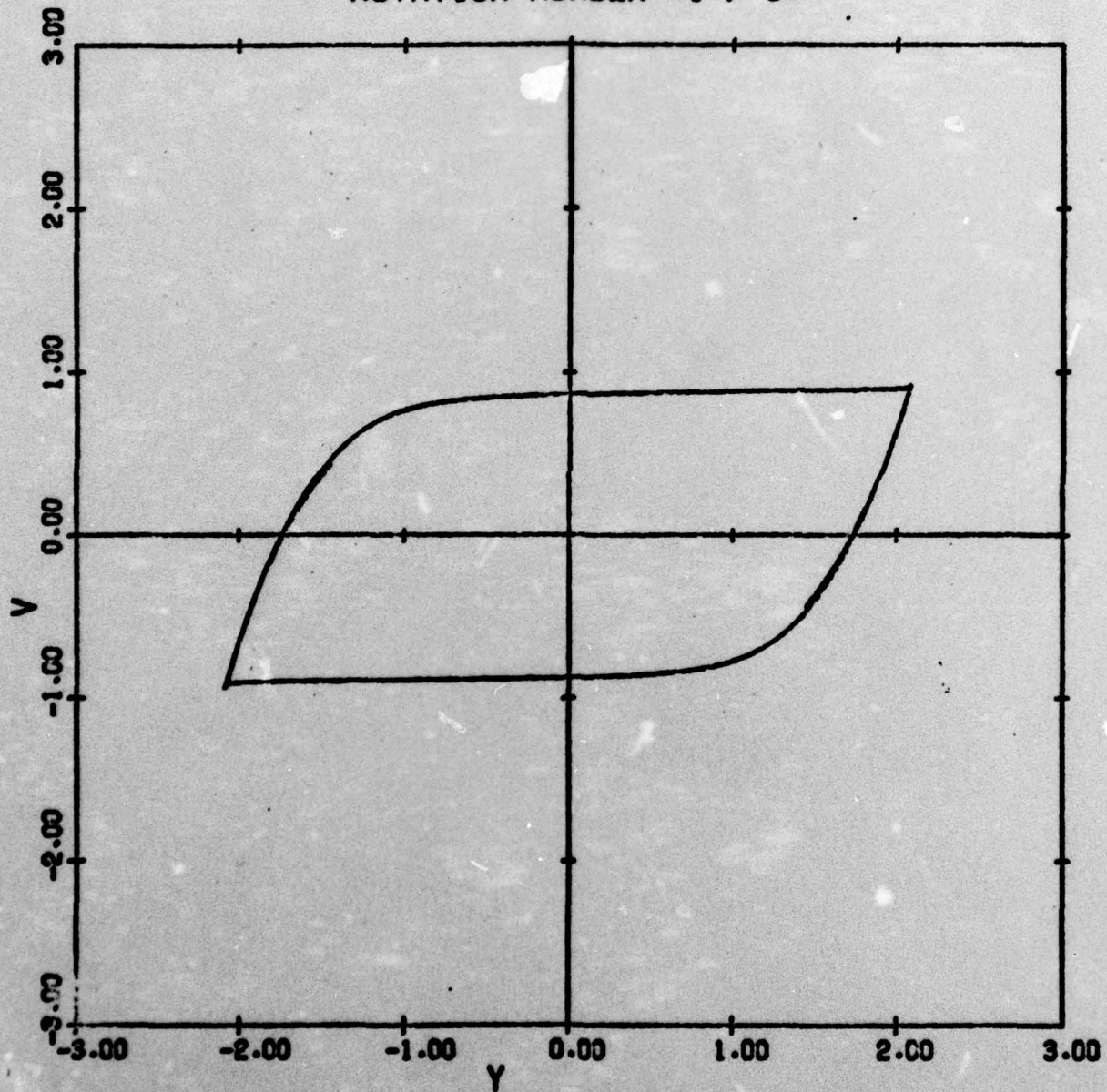
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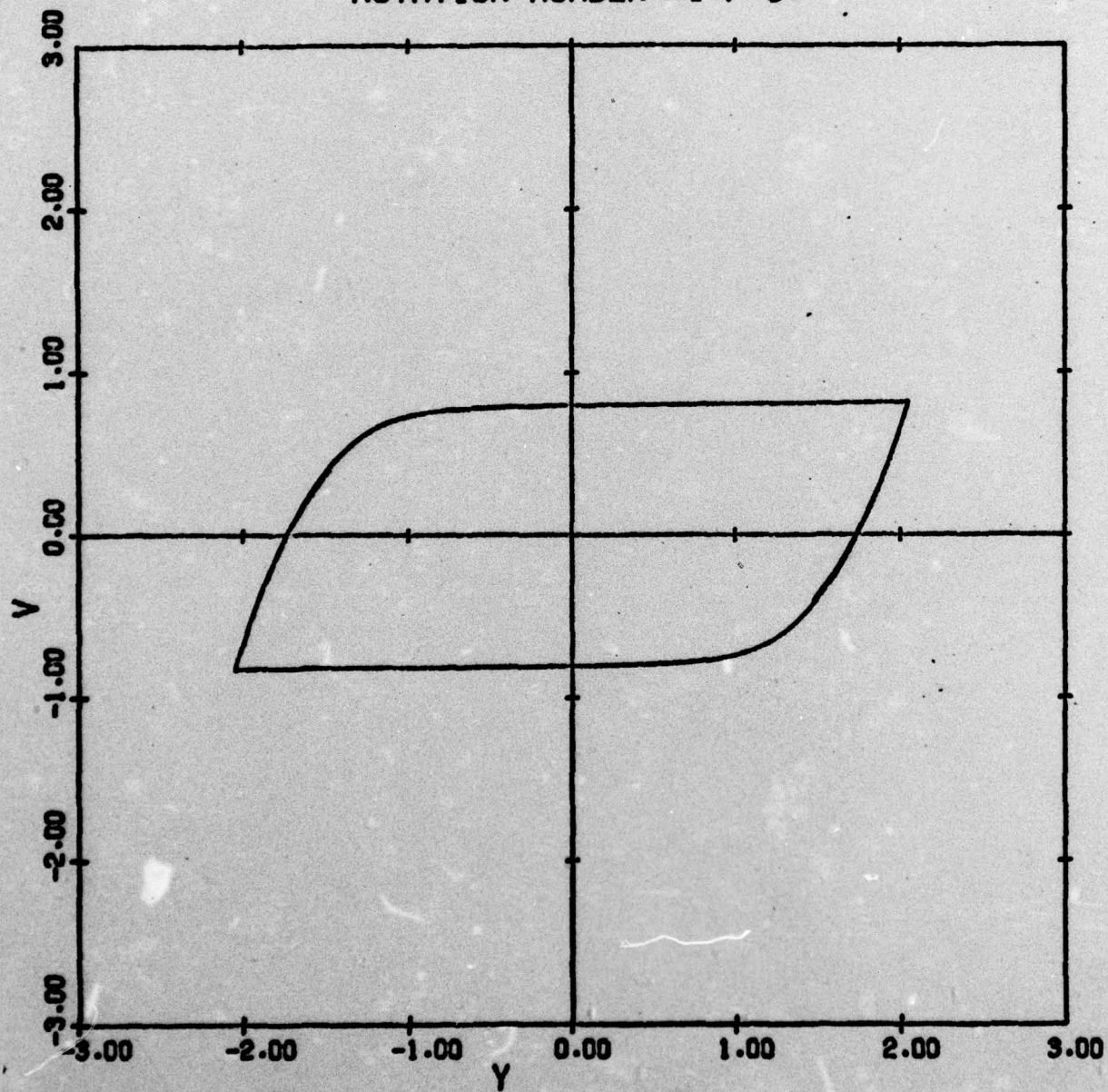
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are obtained
amplitude and oscillator tuning, we obtain descriptions of regions of phase locking, successive bifurcation of stable subharmonic and almost periodic oscillations, and overlap regions where two distinct stable oscillations can coexist.
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